THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH2101: Linear Algebra I

December 14, 2024 2:30p.m. – 5:00p.m.

No calculators are allowed in the examination.

Answer ALL 12 questions

Note: You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**

1. (3 points) Write down *all* possible values of a and b such that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b & 0 \end{pmatrix}$$

is a matrix in reduced row echelon form. NO explanation is required.

2. (4 points) Let $T : \mathbb{R}^7 \to \mathbb{R}^5$ be a linear transformation. If $\operatorname{rank}(T) = 3$, find $\operatorname{nullity}(T)$. Briefly explain your answer.

3. (4 points) Find the dimension of the following vector space:

$$\operatorname{span}\left(\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}\right).$$

Briefly explain your answer.

4. (3 points) Find all unit vectors which are linearly dependent to

$$\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}.$$

NO explanation is required.

5. (9 points) Determine whether the following statements are true or false. Briefly explain your answers. Answers without clearly stating true or false will not receive any credit.

   (a) Let A be a 2024×2025 matrix. The columns of A must form a linearly dependent set in $\mathbb{R}^{2024}$.

   (b) Let $\{v_1, v_2, v_3, v_4, v_5\}$ be a basis for $\mathbb{R}^5$. Then, for any linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^6$, $\{T(v_1), T(v_2), T(v_3), T(v_4), T(v_5)\}$ is linearly independent.

   (c) Let A be an invertible 2024×2024 matrix. Then $\det(-A) = -\det(A)$.

6. (6 points) Determine whether the following sets are vector subspaces in the given vector space. Briefly explain your answers. Answers without clearly stating yes or no will not receive any credit.

(a) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\}$ in $\mathbb{R}^3$;

(b) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\}$ in $\mathbb{R}^3$;

(c) $\{f : \mathbb{R} \to \mathbb{R} : f(2024) = 12\}$ in the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$.

7. (a) (4 points) Let

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix}.$$

Find $\det(A)$.

(b) (8 points) Consider the following system of linear equations:

$$\begin{array}{rrrrrl} x_1 & & +2x_3 & -x_4 & +12345678x_5 & = 2024 \\ -x_1 & & +x_3 & & +43215678x_5 & = 20242024 \\ & x_2 & +2x_3 & & +56781234x_5 & = 2024^2 \\ x_1 & & -x_3 & +x_4 & +87654321x_5 & = 20242024^2 \end{array}$$

What is the number of solution(s) of the system? Explain your answer. (You are not required to give the solution set of the system of linear equations.)

8. Let

$$A = \begin{pmatrix} 0 & 12 & -14 \\ 0 & -4 & 2 \\ 1 & 3 & -6 \end{pmatrix}.$$

(a) (2 points) Show that -2 is an eigenvalue of A.

(b) (9 points) Determine if A is diagonalizable. Explain your answer.

9. (8 points) Define $T : \mathbb{R}^4 \to \mathbb{R}^3$ to be the linear transformation given by:

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 - x_2 \\ x_2 + x_4 \\ x_3 + x_4 \end{pmatrix}.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

$$\beta' = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Find the matrix representation $[T]_{\beta}^{\beta'}$.

10. Let

$$U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : x_1 - x_3 = 0 \right\} \cap \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : x_2 - x_4 = 0 \right\},$$

$$U_2 = \operatorname{span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right).$$

(a) (3 points) Find a basis for U_1.

(b) (3 points) Find the dimension of $U_1 \cap U_2$.

(c) (7 points) Prove or disprove the following statement: For any vector subspace W of $\mathbb{R}^4$, each vector $w \in W$ is equal to $w_1 + w_2$ for some vectors $w_1 \in U_1 \cap W$ and $w_2 \in U_2 \cap W$.

11. (12 points) The 5×5 matrix A is known to take the form:

$$A = \begin{pmatrix} 1 & 2 & * & 0 & 3 \\ 0 & 2 & * & 2 & 2 \\ * & * & * & * & * \\ 1 & 0 & * & -2 & 1 \\ 0 & 1 & * & 1 & 1 \end{pmatrix}$$

Those $*$ denote some unknown entries in A. Find

(a) the maximum possible rank of A; and

(b) the minimum possible rank of A.

Explain your answers.

12. (15 points) For any square matrix A, determine whether A and AA^T *always* have the same

   (a) rank;

   (b) column space;

   (c) row space;

   (d) null space.

   Explain your answers.

******* **END OF PAPER** *******