THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH 2101: LINEAR ALGEBRA I

December 12, 2018 2:30 – 5:00 pm

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer all SIX questions. Each question carries 16 marks.
Another 4 marks is given for clarity of presentation and following instructions.

Notes:

- You should give precise and adequate explanations to support your conclusions unless otherwise specified by the question. Clarity of presentation of your argument counts. So **think carefully before you write.**
- You must start each question on a new page. However do NOT start each part of a question on a new page. You should write down the question number on the top right hand corner of each page. Indicate clearly which part of a question you are answering.
- If your answer to a question spans over more than one page, you must indicate clearly on each page (except the last) that the answer will continue on the subsequent page.
- If you do not otherwise specify, you are assumed to be answering the questions in order. In particular, indicate on each question where your answer to the next question is if it does not follow immediately. (For example you may write 'turn back 3 pages for Question 5' at the end of Question 4.)
- Whenever there is a blank page it will be assumed that all subsequent pages are also blank. If this is not true, indicate clearly on the page which you would otherwise leave blank.

1. Give answers only to the following questions. Explanation is <u>not</u> required.

Given that the reduced row echelon form of $A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 2 & 4 & 6 & 8 \\ 1 & 3 & 5 & 7 \end{bmatrix}$ is $R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Let $\mathbf{a}_i$ denote the i-th column of A.

(a) Express $\mathbf{a}_1$ as a linear combination of $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$.

(b) What is the nullity of A?

(c) What is the rank of $A^T R$?

(d) Let B be a matrix such that $AB = I_n$. What is n?

(e) Continuing the previous part, how many such choices of B are there?

(f) If we add a row of 1's to the bottom of A, what is the determinant of the resulting 4×4 matrix?

(g) Let $\mathcal{B} = \{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ be an ordered basis of $\mathbb{R}^3$, and $\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. What is $[\mathbf{x}]_{\mathcal{B}}$?

(h) Continuing the previous part, what is the projection of $\mathbf{x}$ onto $\mathbf{a}_1$?

2. In each multiple choice question below, some numbered choices are given and some are correct (wordings like *is* and *are* do <u>not</u> indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is <u>not</u> required.

(a) Let A be an upper triangular square matrix. Which of the following must be true?

(1) A is invertible.

(2) A is not symmetric.

(4) A^T is lower triangular.

(8) A^2 is upper triangular.

(b) Which of the following matrices are in reduced row echelon form?

(1) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

(8) $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

(c) Let A be a 5×4 matrix with rank 3. We must be able to find

(1) two columns of A which are linearly independent

(2) three columns of A which are linearly independent

(4) three columns of A which are linearly dependent

(8) four columns of A which are linearly dependent

(d) Let A and B be square matrices of the same size. If A is diagonalisable but B is not, then

(1) $\det A \neq \det B$

(2) A and B are not similar

(4) at least one eigenvalue of A is not an eigenvalue of B

(8) at least one eigenvalue of B is not an eigenvalue of A

(e) Let A be a 3×3 matrix with eigenvalues 1, 1 and -1. If A is diagonalisable, which of the following must be true?

(1) $A^2 = A$

(2) $A^3 = A$

(4) $A^4 = A$

(8) $A^5 = A$

(f) Let $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Then

(1) $\mathbf{x}^T$ is in the row space of A

(2) $\mathbf{x}$ is in the column space of A

(4) $\mathbf{x}$ is in the null space of A

(8) $\mathbf{x}$ is an eigenvector of A

(g) When every entry of a square matrix A is doubled, which of the following must also be doubled?

(1) The rank of A

(2) The determinant of A

(4) The eigenvalues of A

(8) The characteristic polynomial of A

(h) For any matrix A, A and A^T must have the same

(1) rank

(2) nullity

(4) determinant, provided that A is a square matrix

(8) eigenvalues, provided that A is a square matrix

3. For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

(a) Let A and B be square matrices of the same size. If A is invertible and $ABA = A$, then B is invertible.

(b) Let A be an $n \times n$ matrix where $n > 1$. Then A can be written as $A = B + C$ where B and C are $n \times n$ matrices, both of which are non-invertible.

(c) The matrices $\begin{bmatrix} 10 & 9 & 9 & 9 \\ 9 & 0 & 1 & 2 \\ 9 & 3 & 4 & 5 \\ 9 & 6 & 7 & 8 \end{bmatrix}$ and $\begin{bmatrix} 20 & 9 & 9 & 9 \\ 9 & 0 & 1 & 2 \\ 9 & 3 & 4 & 5 \\ 9 & 6 & 7 & 8 \end{bmatrix}$ have the same determinant.

(d) Let A be a square matrix with integer entries. If $\det A = 1$, then all entries of A^{-1} are integers.

(e) Let W be a subspace of $\mathbb{R}^6$ with dimension 4. Then any 3 vectors in W are linearly independent.

(f) If the characteristic polynomial of A is $(t^2 - 4t + 3)(t^2 - 4t + 1)$, then A must be diagonalisable.

(g) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors in $\mathbb{R}^n$, then $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal.

(h) Let $S \subseteq \mathbb{R}^7$. If $|S| = 3$, then $\dim S^\perp = 4$.

4. Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(a) Find an LU decomposition of A.

(b) Find an orthonormal basis for the column space of A.

(c) Given that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent, find the best approximate solution.

5. Let $A = \begin{bmatrix} 2 & -11 & -11 & -3 \\ -3 & 13 & 11 & 5 \\ 5 & -17 & -11 & -9 \\ -7 & 19 & 11 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 14 \\ 13 \\ 4 \\ 7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -180 \\ 206 \\ -258 \\ 242 \end{bmatrix}$.

It is given that $\mathbf{v}_1$ and $\mathbf{v}_2$ are both solutions to the system $A\mathbf{x} = \mathbf{b}$.

(a) Find another solution to $A\mathbf{x} = \mathbf{b}$.

(b) Let $\mathbf{a}_i$ denote the i-th column of A. For each of the following, determine whether the vectors are linearly independent or not. (If you intend to solve this part by direct computation, you must show all the detailed steps to receive credit. In particular you must not make use of R in part (c) unless you derive it by yourself.)

(i) $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$

(ii) $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}$

(c) It is known that A has reduced row echelon form $R = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Using this, or otherwise, find the general solution to $A\mathbf{x} = \mathbf{b}$.

(Warning: Do not solve $R\mathbf{x} = \mathbf{b}$. It is not equivalent to $A\mathbf{x} = \mathbf{b}$!)

6. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **idempotent** if $T(T(\mathbf{x})) = T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. The zero and identity transformations are two trivial examples of idempotent linear transformations.

(a) Give a non-trivial example of an idempotent linear transformation.

(b) Explain why every non-zero idempotent linear transformation has 1 as an eigenvalue.

(c) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a non-zero idempotent linear transformation, and let $\mathcal{A} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a basis for the range of T, where $k < n$.

(i) Based on the above information, what can you say about the injectivity and surjectivity of T?

(ii) We know that $\mathcal{A}$ can be extended to a basis $\mathcal{B} = \mathcal{A} \cup \{\mathbf{y}_{k+1}, \ldots, \mathbf{y}_n\}$ of $\mathbb{R}^n$. For $i = k+1, \ldots, n$, let $\mathbf{z}_i = \mathbf{y}_i - T(\mathbf{y}_i)$. Show that $\mathcal{C} = \mathcal{A} \cup \{\mathbf{z}_{k+1}, \ldots, \mathbf{z}_n\}$ is a basis for $\mathbb{R}^n$.

(iii) Continuing the previous part, what is the $\mathcal{C}$-matrix of T?

******* **END OF PAPER** *******