MATH 2101 LINEAR ALGEBRA I, FALL SEMESTER 2025
CHAPTER 1: MATRIX ALGEBRA AND DETERMINANTS

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Goal of this section:

(1) Basic terminology: matrices, entries, columns, rows, etc
(2) Basic operations: additions, matrix multiplication, matrix addition, transpose
(3) Properties of matrix multiplications
(4) Determinants: definition, rules to compute
(5) Inverse: computing inverse from a determinant, determining invertibility by determinants

1. Matrix Algebra

1.1. Basic terminology.

Definition 1.1. (1) We use $\mathbb{R}$ to represent the set of real numbers. A scalar is usually referred to a number in $\mathbb{R}$ (in this course).

(2) An $n \times m$ matrix A is a rectangular array of the form:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \ddots & \ddots & & \ddots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

where a_{ij} are in $\mathbb{R}$. We sometimes write (a_{ij}) to represent an arbitrary matrix. The (i,j)-entry of A will be referred to the number a_{ij}.

(3) For a matrix A, we also write A_{ij} to be the (i,j)-th entry of A.

(4) The i'-th row of (a_{ij}) is the $1 \times m$ matrix

$$\begin{pmatrix} a_{i'1} & a_{i'2} & \dots & a_{i'm} \end{pmatrix}$$

(5) The j'-th column of (a_{ij}) is the $n \times 1$ matrix

$$\begin{pmatrix} a_{1j'} \\ a_{2j'} \\ \vdots \\ b_{nj'} \end{pmatrix}.$$

Example 1.2. Let $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 7 & 9 \end{pmatrix}$. Then $A_{13} = 1$. The 2nd column of A is $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ and the 1st row of A is $\begin{pmatrix} 2 & 4 & 1 \end{pmatrix}$.

1.2. **Operations of matrices.**

(1) Addition of two $n \times m$ matrices $(a_{ij}) + (b_{ij})$ is defined as:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2m}+b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \dots & a_{nm}+b_{nm} \end{pmatrix}$$

(2) Scalar multiplication on a matrix: for a scalar $c \in \mathbb{R}$ and a $n \times m$ matrix (a_{ij}), $c(a_{ij})$ is defined as:

$$c \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1m} \\ ca_{21} & ca_{22} & \dots & ca_{2m} \\ \vdots & \vdots & & \vdots \\ ca_{n1} & ca_{n2} & \dots & ca_{nm} \end{pmatrix}$$

(3) Matrix multiplication: Let $A = (a_{ij})$ be a $n \times m$ matrix and let $B = (b_{ij})$ be a $m \times p$ matrix. We define the multiplication AB such that the (i,j)-th entry of AB is:

$$(AB)_{ij} = \sum_{x=1}^{m} a_{ix} b_{xj}$$

Remark 1.3. Let A be a $n \times m$ matrix and let B be a $p \times q$ matrix. If $m \neq p$, then AB is *not* well-defined.

(*Curious students may ask why to define matrix multiplication in such way. This is related to how to the concept of linear transformation, which we will come back later.*)

Example 1.4.

(1) $\begin{pmatrix} 1 & 2 & 4 \\ -1 & 7 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 3 & 2 \\ 1 & 3 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 6 \\ 0 & 10 & 10 \end{pmatrix}$.

(2)
$$(-2) \cdot \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0.5 & 8 \end{pmatrix} = \begin{pmatrix} -4 & -6 & 0 \\ -2 & -1 & 0 \end{pmatrix}$$

(3)
$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 0 & 1 \\ 9 & 2 \end{pmatrix} = \begin{pmatrix} 1(4)+2(0)+3(9) & 1(3)+2(1)+3(2) \end{pmatrix} = \begin{pmatrix} 31 & 11 \end{pmatrix}$$

(4)
$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -16 \end{pmatrix}$$

is not well-defined.

1.3. Vectors.

Definition 1.5. (1) A column vector is a $n \times 1$-matrix i.e. of the form

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

(2) A row vector is a $1 \times m$-matirx i.e. of the form

$$(a_1, \ldots, a_m).$$

(3) A vector simply refers to either a row vector or a column vector.
(4) $\mathbb{R}^n$ refers to the set of all column vectors with n real entries.
(5) We sometimes simply write 0 for the row vector or column vector with all entries to be zero. We shall call such vector to be the zero vector.

Exericse: Visualize geometrically the additions and scalar multiplication for vectors in $\mathbb{R}^2$.

1.4. Identity matrix.

The identity matrix I_n is defined as:

$$I_n = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & \vdots \\ \vdots & \ldots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

that is

$$(I_n)_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The main properties of the identity matrix are the following:

Theorem 1.6. *Let A be an $n \times m$ matrix. Then*

(1) $I_n A = A$; *and*
(2) $AI_m = A$.

Proof. Exercise. $\square$

Indeed, one should do the following more general exercise:

Exercise 1.7. An $n \times n$ matrix A is said to be diagonal if for any i, j with $i \neq j$, $A_{ij} = 0$. For example, $\begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$ is diagonal but $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ is not diagonal. The diagonal entries of a $n \times n$ matrix (a_{ij}) are referred to the entries $a_{11}, a_{22}, \ldots, a_{nn}$.

Let A be a $n \times m$ matrix.

(1) For a diagonal $n \times n$ matrix D, describe the matrix multiplication DA;
(2) For a diagonal $m \times m$ matrix E, describe the matrix multiplication AE.

1.5. **Zero matrices.** The $n \times m$ matrix is called the zero matrix $0_{n\times m}$ (or sometimes simply 0) if all its entries are zero. The main properties of the zero matrix is the following:

Theorem 1.8. *Let A be an $n \times m$ matrix. Then*

(1) $0_{p\times n}A = 0_{p\times m}$; *and*
(2) $A0_{m\times q} = 0_{n\times q}$; *and*
(3) $A + 0 = A$.

Proof. Exercise. □

Remark 1.9. For simplicity, we usually write 0 for different things. One may need a bit thought to identify the meaning of 0 in a content. Can you explain the meaning of of 0 in the following equations? Let A be a $n \times p$-matrix.

(1) Then $0_{m\times n}A = 0$.
(2) Let $c = 0$ be the zero scalar. Then $cA = 0$.
(3) $A - A = 0$.

1.6. **Square matrices.**

Definition 1.10. A $n \times m$ matrix is called sqaure if $n = m$.

Example 1.11. (1) Identity matrices are square. Diagonal matrices are also square.
(2) The row vector $(1 \quad 3)$ is not square.
(3) The matrix $\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$ is not square.

1.7. **Transpose of a matrix.**

Definition 1.12. The transpose A^T of an $n \times m$ matrix A is the $m \times n$ matrix obtained from interchanging rows and columns i.e.

$$(A^T)_{ij} = A_{ji}$$

Example 1.13. (1) Let $A = \begin{pmatrix} 3 & 5 & 0 \end{pmatrix}$. Then

$$A^T = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}.$$

(2) Let $B = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$. Then $B^T = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix}$.

There are some basic properties of the transpose:

Theorem 1.14. (1) *(Transpose commutes with addition) For $n \times m$-matrices A and B,*

$$(A+B)^T = A^T + B^T.$$

(2) *(Transpose anti-commutes with matrix multiplication) For an $n \times m$-matrix A and a $m \times p$-matrix B,*

$$(AB)^T = B^T A^T.$$

(3) *(Transpose commutes with scalar multiplication) For an $n \times m$-matrix A and a scalar c,*

$$(cA)^T = cA^T.$$

Example 1.15. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ and let $B = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix},$$

and so

$$(AB)^T = \begin{pmatrix} 1 & -11 \end{pmatrix},$$

$$B^T A^T = \begin{pmatrix} -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -11 \end{pmatrix}.$$

Theorem 1.16. *(Transpose twice) Let A be an $n \times m$ matrix. Then $(A^T)^T = A$.*

Proof.

$$((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}.$$

□

1.8. Symmetric matrices.

Definition 1.17. A matrix A is said to be *symmetric* if $A^T = A$.

Example 1.18. (1) $\begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}$ is symmetric.

(2) $\begin{pmatrix} 1 & 3 \\ -3 & -2 \end{pmatrix}$ is not symmetric.

(3) Any diagonal matrices are symmetric.

Exercise 1.19. Show that if a matrix is symmetric, then A is square.

Exercise 1.20. Write down the form of symmetric 2×2 matrices and the form of symmetric 3×3 matrices.

1.9. **Upper triangular matrices.**

Definition 1.21.
(1) A square matrix (a_{ij}) is said to be an upper triangular matrix if for any $i > j$, $a_{ij} = 0$.
(2) A square matrix (a_{ij}) is said to be a lower triangular matrix if for any $i < j$, $a_{ij} = 0$.

Example 1.22.
(1) The matrices $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -2 \end{pmatrix}$ are upper triangular.
(2) The matrices $\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$ are lower triangular matrices.
(3) The matrix $\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$ is neither upper triangular nor lower triangular.

Exercise 1.23. Show that a square matrix A is both upper triangular and lower triangular if and only if A is diagonal.