MATH 2101 LINEAR ALGEBRA I, FALL 2025 – ASSIGNMENT 2

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Due date: 22 September (Monday), 2025 10:00pm.

- To receive *full credits*, the solution has to be *clear* and provide sufficient explanations.
- All solutions have to be turned in HKU moodle in the format of *PDF file*.
- Please include your *Name, UID, Faculty, Major (if declared)* in your solution.

1. Let A be an invertible 5×5 matrix and let B be an invertible 3×3 matrix. Let C be a 5×3 matrix. Consider the following block matrix:

$$D = \begin{pmatrix} A & C \\ 0_{3\times5} & B \end{pmatrix}.$$

Determine if D is invertible. If D is invertible, find D^{-1} in terms of A^{-1}, B^{-1}, A, B, C. If D is not always invertible, give an example.

2. Let v be an $n \times 1$ matrix. Suppose $n \geq 2$. Use properties of determinants to prove that $\det(vv^T) = 0$.

vector

1.(1) D blockmatrix is upper-triangular $\Rightarrow \det(D) = \underset{\neq 0}{\det(A)} \cdot \underset{\neq 0}{\det(B)} \neq 0$ ∴ D invertible

A, B are square

1.(2) Let $D^{-1} = \begin{pmatrix} E_{5\times5} & F_{5\times3} \\ G_{3\times5} & H_{3\times3} \end{pmatrix}_{8\times8}$

$$I_8 = D \cdot D^{-1} = \begin{pmatrix} A & C \\ 0_{3\times5} & B \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} (AE+CG)_{5\times5} & (AF+CH)_{5\times3} \\ (0 \cdot E + BG)_{3\times5} & (0F+BH)_{3\times3} \end{pmatrix}$$

$$\begin{cases} AE+CG = I_5 \Rightarrow AE + C\cdot 0 = I_5 \Rightarrow AE = I_5 \Rightarrow \cancel{A}E = A^{-1} \\ \cancel{0F}+BH = I_3 \Rightarrow H = B^{-1} \\ \cancel{0E}+BG = 0_{3\times5} \Rightarrow B^{-1}BG = B^{-1}0_{3\times5} \Rightarrow G = 0_{3\times5} \\ AF+CH = 0_{5\times3} \Rightarrow AF = -CH = -CB^{-1} \Rightarrow F = -A^{-1}CB^{-1} \end{cases}$$

$$\therefore D^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0_{3\times5} & B^{-1} \end{pmatrix}$$

2. Let $v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ $vv^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot (a_1 a_2 \cdots a_n) = \begin{pmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & \cdots & \cdots & a_2a_n \\ \vdots & & & \\ a_na_1 & \cdots & \cdots & a_na_n \end{pmatrix} = \left(\prod_{i=1}^{n} a_i\right) \underbrace{\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & & \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}}_{M}$

M with same rows

∴ $\det(M) = 0$

$$\det(vv^T) = \left(\prod_{i=1}^{n} a_i\right) \cdot \det(M) = \left(\prod_{i=1}^{n} a_i\right) \cdot 0 = 0.$$