STAT2602 Assignment 1 Shen Hongshan 3036290936 BEng(DS&E)

Q1 (i) $F(x) = 2\sum_{i=1}^{x}\left(\frac{1}{3}\right)^i = 2\cdot\frac{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^x\right)}{1-\frac{1}{3}} = 1-\left(\frac{1}{3}\right)^x$

$$\Rightarrow F(x) = \begin{cases} 1-\left(\frac{1}{3}\right)^{\lfloor x \rfloor} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

(ii) $M_X(t) = E\left(e^{tX}\right) = 2\sum_{i=1}^{\infty} e^{tx}\cdot\left(\frac{1}{3}\right)^x = 2\sum_{i=1}^{\infty}\left(\frac{e^t}{3}\right)^x = \frac{2e^t}{3}\cdot\frac{1}{1-\frac{e^t}{3}} = \frac{2e^t}{3-e^t}$

for $\frac{e^t}{3} < 1 \Rightarrow t < \ln 3$

(iii) $E(X) = 2\left(1\cdot\frac{1}{3} + 2\left(\frac{1}{3}\right)^2 + \cdots\right)$

$\frac{1}{3}E(X) = 2\left(1\cdot\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 + \cdots\right)$

$\Rightarrow \frac{2}{3}E(X) = \frac{2}{3}\cdot\frac{1}{1-\frac{1}{3}} \Rightarrow E(X) = \frac{3}{2}$ //

$Var(X) = E(X^2) - E^2(X) = 2\sum_{x=1}^{\infty} x^2 f(x) - \frac{9}{4} = 2\times\frac{\frac{1}{3}\left(1+\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)^3} - \frac{9}{4} = \frac{3}{4}$ //

Q2 $X_1 X_2 \cdots X_n \overset{iid}{\sim} Gamma(3, \theta)$ $f(x) = \frac{\lambda^3}{\Gamma(3)} x^{3-1} e^{-\theta x}$

$Y = \sum_{i=1}^{n} X_i$

(i) $M_t(X_i) = \left(\frac{\theta}{\theta-t}\right)^3$ $M_t(Y) = E(e^{Yt}) = E\left(e^{t\sum_{i=1}^{n}X_i}\right) \overset{\text{independence}}{=} \left[E\left(e^{tX_1}\right)\right]^n = \left[M_t(X_1)\right]^n = \left(\frac{\theta}{\theta-t}\right)^{3n}$

(for $t < \theta$)

$\Rightarrow Y \sim Gamma(3n, \theta)$ //

(ii) $E(Y) = \frac{3n}{\theta}$ $\theta^{-1} = E(cY) = cE(Y) = c\frac{3n}{\theta} \Rightarrow c = \frac{1}{3n}$ //

(iii) $M_{3\theta Y+1}(t) = E\left(e^{(3\theta Y+1)t}\right) = E\left(e^t\cdot e^{(3\theta t)Y}\right) = e^t M_Y(3\theta t)$

$= e^t\left(\frac{\theta}{\theta-3\theta t}\right)^{3n} = e^t\left(\frac{1}{1-3t}\right)^{3n}$ (for $t < \frac{1}{3}$)

Q3 (i) $M_X'(t) = -\frac{3}{4}e^{-3t} + \frac{1}{4}e^t \Rightarrow E(X) = M_X'(0) = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$ //

$M_X''(t) = \frac{9}{4}e^{-3t} + \frac{1}{4}e^t \Rightarrow E(X^2) = M_X''(0) = \frac{9}{4} + \frac{1}{4} = \frac{5}{2}$

$Var(X) = E(X^2) - E^2(X) = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$ //

(ii) $M_X(t) = E\left(e^{tX}\right) = \frac{1}{4}e^{-3t} + \frac{1}{4}e^{1t} + \frac{1}{2}e^{0\cdot t}$

x	-3	0	1
Prob	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

pdf: $f(x) = \begin{cases} \frac{1}{4} & x = -3, 1 \\ \frac{1}{2} & x = 0 \end{cases}$

check: $E(X) = -3\times\frac{1}{4} + 0\times\frac{1}{2} + 1\times\frac{1}{4} = -\frac{1}{2}$

$Var(X) = \left(-\frac{5}{2}\right)^2\times\frac{1}{4} + \left(-\frac{1}{2}\right)^2\times\frac{1}{2} + \left(\frac{3}{2}\right)^2\times\frac{1}{4}$

$= \frac{9}{4}$ correct!

Q4 (i)

#swim	frequency
0	1
1	1
2	2
3	2
4	1
6	2
7	1

$\Rightarrow$

x	$F_n(x)$
$(-\infty, 0)$	0
$[0, 1)$	0.1
$[1, 2)$	0.2
$[2, 3)$	0.4
$[3, 4)$	0.6
$[4, 6)$	0.7
$[6, 7)$	0.9
$[7, \infty)$	1

$$F_{10}(x) = \frac{1}{10}\sum_{k=1}^{10} \mathbb{I}(X_k \le x)$$

(ii) $P(X \le 4) \approx F_{10}(4) = 0.7$

$P(4 < X < 7) \approx F_{10}(7^-) - F_{10}(4^+) = 0.9 - 0.1 = 0.2$

Q5 $X = \varepsilon_1 + \varepsilon_2$ $\varepsilon_1 \sim N(\theta, 1)$ $\varepsilon_2 \sim N(\lambda\theta, \lambda^2)$ $\varepsilon_1 \perp\!\!\!\perp \varepsilon_2$ $\lambda \ge 1$.

(i) $M_X(t) = E(e^{tX}) = E(e^{t(\varepsilon_1+\varepsilon_2)}) \underset{\text{independence}}{=} E(e^{t\varepsilon_1}) \cdot E(e^{t\varepsilon_2}) = M_{\varepsilon_1}(t) \cdot M_{\varepsilon_2}(t)$

$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$= e^{\theta t + \frac{1}{2}t^2} \cdot e^{\lambda\theta t + \frac{1}{2}\lambda^2 t^2}$

$= e^{\theta(1+\lambda)t + \frac{1}{2}(1+\lambda^2)t^2}$ //

(ii) $\Rightarrow X \sim N((1+\lambda)\theta, 1+\lambda^2)$ Let $\mu = (1+\lambda)\theta$ $\sigma^2 = 1+\lambda^2$

$E(X) = \mu$ $E(X^2) = Var(X) + E^2(X) = \sigma^2 + \mu^2$

$0 = E((X-\mu)^3) = E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3$

$\Rightarrow E(X^3) = 3\mu\sigma^2 + \cancel{3\mu^3} - \cancel{3\mu^3} + \mu^3$

$= 3\theta(1+\lambda)(1+\lambda^2) + (1+\lambda)^3\theta^3$ //

(iii) $X \sim N((1+\lambda)\theta, 1+\lambda^2)$ Normal dist. //

Q6. $\bar{X}$ $n = 25$ pdf $f(x) = x^3/4$ $0 < x < 2$

$E(\bar{X}) = E(X) = \int_0^2 \frac{x^4}{4}\,dx = \frac{1}{20}\left[x^5\right]_0^2 = \frac{32}{20} = \frac{8}{5}$

$E(X^2) = \int_0^2 \frac{x^5}{4}\,dx = \frac{1}{24} \cdot x^6 \Big|_0^2 = \frac{64}{24} = \frac{8}{3}$

$Var(\bar{X}) = \frac{Var(X)}{25} = \frac{E(X^2) - E(X)}{25} = \frac{\frac{8}{3} - \frac{64}{25}}{25} = \frac{8}{1875}$

by CLT. $\bar{X} \approx N\left(\frac{8}{5}, \frac{8}{1875}\right)$

$P(1.2 \le \bar{X} \le 1.6) = P\left(-6.124 \le \frac{\bar{X} - \frac{8}{5}}{\sqrt{\frac{8}{1875}}} \le 0\right) \approx 0.5$

Q7 $X_1 X_2 \cdots X_{12} \overset{iid}{\sim} U(0,1)$

(i) $Y = X_{(12)}$

$$F_Y(y) = P(X_{(12)} \le y) = P(X_1 \le y, X_2 \le y, \cdots X_{12} \le y) \overset{\text{Independent}}{=} [F(y)]^{12} = y^{12} \mathbb{1}_{[0,1]}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 12y^{11} & y \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

(ii) $Z = \left(\sum_{i=1}^{12} X_i\right) - 6$

$$M_{X_i}(t) = E(e^{tX_i}) = \int_0^1 e^{tx}\,dx = \frac{e^{tx}}{t}\Big|_0^1 = \frac{e^t - 1}{t}$$

$$M_Z(t) = E(e^{tZ}) = E\left(e^{(\sum_{i=1}^{12} X_i)t - 6t}\right) \underset{\text{iid}}{=} e^{-6t}\left[E(e^{X_i t})\right]^{12} = e^{-6t}\left[M_X(t)\right]^{12} = e^{-6t}\left(\frac{e^t - 1}{t}\right)^{12}$$

(iii) $E\left(\sum_{i=1}^{12} X_i\right) = 6 \quad Var\left(\sum_{i=1}^{12} X_i\right) = 12\, Var(X_i) = 12 \times \frac{1}{12} = 1$

By CLT $\sum_{i=1}^{12} X_i \approx N(6,1)$

thus, $Z = \sum_{i=1}^{12} X_i - 6 \overset{\text{approx}}{\approx} N(0,1)$

Q8. (i) ① base case: $n=2$ $\quad \frac{X_1 + X_2}{2} = \frac{1}{2}X_1 + (1-\frac{1}{2})X_2 \sim$ Cauchy.

② Induction: assume when $n=k$ $\quad \frac{1}{k}\sum_{i=1}^{k} X_i \sim$ Cauchy

$n = k+1$ $\quad \frac{1}{k+1}\sum_{j=1}^{k+1} X_j = \frac{k}{k+1}\underbrace{\left[\sum_{i=1}^{k} X_i\right]}_{\sim \text{Cauchy}} + \frac{1}{k+1}\underbrace{X_{k+1}}_{\sim \text{Cauchy}} \sim$ Cauchy

$\Rightarrow$ by ①②. $\bar{X} \sim$ Cauchy

$\Rightarrow$ $\bar{X}$ does not converge in probability to anything

(ii) LLN require finite mean

but Cauchy mean is not defined (no finite mean)

$$E(X) = \int_0^\infty \frac{1}{2\pi(1+x^2)}\,dx \qquad u = 1+x^2 \quad du = 2x\,dx$$

$$= \int_1^\infty \frac{1}{2\pi u (2u^{\frac{1}{2}})}\,du$$

$$= \int_1^\infty \frac{1}{4\pi} u^{-\frac{3}{2}}\,du$$

$$= -\frac{1}{2\pi} u^{-\frac{1}{2}}\Big|_0^\infty$$

$$= -\frac{1}{2\pi}\lim_{u\to\infty} u^{-\frac{1}{2}} \to \infty$$ is not finite mean