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MATH2101 Assignment 1

1. U upper triangular matrix, diagonal 0, dimension $n\times n$.

1° ~~base: $n=2$~~ $U=\begin{pmatrix}0 & a_{12}\\ 0 & 0\end{pmatrix}$ $U^2=\begin{pmatrix}0 & a_{12}\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & a_{12}\\ 0 & 0\end{pmatrix}=0$

2° induction: $\cancel{\exists (n-1)\in\{2,}$ $\exists\, n\in\{3,4,5,\dots\}$, s.t. for $\forall\, U_{(n-1)\times(n-1)}$, $U^{(n-1)}=0$,

then for $U^{N}_{n\times n}$, it could be rewrited as separate to 4 block matrices $\begin{pmatrix}\widetilde{U}_{nn} & \vec{V}_{n-1}^{T}\\ \vec{0}_{n-1} & 0\end{pmatrix}$, in which $\widetilde{U}_{nn}$ is upper triangular matrix with diagonal 0's

$$U^2=\begin{pmatrix}\widetilde{U}_{nn} & \vec{V}_{n-1}^{T}\\ \vec{0}_{n-1} & 0\end{pmatrix}\begin{pmatrix}\widetilde{U}_{nn} & \vec{V}_{n-1}^{T}\\ \vec{0}_{n-1} & 0\end{pmatrix}=\begin{pmatrix}(\widetilde{U}_{nn})^2 & \widetilde{U}_{nn}\vec{V}_{n-1}^{T}\\ \vec{0}_{n-1} & 0\end{pmatrix}$$ $\boxed{\therefore (\widetilde{U}_{nn})^{n-1}=0.}$

$$U^3=\begin{pmatrix}(\widetilde{U}_{nn})^2 & \widetilde{U}_{nn}\vec{V}_{n-1}^{T}\\ \vec{0}_{n-1} & 0\end{pmatrix}\begin{pmatrix}\widetilde{U}_{nn} & \vec{V}_{n-1}^{T}\\ \vec{0}_{n-1} & 0\end{pmatrix}=\left(\begin{array}{c|c}(\widetilde{U}_{nn})^3 & (\widetilde{U}_{nn})^2\vec{V}_{n-1}^{T}\\ \hline \vec{0}_{n-1} & 0\end{array}\right)$$

$\vdots$

$$U^n=\left(\begin{array}{c|c}(\widetilde{U}_{nn})^n & (\widetilde{U}_{nn})^{n-1}\vec{V}_{n-1}^{T}\\ \hline \vec{0}_{n-1} & 0\end{array}\right)= \cancel{(\widetilde{U}_{nn})^{n-1}\cdot(\widetilde{U}_{nn})}$$

$\because (\widetilde{U}_{nn})^{n-1}=0$

$\therefore (\widetilde{U}_{nn})^{n}=(\widetilde{U}_{nn})^{n-1}(\widetilde{U}_{nn})=0$

$(\widetilde{U}_{nn})^{n-1}\vec{V}_{n-1}=0\cdot\vec{V}_{n-1}=0$

$\therefore U^n=0$

□.

~~8.~~ Let N ~~be~~ the dimension of U

then for $\forall\, k>N$, $U^k=U^N\cdot U^{k-N}=0\cdot U^{k-N}=0$ □.

2. $K_{ij}=\begin{cases}0 & \cancel{若}\ i+j=n+1\\ 1 & \text{otherwise}\end{cases}$ both size $n\times n$.

~~&~~ $U_{ij}=\begin{cases}0 & i<j\\ a_{ij} & \text{otherwise}\end{cases}$

a_{ij} can take 0.

$$(KU)_{ij}=(0\,0\cdots 0\,1\,0\cdots 0)\begin{pmatrix}0\\0\\ \vdots\\ a_{jj}\\ a_{(j+1)j}\\ \vdots\\ a_{nj}\end{pmatrix}=\begin{cases}0, & n+1-i<j \Leftrightarrow i+j>n+1.\\ a_{(n+1-i),j} & \text{otherwise.}\end{cases}$$

(the 1 is the $(n+1-i)$th entry; denote $(KU)_{ij}$ as b_{ij}.)

$$\therefore (KUK)_{xy}=[(KU)K]_{xy}=(b_{x1}\,b_{x2}\cdots b_{x(n+1-x)}\cdots 0)\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}\leftarrow (n+1-y)\text{th}.$$

$$=\begin{cases}0 & n+1-y>\cancel{x}\Leftrightarrow \cancel{x+y<n+1}\ \ n+1<x\ \ y<x.\\ b_{x,(n+1-y)} & \text{otherwise}\end{cases}$$

~~$\therefore (KUK)_{xy}=0$ if $y<x$. $\therefore$ KUK is upper triangular matrix~~

3. denote the position of 1's in $\underset{n\times n}{P}$ (i,j), s.t. $a_{ij}=1$

Let the set of these positions $S=\{(1,x_1),(2,x_2),\cdots(n,x_n)\}$ ~~where x_i takes value $\{1,2,\cdots n\}$. $\forall i\neq j$, $x_i\neq x_j$. $y_i\neq y_j$.~~

rewrite the set $S=\{(y_1,1),(y_2,2),\cdots(y_n,n)\}$.

x_i, y_i are permutations of $1\sim n$.

Let $\underset{n\times n}{A}=\begin{pmatrix} V_1\\ V_2\\ \vdots \\ V_n\end{pmatrix}$ where V_i is row vector with size n,

~~then~~ $\Rightarrow PA=\begin{pmatrix} V_{x_1}\\ V_{x_2}\\ \vdots \\ V_{x_3}\end{pmatrix}$ given position set $\{(1,x_1),(2,x_2),\cdots(n,x_n)\}$.

$A\to PA$ is only change the order of rows.

$$\Rightarrow PAP=(PA)P=\begin{pmatrix} V_{x_1}\\ V_{x_2}\\ \vdots \\ V_{x_3}\end{pmatrix}P=\begin{pmatrix} a_{x_1y_1} & a_{x_1y_2} & \cdots & a_{x_1y_n}\\ a_{x_2y_1} & a_{x_2y_2} & \cdots & a_{x_2y_n}\\ \vdots & & & \\ a_{x_ny_1} & \cdots & \cdots & a_{x_ny_n}\end{pmatrix}.$$

$i,j\in\{1,2,\cdots n\}$.

$\because$ $\forall i\neq j$, $x_i\neq x_j$. $y_i\neq y_i$.

$\therefore$ ~~no two entries in PAP~~

$(PAP)_{ij}$ is permutation of $\{a_{xy}\}$ $i,j,x,y\in\{1,2,\cdots n\}$.

$\therefore$ #non-zero entries in A is the same as #non-zero entries in PAP.