MATH 2101 LINEAR ALGEBRA I, FALL SEMESTER 2025
CHAPTER 1: MATRIX ALGEBRA AND DETERMINANTS

K.Y. CHAN

Goal of this section:

(1) Basic terminology: matrices, entries, columns, rows, etc
(2) Basic operations: additions, matrix multiplication, matrix addition, transpose
(3) Properties of matrix multiplications
(4) Determinants: definition, rules to compute
(5) Inverse: computing inverse from a determinant, determining invertibility by determinants

1. Matrix Algebra

1.1. Basic terminology.

Definition 1.1. (1) We use $\mathbb{R}$ to represent the set of real numbers. A scalar is usually referred to a number in $\mathbb{R}$ (in this course).

(2) An $n \times m$ matrix A is a rectangular array of the form:
$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \ddots & \ddots & & \ddots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$
where a_{ij} are in $\mathbb{R}$. We sometimes write (a_{ij}) to represent an arbitrary matrix. The (i,j)-entry of A will be referred to the number a_{ij}.

(3) For a matrix A, we also write A_{ij} to be the (i,j)-th entry of A.

(4) The i'-th row of (a_{ij}) is the $1 \times m$ matrix
$$\begin{pmatrix} a_{i'1} & a_{i'2} & \dots & a_{i'm} \end{pmatrix}$$

(5) The j'-th column of (a_{ij}) is the $n \times 1$ matrix
$$\begin{pmatrix} a_{1j'} \\ a_{2j'} \\ \vdots \\ b_{nj'} \end{pmatrix}.$$

Example 1.2. Let $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 7 & 9 \end{pmatrix}$. Then $A_{13} = 1$. The 2nd column of A is $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ and the 1st row of A is $\begin{pmatrix} 2 & 4 & 1 \end{pmatrix}$.

1.2. **Operations of matrices.**

(1) Addition of two $n \times m$ matrices $(a_{ij}) + (b_{ij})$ is defined as:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2m}+b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \dots & a_{nm}+b_{nm} \end{pmatrix}$$

(2) Scalar multiplication on a matrix: for a scalar $c \in \mathbb{R}$ and a $n \times m$ matrix (a_{ij}), $c(a_{ij})$ is defined as:

$$c \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1m} \\ ca_{21} & ca_{22} & \dots & ca_{2m} \\ \vdots & \vdots & & \vdots \\ ca_{n1} & ca_{n2} & \dots & ca_{nm} \end{pmatrix}$$

(3) Matrix multiplication: Let $A = (a_{ij})$ be a $n \times m$ matrix and let $B = (b_{ij})$ be a $m \times p$ matrix. We define the multiplication AB such that the (i, j)-th entry of AB is:

$$(AB)_{ij} = \sum_{x=1}^{m} a_{ix} b_{xj}$$

Remark 1.3. Let A be a $n \times m$ matrix and let B be a $p \times q$ matrix. If $m \neq p$, then AB is *not* well-defined.

(*Curious students may ask why to define matrix multiplication in such way. This is related to how to the concept of linear transformation, which we will come back later.*)

Example 1.4. (1) $\begin{pmatrix} 1 & 2 & 4 \\ -1 & 7 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 3 & 2 \\ 1 & 3 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 6 \\ 0 & 10 & 10 \end{pmatrix}$.

(2)
$$(-2) \cdot \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0.5 & 8 \end{pmatrix} = \begin{pmatrix} -4 & -6 & 0 \\ -2 & -1 & 0 \end{pmatrix}$$

(3)
$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 0 & 1 \\ 9 & 2 \end{pmatrix} = \begin{pmatrix} 1(4)+2(0)+3(9) & 1(3)+2(1)+3(2) \end{pmatrix} = \begin{pmatrix} 31 & 11 \end{pmatrix}$$

(4)
$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -16 \end{pmatrix}$$
is not well-defined.

1.3. **Vectors.**

Definition 1.5. (1) A column vector is a $n \times 1$-matrix i.e. of the form
$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$
(2) A row vector is a $1 \times m$-matirx i.e. of the form
$$(a_1, \dots, a_m).$$
(3) A vector simply refers to either a row vector or a column vector.
(4) $\mathbb{R}^n$ refers to the set of all column vectors with n real entries.
(5) We sometimes simply write 0 for the row vector or column vector with all entries to be zero. We shall call such vector to be the zero vector.

Exericse: Visualize geometrically the additions and scalar multiplication for vectors in $\mathbb{R}^2$.

1.4. **Identity matrix.** The identity matrix I_n is defined as:
$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$
that is
$$(I_n)_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$
The main properties of the identity matrix are the following:

Theorem 1.6. *Let A be an $n \times m$ matrix. Then*
(1) $I_n A = A$; *and*
(2) $AI_m = A$.

Proof. Exercise. □

Indeed, one should do the following more general exercise:

Exercise 1.7. An $n \times n$ matrix A is said to be diagonal if for any i, j with $i \neq j$, $A_{ij} = 0$. For example, $\begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$ is diagonal but $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ is not diagonal. The diagonal entries of a $n \times n$ matrix (a_{ij}) are referred to the entries $a_{11}, a_{22}, \dots, a_{nn}$.

Let A be a $n \times m$ matrix.
(1) For a diagonal $n \times n$ matrix D, describe the matrix multiplication DA;
(2) For a diagonal $m \times m$ matrix E, describe the matrix multiplication AE.

1.5. **Zero matrices.** The $n \times m$ matrix is called the zero matrix $0_{n\times m}$ (or sometimes simply 0) if all its entries are zero. The main properties of the zero matrix is the following:

Theorem 1.8. *Let A be an $n \times m$ matrix. Then*

(1) $0_{p\times n}A = 0_{p\times m}$*; and*
(2) $A0_{m\times q} = 0_{n\times q}$*; and*
(3) $A + 0 = A$.

Proof. Exercise. $\square$

Remark 1.9. For simplicity, we usually write 0 for different things. One may need a bit thought to identify the meaning of 0 in a content. Can you explain the meaning of of 0 in the following equations? Let A be a $n \times p$-matrix.

(1) Then $0_{m\times n}A = 0$.
(2) Let $c = 0$ be the zero scalar. Then $cA = 0$.
(3) $A - A = 0$.

1.6. **Square matrices.**

Definition 1.10. A $n \times m$ matrix is called sqaure if $n = m$.

Example 1.11. (1) Identity matrices are square. Diagonal matrices are also square.
(2) The row vector $\begin{pmatrix} 1 & 3 \end{pmatrix}$ is not square.
(3) The matrix $\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$ is not square.

1.7. **Transpose of a matrix.**

Definition 1.12. The transpose A^T of an $n \times m$ matrix A is the $m \times n$ matrix obtained from interchanging rows and columns i.e.

$$(A^T)_{ij} = A_{ji}$$

Example 1.13. (1) Let $A = \begin{pmatrix} 3 & 5 & 0 \end{pmatrix}$. Then

$$A^T = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}.$$

(2) Let $B = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$. Then $B^T = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix}$.

There are some basic properties of the transpose:

Theorem 1.14. (1) *(Transpose commutes with addition) For $n \times m$-matrices A and B,*
$$(A+B)^T = A^T + B^T.$$
(2) *(Transpose anti-commutes with matrix multiplication) For an $n \times m$-matrix A and a $m \times p$-matrix B,*
$$(AB)^T = B^T A^T.$$
(3) *(Transpose commutes with scalar multiplication) For an $n \times m$-matrix A and a scalar c,*
$$(cA)^T = cA^T.$$

Example 1.15. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ and let $B = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. Then
$$AB = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix},$$
and so
$$(AB)^T = \begin{pmatrix} 1 & -11 \end{pmatrix},$$
$$B^T A^T = \begin{pmatrix} -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -11 \end{pmatrix}.$$

Theorem 1.16. *(Transpose twice) Let A be an $n \times m$ matrix. Then $(A^T)^T = A$.*

Proof.
$$((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}.$$
□

1.8. Symmetric matrices.

Definition 1.17. A matrix A is said to be *symmetric* if $A^T = A$.

Example 1.18. (1) $\begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}$ is symmetric.
(2) $\begin{pmatrix} 1 & 3 \\ -3 & -2 \end{pmatrix}$ is not symmetric.
(3) Any diagonal matrices are symmetric.

Exercise 1.19. Show that if a matrix is symmetric, then A is square.

Exercise 1.20. Write down the form of symmetric 2×2 matrices and the form of symmetric 3×3 matrices.

1.9. Upper triangular matrices.

Definition 1.21. (1) A square matrix (a_{ij}) is said to be an upper triangular matrix if for any $i > j$, $a_{ij} = 0$.
(2) A square matrix (a_{ij}) is said to be a lower triangular matrix if for any $i < j$, $a_{ij} = 0$.

Example 1.22. (1) The matrices $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -2 \end{pmatrix}$ are upper triangular.
(2) The matrices $\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$ are lower triangular matrices.
(3) The matrix $\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$ is neither upper triangular nor lower triangular.

Exercise 1.23. Show that a square matrix A is both upper triangular and lower triangular if and only if A is diagonal.

2. More on matrix multiplications

2.1. The slogan in matrix multiplication. Let $A = (a_{ij})$ be a $n \times m$ matrix and let $B = (b_{ij})$ be a $m \times p$ matrix. Then one may say that: *the (i, j)-th entry of AB is obained by the i-th row of A times the j-th column of B.*

The meaning of the above slogan is that:

$$(AB)_{ij} = \begin{pmatrix} a_{i1} & \dots & a_{im} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix}.$$

2.2. Associativity. Here are some properties of the matrix multiplication:

Theorem 2.1. *(Associativity) The matrix multiplication is an associative operation, meaning that: Let A be a $n \times m$ matrix. Let B be a $m \times p$ matrix. Let C be a $p \times q$ matrix. Then*

$$A(BC) = (AB)C.$$

Proof. Write $A = (a_{ij})$, $B = (b_{jk})$ and $C = (c_{kl})$.

<u>Compute $(AB)C$</u>:

$$(AB)_{ij} = \sum_{x=1}^{m} a_{ix} b_{xj},$$

$$((AB)C)_{il} = \sum_{y=1}^{p} (AB)_{iy} C_{yq} = \sum_{y=1}^{p} \sum_{x=1}^{m} a_{ix} b_{xy} c_{yl}$$

Compute $A(BC)$:

$$(BC)_{kl} = \sum_{y=1}^{p} b_{ky} c_{yl},$$

$$A(BC)_{il} = \sum_{x=1}^{m} a_{ix} (BC)_{xl} = \sum_{x=1}^{m} \sum_{y=1}^{p} a_{ix} b_{xy} c_{yl}.$$

Thus, we have two expressions coincide. □

2.3. **Distributive.**

Theorem 2.2. *(Distributive property) For a $n \times m$ matrix A and $m \times p$ matrices B, C, we have:*

$$A(B + C) = AB + AC.$$

Exercise 2.3. Verify that $A(B + C) = AB + AC$ by using definitions.

2.4. **Non-commutativity.** For an $n \times m$ matrix A and a $m \times n$ matrix B, we *do not always* have $AB = BA$.

Example 2.4. Let $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 1 + ab & a \\ b & 1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 1 & a \\ b & 1 + ab \end{pmatrix}.$$

So $AB \neq BA$ when $a \neq 0$ and $b \neq 0$.

Exercise 2.5. Let A and B be two $n \times n$ symmetric matrices. Show that $AB = BA$ if and only if AB is also symmetric..

2.5. **Block multiplication.** To motivate the term 'block multiplication', it is better to illustrate first with an example:

Example 2.6. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$. Let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

Let

$$D = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Suppose we have already known that

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \quad AC = \begin{pmatrix} 2 & -2 \\ -1 & -6 \end{pmatrix}.$$

Can you tell the answer the matrix multiplication:

$$AD$$

without direct computations?

Indeed, the answer is

$$AD = \begin{pmatrix} 1 & 2 & 2 & -2 \\ 3 & -1 & -1 & -6 \end{pmatrix}!$$

This is done by investigating how each entry is obtained in the matrix multiplication by using the above slogan!

The above example suggests that one can partition a matrix into blocks suitably in order to carry our the matrix multiplication.

Proposition 2.7. *(Block multiplication) Let A be an $n\times m$-matrix. Let $B_1, B_2, \dots, B_r$ be $m \times p_1$, $m \times p_2$, ..., $m \times p_r$ matrices respectively. Let*

$$C = (B_1 \ B_2 \ \dots B_r)$$

which is a $m \times (p_1 + \dots + p_r)$-matrix. Then

$$AC = (AB_1 \ AB_2 \ \dots AB_r).$$

The following is a particular case that we shall use few times later:

Corollary 2.8. *Let A be an $n \times m$ matrix. Let $v_1, \dots, v_k$ be column vectors in $\mathbb{R}^m$. Let*

$$B = \begin{pmatrix} v_1 & v_2 & \dots & v_k \end{pmatrix},$$

which is a $m \times k$ matrix. Then

$$AB = \begin{pmatrix} Av_1 & Av_2 & \dots & Av_k \end{pmatrix}.$$

We can also consider some other variations of block matrix multiplication:

Exercise 2.9. Let A_1, A_2 be 2×2 matrices, B_1, B_2 be a 2×3 matrices, C_1, C_2 be 3×2 matrices and D_1, D_2 be 3×3 matrices. Write the multiplication of two block matrices

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

in the form of one matrix.

Exercise 2.10. Formulate a general version of block multiplication.

3. Determinant of a matrix

The determinant will be defined for square matrices.

3.1. Definition.

Definition 3.1. (2×2 matrix) The determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad-bc$.

Example 3.2. $\det\begin{pmatrix} 3 & 4 \\ 2 & 9 \end{pmatrix} = 27-8 = 19$

Definition 3.3. ($n\times n$ matrix) Let $A=(a_{ij})$ be an $n\times n$ matrix.

(1) Define the (i,j)-minor, denoted by $\widetilde{A}_{ij}$, of A to be the matrix obtained by deleting the i-th row and j-th column.
(2) Define the (i,j)-cofactor C_{ij} of A to be the scalar $(-1)^{i+j}\det \widetilde{A}_{ij}$
(3) Fix certain j^*. Define *recursively* the determinant, denoted by $\det A$, of A to be a scalar

$$\det(A) = \sum_{i=1}^{n} a_{ij^*}C_{ij^*} = \sum_{i=1}^{n}(-1)^{i+j^*}a_{ij^*}\det(\widetilde{A}_{ij^*}) \tag{3.1}$$

or fixing certain i^*, define $\det A$ as:

$$\det(A) = \sum_{j=1}^{n} a_{i^*,j}C_{i^*j} = \sum_{j=1}^{n}(-1)^{i^*+j}a_{i^*,j}\det(\widetilde{A}_{i^*,j}). \tag{3.2}$$

The above definition is *independent of a choice of a row or a column.* We shall call (3.1) (respectively (3.2)) to be the cofactor expansion of $\det(A)$ along j^*th-column (respectively along i^*th-row).

Example 3.4. Let $A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 9 & 1 \\ 0 & -1 & -5 \end{pmatrix}$. The minors for the second column is:

$$\widetilde{A}_{12} = \begin{pmatrix} 2 & 1 \\ 0 & -5 \end{pmatrix}, \quad \widetilde{A}_{22} = \begin{pmatrix} 3 & 1 \\ 0 & -5 \end{pmatrix}, \quad \widetilde{A}_{32} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

Then

$$\begin{aligned}\det(A) &= (-1)^{1+2}(4)\det\begin{pmatrix} 2 & 1 \\ 0 & -5 \end{pmatrix} + (-1)^{2+2}(9)\det\begin{pmatrix} 3 & 1 \\ 0 & -5 \end{pmatrix} + (-1)^{2+3}(-1)\det\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\ &= (-1)(4)(-10) + (9)(-15) - 1(-1)\end{aligned}$$

Exercise 3.5. Find

$$\det\begin{pmatrix} 3 & 4 & 0 & 0 \\ 2 & 9 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 0 & -5 \end{pmatrix}$$

by using the cofactor expansion.

Exercise 3.6. (a) Write down the determinant of a 3×3-matrix:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

View the formula by a 'diagonal multiplication'.

(b) Do you think a similar 'diagonal multiplication' can hold for 4×4-matrices? (Hint: consider

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 3.7. Let A be a 3×3 matrix. Prove that there exists $t \in \mathbb{R}$ such that $\det(A - tI_3) = 0$. (Hint: Consider $\det(A - xI_3)$ as a polynomial in x and solve for x.)

3.2. **Rules and properties for computing determinants.**

Theorem 3.8. *(Property 1: switching rows) Let A be an $n \times n$ matrix. If B is obtained from A by switching two rows, then*

$$\det(A) = -\det(B).$$

Proof. (Sketch of the proof) First consider that B is obtained from A by switching the i-th and $(i+1)$-th rows.

Relation of the (i, j)-minor of A and the $(i+1, j)$-minor of B

$$\widetilde{A}_{ij} = \widetilde{B}_{i+1,j} \tag{3.3}$$

Compute $\det(B)$ by the cofactor expansion along the $(i+1)$-th row:

$$\det(B) = \sum_{j=1}^{n} (-1)^{(i+1)+j} b_{i+1,j} \det \widetilde{B}_{i+1,j} = (-1) \sum_{j=1}^{n} (-1)^{i+j} b_{i+1,j} \det \widetilde{B}_{i+1,j}. \tag{3.4}$$

Compute $\det(A)$ by the cofactor expansion along the i-th row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \widetilde{A}_{ij}. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we have

$$\det(A) = -\det(B).$$

The general case follows from switching two consecutive rows multiple times. □

Example 3.9. $\det \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} = -\det \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}.$

Corollary 3.10. *(Property 2: zero determinant for matrix with two identical rows) Let A be an $n \times n$ matrix. If A has two identical rows, then* $\det(A) = 0$.

Proof. By using the previous theorem, we have $\det(A) = -\det(A)$. Hence $2\det(A) = 0$ and so $\det(A) = 0$. □

Example 3.11. $\det \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} = 0.$

Theorem 3.12. *(Property 3: Adding a scalar multiple on a row vector) Let $v_1, \ldots, v_n$ be row vectors with n-entries and let u be another row vector with n entries. Let $c \in \mathbb{R}$. Then*

$$\det \begin{pmatrix} v_1 \\ \vdots \\ v_{r-1} \\ v_r + cu \\ v_{r+1} \\ \vdots \\ v_n \end{pmatrix} = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{r-1} \\ v_r \\ v_{r+1} \\ \vdots \\ v_n \end{pmatrix} + c\det \begin{pmatrix} v_1 \\ \vdots \\ v_{r-1} \\ u \\ v_{r+1} \\ \vdots \\ v_n \end{pmatrix}$$

Proof. (Sketch) One uses the cofactor expansion along the rth row. □

Example 3.13.

$$\begin{aligned} \det \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} &= \det \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + 0 \\ &= \det \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + \det \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \\ &= \det \begin{pmatrix} 2 & 3 \\ 1+2 & 4+3 \end{pmatrix} \\ &= \det \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix} \end{aligned}$$

Exercise 3.14. Try to use Corollary 3.10 and Theorem 3.12 to show

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} = 0.$$

Corollary 3.15. *Let A be an $n \times n$ matrix.*

(1) *If one row of A is multiplied by a scalar c to obtain A', then* $\det(A') = c \cdot \det(A)$.
(2) *If one row of A is scalar multiple of another row of A, then* $\det(A) = 0$.

Theorem 3.16. *(Property 4: Taking transpose) Let A be an $n \times n$ matrix. Then* $\det(A^T) = \det(A)$.

Proof. Write $A = (a_{ij})$. By the cofactor expansion for $\det(A)$ along the first row,

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(\widetilde{A}_{1j}) \tag{3.6}$$

Write $A^T = (b_{ij})$. By the cofactor expansion for $\det(A^T)$ and we have:

$$\det(A^T) = \sum_{i=1}^{n} (-1)^{i+1} b_{i1} \det(\widetilde{A^T}_{i1}). \tag{3.7}$$

By the definition of minors,

$$(\widetilde{A}_{1x})^T = \widetilde{A^T}_{x1}.$$

Now, by the mathematical induction on n, we have that

$$\det(\widetilde{A}_{1x}) = \det((\widetilde{A}_{1x})^T) = \det((\widetilde{A^T})_{x1}). \tag{3.8}$$

By the definition of transpose,

$$a_{1x} = b_{x1} \tag{3.9}$$

Combining (3.6), (3.7), (3.8) and (3.9), we have:

$$\det(A) = \det(A^T).$$

$\square$

Recall that transpose takes a row to a column and so one may apply Theorem 3.16 to deduce the column versin of Properties 1-3.

Theorem 3.17. *(Column version of properties 1-3) Let A be an $n \times n$ matrix. Then*

(1) *If two columns of A are switched to obtain B, then $\det(B) = -\det(A)$.*
(2) *If two columns of A are identical, or more generally one column of A is a scalar multiple of another column, then $\det(A) = 0$.*
(3) *Let $v_1, \ldots, v_n$ be column vectors in $\mathbb{R}^n$ and let u be another column vector in $\mathbb{R}^n$. Let $c \in \mathbb{R}$. Then*

$$\begin{aligned} &\det \begin{pmatrix} v_1 & \ldots & v_{r-1} & v_r + cu & v_{r+1} & \ldots & v_r \end{pmatrix} \\ =&\det \begin{pmatrix} v_1 & \ldots & v_{r-1} & v_r & v_{r+1} & \ldots & v_n \end{pmatrix} + c \cdot \det \begin{pmatrix} v_1 & \ldots & v_{r-1} & u & v_{r+1} & \ldots & v_n \end{pmatrix} \end{aligned}$$

Theorem 3.18. *(Property 5: Scalar multiplication) For an $n \times n$ matrix A and a scalar c,*

$$\det(cA) = c^n \det(A).$$

Proof. We write

$$A = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix},$$

for column vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$.

Then,

$$cA = \begin{pmatrix} cv_1 & cv_2 & \dots & cv_n \end{pmatrix}.$$

Applying Theorem 3.17(3) n-times,

$$\begin{aligned} \det(cA) &= \det(\begin{pmatrix} cv_1 & cv_2 & \dots & cv_n \end{pmatrix}) \\ &= c\det(\begin{pmatrix} v_1 & cv_2 & \dots & cv_n \end{pmatrix}) \\ &= \vdots & &= c^n\det(\begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}) \\ &= c^n\det(A) \end{aligned}$$

□

Theorem 3.19. *(Property 6: Multiplicative property) Let A, B be two $n \times n$-matrices. Then* $\det(AB) = \det(A) \cdot \det(B)$.

We shall not discuss a proof here. But one may readily verifies the theorem for some examples.

Exercise 3.20. An $n \times n$ matrix A is said to be *orthogonal* if $AA^T = I_n$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

3.3. **Simplifying computation of determinants by using rules.** We can write a matrix A in the form of rows:

$$A = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix},$$

where $u_1, \dots, u_n$ are row vectors with n entries. Suppose B is obtained from a matrix A by subtracting the i-th row of A by the c-multiple of the j-the row of A.

Then

$$\det(B) = \det\begin{pmatrix} u_1 \\ \vdots \\ u_i - c\cdot u_j \\ \vdots \\ u_n \end{pmatrix}$$

$$= \det\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} - c\cdot\det\begin{pmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_j \\ \vdots \\ u_n \end{pmatrix} \quad \text{by Theorem 3.12}$$

$$= \det(A) - 0 \text{by Corollary 3.10}$$

$$= \det(A)$$

Such rule is useful in simplifying computing some determinants.

Example 3.21.

$$\det\begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & -1 \\ 3 & 1 & -2 & 1 \\ -1 & 1 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$= (1)\det\begin{pmatrix} 1 & 1 & -3 \\ 1 & -5 & -2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{by cofactor expansion along 1st column}$$

$$= \det\begin{pmatrix} 1 & 1 & -3 \\ 0 & -6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

$$= (1)\det\begin{pmatrix} -6 & 1 \\ 1 & 6 \end{pmatrix} \quad \text{by cofactor expansion along 1st column}$$

$$= -36 - 1$$

$$= -37$$

4. Inverse

4.1. Definition.

Definition 4.1. An $n \times n$ matrix A is said to be invertible (or non-singular) if there exists a matrix B such that $AB = BA = I_n$. We say B to be the inverse of A. We shall denote such matrix by A^{-1}.

Remark 4.2. The inverse of an invertible matrix A is unique if it exists, which means that there is *at most one* matrix B such that $AB = BA = I_n$.

4.2. Relation to determinants and a formula for the inverse.

Theorem 4.3. *(Determinant of an inverse) Let A be an invertible $n \times n$-matrix. Then $\det(A) \neq 0$. Moreover, $\det(A^{-1}) = \frac{1}{\det(A)}$.*

Proof.

$$
\begin{aligned}
AA^{-1} = I_n &\overset{\text{taking det}}{\Longrightarrow} \det(AA^{-1}) = \det(I_n) \overset{\det(I_n)=1}{\Longrightarrow} \det(AA^{-1}) = 1 \\
&\overset{\text{use } \det(AB) = \det(A)\det(B)}{\Longrightarrow} \det(A)\det(A^{-1}) = 1
\end{aligned}
$$

Hence, $\det(A) \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$. ☐

Definition 4.4. Let A be an $n \times n$ matrix. Define the adjugate of A as:

$$
\operatorname{adj}(A) = \begin{pmatrix} (-1)^{1+1}\det(\widetilde{A}_{11}) & \dots & (-1)^{1+n}\det(\widetilde{A}_{1n}) \\ \vdots & & \vdots \\ (-1)^{n+1}\det(\widetilde{A}_{n1}) & \dots & (-1)^{n+n}\det(\widetilde{A}_{nn}) \end{pmatrix}^T
$$

Theorem 4.5. *(Formula for the inverse) Let A be an $n \times n$ matrix. If $\det(A) \neq 0$, then*

$$
\frac{1}{\det A} \cdot \operatorname{adj}(A)
$$

is the inverse of A.

Proof. We compute $A(\operatorname{adj}(A))$.

<u>Compute the (i,i)-entry on $A(\operatorname{adj}(A))$</u>

The i-th row of A is

$$
\begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix}
$$

and the i-th column of $\operatorname{adj}(A)$ is

$$
\begin{pmatrix} (-1)^{i+1}\det\widetilde{A}_{i1} \\ \vdots \\ (-1)^{i+n}\det\widetilde{A}_{in} \end{pmatrix}.
$$

According to Section 2.1, the (i,i)-th entry of $A(\operatorname{adj}(A))$ is

$$
\sum_{x=1}^{n} (-1)^{i+x} a_{ix} \det\widetilde{A}_{ix}. \tag{4.10}
$$

Note that the term (4.10) is the cofactor expansion for $\det(A)$ along the i-th row, and hence is equal to $\det(A)$. Thus, all the diagonal entries in $A(\operatorname{adj}(A))$ are $\det(A)$.

<u>Compute the (i,i')-entry of $A\operatorname{adj}(A)$ with $i \neq i'$</u>

Again, according to Section 2.1, the (i, i') entry of $A\text{adj}(\widetilde{A})$ is:

$$\sum_{x=1}^{n}(-1)^{i'+x}a_{ix}\det\widetilde{A}_{i'x} \tag{4.11}$$

Let B be the matrix obtained by replacing the i'-th row of A with the i-th row of A. However, the term in (4.11) is simply the cofactor expansion for $\det(B)$ along the i'-th row of B, and so is $\det(B)$. Since B has two identical rows, $\det(B) = 0$ by Corollary 3.10. In other words, all the (i, i')-entry $(i \neq i')$ of $A\text{adj}(A)$ is zero.

We combine the above two cases to have:

$$A(\text{adj}(A)) = \det(A)I_n.$$

When $\det(A) \neq 0$, we have:

$$A\left(\frac{1}{\det(A)}\text{adj}(A)\right) = I_n.$$

□

Exercise 4.6. Suppose the 5-th column of a 6×6-matrix A is

$$\begin{pmatrix}1\\2\\3\\4\\5\\6\end{pmatrix}$$

Find

$$2\det(\widetilde{A}_{13}) - 4\det(\widetilde{A}_{23}) + 6\det(\widetilde{A}_{33}) - 8\det(\widetilde{A}_{43}) + 10\det(\widetilde{A}_{53}) - 12\det(\widetilde{A}_{63})$$

Example 4.7. Find the inverse of the matrix

$$A = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$$

First, $\det(A) = 4 - 6 = -2$. Next we have:

$$\text{adj}(A) = \begin{pmatrix}4 & -3\\-2 & 1\end{pmatrix}^T = \begin{pmatrix}4 & -2\\-3 & 1\end{pmatrix}.$$

Hence, $A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A) = \begin{pmatrix}-2 & 1\\ \frac{3}{2} & \frac{-1}{2}\end{pmatrix}$.

4.3. **Criteria for the invertibility of a matrix.** Combining Theorems 4.3 and 4.5, one can check the invertibility of a matrix as follows:

Theorem 4.8. *(Invertibility by computing determinant) Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$.*

Example 4.9. Determine if the following matrices are invertible. If it is invertible, find the inverse.

- $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & 5 & 7 \end{pmatrix}$
- $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}$
- $C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$.

Solution:

- Compute $\det(A) = \det(B) = 0$ and so A and B are not invertible.
- Compute $\det(C) = 1 \neq 0$ and so C is invertible. Then

$$\operatorname{adj}(C) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}^T, C^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}.$$

4.4. **Inverse with other operations.**

Theorem 4.10. *Let A, B be invertible $n \times n$ matrices. Let $c \in \mathbb{R}$. Then*

(1) *(Inverse anti-commutes with multiplication) $(AB)^{-1} = B^{-1}A^{-1}$;*
(2) *(Inverse commutes with scalar multiplication) $(cA)^{-1} = c^{-1}A^{-1}$;*
(3) *(Inverse commutes with transpose) $(A^T)^{-1} = (A^{-1})^T$;*
(4) *(Inverse of inverse is identity) $(A^{-1})^{-1} = A$.*

Proof. We only check for the left inverse.

(1) $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$
(2) $(c^{-1}A^{-1})(cA) = c^{-1}cA^{-1}A = I_n$
(3) $(A^{-1})^TA^T = (AA^{-1})^T = I_n$
(4) $AA^{-1} = I_n$

$\square$