MATH 2101 Assignment 2 Shen Hongshan 3036290936 BEng (DS&E)

1. 1° $D = \begin{pmatrix} A_{5\times5} & C_{5\times3} \\ 0_{3\times5} & B_{3\times3} \end{pmatrix} = \begin{pmatrix} I_5 & 0_{5\times3} \\ 0_{3\times5} & B \end{pmatrix} \cdot \begin{pmatrix} I_5 & C \\ 0_{3\times5} & I_3 \end{pmatrix} \cdot \begin{pmatrix} A & 0_{5\times3} \\ 0_{3\times5} & I_3 \end{pmatrix}$

$\det(D) = \det\begin{pmatrix} I_5 & 0_{5\times3} \\ 0_{3\times5} & B \end{pmatrix} \cdot \det\begin{pmatrix} I_5 & C \\ 0_{3\times5} & I_3 \end{pmatrix} \cdot \det\begin{pmatrix} A & 0_{5\times3} \\ 0_{3\times5} & I_3 \end{pmatrix}$

upper triangular matrix. with diagonal all '1'

determinant = 1

Induction: denote: $M_k = \begin{pmatrix} 1 & 0_{1\times(k-1)} \\ 0_{(k-1)\times1} & N \end{pmatrix}$ matrix transformed from B by adding $1_{1\times1}$ and $\vec{0}, \vec{0}^T$

proof:

① $k=4$

$\det(M_4) = \det\begin{pmatrix} 1 & 0_{1\times3} \\ 0_{3\times1} & B \end{pmatrix} = 1 \cdot \det(B) = \det(B)$

② assume $\det(M_k) = \det(M_{k-1})$

$\det(M_{k+1}) = \det\begin{pmatrix} 1 & 0_{1\times k} \\ 0_{k\times1} & M_k \end{pmatrix} = 1 \cdot \det(M_k) = \det(M_k)$

$\Rightarrow \det\begin{pmatrix} I_5 & 0_{5\times3} \\ 0_{3\times5} & B \end{pmatrix} = \det(M_8) = \det(M_7) = \cdots = \det(B)$

$\Rightarrow \det(D) = \det(B) \neq 0$. invertible

$\Rightarrow D$ is invertible

2° Let $D^{-1} = \begin{pmatrix} E_{5\times5} & F_{5\times3} \\ G_{3\times5} & H_{3\times3} \end{pmatrix}$

$I_8 = D \cdot D^{-1} = \begin{pmatrix} A & C \\ 0_{3\times5} & B \end{pmatrix} \cdot \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} (AE+CG)_{5\times5} & (AF+CH)_{5\times3} \\ (\cancel{0E}+BG)_{3\times5} & (\cancel{0F}+BH)_{3\times3} \end{pmatrix}$

$\therefore \begin{cases} AE+CG = I_5 \\ BH = I_3 \Rightarrow H = B^{-1} \\ AF+CH = 0_{5\times3} \Rightarrow AF = -CH = -CB^{-1} \Rightarrow A^{-1}AF = A^{-1}(-CB^{-1}) \Rightarrow F = -A^{-1}CB^{-1} \\ BG = 0_{3\times5} \Rightarrow B^{-1}BG = B^{-1}0_{3\times5} \Rightarrow G = 0_{3\times5} \end{cases}$

$\Rightarrow AE + C0 = I_5 \Rightarrow AE = I_5 \Rightarrow A^{-1}AE = A^{-1}I_5 \Rightarrow E = A^{-1}$

$\therefore D^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0_{3\times5} & B^{-1} \end{pmatrix}$

2. Let $v = (a_1\ a_2\ \cdots\ a_n)$

$$v \cdot v^T = (a_1\ a_2\ \cdots\ a_n)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & & & \\ a_na_1 & a_na_2 & \cdots & a_na_n \end{pmatrix}$$

$$\det(v \cdot v^T) = \det\begin{pmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & & & \\ a_na_1 & a_na_2 & \cdots & a_na_n \end{pmatrix}$$

note that 1^{st} row has common factor a_1
2^{nd} row has common factor a_2

$$= a_1 \cdot a_2 \det\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ a_3a_1 & a_3a_2 & \cdots & a_3a_n \\ \vdots & & & \\ a_na_1 & a_na_2 & \cdots & a_na_n \end{pmatrix} \to 0$$

← 1^{st} 2^{nd} rows are identical

$$= 0$$