MATH2101 A3 Shen Hongshan 3036290936 Engineering BEng (DS&E)

1.
$$\left(\begin{array}{ccccc|c} -1 & 2 & 1 & -1 & 3 & 7 \\ 1 & -4 & 0 & 5 & -5 & -7 \\ -2 & 6 & -1 & -5 & 7 & 6 \\ -1 & -4 & 4 & 11 & -2 & 11 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & -2 & -1 & 1 & -3 & -7 \\ 0 & \cancel{1} & \cancel{-1/2} & \cancel{-2} & \cancel{1} & 0 \\ 0 & 2 & -2 & -3 & 1 & -8 \\ 0 & -6 & 3 & 12 & -5 & 4 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & -2 & -1 & 1 & -3 & -7 \\ 0 & 1 & -1/2 & -2 & 1 & 0 \\ 0 & 0 & +1 & -1 & +1 & +8 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & -2 & -1 & 1 & 0 & 5 \\ 0 & 1 & -1/2 & -2 & 0 & -4 \\ 0 & 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -4 & 0 & 1 \\ 0 & 1 & 0 & -\frac{9}{4} & 0 & -3 \\ 0 & 0 & 1 & -1/2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right) \quad \begin{cases} x_4 = t \\ x_1 = 1+4t \\ x_2 = -3+\frac{9}{4}t \\ x_3 = 2+\frac{1}{2}t \\ x_5 = 4 \end{cases} \Rightarrow x \in \left\{ \left(t, \left(-3+\frac{9}{4}t\right), 2+\frac{1}{2}t, t, 4\right)^T \middle| t \in \mathbb{R} \right\}$$

2.
$$\det\begin{pmatrix} c & 0 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2\begin{vmatrix} c & 0 & 2 \\ 1 & 2 & 3 \\ 2 & 4 & 8 \end{vmatrix} = 2\cdot\begin{vmatrix} c & 0 & 2 \\ 1 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 4(2c) = 8c$$

① $c \neq 0$. determinent $\neq 0 \Rightarrow$ RREF $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

② $c = 0$
$$\begin{pmatrix} 0 & 0 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 0 & 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. $\underset{n\times m}{A}$ $\underset{n\times 1}{b}$

(a) True

$Ax=b$ has m leading variable $\Rightarrow$ REF of A has m leading variable 0 free variable, $m \geq n$

if $Ax=b$ consistent, then all-zero row placed at the bottom (REF) provides no information, thus could be dismissed.

Let $\underset{m\times m}{D}$ be the matrix by canceling bottom $(m-n)$ 0 rows of REF(A)

$\underset{m\times 1}{d}$ be the column vector by canceling bottom $(m-n)$ 0-entries of b.

$\Rightarrow \underbrace{D}_{\text{full rank} \Leftrightarrow \text{invertible}} x = d \Rightarrow x = \underbrace{D^{-1}d}_{\text{all operation is } +/*}$ original entries are integer

$\Rightarrow$ resulting entries are rational

(b) False

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$ is one possible solution with irrational entry.

4(a) Let $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{pmatrix}$ $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}$ where $x = \frac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{-a_{22} + \frac{a_{21}}{a_{11}} a_{12}}$

then $E_3 E_2 E_1 A$ is in echelon form in order words upper triangular matrix.

Let $\boxed{U = E_3 E_1 E_2 A}$

$$E_3 E_2 E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} - \frac{x a_{21}}{a_{11}} & x & 1 \end{pmatrix}$$ is lower triangular

1° Let $\underset{3\times 3}{M}$ be a lower triangular matrix. & invertible

$M = \begin{pmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$

$\det(M) \neq 0$

then $(M^{-1})_{12} = \frac{\det(\widetilde{M_{21}})}{\det(M)} = 0$

$(M^{-1})_{13} = \frac{\det(\widetilde{M_{31}})}{\det(M)} = 0$

$(M^{-1})_{23} = \frac{\det(\widetilde{M_{32}})}{\det(M)} = 0$

$\Rightarrow M^{-1}$ has the form $\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$ is lower triangular

i.e. M lower triangular $\Rightarrow M^{-1}$ lower triangular

2° similary, we can prove that

M upper triangular $\Rightarrow M^{-1}$ upper triangular

$E_3 E_2 E_1 A = U$

Let $\boxed{L = E_1^{-1} E_2^{-1} E_3^{-1} = (E_3 E_2 E_1)^{-1}}$

(by 1° $\Rightarrow$ L is lower triangular matrix.)

Let $P = I_3$.

then $A = UPL$ holds. □

4.(b) $A = I$

① $U = P = L = I$ check $A = UPL$ true

② $P = I$ $U = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $L = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$UPL = UIL = UL = I = A$ true

$\Rightarrow$ Not unique