THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

May 22, 2017 9:30 am – 12:00 noon

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all SIX questions. Each question carries 16 marks.
Another 4 marks is given for clarity of presentation and following instructions.

Notes:

- You should give precise and adequate explanations to support your conclusions unless otherwise specified by the question. Clarity of presentation of your argument counts. So **think carefully before you write.**
- You must start each question on a new page. However do NOT start each part of a question on a new page. You should write down the question number on the top right hand corner of each page. Indicate clearly which part of a question you are answering.
- If your answer to a question spans over more than one page, you must indicate clearly on each page (except the last) that the answer will continue on the subsequent page.
- If you do not otherwise specify, you are assumed to be answering the questions in order. In particular, indicate on each question where your answer to the next question is if it does not follow immediately. (For example you may write 'turn back 3 pages for Question 5' at the end of Question 4.)
- Whenever there is a blank page it will be assumed that all subsequent pages are also blank. If this is not true, indicate clearly on the page which you would otherwise leave blank.

1. Give answers only to the following questions. Explanation is not required.

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -2 & 12 \\ 0 & 2 & -10 \end{bmatrix}$.

(a) Find the reduced row echelon form of A.

(b) Find an LU decomposition of A.

(c) Find the characteristic polynomial of A.

(d) If T denotes the linear transformation induced by A, what is the range of T?

(e) Let B be the 4×4 matrix obtained from A by appending a row of 1's at the bottom and a column of 1's on the right. What is $\det B$?

(f) What is $(\operatorname{Null} A)^{\perp}$?

(g) If $\mathcal{A}$ denotes the ordered basis of $\mathbb{R}^3$ formed by the columns of A, what is $[\mathbf{e}_1]_{\mathcal{A}}$?

(h) With T and $\mathcal{A}$ defined as above, what is $[T]_{\mathcal{A}}$?

2. In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do not indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is not required.

(a) Let A and B be invertible square matrices of the same size. Which of the following must be true?

(1) $AB = BA$

(2) $A^2 - B^2 = (A+B)(A-B)$

(4) $(AB)^{-1} = (BA)^{-1}$

(8) If $AB = I$, then $BA = I$.

(b) Let A be a matrix and R be its reduced row echelon form. Which of the following must be equal to $\operatorname{rank} A$?

(1) The number of non-zero rows in R

(2) The number of non-zero columns in R

(4) The maximum number of linearly independent rows in A

(8) The maximum number of linearly independent columns in A

(c) Let A be an $m \times n$ matrix. If the system $A\mathbf{x} = \mathbf{b}$ has a unique solution, which of the following is possible?

(1) $m < n$

(2) $m = n$ and A is invertible.

(4) $m = n$ and A is non-invertible.

(8) $m > n$

(d) Let A be an $m \times n$ matrix where $m \neq n$. Which of the following is a subspace of $\mathbb{R}^m$?

(1) The row space of A

(2) The column space of A

(4) The null space of A

(8) An eigenspace of A

(e) Let W be a subspace of $\mathbb{R}^4$. Which of the following is possible?

(1) $W^\perp$ is empty.

(2) $W^\perp$ is a non-empty finite set.

(4) $W^\perp$ is not a subspace of $\mathbb{R}^4$.

(8) $W^\perp$ has the same dimension as W.

(f) Let A be a square matrix. Which of the following implies A is invertible?

(1) $\det A \neq 0$

(2) 0 is not an eigenvalue of A.

(4) The system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

(8) The reduced row echelon form of A has no zero row.

(g) Let $S \subseteq T \subseteq \mathbb{R}^n$. Which of the following must be true?

(1) $\operatorname{Span} S \subseteq \operatorname{Span} T$

(2) $S^\perp \subseteq T^\perp$

(4) If S is linearly independent, so is T.

(8) If S and T are subspaces of $\mathbb{R}^n$, then $\dim S \leq \dim T$.

(h) Let S be a finite subset of $\mathbb{R}^n$. Which of the following must be well-defined?

(1) $S^\perp$

(2) $\dim S$

(4) $\det S$

(8) $\operatorname{Span} S$

3. Let A and B be square matrices of the same size. For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

(a) It is possible that a system of linear equations has infinitely many solutions but at the same time the solution to one of the variables is unique.

(b) The diagonal entries of an elementary matrix must be non-zero.

(c) If A^3 is invertible, so is A^4.

(d) A matrix with positive entries must have positive determinant.

(e) If $\det A = \det B$, then A can be obtained from B by a sequence of elementary row operations.

(f) A subset of $\mathbb{R}^n$ which is closed under addition must also be closed under scalar multiplication.

(g) If λ is an eigenvalue of A^3, then it is also an eigenvalue of A^4.

(h) Let U and V be subspaces of $\mathbb{R}^n$. Then $U^\perp \cap V^\perp = (U \cap V)^\perp$.

4. Let $A = \begin{bmatrix} 6 & -2 & -4 \\ k & 0 & -4 \\ 4 & -2 & -2 \end{bmatrix}$ where $k \in \mathbb{R}$.

(a) Show that 2 is an eigenvalue of A.

(b) Suppose A is non-diagonalisable. Find, with careful justifications,

(i) the value of k;

(ii) an orthonormal basis for the column space of $A - I$.

5. Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ be a subset of V.

(a) Prove the following statements. (Note: Take a look at part (b) first.)

(i) If S generates V, then any set of more than k vectors in V is linearly dependent.

(ii) If S is linearly independent, then any set of fewer than k vectors in V cannot generate V.

(b) The statements in (a) imply an important result in linear algebra. State the result and derive it from the statements in (a).

(Note: To avoid circular reasoning, make sure that you did not use this result itself in proving the statements in (a)!)

(c) Suppose $\dim V = k$. Show that S generates V if and only if it is linearly independent.

6. Write an essay on matrix multiplication based on the following story and the guidelines that follow.

> A student had to evaluate a matrix product AB in a linear algebra examination. However, in copying the matrices, he has accidentally got the $(2,3)$-entry of A wrong. But apart from this he carried out the correct computations, and luckily he ended up with the correct answer.

Here are some guidelines for your essay:

- There is no 'word limit', but try to be concise. A reference length for the essay is around 2 pages.
- You may assume all definitions and related theorems are known to the reader.
- Try to relate your essay to the story, and discuss what made the student 'lucky'.
- In addition, you may discuss the various ways of viewing and various approaches to defining matrix multiplication and the underlying reasons/rationle for such definitions/approaches, with applications in the real world as well as in other aspects of linear algebra (e.g. linear transformation).

Marks will be given for the relevance and accuracy of the contents, demonstration of the insights to the concepts, as well as organisation and presentation including the proper use of (both English and mathematical) language.

******* **END OF PAPER** *******