THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

May 17, 2019 2:30 – 5:00 pm

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all SIX questions. Each question carries 16 marks.
Another 4 marks is given for clarity of presentation and following instructions.

Notes:

- You should give precise and adequate explanations to support your conclusions unless otherwise specified by the question. Clarity of presentation of your argument counts. So **think carefully before you write.**
- You must start each question on a new page. However do NOT start each part of a question on a new page. You should write down the question number on the top right hand corner of each page. Indicate clearly which part of a question you are answering.
- If your answer to a question spans over more than one page, you must indicate clearly on each page (except the last) that the answer will continue on the subsequent page.
- If you do not otherwise specify, you are assumed to be answering the questions in order. In particular, indicate on each question where your answer to the next question is if it does not follow immediately. (For example you may write 'turn back 3 pages for Question 5' at the end of Question 4.)
- Whenever there is a blank page it will be assumed that all subsequent pages are also blank. If this is not true, indicate clearly on the page which you would otherwise leave blank.

1. Give answers only to the following questions. Explanation is <u>not</u> required.

(a) If $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, compute the reduced row echelon form of A.

(b) Continuing the previous part, find a non-trivial solution to the system $A\mathbf{x} = \mathbf{0}$.

(c) With the same matrix A as above, find an eigenvector of A.

(d) If the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 8 \\ x \end{bmatrix}$ are linearly independent, find all possible values of x.

(e) Compute the determinant of the matrix $\begin{bmatrix} 2 & 4 & 6 & 9 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix}$.

(f) Let $B = \begin{bmatrix} * & * & * \\ 1 & 2 & 3 \\ * & * & * \end{bmatrix}$, where the $*$'s denote (possibly the same or different) real numbers. If $\operatorname{adj} B = \begin{bmatrix} * & 5 & * \\ 4 & 3 & 2 \\ * & 1 & * \end{bmatrix}$, find $\det B$.

(g) Consider the subspace $V = \operatorname{Span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$ of $\mathbb{R}^4$. Find $\dim V$.

(h) What is the projection of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on the vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$?

2. In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do not indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is not required.

(a) Let A and B be invertible square matrices of the same size. Which of the following must be true?

(1) $(2A)(6B) = (3B)(4A)$

(2) $(2AB)^{-1} = (2B)^{-1}A^{-1}$

(4) $(AB)^3 = (A^2B)(AB^2)$

(8) $(A+B)(A-B) = (A-B)(A+B)$

(b) Let $A = [\mathbf{a}_1 \; \mathbf{a}_2 \; \cdots \; \mathbf{a}_n]$ be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. If $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ in two different ways, which of the following must be true?

(1) The system $A\mathbf{x} = \mathbf{b}$ is consistent.

(2) The system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(4) The rank of A is less than n.

(8) $\mathbf{a}_n$ can be expressed as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_{n-1}$ and $\mathbf{b}$.

(c) In which of the following cases is T a linear transformation?

(1) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + y$

(2) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2xy$

(4) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x \end{bmatrix}$

(8) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} |x| \\ 0 \end{bmatrix}$

(d) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a basis of a subspace V of $\mathbb{R}^n$, so is

(1) $\{2\mathbf{x}, 2\mathbf{y}, 2\mathbf{z}\}$

(2) $\{2\mathbf{x}, 3\mathbf{y}, 4\mathbf{z}\}$

(4) $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x}+\mathbf{y}+\mathbf{z}\}$

(8) $\{\mathbf{x}+\mathbf{y}, \mathbf{y}+\mathbf{z}, \mathbf{z}+\mathbf{x}\}$

(e) A 4×4 matrix whose eigenvalues are all real must be diagonalisable if

(1) it has 4 distinct eigenvalues

(2) its eigenvalues are all positive

(4) all its eigenvalues have algebraic multiplicity 1

(8) all its eigenvalues have geometric multiplicity 1

(f) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Which of the following must be true?

(1) $\mathbf{u} \cdot \mathbf{v} \geq 0$

(2) $\|\mathbf{u}\| \cdot \|\mathbf{v}\| \geq \mathbf{u} \cdot \mathbf{v}$

(4) $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u}+\mathbf{v}\|$

(8) $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \geq \|\mathbf{u}+\mathbf{v}\|^2$

(g) Let S be an orthonormal subset of $\mathbb{R}^n$. Which of the following must be true?

(1) $(S^\perp)^\perp = S$

(2) S is a basis for $\mathbb{R}^n$

(4) S is an orthogonal set

(8) $S^\perp$ is a subspace of $\mathbb{R}^n$

(h) Let S be a subset of $\mathbb{R}^n$ and T be the orthogonal complement of S. Which of the following must be true?

(1) T is a subspace of $\mathbb{R}^n$

(2) $\dim T = n - |S|$

(4) $S \cap T = \{\mathbf{0}\}$

(8) $S \cup T = \mathbb{R}^n$

3. For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

(a) Let A be a 3×4 reduced row echelon matrix. Then A has at least 6 zero entries.

(b) There exists matrix A with no zero entries such that $A^3 = 8A$ but $A^2 \neq 8I$.

(c) There exists a 4×4 matrix A such that $\operatorname{adj} A = O$ and no two entries of A are the same.

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation by an angle θ about the origin. If θ is not an integer multiple of $180°$, then T has no real eigenvalue.

(e) An elementary row operation will not change the column space of a matrix.

(f) If A has two distinct eigenvalues 3 and 4, then the intersection of the 3-eigenspace and the 4-eigenspace of A is a subspace of $\mathbb{R}^n$.

(g) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. If $\mathbf{u} \cdot \mathbf{v} = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

(h) A set of n orthogonal vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

4. Let

$$A = \begin{bmatrix} 1 & 4 & * & -1 \\ 2 & 3 & * & -1 \\ 2 & 4 & * & -1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & * & * & * \end{bmatrix}.$$

Here the $*$'s denote (possibly the same or different) real numbers. It is known that R is the reduced row echelon form of A.

(a) Show that $\operatorname{rank} A = 3$.

(b) Find, with careful explanation, all the unknown entries in A and R.

(c) Which columns of A form a basis for $\operatorname{Col} A$? Using the Gram-Schmidt process, turn this basis into an orthonormal basis. Can you find another (much simpler) orthonormal basis?

5. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation which preserves norm, i.e.

$$\|T(\mathbf{x})\| = \|\mathbf{x}\| \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Let also A be the standard matrix of T.

(a) Show that the columns of A form an orthonormal basis for $\mathbb{R}^n$.

Hence deduce that $A^T A = I$.

(b) If $\det A = d$ and λ is an eigenvalue of A, find all possible values of d and λ.

(c) Must T also preserve dot product? That is, must it be true that

$$T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n?$$

6. Let A and B be similar square matrices, with $A = P^{-1}BP$.

Suppose that λ is an eigenvalue of A with algebraic multiplicity a and geometric multiplicity g.

(a) Explain why λ is also an eigenvalue of B with algebraic multiplicity a.

(b) If $\mathbf{x}$ is a λ-eigenvector of A, show that $P\mathbf{x}$ is a λ-eigenvector of B.

(c) Prove or disprove the following statements.

(i) A and B must be row equivalent.

(ii) λ must be an eigenvalue of B with geometric multiplicity g.

******* **END OF PAPER** *******