

$$1. \quad 1^{\circ} \quad D = \begin{pmatrix} I_5 & C_{5 \times 3} \\ 0_{5 \times 5} & B_{3 \times 3} \end{pmatrix} = \begin{pmatrix} I_5 & 0_{5 \times 3} \\ 0_{3 \times 5} & B \end{pmatrix} \cdot \begin{pmatrix} I_5 & C \\ 0_{3 \times 5} & I_3 \end{pmatrix} \cdot \begin{pmatrix} A & 0_{3 \times 3} \\ 0_{3 \times 5} & I_3 \end{pmatrix}$$

$$\det(D) = \det \begin{pmatrix} I_5 & 0_{5 \times 3} \\ 0_{3 \times 5} & B \end{pmatrix} \cdot \det \begin{pmatrix} I_5 & C \\ 0_{3 \times 5} & I_3 \end{pmatrix} \cdot \det \begin{pmatrix} I_5 & 0_{3 \times 3} \\ 0_{3 \times 5} & I_3 \end{pmatrix}$$

Induction: denote: $M_k = \begin{pmatrix} I & 0_{1 \times (k-1)} \\ 0 & M_{k-1} \end{pmatrix}$

proof:

① $k=4$

$$\det(M_4) = \det \begin{pmatrix} I & 0_{1 \times 3} \\ 0 & B \end{pmatrix} = 1 \cdot \det(B) = \det(B)$$

② assume $\det(M_k) = \det(M_{k-1})$

$$\det(M_{k+1}) = \det \begin{pmatrix} I & 0_{1 \times k} \\ 0 & M_k \end{pmatrix} = 1 \cdot \det(M_k) = \det(M_k)$$

$$\Rightarrow \det \begin{pmatrix} I_5 & 0_{5 \times 3} \\ 0_{3 \times 5} & B \end{pmatrix} = \det(M_8) = \det(M_7) = \dots = \det(B)$$

$\Rightarrow \det(D) = \det(B) \neq 0$.

$\Rightarrow D$ is invertible

$$2^{\circ} \quad \text{Let } D^{-1} = \begin{pmatrix} E & F \\ 0_{3 \times 5} & G \end{pmatrix}$$

$$I_8 = D \cdot D^{-1} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} E & F \\ 0 & G \end{pmatrix} = \begin{pmatrix} (AE+CG)_{5 \times 5} & (AF+CH)_{5 \times 3} \\ (DE+BG)_{3 \times 5} & (DF+BH)_{3 \times 3} \end{pmatrix}$$

$$\therefore \begin{cases} AE+CG = I_5 \\ BH = I_3 \end{cases}$$

$$\Rightarrow H = B^{-1} \quad \text{and} \quad AF+CH = 0_{5 \times 3} \Rightarrow AF = -CH = -CB^{-1} \Rightarrow A^{-1}AF = A^{-1}(-CB^{-1}) \Rightarrow F = -A^{-1}CB^{-1}$$

$$\Rightarrow BG = 0_{3 \times 5} \Rightarrow B^{-1}BG = B^{-1}0_{3 \times 5} \Rightarrow G = 0_{3 \times 5}$$

$$\Rightarrow AE+CO = I_5 \Rightarrow AE = I_5 \Rightarrow A^{-1}AE = A^{-1}I_5 \Rightarrow E = A^{-1}$$

$$\therefore D^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}$$

2. Let $V = (a_1 \ a_2 \ \dots \ a_n)$

$$V \cdot V^T = (a_1 \ a_2 \ \dots \ a_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 a_1 & a_1 a_2 \dots a_1 a_n \\ a_2 a_1 & a_2 a_2 \dots a_2 a_n \\ \vdots & \vdots \\ a_n a_1 & a_n a_2 \dots a_n a_n \end{pmatrix}$$

$$\det(V \cdot V^T) = \det \begin{pmatrix} a_1 a_1 & a_1 a_2 \dots a_1 a_n \\ a_2 a_1 & a_2 a_2 \dots a_2 a_n \\ \vdots & \vdots \\ a_n a_1 & a_n a_2 \dots a_n a_n \end{pmatrix}$$

Note that 1st row has common factor a_1
2nd row has common factor a_2

$$= a_1 \cdot a_2 \det \begin{pmatrix} a_1 & a_2 \dots a_n \\ a_2 & a_2 \dots a_n \\ \cancel{a_3 a_1} & \cancel{a_3 a_2} \dots \cancel{a_3 a_n} \\ \vdots & \vdots \\ a_n a_1 & a_n a_2 \dots a_n a_n \end{pmatrix} \leftarrow \begin{matrix} 1^{\text{st}} & 2^{\text{nd}} \\ \text{rows are identical} \end{matrix}$$

$$= 0$$