MATH2101 A3 Shen Hongshan 3036290936 Engineering BEng (DS&E)

1.
$$\left(\begin{array}{ccccc|c} -1 & 2 & 1 & -1 & 3 & 7 \\ 1 & -4 & 0 & 5 & -5 & -7 \\ -2 & 6 & -1 & -5 & 7 & 6 \\ -1 & -4 & 4 & 11 & -2 & 11 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & -2 & -1 & 1 & -3 & -7 \\ 0 & \cancel{1} & \cancel{-1/2} & \cancel{-2} & \cancel{1} & 0 \\ 0 & 2 & -2 & -3 & 1 & -8 \\ 0 & -6 & 3 & 12 & -5 & 4 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & -2 & -1 & 1 & -3 & -7 \\ 0 & 1 & -1/2 & -2 & 1 & 0 \\ 0 & 0 & +1 & -1 & +1 & +8 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & -2 & -1 & 1 & 0 & 5 \\ 0 & 1 & -1/2 & -2 & 0 & -4 \\ 0 & 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -4 & 0 & 1 \\ 0 & 1 & 0 & -\frac{9}{4} & 0 & -3 \\ 0 & 0 & 1 & -1/2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right) \quad \begin{cases} x_4 = t \\ x_1 = 1+4t \\ x_2 = -3+\frac{9}{4}t \\ x_3 = 2+\frac{1}{2}t \\ x_5 = 4 \end{cases} \Rightarrow x \in \left\{ \left(t, \left(-3+\frac{9}{4}t\right), 2+\frac{1}{2}t, t, 4\right)^T \middle| t \in \mathbb{R} \right\}$$

2.
$$\det\begin{pmatrix} c & 0 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2\begin{vmatrix} c & 0 & 2 \\ 1 & 2 & 3 \\ 2 & 4 & 8 \end{vmatrix} = 2\cdot\begin{vmatrix} c & 0 & 2 \\ 1 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 4(2c) = 8c$$

① $c \neq 0$. determinent $\neq 0 \Rightarrow$ RREF $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

② $c = 0$
$$\begin{pmatrix} 0 & 0 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 0 & 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. $\underset{n\times m}{A}$ $\underset{n\times 1}{b}$

(a) True

$Ax=b$ has m leading variable $\Rightarrow$ REF of A has m leading variable 0 free variable, $m \geq n$

if $Ax=b$ consistent, then all-zero row placed at the bottom (REF) provides no information, thus could be dismissed.

Let $\underset{m\times m}{D}$ be the matrix by canceling bottom $(m-n)$ 0 rows of REF(A)

$\underset{m\times 1}{d}$ be the column vector by canceling bottom $(m-n)$ 0-entries of b.

$\Rightarrow \underbrace{D}_{\text{full rank} \Leftrightarrow \text{invertible}} x = d \Rightarrow x = \underbrace{D^{-1}d}_{\text{all operation is } +/*}$ original entries are integer

$\Rightarrow$ resulting entries are rational

(b) False

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$ is one possible solution with irrational entry.

4. $\underset{3\times 3}{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\cdot\ A = UPL \Rightarrow \det(A) = \det(U)\cdot\det(P)\det(L) \neq 0$

all invertible

$\therefore$ Aim: $\underbrace{P^{-1}U^{-1}A = L}_{\text{target}}$

① $a_{33} \neq 0$, $a_{22}\cdot a_{33} \neq a_{32}\cdot a_{23}$

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{a_{23}}{a_{33}} \\ 0 & 0 & 1 \end{pmatrix}$ $E_2 = \begin{pmatrix} 1 & 0 & -\frac{a_{13}}{a_{33}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $E_3 = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

where $x = -\dfrac{a_{12} - a_{32}\cdot\frac{a_{13}}{a_{33}}}{a_{22} - a_{32}\cdot\frac{a_{23}}{a_{33}}}$

$E_3E_2E_1A = \begin{pmatrix} m_{11} & 0 & 0 \\ m_{12} & m_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ m_{11}, m_{12}, m_{22} are polynomial of a_{ij}.

is lower triangular matrix $\Rightarrow$ Let $L = E_3E_2E_1A$

$E_3E_2E_1 = \begin{pmatrix} 1 & x & -\frac{a_{13}}{a_{33}} \\ 0 & 1 & -\frac{a_{23}}{a_{33}} \\ 0 & 0 & 1 \end{pmatrix}$ is upper triangular

$\|$

Let M

then $(M^{-1})_{21} = \dfrac{\det(\widetilde{M}_{12})}{\det(M)} = 0$

$(M^{-1})_{31} = \dfrac{\det(\widetilde{M}_{13})}{\det(M)} = 0$

$(M^{-1})_{32} = \dfrac{\det(\widetilde{M}_{23})}{\det(M)} = 0$

$\Rightarrow (E_3E_2E_1)^{-1}$ is upper triangular

Let $U = (E_3E_2E_1)^{-1}$, $P = I_3$, $L = E_3E_2E_1A$

then $P^{-1}U^{-1}A = L \Leftrightarrow U^{-1}A = PL \Leftrightarrow A = UPL$

② $a_{33}=0$ or $a_{23}a_{32}=a_{22}a_{33}$ (adjust A)

1° A invertible $\Rightarrow$ last column not all zero.

if $a_{13}\neq 0$ Let $E_4=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$ $P=E_4$

if $a_{23}\neq 0$ Let $E_4=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$ $P=E_4$

$\Rightarrow P^{-1}U^{-1}\underbrace{E_4A}_{A'}=L$

A' same as case①

2° after adjustment to $A'=\begin{pmatrix}a'_{11}&a'_{12}&a'_{13}\\a'_{21}&a'_{22}&a'_{23}\\a'_{31}&a'_{32}&a'_{33}\end{pmatrix}$

If $a'_{22}a'_{33}=a'_{23}\cdot a'_{32}$

note that $\det(A')=a'_{11}\begin{vmatrix}a'_{22}&a'_{23}\\a'_{32}&a'_{33}\end{vmatrix}-a'_{21}\begin{vmatrix}a'_{12}&a'_{13}\\a'_{32}&a'_{33}\end{vmatrix}+a'_{31}\begin{vmatrix}a'_{12}&a'_{13}\\a'_{22}&a'_{23}\end{vmatrix}$

not all zero

apply similar row operation E_5

s.t. $(E_5A')_{22}\cdot(E_5A')_{33}\neq(E_5A')_{23}\cdot(E_5A')_{32}$

$\Rightarrow P=E_5E_4$ s.t. $P^{-1}U^{-1}E_5E_4A=L$

$\Downarrow$

E_5A' same as case①

4.(b) $A=I$

① $U=P=L=I$ check $A=UPL$ true

② $P=I$ $U=\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $L=\begin{pmatrix}\frac{1}{2}&0&0\\0&1&0\\0&0&1\end{pmatrix}$

$UPL=UIL=UL=I=A$ true

$\Rightarrow$ Not unique