THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH2101: Linear Algebra I

December 20, 2021 9:30a.m. – 12:00noon

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer ALL SEVEN questions

Note: You should always give precise and adequate explanations to support your conclusions except question 1 and question 2. Clarity of presentation of your argument counts. So **think carefully before you write.**

Short questions. Put down the answer only.

1. (a) (2 points) What is the smallest possible rank of a nonzero 4×3 matrix?

   (b) (2 points) Let I be the 2×2 identity matrix, and let $A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$. Find the $(2,3)$th entry of $\begin{pmatrix} I & I \\ I & I \end{pmatrix}\begin{pmatrix} A & A \\ A & A \end{pmatrix}$.

   (c) (2 points) Find a matrix A such that $A\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & -4 & -9 \end{pmatrix} = \begin{pmatrix} -1 & -4 & -9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$.

   (d) (2 points) Find an LU decomposition of $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$.

   (e) (2 points) Let $\beta = \{(2,3)^T, (1,-2)^T\}$ be an ordered basis for $\mathbb{R}^2$. If the coordinate vector of $\mathbf{v}$ relative to β is $(3,1)^T$, what is $\mathbf{v}$?

   (f) (2 points) Given that $(2,1,0,1)^T$ is an eigenvector of $\begin{pmatrix} -4 & 3 & 5 & -1 \\ 2 & -6 & -3 & -1 \\ -1 & -1 & 2 & 3 \\ 2 & -5 & -7 & -2 \end{pmatrix}$, what is the corresponding eigenvalue?

   (g) (2 points) Find the orthogonal projection of the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ onto $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

   (h) (2 points) Find the equation of the best fit line in the least square sense for the data points $(0,0), (2,0), (3,4)$.

Multiple choice questions. Add up the numbers of all the correct options and put down the result only. In case all options are wrong, answer 0.

2. (a) (2 points) Consider a system of 4 linear equations in 3 unknowns with real coefficients. Which of the following are possible?

   (1) The system has no solution.

   (2) The system has exactly 1 solution.

   (4) The system has exactly 4 solutions.

   (8) The system has infinitely many solutions.

(b) (2 points) Let A be a square matrix. In which of the following cases must A be invertible?

(1) $\det A = 0$

(2) $\det(A^2) = 1$

(4) $\det(-A) = 2$

(8) $\det(A + I) = 3$

(c) (2 points) Let A be a 4×4 matrix. Which of the following operations do not change the determinant of A?

(1) taking the transpose of A

(2) multiply the first two rows of A by -1

(4) adding -1 times the first row of A to the second row

(8) replacing the first three rows $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ of A by $\mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_1$ simultaneously

(d) (2 points) Let V be a subspace of $\mathbb{R}^4$ with dimension 3. Which of the following are correct?

(1) $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ belong to V

(2) all bases for V have exactly 3 vectors

(4) V contains 2 linearly independent vectors

(8) all generating sets of V have exactly 3 vectors

(e) (2 points) Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x, 2y)^T$. Which of the following are matrix representations of T with respect to some ordered bases?

(1) $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

(2) $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$

(4) $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

(8) $\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$

(f) (2 points) Which of the following subspaces must have dimension 2?

(1) $\text{span}\{(1,1,1)^T, (2,3,4)^T\}$

(2) $\text{Col}\, A$ where A is a 3×2 matrix

(4) $\text{Row}\, A$ where A is a 3×2 matrix

(8) $\text{Null}\, A$ where A is a 2×2 non-invertible matrix

(g) (2 points) Let A be a square matrix with $-t^3(t-1)^2$ as the characteristic polynomial. Let E_0 and E_1 be the eigenspaces of A corresponding to 0 and 1 respectively. Suppose $\dim E_0 = \dim E_1 = 2$. Which of the following are correct?

(1) $\text{rank}\, A = 3$

(2) A is invertible

(4) $\text{nullity}(A - I) = 1$

(8) A is diagonalizable

(h) (2 points) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Which of the following are identities?

(1) $|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\| \|\mathbf{v}\|$

(2) $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$

(4) $\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}$

(8) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$

True or false questions. Explanations are required.

3. (a) (4 points) Let A be a square matrix. If A is in reduced row echelon form, then A^T is in reduced row echelon form.

(b) (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If there exist linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ such that $T(\mathbf{u}) = T(\mathbf{v})$, then the standard matrix of T is non-invertible.

(c) (4 points) We have $\operatorname{adj}(A+B) = \operatorname{adj} A + \operatorname{adj} B$ for any $n \times n$ matrices A and B where $n \geqslant 2$.

(d) (4 points) Let A be a square matrix with $(t^2-1)^2$ as the characteristic polynomial. Then A has at least 3 linearly independent eigenvectors.

Long questions. Explanations are required.

4. Let $V = \{(x,y,z)^T \in \mathbb{R}^3 : 2x + y - z = 0\}$ be a subset of $\mathbb{R}^3$.

   (a) (5 points) Prove that V is a subspace of $\mathbb{R}^3$.

   (b) (8 points) Find an orthonormal basis for V and find the dimension of V.

5. Let $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$.

   (a) (9 points) **Using diagonalization**, compute A^n where n is a positive integer.

   (b) (4 points) Define a sequence $\{x_n\}$ by $x_1 = 0$, $x_2 = 2$ and $x_{n+2} = -3x_{n+1} - 2x_n$ for any positive integer n. Find a general formula for x_n **by using the matrix** A. (Hint: Consider $A\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$.)

6. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a subset of $\mathbb{R}^n$, and let $S' = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$.

   (a) (9 points) If T is injective, prove that S is linearly independent if and only if S' is linearly independent.

   (b) (4 points) Give an example for which S is linearly independent but S' is linearly dependent.

7. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}\}$ be a set of $n-1$ vectors in $\mathbb{R}^n$.

(a) (3 points) Prove that $S^\perp$ contains a nonzero vector.

(b) (3 points) If $S^\perp$ contains two linearly independent vectors, what can you say about S?

(c) (7 points) Define an $n \times n$ matrix A such that the first column of A is $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)^T$ and the ith column of A is $\mathbf{v}_{i-1}$ for $i = 2, 3, \ldots, n$. We regard $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ as some variables and compute $\det A$. The result obtained is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, which can be interpreted as a vector in $\mathbb{R}^n$. Prove that this vector belongs to $S^\perp$.

(For example, if $n = 3$, $\mathbf{v}_1 = (0, 1, 2)^T$ and $\mathbf{v}_2 = (3, 0, 4)^T$, then we have $\det A = \begin{vmatrix} \mathbf{e}_1 & 0 & 3 \\ \mathbf{e}_2 & 1 & 0 \\ \mathbf{e}_3 & 2 & 4 \end{vmatrix} = 4\mathbf{e}_1 + 6\mathbf{e}_2 - 3\mathbf{e}_3 = (4, 6, -3)^T$.)

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