

THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

May 16, 2018

2:30 – 5:00 pm

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all SIX questions. Each question carries 16 marks.

Another 4 marks is given for clarity of presentation and following instructions.

Notes:

- You should give precise and adequate explanations to support your conclusions unless otherwise specified by the question. Clarity of presentation of your argument counts. So **think carefully before you write.**
- You must start each question on a new page. However do NOT start each part of a question on a new page. You should write down the question number on the top right hand corner of each page. Indicate clearly which part of a question you are answering.
- If your answer to a question spans over more than one page, you must indicate clearly on each page (except the last) that the answer will continue on the subsequent page.
- If you do not otherwise specify, you are assumed to be answering the questions in order. In particular, indicate on each question where your answer to the next question is if it does not follow immediately. (For example you may write 'turn back 3 pages for Question 5' at the end of Question 4.)
- Whenever there is a blank page it will be assumed that all subsequent pages are also blank. If this is not true, indicate clearly on the page which you would otherwise leave blank.

1. Give answers only to the following questions. Explanation is not required.

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & 5 \\ 1 & 2 & -6 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Let also \mathbf{a}_i and \mathbf{b}_i denote the i -th column of A and B respectively.

- (a) Compute $A^T \mathbf{e}_2$.
 - (b) Find a vector \mathbf{v} for which $[\mathbf{v}]_A = \mathbf{x}$.
 - (c) Find $\mathbf{a}_1 \cdot \mathbf{a}_2 + \mathbf{a}_2 \cdot \mathbf{a}_3 + \mathbf{a}_3 \cdot \mathbf{a}_1$.
 - (d) If $\mathbf{x} = c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + c_3 \mathbf{a}_3$ where c_1, c_2 and c_3 are scalars, what is c_1 ?
 - (e) Find $\det B$.
 - (f) What is the nullity of B ?
 - (g) It is known that \mathbf{y} is a λ -eigenvector of B . Find the value of λ .
 - (h) By applying the Gram-Schmidt process on $\{\mathbf{b}_1, \mathbf{b}_2\}$, one gets $\{\mathbf{b}_1, \mathbf{w}\}$. What is \mathbf{w} ?
2. In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do not indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is not required.
- (a) Let $C = AB$. If the first three rows of A are linearly dependent, the first three columns of A are linearly dependent, the last three rows of B are linearly dependent and the last three columns of B are linearly dependent, which of the following must be linearly dependent?
 - (1) The first three rows of C
 - (2) The first three columns of C
 - (4) The last three rows of C
 - (8) The last three columns of C

(b) An $n \times n$ elementary matrix must

- (1) be symmetric
- (2) be triangular
- (4) have rank n
- (8) have positive determinant

(c) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation. Which of the following is true?

- (1) If $T \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, then $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} y \\ x \end{bmatrix}$ for all $x, y \in \mathbb{R}$.
- (2) If $T \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $T \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, then $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ for all $x, y \in \mathbb{R}$.
- (4) If $T \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $T \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, then $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$ for all $x, y \in \mathbb{R}$.
- (8) If $T \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $T \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$, then $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} -y \\ x \end{bmatrix}$ for all $x, y \in \mathbb{R}$.

(d) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation corresponding to reflection across a line L passing through the origin. Which of the following must be true?

- (1) The standard matrix of T has determinant 1.
- (2) The \mathcal{B} -matrix of T has determinant 1 for any ordered basis \mathcal{B} of \mathbb{R}^2 .
- (4) 1 is an eigenvalue of T .
- (8) -1 is an eigenvalue of T .

(e) Let A be a square matrix and B be row equivalent to A . Then A and B have the same

- (1) rank
- (2) row space
- (4) determinant
- (8) eigenvalues

(f) Let A be a square matrix. Which of the following is a sufficient condition for A to be diagonalisable?

- (1) $\det A = 1$
- (2) A is an elementary matrix
- (4) A is a triangular matrix
- (8) A is a diagonal matrix

(g) Let \mathbf{u} and \mathbf{v} be vectors in \mathbb{R}^n , and c be a scalar. Which of the following must be true?

- (1) $\mathbf{u} \cdot (c\mathbf{v}) = \mathbf{v} \cdot (c\mathbf{u})$
- (2) $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$
- (4) $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$
- (8) $\|c\mathbf{u}\| = c\|\mathbf{u}\|$

(h) Which of the following subspaces of \mathbb{R}^n (where $n > 1$) must have an orthonormal basis?

- (1) \mathbb{R}^n
- (2) The null space of an invertible $n \times n$ matrix
- (4) The orthogonal complement of a set of $n - 1$ vectors in \mathbb{R}^n
- (8) The span of $n - 1$ distinct vectors in \mathbb{R}^n

3. For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

- (a) If the system $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{c}$ are both consistent, then $A\mathbf{x} = \mathbf{d}$ is consistent for any linear combination \mathbf{d} of \mathbf{b} and \mathbf{c} .
- (b) If $\{\mathbf{u}, \mathbf{v}\}$ and $\{\mathbf{u}, \mathbf{w}\}$ are both linearly dependent subsets of \mathbb{R}^n , so is $\{\mathbf{v}, \mathbf{w}\}$.
- (c) Let A and B be $n \times n$ matrices, where $n > 1$. If $MA = MB$ for any non-invertible $n \times n$ matrix M , then $A = B$.

- (d) Let A be a lower triangular square matrix with LU decomposition $A = LU$. Then $U = I$.
- (e) There exists a non-zero matrix A such that $A^3 = 8A$ but $A^2 \neq 8I$.
- (f) If \mathbf{u} and \mathbf{v} are nonzero orthogonal vectors in \mathbb{R}^n , then \mathbf{u} and $\mathbf{u} + \mathbf{v}$ must not be orthogonal.
- (g) For any subspace V of \mathbb{R}^n , we have $(V^\perp)^\perp = V$.
- (h) If $A, B \subseteq \mathbb{R}^n$ and $A^\perp = B^\perp$, then $A = B$.

4. Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 12 & 2 & -6 & 0 \\ 4 & 0 & 0 & 0 \\ 6 & 3 & -9 & -1 \end{bmatrix}$.

- (a) Find a basis for each of Row A , Col A and Null A .
- (b) Using one of the results of (a), find the general solution to the system

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 3 \end{bmatrix}.$$

Justify your answer carefully (in particular you should demonstrate how one of the results of (a) helped you find the answer).

- (c) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be a linear transformation such that $T(\mathbf{x})$ is the vector in Col A for which

$$\|T(\mathbf{x}) - \mathbf{x}\|$$

is minimised. Find the standard matrix of T . (To avoid clumsy computations, you may leave your answer as the product and/or inverse of some matrices instead of evaluating it.)

5. Let A be a matrix. Consider the following four statements:

- (1) The reduced row echelon form of A has no zero row.
- (2) The system $A\mathbf{x} = \mathbf{0}$ only has the trivial solution.
- (3) $A^T A$ is invertible.
- (4) The eigenvalues of $A^T A$ are all positive real numbers.

Three of the above statements are equivalent. Pick (with justifications) the odd one out, and show that the other three statements are equivalent.

6. The **trace** of a square matrix A , denoted by $\text{tr } A$, is the sum of the diagonal entries of A .

- (a) If A and B are square matrices of the same size, show that $\text{tr}(AB) = \text{tr}(BA)$.
- (b) Using (a), show that similar matrices have the same trace.
- (c) Is the converse of (b) true?
- (d) Show that $\text{tr } A$ is equal to the sum of the eigenvalues of A (counting multiplicities).

(Hint: You may use the formula

$$\det A = \sum_{\sigma \in S(n)} \text{sgn}(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

discussed in Tutorial 6 as the definition of determinant.)

***** END OF PAPER *****