THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

May 8, 2023 2:30 p.m. – 5:00 p.m.

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all SIX questions.

Notes:

- Write down your university number, seat number and model of calculator used on the cover of the answer book.
- The blank pages in the answer book can be used for rough work. You can also use them as additional answer space provided that you make a clear indication in the original answer space.
- Unless otherwise specified, you should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**

1. (38 marks) Give answers only to the following questions. Explanation is <u>not</u> required.

Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$. Find the following.

(a) The reduced row echelon form of A.

(b) The LU decomposition of A.

(c) $T_A(3\mathbf{e}_1 + 2\mathbf{e}_2)$, where T_A denotes the linear transformation induced by A.

(d) Find the orthogonal complement of $S = \{T_A(3\mathbf{e}_1 + 2\mathbf{e}_2)\}$.

(e) A basis for the row space of A.

(f) $\mathrm{Col}(A)$.

(g) The nullity of A.

(h) Let $\mathcal{B} = \{\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2\}$ be an ordered basis. If $\mathbf{v} = A \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ which is in $\mathrm{Span}\{\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2\}$, find $[\mathbf{v}]_{\mathcal{B}}$.

(i) The determinant of the 4×4 matrix B, where

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i < j \text{ and } i, j \in \{1, 2, 3\} \\ 1 & \text{otherwise} \end{cases}$$

and a_{ij} and b_{ij} denote the (i, j)-entry of A and B respectively.

(j) The largest (real) eigenvalue of A.

(k) The eigenvector corresponding to the largest (real) eigenvalue of A. (Let the first nonzero entry to be 1).

Let C be a 5×5 matrix with determinant 3 and an eigenvalue 2. Find the following.

(l) $\det(C^{-1})$.

(m) $\det(C^2)$.

(n) $\det(2C)$.

(o) $\det(adj(C^T))$.

(p) An eigenvalue of C^T.

(q) An eigenvalue of C^3.

(r) An eigenvalue of $3C$.

(s) An eigenvalue of $adj(C^T)$.

2. (22 marks) In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do not indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is not required.

(a) Let X and Y be subspaces of $\mathbb{R}^2$. Which of the following are possible?

(1) $X \cap Y$ is a subspace of $\mathbb{R}^2$.

(2) All vectors in X have integer entries.

(4) The complement of X is a subspace of $\mathbb{R}^2$.

(8) X is a proper subset of Y.

(b) Which of the following are bases for $\mathbb{R}^3$?

(1) $\{\mathbf{0}\}$

(2) $\{\mathbf{e}_1, \mathbf{e}_2\}$

(4) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$

(8) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(c) Let V and W be two subspaces of $\mathbb{R}^6$. If $\dim V = 2$ and $\dim W = 4$, which of the following number is possible for dimension of the subspace $V \cap W$?

(1) 0

(2) 1

(4) 2

(8) 4

(d) Which of the following are subspaces of $\mathbb{R}^3$?

(1) $\emptyset$

(2) $\mathbb{R}^2$

(4) $\text{Span}\{(1,2,3)^T\} \cap \text{Span}\{(1,3,5)^T\}$

(8) $\{(x,y,z)^T : 2x - 3y + 4z = 5\}$

(e) Let $\mathbf{u}$ be a vector in $\mathbb{R}^3$, and let $\mathcal{B}$ be a basis for $\mathbb{R}^3$. Which of the following must be true?

(1) There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\mathbf{w}) = [\mathbf{w}]_{\mathcal{B}}$ for any $\mathbf{w}$ in $\mathbb{R}^3$.

(2) If $[\mathbf{u}]_{\mathcal{B}} = \mathbf{u}$, then $\mathcal{B}$ is the standard ordered basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(4) $[\mathbf{u}]_{\mathcal{B}} = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$.

(8) $[a\mathbf{u}]_{\mathcal{B}} = a[\mathbf{u}]_{\mathcal{B}}$ for any scalar a.

(f) Let S be a set of 4 vectors in $\mathbb{R}^3$. Which of the following is possible?

(1) S is linearly dependent.

(2) S is linearly independent.

(4) S is a basis for $\mathbb{R}^3$.

(8) S is a generating set for $\mathbb{R}^3$.

(g) Let A be the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$. Which of the following has dimension 3?

(1) The row space of A.

(2) The column space of A^T.

(4) The null space of A.

(8) The null space of A^T.

(h) Which of the following matrices has 1 as an eigenvalue?

(1) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(8) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(i) Let A be a 4×4 matrix with eigenvector $\mathbf{e}_2$. Which of the following must be 0?

(1) The $(1,1)$-entry of A.

(2) The $(1,2)$-entry of A.

(4) The $(2,1)$-entry of A.

(8) The $(2,2)$-entry of A.

(j) Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors of $\mathbb{R}^n$, and let a be a scalar. Which of the following are identities?

(1) $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|$.

(2) $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.

(4) $\mathbf{v}\cdot\mathbf{v} = \sqrt{\|\mathbf{v}\|}$.

(8) $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$.

(k) A 3×3 matrix A is non-diagonalisable over $\mathbb{C}$. If A has an eigenvalue λ with algebraic multiplicity 2, which of the following must be true?

(1) All eigenvalues of A are real numbers.

(2) The geometric multiplicity of λ is 1.

(4) A has 3 linearly independent eigenvectors over $\mathbb{C}$.

(8) There are two eigenvectors of A corresponding to different eigenvalues that are linearly dependent.

3. (10 marks) For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

(a) Let A and B be $m \times n$ matrices. If A is row equivalent to B, then $\mathrm{Col}(A) = \mathrm{Col}(B)$.

(b) If 0 is an eigenvalue of an $n \times n$ matrix A, the $\mathrm{rank}(A) < n$.

(c) There exists a vector in $\mathbb{R}^3$ that are orthogonal to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

(d) The set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ where $y = x + 1$ is a subspace of $\mathbb{R}^2$.

(e) If S is a basis for a subspace V of $\mathbb{R}^n$, then $\mathrm{Span}\, S = V$.

4. (8 marks) Let $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$. Determine if A is diagonalizable. If yes, diagonalize it; if not, state the reason.

5. (11 marks) Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : 2x_1 - x_3 + x_4 = 0, \ x_1, x_3, x_4 \in \mathbb{R} \right\}$.

(a) (3 marks) Prove that W is a subspace of $\mathbb{R}^4$.

(b) (8 marks) Find an orthonormal basis for W.

6. (11 marks) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation giving by rotating the vectors about the origin by $\frac{\pi}{6}$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation giving by reflecting vectors over the line $y = 2x$.

(a) (3 marks) Find the matrix A_1 corresponding to T_1.

(b) (3 marks) Find the matrix A_2 corresponding to T_2.

(c) (3 marks) Find the matrix of linear transformation T which is obtained by first rotating all vectors about the origin by $\frac{\pi}{6}$ and then reflecting over the line $y = 2x$.

(d) (2 marks) What is the image of the vector $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ under the linear transformation T?

******* **END OF PAPER** *******