THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

December 17, 2015 9:30 am – 12:00 noon

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all SIX questions. Each question carries 16 marks.
Another 4 marks is given for clarity of presentation and following instructions.

Notes:

- You should give precise and adequate explanations to support your conclusions unless otherwise specified by the question. Clarity of presentation of your argument counts. So **think carefully before you write.**
- You must start each question on a new page. However do NOT start each part of a question on a new page. You should write down the question number on the top right hand corner of each page. Indicate clearly which part of a question you are answering.
- If your answer to a question spans over more than one page, you must indicate clearly on each page (except the last) that the answer will continue on the subsequent page.
- If you do not otherwise specify, you are assumed to be answering the questions in order. In particular, indicate on each question where your answer to the next question is if it does not follow immediately. (For example you may write 'turn back 3 pages for Question 5' at the end of Question 4.)
- Whenever there is a blank page it will be assumed that all subsequent pages are also blank. If this is not true, indicate clearly on the page which you would otherwise leave blank.

1. Give answers only to the following questions. Explanation is <u>not</u> required.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find the following.

(a) The reduced row echelon form of A

(b) The solution set of the system $A\mathbf{x} = \mathbf{0}$, using x_3 as free variable

(c) An LU decomposition of A

(d) $T_A(2\mathbf{e}_1 + 3\mathbf{e}_2)$, where T_A denotes the linear transformation induced by A

(e) The determinant of the 4×4 matrix B, where

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i, j \in \{1, 2, 3\} \\ 1 & \text{otherwise} \end{cases}$$

and a_{ij} and b_{ij} denote the (i, j)-entry of A and B respectively

(f) A basis for the row space of A

(g) The dimension of the column space of A

(h) The smallest (real) eigenvalue of A

2. In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do <u>not</u> indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is <u>not</u> required.

(a) In which of the following cases are AB and BA both undefined?

(1) A and B are both 2×3

(2) A is 3×2 and B is 2×3

(4) A is 4×4 and B is 2×2

(8) A is 3×5 and B is 2×3

(b) If A, B and C are matrices such that $AB = C$, which of the following is true?

(1) If A has a zero row, so does C.

(2) If B has a zero row, so does C.

(4) If A has a zero column, so does C.

(8) If B has a zero column, so does C.

(c) Let A be an $n \times n$ matrix. Which of the following is equivalent to A being non-invertible?

(1) A has a zero row.

(2) 0 is an eigenvalue of A.

(4) The determinant of A is 0.

(8) $A\mathbf{x} = \mathbf{0}$ for some nonzero $\mathbf{x}$.

(d) Let S be a set of 4 vectors in $\mathbb{R}^5$. Which of the following is possible?

(1) S is linearly dependent.

(2) S is linearly independent.

(4) S is a basis for $\mathbb{R}^5$.

(8) S is a generating set for $\mathbb{R}^5$.

(e) Let A be the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$. Which of the following has dimension 3?

(1) The row space of A

(2) The column space of A^T

(4) The null space of A

(8) The null space of A^T

(f) Which of the following matrices has 3 as an eigenvalue?

(1) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(4) $\begin{bmatrix} 0 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 \\ 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 0 \end{bmatrix}$

(8) $\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$

(g) Let A be a 5×5 matrix with eigenvector $\mathbf{e}_1$. Which of the following must be true?

(1) A is non-invertible.

(2) $A - I$ is non-invertible.

(4) The $(1, 1)$-entry of A is zero.

(8) The $(2, 1)$-entry of A is zero.

(h) Let A be a square matrix. Which of the following statements make sense mathematically?

(1) A has a basis.

(2) The number of nullity of A is 4.

(4) The eigenvalues of A generate $\mathbb{R}^n$.

(8) The column space of A is linearly independent.

3. For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

   (a) The coefficient matrix and the augmented matrix of a system of linear equations must have the same rank.

   (b) If m and n are positive integers with $m < n$, then a system of m linear equations in n unknowns must be consistent.

   (c) There exists an injective linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$.

   (d) All elementary matrices have positive determinant.

   (e) If W is a subspace of $\mathbb{R}^5$ and S is a set of three vectors in $\mathbb{R}^5$, then S is not a generating set for W.

   (f) The $\mathcal{B}$-matrix of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, where $\mathcal{B}$ is an ordered basis of $\mathbb{R}^2$. Then 2 is an eigenvalue of the standard matrix of T.

   (g) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly independent vectors in $\mathbb{R}^n$ and $\mathcal{B}$ is an ordered basis of $\mathbb{R}^n$, then $[\mathbf{x}_1]_{\mathcal{B}}, [\mathbf{x}_2]_{\mathcal{B}}, \ldots, [\mathbf{x}_m]_{\mathcal{B}}$ are also linearly independent.

   (h) It is possible for a square matrix A to have two linearly independent eigenvectors corresponding to the same eigenvalue.

4. Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

   Carry out the following computations with careful explanations to your steps. Do not just write down some expressions and the final answers.

   (a) Find an orthonormal basis $\mathcal{B}$ for the column space of A.

   (b) Is $\mathbf{b}$ in the column space of A?

   (c) Make good use of the orthonormal basis $\mathcal{B}$ to do one of the following.

   - If the answer to (b) is yes, find $\mathbf{b}$ as a linear combination of the vectors in $\mathcal{B}$.
   - If the answer to (b) is no, find the distance from $\mathbf{b}$ to the column space of A.

5. Recall that a subset W of $\mathbb{R}^n$ is said to be a *subspace* of $\mathbb{R}^n$ if the three conditions

(O) W contains the zero vector;

(A) W is closed under addition; and

(M) W is closed under scalar multiplication

all hold.

(a) A student makes the following remark about the definition.

> I think condition (O) can be omitted without affecting the definition. This is because if (M) holds, then we can multiply a vector by the scalar 0 to get the zero vector, which must be in W by (M). Hence if (M) holds, then (O) must hold.

Do you agree with the student?

(b) Can each of (A) and (M) be omitted without affecting the definition?

(c) Consider a new condition

(N) For any $\mathbf{u}, \mathbf{v} \in W$ and $c \in \mathbb{R}$, we have $c\mathbf{u} + \mathbf{v} \in W$.

Show that W is a subspace of $\mathbb{R}^n$ if and only if (O) and (N) hold.

(d) For subsets V, W of $\mathbb{R}^n$ and real number k, define

$$kV + W = \{k\mathbf{v} + \mathbf{w} : \mathbf{v} \in V,\ \mathbf{w} \in W\}.$$

If V and W are subspaces of $\mathbb{R}^n$, show that $kV + W$ is also a subspace of $\mathbb{R}^n$.

Is the converse true?

6. In a linear algebra quiz, a question asks students to compute the inverse of an invertible matrix P.

   (a) A student Ann mistakenly swapped the second and third rows of P, but apart from this she had carried out the correct computations. Show that Ann's answer can be obtained from the correct answer (say, Q) by exchanging the second and third columns of Q.

   (b) Work out a result similar to (a) for a student Ben who had mistakenly doubled the second row of P.

   (c) Work out a result similar to (a) for a student Cat who had mistakenly added two times Row 1 to Row 2 of P.

******* **END OF PAPER** *******