MATH 2101 Linear Algebra I, FALL 2025

Tutorial 1 Solution

1. We claim that B is of the form $\begin{pmatrix} r & r \\ r & r \end{pmatrix}$ for $r \neq 0$.

   - Suppose $A + B = kB$ for some scalar k. This can be rewritten as $A = (k-1)B$. Since A is not the zero matrix, we have $k \neq 1$, and hence $B = \frac{1}{k-1}A$. This shows B is a scalar multiple of A. The claim holds by taking $r = \frac{1}{k-1} \neq 0$.
   - Conversely, suppose that $B = rA$ for some nonzero scalar r. Then we have

   $$A + B = \frac{1}{r}B + B = \left(\frac{1}{r} + 1\right)B,$$

   which shows that $A + B$ is a scalar multiple of B.

2. - If A is symmetric, then $A = A^T$, which is equivalent to

   $$\begin{cases} x + y = 0, \\ x^2 = y + 6, \\ z = 2z - 1. \end{cases}$$

   The last equation implies $z = 1$. From the first equation, we have $y = -x$. Substituting into the second equation, we obtain $x^2 = y + 6 = -x + 6$. The roots to the quadratic equation $x^2 + x - 6 = 0$ are $x = 2$ and $x = -3$. So we get $(x, y, z) = (2, -2, 1)$ or $(x, y, z) = (-3, 3, 1)$.
   - Conversely, if $(x, y, z) = (2, -2, 1)$ or $(-3, 3, 1)$, then

   $$A = \begin{pmatrix} -2 & 0 & 4 \\ 0 & 0 & 1 \\ 4 & 1 & -3 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 0 & 9 \\ 0 & 0 & 1 \\ 9 & 1 & 2 \end{pmatrix}.$$

   Clearly, for both cases, A is symmetric.

3. (a)
   - Suppose A is both symmetric and skew-symmetric. Then $A^T = A$ and $A^T = -A$. Thus $2A = 0_{n\times n}$ and hence $A = 0_{n\times n}$.
   - Conversely, suppose $A = 0_{n\times n}$. Clearly, A is both symmetric and skew-symmetric.

   To conclude, A must be the zero matrix $0_{n\times n}$.

   (b) As $\det K = \det K^T = \det(-K) = (-1)^7 \det K = -\det K$, we have $2 \det K = 0$ and hence $\det K = 0$.

   (c) *(Existence)*
   Let $S = \frac{1}{2}(A + A^T)$ and $K = \frac{1}{2}(A - A^T)$. Clearly, $A = S + K$, and S is symmetric since

   $$S^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = S.$$

   Also K is skew-symmetric since

   $$K^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -K.$$

(Uniqueness)
Suppose there exist symmetric matrices S and S' and skew-symmetric matrices K and K' such that $A = S + K = S' + K'$. Then we have $S - S' = K' - K$. Note that

$$(S - S')^T = S^T - (S')^T = S - S',$$

so $S - S'$ is symmetric. On the other hand, we have

$$(K' - K)^T = (K')^T - K^T = -K' - (-K) = -(K' - K),$$

so $K' - K$ is skew-symmetric. Let $B = S - S' = K' - K$. It follows from (a) that $B = 0_{n\times n}$ and hence we must have $S = S'$ and $K = K'$, proving the uniqueness of S and K.

4. (a) Disproof by counter-example:
   Take $n = 2$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then clearly $A^2 = A$. But A is different from I_2 and $0_{2\times 2}$.

   (b) Disproof by counter-example:
   Take $n = 3$ and $A = I_3$, then $A(A - I_3) = 0_{3\times 3}$ and

$$\det(AA^T) = \det I_3 = 1 \neq -1 = \det I_3 = \det(-A^T).$$

   (c) Proof:
   Note that $(I - A)^2 = I - 2A + A^2 = I - A$.

$$\det[(I - A)^2] = \det(I - A) \Rightarrow \det(I - A)[\det(I - A) - 1] = 0.$$

   So $\det(I - A) = 0$ or 1.