

THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

May 23, 2016

2:30 – 5:00 pm

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all SIX questions. Each question carries 16 marks.
Another 4 marks is given for clarity of presentation and following instructions.

Notes:

- You should give precise and adequate explanations to support your conclusions unless otherwise specified by the question. Clarity of presentation of your argument counts. So think carefully before you write.
- You must start each question on a new page. However do NOT start each part of a question on a new page. You should write down the question number on the top right hand corner of each page. Indicate clearly which part of a question you are answering.
- If your answer to a question spans over more than one page, you must indicate clearly on each page (except the last) that the answer will continue on the subsequent page.
- If you do not otherwise specify, you are assumed to be answering the questions in order. In particular, indicate on each question where your answer to the next question is if it does not follow immediately. (For example you may write 'turn back 3 pages for Question 5' at the end of Question 4.)
- Whenever there is a blank page it will be assumed that all subsequent pages are also blank. If this is not true, indicate clearly on the page which you would otherwise leave blank.

1. Give answers only to the following questions. Explanation is not required.

Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

- (a) What is the reduced row echelon form of A ?
- (b) What is the inverse of A ?
- (c) What is the rank of A ?
- (d) Let T_A denote the linear transformation induced by A . Find a vector \mathbf{x} such that $T_A(\mathbf{x}) = \mathbf{e}_1$.
- (e) Find all eigenvalues of A .
- (f) Let $\mathcal{A} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathcal{B} = \{\mathbf{u}_2, \mathbf{u}_2 + \mathbf{u}_3, \mathbf{u}_1\}$ be two ordered bases for \mathbb{R}^3 . If $[\mathbf{v}]_{\mathcal{A}} = \mathbf{w}$, find $[\mathbf{v}]_{\mathcal{B}}$.
- (g) Find a matrix with \mathbf{w} as an eigenvector.
- (h) Find the projection of $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ on $\text{Span}\{\mathbf{w}\}$.

2. In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do not indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is not required.

- (a) Let A be an $m \times n$ matrix. Which of the following is possible?
 - (1) The rank of A is $m + n$.
 - (2) The nullity of A is m .
 - (4) The system $A\mathbf{x} = \mathbf{0}$ has no solution.
 - (8) The span of the rows of A is the same as the span of the columns of A .

(b) Which of the following matrices is invertible?

(1)
$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & 5 & 9 & 0 \\ 3 & 7 & 11 & 0 \\ 4 & 9 & 15 & 0 \end{bmatrix}$$

(2)
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

(4)
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

(8)
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

(c) The 4×4 zero matrix is

- (1) invertible
- (2) an elementary matrix
- (4) in reduced row echelon form
- (8) similar to any 4×4 matrix with determinant 0

(d) Let V, W be subspaces of \mathbb{R}^n , and $S, T, \mathcal{B}_1, \mathcal{B}_2$ be subsets of both V and W . Which of the following is true?

- (1) If S generates both V and W , then $V = W$.
- (2) If S and T both generate V , then $S = T$.
- (4) If \mathcal{B} is a basis of both V and W , then $V = W$.
- (8) If \mathcal{B}_1 and \mathcal{B}_2 are both bases of V , then $\mathcal{B}_1 = \mathcal{B}_2$.

- (e) If A and B are both upper triangular square matrices, then
- (1) AB is also upper triangular
 - (2) A^T is lower triangular
 - (4) the diagonal entries of A are eigenvalues of A
 - (8) $\det A$ is equal to the product of the diagonal entries of A
- (f) Let W be a subspace of \mathbb{R}^3 . Which of the following is true about W^\perp ?
- (1) It must be non-empty.
 - (2) It must be an infinite set.
 - (4) It must be a subspace of \mathbb{R}^3 .
 - (8) It must have the same dimension as W .
- (g) If S is an orthonormal set of nonzero vectors in \mathbb{R}^n , then
- (1) S is ~~an~~ an orthogonal set
 - (2) S has at most n vectors
 - (4) each vector in S has norm 1
 - (8) if we apply the Gram-Schmidt process on S , we will get back the set S
- (h) Let A be a matrix. Which of the following must be well-defined?
- (1) The eigenvalues of A
 - (2) The determinant of AA^T
 - (4) The column space of A
 - (8) The rank of A^2

3. For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

- (a) If A is an elementary matrix, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$.
- (b) Define a function T by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 1 \end{bmatrix}$. Then T is a linear transformation.
- (c) Let A be a non-zero square matrix. If $\text{adj } A$ is invertible, then A is also invertible.
- (d) Applying an elementary row operation on a square matrix will not change its determinant.
- (e) If \mathbf{v} is an eigenvector of a square matrix A , then \mathbf{v} is also an eigenvector of A^2 .
- (f) If \mathbf{v} is an eigenvector of a square matrix A with algebraic multiplicity 1, then its geometric multiplicity must also be 1.
- (g) Let S be a subset of \mathbb{R}^n . Then $S \cap S^\perp$ contains at most one vector.
- (h) If \mathbf{u} and \mathbf{v} are orthogonal, then $a\mathbf{u}$ and $b\mathbf{v}$ are orthogonal for any scalars a and b .

4. Let W be the plane $x + 2y - 3z = 0$ in \mathbb{R}^3 .

- (a) Find a basis \mathcal{B}_1 of W and a basis \mathcal{B}_2 of W^\perp . Justify your answers carefully.
- (b) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation corresponding to reflection across W , and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Find
 - (i) the \mathcal{B} -matrix of T (make clarifications on \mathcal{B} where necessary);
 - (ii) the standard matrix of T ;
 - (iii) the eigenvalues of the matrix in (ii).

5. Give brief answers (say, one sentence each; details are not required) to the following questions.

- (a) Let A and B be square matrices with the same size. We know that if AB is invertible, then so are A and B . However, we cannot prove this using determinants. Why not?
- (b) Describe all linear transformations $T : \mathbb{R} \rightarrow \mathbb{R}$.
- (c) If T is a linear transformation, why must we have $T(0) = 0$?
- (d) If a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is injective, then its standard matrix A has rank n . Outline the main step of the proof in one sentence.
- (e) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation with standard matrix A . Construct a sentence involving T and $\det A$, without including any other variable or symbol.
- (f) For any $\mathbf{v} \in \mathbb{R}^n$, we have the formula $[\mathbf{v}]_B = B^{-1}\mathbf{v}$. What is the difference between B and B^{-1} ?
- (g) In the previous question, why does B^{-1} exist?
- (h) What is the difference between *null space* and *nullity*?

6. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ be distinct vectors in \mathbb{R}^n .

- (a) Suppose $\mathbf{x}_i \cdot \mathbf{x}_i = q$ for all i and $\mathbf{x}_i \cdot \mathbf{x}_j = p$ for all $i \neq j$, where p and q are constants with $q > p > 0$. Using mathematical induction on m , or otherwise, show that $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ are linearly independent.
- (b) If the distance between any two distinct \mathbf{x}_i 's is the same (note that we no longer assume $\mathbf{x}_i \cdot \mathbf{x}_i = q$ for all i and $\mathbf{x}_i \cdot \mathbf{x}_j = p$ for all $i \neq j$), show that $m \leq n + 1$.
(Hint: Let $\mathbf{y}_i = \mathbf{x}_i - \mathbf{x}_m$ for $1 \leq i \leq m - 1$ and apply the result of (a).)
- (c) In each of the cases $n = 2$ and $n = 3$, give an example to show that equality may hold in (b).

***** END OF PAPER *****