THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

May 8, 2024 6:30 p.m. – 9:00 p.m.

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all FIVE questions.

Notes:

- Write down your university number, seat number and model of calculator used on the cover of the answer book.
- The blank pages in the answer book can be used as additional answer space provided that you make a clear indication in the original answer space.
- Unless otherwise specified, you should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**

1. (26 marks) Give answers only to the following questions. Explanation is <u>not</u> required. Each question carries 2 marks.

Let $A = \begin{bmatrix} a & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ where $a \neq 0$. Answer Questions (a)-(g).

(a) Find the LU decomposition of A.

(b) Find the orthogonal complement of $S = \{T_A(\mathbf{e}_1 + 2\mathbf{e}_3)\}$, where T_A denotes the linear transformation induced by A.

(c) Let $a = -1$, find a basis for the column space of A.

(d) Let $a = -1$, find the null space of A.

(e) Let $a = -1$, find the determinant of the 4×4 matrix B, where

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \geq j \text{ and } i, j \in \{1, 2, 3\} \\ 1 & \text{otherwise} \end{cases}$$

and a_{ij} and b_{ij} denote the (i, j)-entry of A and B respectively.

(f) Let $a = -1$. Find the sum of all the eigenvalues of A.

(g) Let $a = -1$. Find the eigenvector corresponding to the second smallest (real) eigenvalue of A. (Let the first nonzero entry to be 1).

(h) Find the image of the reflection of $\begin{bmatrix} 9 \\ 9 \\ -18 \end{bmatrix}$ across the plane $2x - y + 2z = 0$.

(i) Consider the block matrix $M = \begin{bmatrix} S & P \\ Q & T \end{bmatrix}$, where S is a 3×2 zero matrix and T is a 2×3 zero matrix. If $\det P = a$ and $\det Q = b$, find $\det M$.

(j) Find the new coordinates of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ if the coordinate axes in $\mathbb{R}^2$ are rotated clockwise by $120°$. (Correct your answer to two decimal places.)

(k) A curve in $\mathbb{R}^2$ has equation $13x^2 - 10xy + 13y^2 = 72$. The coordinate axes are rotated anticlockwise by $45°$. The new equation of the curve is $ax^2 + bxy + cy^2 = 1$. Find the value of a, b, c. (Keep exact asnwer.)

(l) Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{v}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \mathbf{v}$, and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$. Find $[T]_{\mathcal{B}}$.

(m) Let $\mathbf{u} = (-1, 4, 2)^T$ and $\mathbf{v} = (1, 0, 3)^T$. Then the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is $k\mathbf{v}$. Find k.

2. (33 marks) In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do not indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is not required. Each question carries 3 marks.

(a) For each of the following, suppose a square matrix A satisfies the following conditions where $f(t)$ denotes the characteristic polynomial. In which cases can A be diagonalizable over $\mathbb{R}$?

(1) $f(t) = t^4 - 1$ and the geometric multiplicity of 1 and -1 is 1.

(2) $f(t) = -t^3 + t^2 + 8t - 12$ and the geometric multiplicity of each eigenvalue is 1.

(4) $f(t) = t^4$ and $A \neq O$.

(8) $f(t) = t^2(1 - t)$ and rank $A = 1$.

(b) Which of the following are bases for $\mathbb{R}^3$?

(1) $\{\mathbf{e}_1, 2\mathbf{e}_2, 3\mathbf{e}_3\}$

(2) $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \right\}$

(4) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$

(8) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(c) Let V and W be two subspaces of $\mathbb{R}^5$. If $\dim V = 2$ and $\dim W = 4$, which of the following number is possible for dimension of the subspace $V \cap W$?

(1) 2

(2) 4

(4) 5

(8) 6

(d) Which of the following are subspaces of $\mathbb{R}^3$?

(1) $\emptyset$

(2) $\text{Span}\{(1,0,0)^T\}$

(4) $\text{Span}\{(1,3,5)^T\} \cap \text{Span}\{(2,4,6)^T\}$

(8) $\{(x,y,z)^T : 2x - 3y + 4z = 5\}$

(e) Let A and B be $m \times n$ matrices.

(1) If $A\mathbf{v} = B\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$, then $A = B$.

(2) If $A\mathbf{v} = B\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^n$, then $A = B$.

(4) There exists a $\mathbf{v} \in \mathbb{R}^n$, such that $A\mathbf{v} = B\mathbf{v}$.

(8) For all $\mathbf{v} \in \mathbb{R}^n$, $A\mathbf{v} = B\mathbf{v}$.

(f) Let S be a set of 4 vectors in $\mathbb{R}^3$. Which of the following is possible?

(1) S is linearly dependent.

(2) S is linearly independent.

(4) S is a basis for $\mathbb{R}^3$.

(8) S is a generating set for $\mathbb{R}^3$.

(g) Let A be the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix}$. Which of the following has dimension 3?

(1) The row space of A.

(2) The column space of A^T.

(4) The null space of A.

(8) The null space of A^T.

(h) Let A be a 3×3 matrix with eigenvectors $\mathbf{e}_2$ and $\mathbf{e}_3$. Which of the following must be 0?

(1) The $(1,2)$-entry of A.

(2) The $(2,1)$-entry of A.

(4) The $(2,3)$-entry of A.

(8) The $(3,2)$-entry of A.

(i) Which of the following matrices has 1 as an eigenvalue?

(1) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(8) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(j) Let C be a 4×4 matrix with determinant 3. Which of the following is correct?

(1) $\det(C^{-1}) = 1/3$

(2) $\det(C^2) = 9$

(4) $\det(2C) = 6$

(8) $\det(adj(C^T)) = 81$

(k) Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal vectors in $\mathbb{R}^n$. Which of the following must be correct?

(1) $\|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$

(2) $\|\mathbf{u} + 2\mathbf{v}\| = \|\mathbf{u} - 2\mathbf{v}\|$

(4) $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

(8) $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$

3. (21 marks) For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation. Each question carries 3 marks.

(a) If a matrix A is both upper triangular and lower triangular, then it is a diagonal matrix.

(b) If S is a subset of $\mathbb{R}^n$, then $S \bigcap S^{\perp} = \{\mathbf{0}\}$.

(c) Given a square matrix A, let λ be an eigenvalue of A, and let $\mathbf{v}$ be a λ-eigenvector of A, then $\mathbf{v}$ is also an eigenvector of $-A$.

(d) Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be linearly independent vectors in $\mathbb{R}^3$, then $\text{Span}\{\mathbf{x}, \mathbf{y}\}$ must be $\mathbb{R}^2$.

(e) For any matrices A and B, if $|AB| = 0$, then either $|A| = 0$ or $|B| = 0$.

(f) Any invertible matrix has LU decomposition.

(g) Let A and B be $n \times n$ matrices, then AB and BA have the same eigenvalues.

4. (10 marks) Let $A = \begin{bmatrix} 1 & 4 & 4 & 4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Determine if A is diagonalizable. If yes, diagonalize it and find A^n where $n \in \mathbb{N}$; if not, state the reason.

5. (10 marks) The number of students getting an A in the final exam of Math2101 is as follows:

Year	2020	2021	2022	2023
A's	20	10	40	60

Represent the years $2020, 2021, 2022, 2023$ as $0, 1, 2, 3$, respectively, and let t denote the year (after 2020). Let y denote the number of A's.

(a) (8 marks) Find the line $y = mt + b$ that best fits the above data points, using the least squares method.

(b) (2 marks) Use the equation obtained in part (a) to estimate how many students will get an A in Linear Algebra in 2024.

******* **END OF PAPER** *******