THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

December 19, 2016 9:30 am – 12:00 noon

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all SIX questions. Each question carries 16 marks.
Another 4 marks is given for clarity of presentation and following instructions.

Notes:

- You should give precise and adequate explanations to support your conclusions unless otherwise specified by the question. Clarity of presentation of your argument counts. So **think carefully before you write.**
- You must start each question on a new page. However do NOT start each part of a question on a new page. You should write down the question number on the top right hand corner of each page. Indicate clearly which part of a question you are answering.
- If your answer to a question spans over more than one page, you must indicate clearly on each page (except the last) that the answer will continue on the subsequent page.
- If you do not otherwise specify, you are assumed to be answering the questions in order. In particular, indicate on each question where your answer to the next question is if it does not follow immediately. (For example you may write 'turn back 3 pages for Question 5' at the end of Question 4.)
- Whenever there is a blank page it will be assumed that all subsequent pages are also blank. If this is not true, indicate clearly on the page which you would otherwise leave blank.

1. Give answers only to the following questions. Explanation is not required.

Let $A = \begin{bmatrix} 10 & 20 & 30 & 40 \\ 2 & 4 & 6 & 9 \\ 1 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$.

The three real eigenvalues of B are denoted by λ_1, λ_2 and λ_3 in ascending order (i.e. $\lambda_1 \leq \lambda_2 \leq \lambda_3$).

(a) Find $\det A$.

(b) What is the null space of A?

(c) What is the $(2,1)$-entry of the reduced row echelon form of A?

(d) What is the $(2,1)$-entry of A^{-1}?

(e) Find the value of λ_1.

(f) Find a λ_1-eigenvector of B.

(g) Find a basis for the λ_2-eigenspace of B.

(h) What is the dimension of the column space of B?

2. In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do not indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is not required.

(a) Which of the following matrices are in reduced row echelon form?

(1) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

(8) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

(b) Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be linearly dependent vectors. Which of the following must be true?

(1) $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

(2) If $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$, then $a \neq 0$.

(4) $\mathbf{x}$ is a linear combination of $\mathbf{y}$ and $\mathbf{z}$.

(8) $\text{Span}\{\mathbf{x}, \mathbf{y}\} = \text{Span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$

(c) Let A, B, C be matrices such that $AB = C$. If A has __________, so has C.

(1) a zero row

(2) a zero column

(4) full row rank

(8) full column rank

(d) Let A, B, C be matrices such that $AB = C$. Which of the following is true?

(1) If some rows of B are linearly dependent, so are some rows of C.

(2) If some columns of B are linearly dependent, so are some columns of C.

(4) If some rows of C are linearly dependent, so are some rows of B.

(8) If some columns of C are linearly dependent, so are some columns of B.

(e) Let W be a subspace of $\mathbb{R}^3$. Which of the following is possible?

(1) $W^\perp$ is empty.

(2) $W^\perp$ is a non-empty finite set.

(4) $W^\perp$ is a not subspace of $\mathbb{R}^3$.

(8) $W^\perp$ has the same dimension as W.

(f) Let A be a square matrix. Which of the following is a scalar?

(1) The row space of A

(2) The nullity of A

(4) An eigenvalue of A

(8) An eigenvector of A

(g) Let S be a finite subset of $\mathbb{R}^n$. Which of the following must be well-defined?

(1) $S^{\perp}$

(2) $\dim S$

(4) $\det S$

(8) $\operatorname{Span} S$

(h) Let A be a square matrix. Then A and $2A$ must have the same ____________.

(1) row space

(2) column space

(4) null space

(8) eigenspaces

3. For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

Unless otherwise specified, A and B are both $n \times n$ matrices in this question.

(a) It is possible for a system of 3 linear equations in 4 unknowns to have a unique solution.

(b) If $A^5 = O$, then $A^4 = O$.

(c) If $AB = O$ and that A, B are both invertible, then $A = B = O$.

(d) If $\det A = \det B$, then A and B are similar.

(e) If A is diagonalisable, then every vector in $\mathbb{R}^n$ is an eigenvector of A.

(f) If A is invertible, then 0 is not an eigenvalue of A.

(g) There exists an orthogonal set of 6 vectors in $\mathbb{R}^5$.

(h) There exists a generating set for $\mathbb{R}^5$ which contains 6 vectors.

4. (a) (i) Show that if $\mathcal{P}$ is a parallelogram (including its interior) in $\mathbb{R}^2$ with one vertex at the origin, and that $T: \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear transformation with standard matrix A, then $T(\mathcal{P})$ is also a parallelogram with one vertex at the origin.

(ii) In (i), derive the relationship between the areas of $\mathcal{P}$ and $T(\mathcal{P})$ in terms of the standard matrix A of T.

(iii) State (without proof) a three-dimensional version of your statement in (ii).

(b) Let $B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -1 & 4 \\ 2 & 0 & -5 \end{bmatrix}$.

(i) State (without proof) a geometric meaning of $\det B$ which does not involve linear transformation.

(ii) Show that the three columns of B form an orthogonal set. How does this give an alternative way of evaluating $\det B$?

(iii) Use a non-trivial example to illustrate your statement in (a)(iii).

5. Prove the following statements.

(a) Let $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_k$ be linearly independent vectors in $\mathbb{R}^n$. Then for any $\mathbf{x}_{k+1} \in \mathbb{R}^n$,

$$\text{Span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k\} = \text{Span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k,\ \mathbf{x}_{k+1}\}$$

if and only if $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_k$, $\mathbf{x}_{k+1}$ are linearly dependent.

(b) Let A be an $m \times m$ matrix and B be an $m \times n$ matrix. Then $\text{Row}\, AB \subseteq \text{Row}\, B$, and equality holds if A is invertible.

(c) Let A, B, S be square matrices such that $A = S^{-1}BS$. If $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_k$ are linearly independent eigenvectors of A corresponding to the same eigenvalue λ, then $S\mathbf{x}_1$, $S\mathbf{x}_2$, ..., $S\mathbf{x}_k$ are linearly independent eigenvectors of B corresponding to λ.

6. Recall that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a *linear transformation* if the following two conditions hold:

(A) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

(M) $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

(a) A student Ann makes the following remark about the definition.

> I think condition (M) can be omitted without affecting the definition because (A) implies (M). Indeed, if (A) holds, then we have
>
> $$T(c\mathbf{x}) = T(\underbrace{\mathbf{x} + \cdots + \mathbf{x}}_{c \text{ times}}) = \underbrace{T(\mathbf{x}) + \cdots + T(\mathbf{x})}_{c \text{ times}} = cT(\mathbf{x})$$
>
> i.e. (M) must also hold.

(i) Explain why Ann's argument is incorrect.

(ii) Is it true that condition (M) can be omitted without affecting the definition? (Hint: You may use, without proof, the fact that there exists $S \subseteq \mathbb{R}$ containing π such that every real number x can be written uniquely as a (possibly infinite) linear combination of the elements in S with rational coefficients.)

(b) Consider a new condition

(N) $T(c\mathbf{x} + \mathbf{y}) = cT(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

A student Ben claims that T is a linear transformation if and only if (N) holds. Is Ben's claim true?

(c) A student Cat defines a function $T : \mathbb{R}^n \to \mathbb{R}^n$ as follows.

> Fix two ordered bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathbb{R}^n$. For any $\mathbf{x} \in \mathbb{R}^n$, define
>
> $$T(\mathbf{x}) = [\mathbf{y}]_{\mathcal{B}_2},$$
>
> where $\mathbf{y}$ is the vector satisfying $[\mathbf{y}]_{\mathcal{B}_1} = \mathbf{x}$.

Is Cat's function a well-defined linear transformation?

******* **END OF PAPER** *******