THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

December 8, 2017 9:30 am – 12:00 noon

*Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.*

**Answer all SIX questions. Each question carries 16 marks.**
**Another 4 marks is given for clarity of presentation and following instructions.**

Notes:

- You should give precise and adequate explanations to support your conclusions unless otherwise specified by the question. Clarity of presentation of your argument counts. So **think carefully before you write.**
- You must start each question on a new page. However do NOT start each part of a question on a new page. You should write down the question number on the top right hand corner of each page. Indicate clearly which part of a question you are answering.
- If your answer to a question spans over more than one page, you must indicate clearly on each page (except the last) that the answer will continue on the subsequent page.
- If you do not otherwise specify, you are assumed to be answering the questions in order. In particular, indicate on each question where your answer to the next question is if it does not follow immediately. (For example you may write 'turn back 3 pages for Question 5' at the end of Question 4.)
- Whenever there is a blank page it will be assumed that all subsequent pages are also blank. If this is not true, indicate clearly on the page which you would otherwise leave blank.

1. Give answers only to the following questions. Explanation is <u>not</u> required.

Let $A = \begin{bmatrix} 0 & 2 & 4 & 6 \\ 0 & 1 & 2 & 3 \\ 1 & -2 & 1 & 0 \end{bmatrix}$ and $T$ be the linear transformation induced by $A$.

(a) Find the reduced row echelon form of $A$.

(b) Find a PLU decomposition of $A$.

(c) What is the dimension of the natural domain of $T$?

(d) What is the dimension of the range of $T$?

(e) Find a basis for the column space of $A$.

(f) Find a basis for the kernel of $T$.

(g) Let $B$ be the square matrix obtained by appending a row of 1's at the bottom of $A$, and $C$ be the matrix obtained by adding 1 to each entry in the first column of $B$. Find $\det C$.

(h) With $C$ as defined above, what is the $(2,3)$-entry of $C^{-1}$?

2. In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do <u>not</u> indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is <u>not</u> required.

(a) If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, which of the following is a linear combination of $\mathbf{x}$ and $\mathbf{y}$?

(1) $(-1, 0, 6)^T$

(2) $(0, 1, 4)^T$

(4) $(1, 0, 2)^T$

(8) $(2, 3, 4)^T$

(b) Let $A$ be a $4 \times 6$ matrix. Which of the following is possible?

(1) The rank of $A$ is 5.

(2) The nullity of $A$ is 5.

(4) There exists $\mathrm{b} \in \overset{\mathbb{R}^4}{\cancel{\mathbb{R}^6}}$ for which the system $A\mathrm{x} = \mathrm{b}$ has a unique solution.

(8) There exists $\mathrm{b} \in \overset{\mathbb{R}^4}{\cancel{\mathbb{R}^6}}$ for which the system $A\mathrm{x} = \mathrm{b}$ has infinitely many solutions.

(c) Let $A$ be a $6 \times 4$ matrix. If $\mathbf{b} \in \overset{\mathbb{R}^6}{\cancel{\mathbb{R}^4}}$ and $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^4$, which of the following is possible?

(1) The system $A\mathbf{x} = \mathbf{0}$ is inconsistent.

(2) The system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

(4) The system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(8) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(d) Let $A$ be a $5 \times 4$ matrix with rank 3. Which of the following must be true?

(1) There exist 2 columns of $A$ which are linearly dependent.

(2) There exist 2 columns of $A$ which are linearly independent.

(4) There exist 3 columns of $A$ which are linearly dependent.

(8) There exist 3 columns of $A$ which are linearly independent.

(e) If $A$, $B$ and $C$ are matrices such that $AB = C$, which of the following is true?

(1) If $C$ has a zero row, so does $A$.

(2) If $C$ has a zero row, so does $B$.

(4) If $C$ has a zero column, so does $A$.

(8) If $C$ has a zero column, so does $B$.

(f) Let $A$ be an invertible matrix. Which of the following must be symmetric?

(1) $AA^T$

(2) $AA^{-1}$

(4) $A + A^T$

(8) $A + A^{-1}$

(g) Let $A$ be an $n \times n$ matrix. Which of the following is a subspace of $\mathbb{R}^n$?

(1) The nullity of $A$

(2) The set of all linear combinations of the columns of $A$

(4) The set of all solutions to the system $A\mathbf{x} = \mathbf{0}$

(8) The set of all solutions to the system $A^2\mathbf{x} = \mathbf{0}$

(h) Let $V$ be a subspace of $\mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. If $\mathbf{y}$ is the orthogonal projection of $\mathbf{x}$ onto $V$, which of the following must be true?

(1) $\mathbf{x} - \mathbf{y}$ is orthogonal to $V$.

(2) The choice of $\mathbf{y}$ is unique.

(4) $\mathbf{y}$ is the vector in $V$ with the least norm.

(8) $\mathbf{y}$ is the vector in $V$ that is closest to $\mathbf{x}$.

3. ~~Let $A$ and $B$ be square matrices of the same size.~~ For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

(a) Suppose the reduced row echelon form of $A$ has a zero row. If the system $A\mathbf{x} = \mathbf{b}$ is consistent, then it has infinitely many solutions.

(b) Let $B$ be an $n \times n$ matrix. If $BA = A$ for all $n \times n$ matrix $A$, then $B = I$.

(c) Let $A$ be a square matrix with real entries. Then the entries of $A^2$ are all non-negative.

(d) There exists a matrix $A$ such that $A^3 = 8A$ but $A^2 \neq 8I$.

(e) The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$ is closed under addition.

(f) Let $V$ be a 3-dimensional subspace of $\mathbb{R}^4$. Then any two vectors in $\mathbb{R}^4$ which do not belong to $V$ must be linearly dependent.

(g) If $S$ is a subspace of $\mathbb{R}^n$, so is $(\mathbb{R}^n \setminus S) \cup \{\mathbf{0}\}$.

(h) For any subset $S$ of $\mathbb{R}^n$, we have $S \neq S^\perp$.

4. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Let also $\mathcal{A}$ be the ordered basis of $\mathbb{R}^3$ whose elements are the columns of $A$, and similarly for $\mathcal{B}$. Define a function $T : \mathbb{R}^3 \to \mathbb{R}^3$ as follows. For each $\mathbf{x} \in \mathbb{R}^3$, if $\mathbf{y}$ is the vector in $\mathbb{R}^3$ satisfying $[\mathbf{y}]_{\mathcal{A}} = \mathbf{x}$, then $T(\mathbf{x}) = [\mathbf{y}]_{\mathcal{B}}$.

(a) Explain why such $T$ is well-defined.

(b) Find $T(\mathbf{v})$.

(c) Is $T$ a linear transformation?

(d) Describe the mathematical meaning of $T$.

5. Let $U$, $V$ be subspaces of $\mathbb{R}^n$. We define the *sum* of $U$ and $V$, denoted by $U + V$, by

$$U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \ \mathbf{v} \in V\}.$$

The sum is said to be a *direct sum*, denoted by $U \oplus V$, if $U \cap V = \{0\}$.

(a) Show that $U + V$ is a subspace of $\mathbb{R}^n$.

(b) If $W = U \oplus V$, show that every vector in $\mathbf{w} \in W$ can be decomposed uniquely into the form $\mathbf{w} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in U$ and $\mathbf{v} \in V$.

(c) Formulate and prove a converse of (b).

(d) With the above notations and results, the orthogonal decomposition theorem can be rephrased as $\mathbb{R}^n = U \oplus U^{\perp}$. If $\mathbb{R}^n = U \oplus V$, is it true that we must have $V = U^{\perp}$?

(a)

6. Let $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 4 & -2 & 0 \\ 4 & 8 & 0 & -2 \end{bmatrix}$.

~~(a)~~ Show that $A$ is diagonalisable over $\mathbb{C}$ but not over $\mathbb{R}$.

(Hint: First show that $A$ has two equal real eigenvalues and a pair of complex eigenvalues that are conjugates of each other.)

(b) Suppose you are the teacher of a linear algebra course, and you only have pen and paper with you. Now you want to set an examination question involving a $4 \times 4$ matrix $A$ with real entries, for which students need to find the eigenvalues. Discuss methods that you can apply to generate such $A$ for different forms of eigenvalues you may have in mind (e.g. two equal real eigenvalues and a pair of complex conjugates), how the difficulty of the question may be adjusted (for a fixed set of eigenvalues), and ways to make the entries of $A$ integral (provided that the desired eigenvalues are sufficiently nice).

******* END OF PAPER *******