THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH2101: Linear Algebra I

December 22, 2022 2:30pm – 5:00pm

No calculators are allowed in the examination.

Answer all questions. The full score of this paper is 100.

Note: You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**

Part A. Ten true or false questions (2 points each). No explanation is needed.

(1) If $T : V \to W$ is a linear transformation, then T is injective if and only if the kernel of T is trivial.

(2) There exists $A \in M_2(\mathbb{R})$ such that A^2, A, and I_2 are linearly independent in the vector space $M_2(\mathbb{R})$.

(3) If $T : \mathbb{R}^9 \to \mathbb{R}^8$ is linear, then the kernel of T is of odd dimension if and only if the image of T is of even dimension.

(4) There exists an orthogonal matrix $Q \in M_n(\mathbb{R})$ having 2 as an eigenvalue.

(5) If $A \in M_n(\mathbb{R})$ is diagonalizable, then $22I_n + 12A + 2022A^2$ is diagonalizable.

(6) Every real symmetric matrix is diagonalizable.

(7) Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the transpose map, i.e., $T(A) = A^t$. Let β be an ordered basis of $M_2(\mathbb{R})$. The characteristic polynomial of the matrix representation $[T]_\beta$ is $(t-1)^3(t+1)$.

(8) Let V be a finite dimensional vector space and $T : V \to V$ a linear map. If β_1 and β_2 are two ordered bases of V, then the matrix representations $[T]_{\beta_1}$ and $[T]_{\beta_2}$ have the same reduced row echelon form.

(9) If $S \subset \mathbb{R}^n$ is a non-empty orthogonal subset, then S is linearly independent.

(10) There are only finitely many orthogonal matrices Q in $M_{100}(\mathbb{R})$ such that all the entries of Q are integers.

Part B. Seven long questions. Show your steps or give adequate explanations to support your conclusions.

(11) (5 pts) Solve the following linear system by Gaussian elimination:

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 5 & 4 & 5 \\ 1 & 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix}.$$

(12) (10 pts) Let $\beta = \{(2,0,1,1)^t, (2,1,3,1)^t, (1,1,3,1)^t, (1,1,1,0)^t\}$ be an ordered basis of $\mathbb{R}^4$. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map defined by

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a-b \\ b-c \\ c-d \\ d-a \end{pmatrix}.$$

Find the matrix representation $[T]_\beta$.

(13) Let $A = \begin{pmatrix} 9 & -6 & -6 \\ 4 & -1 & -4 \\ 4 & -4 & -1 \end{pmatrix}$.

(a) (5 pts) Find the characteristic polynomial of A.

(b) (10 pts) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(14) Let $\mathbf{w} = (1, 3, -1, 1, 0, 2)^t \in \mathbb{R}^6$ and $V \subset \mathbb{R}^6$ be the subspace spanned by $S = \{(1, 1, 1, 1, 1, 1)^t,\ (2, 2, 0, 0, 1, 1)^t,\ (0, 2, 2, 1, 1, 0)^t\}$

(a) (10 pts) Find an orthogonal basis of V by applying Gram-Schmidt process on S.

(b) (5 pts) Find the distance between $\mathbf{w}$ and V.

(15) (10 pts) Let $F : V \to W$ and $G : W \to U$ be linear transformations. Prove that

$$\dim(\mathrm{Im}(F) \cap \mathrm{Ker}(G)) = \dim(\mathrm{Ker}(G \circ F)) - \dim(\mathrm{Ker}(F)).$$

(16) (10 pts) Let $A \in M_n(\mathbb{R})$ be a singular matrix such that the ranks of A and A^2 are different. Prove that the geometric multiplicity of 0 is smaller than the algebraic multiplicity of 0.

(17) (a) (5 pts) Prove that $x(x-1)$, $x(x+1)$, and $(x-1)(x+1)$ are linearly independent in $\mathbb{R}[x]$, the vector space of real polynomials.

(b) (10 pts) Let $A \in M_n(\mathbb{R})$ such that $A^3 = A$. Prove that A is diagonalizable.

******* **END OF PAPER** *******