THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH2101: LINEAR ALGEBRA I

December 16, 2023 2:30 pm – 5:00 pm

No calculators are allowed in the examination.

Answer ALL EIGHT questions

Note:

- You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**
- If you write more than one solutions for a question, only the first one will be graded.

1. (24 points) Write down answers to the following questions. **No** explanation is required.

   (a) Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$ and let $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find the solution set for the system of linear equations $Ax = b$.

   (b) Find all eigenvalues of the matrix $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$.

   (c) Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & a \end{pmatrix}$$
   for some $a \in \mathbb{R}$. It is known that $\det(A) = 3$. Find
$$\det \begin{pmatrix} 0 & 0 & a \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

   (d) Find the dimension of the vector subspace:
$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = x_3 = 0\}.$$

   (e) Let A be a 5×4 matrix. If the reduced row echelon form of A has exactly 3 zero rows, find $\operatorname{rank}(A)$.

   (f) Find all values of x in $\mathbb{R}$ such that the matrix
$$\begin{pmatrix} 2 & x \\ 0 & 2 \end{pmatrix}$$
   is diagonalizable.

   (g) Find all unit vectors v in $\mathbb{R}^3$ such that v and the vector $(2, 2, -1)^T$ are linearly dependent.

   (h) Express the vector $(3, 1, 0)^T$ as a linear combination of the following two vectors
$$(2, 1, 1)^T, \quad (1, 0, -1)^T.$$

2. Let β be an orthonormal basis for $\mathbb{R}^n$. Let P be the change of coordinate matrix from β to the standard basis.

(a) (6 points) Prove that $\langle Px, Py\rangle = \langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$.

(b) (4 points) Prove that $\det(P) = \pm 1$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by:

$$T(x_1, x_2, x_3) = (x_1 - 2x_2 - 3x_3, x_1 - x_3, ax_2 + x_3)$$

for some $a \in \mathbb{R}$.

(a) (5 points) Determine all values of a so that T is invertible.

(b) (5 points) For $a = 0$, find an expression for the inverse $T^{-1}(x_1, x_2, x_3)$.

4. (10 points) Express the matrix

$$\begin{pmatrix} 8 & -10 \\ 5 & -7 \end{pmatrix}^{100}$$

into the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

5. Let

$$A = \begin{pmatrix} 1 & 6 & 4 & 1 \\ 2 & 6 & 5 & -1 \\ 1 & 2 & 2 & -1 \\ 2 & 0 & 2 & -4 \end{pmatrix}.$$

(a) (5 points) Find a basis for the column space of A.

(b) (5 points) Find an orthonormal basis for $\mathrm{col}(A)$ by using the Gram-Schmidt orthogonalization process on the basis found in (a).

6. Let

$$A = \begin{pmatrix} 107 & -214 & 1 & 109 \\ 50 & -100 & 31 & 112 \\ 31 & -62 & 29 & 89 \\ 138 & -276 & 28 & 194 \end{pmatrix}, \quad b = \begin{pmatrix} 217 \\ 193 \\ 149 \\ 360 \end{pmatrix}.$$

It is known that $(3, 1, 1, 1)^T$, $(3, 2, -3, 3)^T$, $(2, 0, 3, 0)^T$ are some solutions for the system of linear equations

$$Ax = b.$$

(a) (3 points) What can you say about the number of solutions for the system $Ax = b$? No explanation is required.

(b) (9 points) Find the solution set for the system $Ax = b$. Explain your answer.

7. (12 points) Let v_1, v_2, v_3 be vectors in a vector space V. Prove or disprove the following statement: v_1, v_2, v_3 are linearly independent vectors if and only if $v_1 + v_2, v_1 - v_2 + v_3, v_2 + v_3$ are linearly independent vectors.

8. (12 points) Let A be an $n \times n$ matrix. Suppose $A^2 = A$. Prove that A is diagonalizable. (Hints: Consider possible eigenvalues of A and consider the subspaces $\mathrm{col}(A - I_n)$ and $\mathrm{col}(A)$.)

* * * * * * * **END OF PAPER** * * * * * * *