THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

May 16, 2015 2:30 pm – 5:00 pm

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all SIX questions. Each question carries 16 marks.
Another 4 marks is given for clarity of presentation and following instructions.

Notes:

- You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**
- You must start each question on a new page. However do NOT start each part of a question on a new page. You should write down the question number on the top right hand corner of each page. Indicate clearly which part of a question you are answering.
- If your answer to a question spans over more than one page, you must indicate clearly on each page (except the last) that the answer will continue on the subsequent page.
- If you do not otherwise specify, you are assumed to be answering the questions in order. In particular, indicate on each question where your answer to the next question is if it does not follow immediately. (For example you may write 'turn back 3 pages for Question 5' at the end of Question 4.)
- Whenever there is a blank page it will be assumed that all subsequent pages are also blank. If this is not true, indicate clearly on the page which you would otherwise leave blank.

1. Give answers only to the following questions. Explanation is <u>not</u> required.

   (a) Compute $\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

   (b) Solve the following system of equations:

$$\begin{cases} x + 2y = 1 \\ 3x + 2y + 4z = 7 \\ -2x - 4z = -6 \end{cases}$$

   (c) If the null space of a 7×11 matrix has dimension 5, what is the dimension of its column space?

   (d) Compute the determinant of the matrix $\begin{bmatrix} 2 & 0 & 1 & 5 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$.

   (e) If the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ has determinant 1, what is the determinant of the matrix below?

$$\begin{bmatrix} a & b & c \\ -d & -e & -f \\ 2a+g & 2b+h & 2c+i \end{bmatrix}^{-1}$$

   (f) Find all real numbers c for which the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix} \right\}$ is linearly dependent.

   (g) Find a basis for the subspace $\{(k, k, k)^T : k \in \mathbb{R}\}$ of $\mathbb{R}^3$.

   (h) Find a basis for the row space of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

2. In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do <u>not</u> indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is <u>not</u> required.

(a) Which of the following is possible for a system of 3 linear equations in 4 unknowns?

(1) There is no solution.

(2) There is exactly one solution.

(4) There are exactly two solutions.

(8) There are infinitely many solutions.

(b) Which of the following is a linear combination of the vectors $(1, 2, 0)^T$ and $(0, 1, 2)^T$?

(1) $(0, 0, 0)^T$

(2) $(1, 3, 2)^T$

(4) $(1, 0, -4)^T$

(8) $(1, 0, 2)^T$

(c) Which of the following matrices are in reduced row echelon form?

(1) The 3×4 zero matrix

(2) The 5×5 identity matrix

(4) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(8) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

(d) For which of the following values of t is the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & -1 & t^2 \end{bmatrix}$ invertible?

(1) $t = 1$

(2) $t = -1$

(4) $t = \sqrt{3}$

(8) $t = -\sqrt{3}$

(e) Which of the following matrices is similar to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & -3 & -5 \end{bmatrix}$?

(1) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & -3 & -5 \end{bmatrix}$

(4) $\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & -3 & -5 \end{bmatrix}$

(8) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

(f) Let $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$ be vectors in a subspace U of $\mathbb{R}^3$. Which of the following must be in U?

(1) $(0, 0, 0)^T$

(2) $(x_1 + 1, x_2 + 1, x_3 + 1)^T$

(4) $(x_1y_1, x_2y_2, x_3y_3)^T$

(8) $(2x_1 - y_1, 2x_2 - y_2, 2x_3 - y_3)^T$

(g) Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ be a generating set for $\mathbb{R}^3$, and i, j, k be distinct elements from the set $\{1, 2, 3, 4\}$. Which of the following must be true?

(1) There exist choices of i, j such that the set $\{\mathbf{x}_i, \mathbf{x}_j\}$ is linearly dependent.

(2) For any choice of i, j, the set $\{\mathbf{x}_i, \mathbf{x}_j\}$ is linearly dependent.

(4) There exist choices of i, j, k such that $\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\}$ is a generating set for $\mathbb{R}^3$.

(8) For any choice of i, j, k, the set $\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\}$ is a generating set for $\mathbb{R}^3$.

(h) Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ be a linearly dependent set in $\mathbb{R}^4$, and i, j, k be distinct elements from the set $\{1, 2, 3, 4\}$. Which of the following must be true?

(1) There exist choices of i, j such that the set $\{\mathbf{x}_i, \mathbf{x}_j\}$ is linearly dependent.

(2) For any choice of i, j, the set $\{\mathbf{x}_i, \mathbf{x}_j\}$ is linearly dependent.

(4) There exist choices of i, j, k such that the set $\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\}$ is linearly dependent.

(8) For any choice of i, j, k, the set $\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\}$ is linearly dependent.

3. For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

(a) If a square matrix A is its own inverse (i.e. $A = A^{-1}$), then $A = \pm I$.

(b) If A is a square matrix, then the block matrix $\begin{bmatrix} A & O \\ O & A \end{bmatrix}$ must be symmetric.

(c) If the row space of A is equal to the column space of A, then A is a symmetric matrix.

(d) There exists no matrix A with real entries such that $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$.

(e) If A is a 2×2 matrix whose only eigenvalue is 1 with multiplicity 2, then A must be the identity matrix.

(f) The set $\{(x, y, z)^T : x, y, z \in \mathbb{R},\ x + y + z \geq 0\}$ is a subspace of $\mathbb{R}^3$.

(g) If two matrices have the same characteristic polynomial, then they must be similar.

(h) It is possible for a matrix to have an eigenvalue with algebraic multiplicity 3 and geometric multiplicity 1.

4. Let $P = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$.

(a) Is P diagonalisable?

(b) Show that for any positive integer n, there exist unique real numbers a_n and b_n such that $P^n = a_n P + b_n I$.

(Suggestion: Find a_1, b_1, a_2 and b_2 first.)

(c) For positive integer n, find P^n in terms of n.

5. For real number m, let $T_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation corresponding to projection onto the line $y = mx$, and A_m denote the standard matrix of T_m.

(a) Find the following matrices.

(i) A_1

(ii) The matrix representation of T_2 with respect to the ordered basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

(b) Prove that the following statements hold for all real numbers m.

(i) The matrix representation of T_m with respect to any ordered basis of $\mathbb{R}^2$ is non-invertible.

(ii) There exists a linearly independent subset $S \subseteq \mathbb{R}^2$ for which $T_m(S)$ is linearly dependent.

(iii) A_0 is similar to A_m.

6. In the game *Lights Out*, there is a 5×5 square grid, and inside each cell there is a light bulb. Initially, some bulbs are on. Each light bulb comes with a switch, and when the switch is pressed, it changes the state (i.e. from on to off, or from off to on) of the light bulb as well as those of its neighbours. Two cells are neighbours if they share a common side. For example:

- if the button in the $(1,1)$-cell is pressed, the bulbs in the $(1,1)$-, $(1,2)$- and $(2,1)$-cells are affected; while
- if the button in the $(3,3)$-cell is pressed, the bulbs in the $(3,3)$-, $(3,2)$-, $(3,4)$-, $(2,3)$- and $(4,3)$-cells are affected.

The purpose of the game, as its name reflects, is to press the buttons so that all the bulbs can be turned off.

Apply your knowledge in linear algebra to model the game. (Instead of working on $\mathbb{R}$, you may wish to work on the set $\{0,1\}$ instead — explain why and explain how the arithmetic on this set works.) Then, apply your model to analyse the modified versions of the game where the grid is 2×2 and 3×3 (instead of the original 5×5, as it is too large for computations without a computer). You may wish to touch upon points like how to find a way to win the game (i.e. to turn all bulbs off), whether one can always win with any initial configuration, and so on.

$* * * * * * *$ *End of Paper* $* * * * * * *$