THE UNIVERSITY OF HONG KONG
DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

STAT2602   PROBABILITY AND STATISTICS II

13 Dec, 2022   Time: 9:30 a.m. - 11:30 a.m.

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer ALL FIVE questions. Marks are shown in square brackets.

1. (True-false question)

   (i) If $Z \sim N(0,1)$, $V \sim \chi_\nu^2$, and Z and V are independent, then $\dfrac{Z}{\sqrt{V/\nu}}$ is t_ν.

   [3 marks]

   A. True   B. False

   (ii) If $\hat{\theta}$ is the MLE of θ, then $\hat{\theta}^2$ is the MLE of θ^2.   [3 marks]

   A. True   B. False

   (iii) If both $\hat{\theta}$ and $\tilde{\theta}$ are UMVUEs of θ, then $\hat{\theta} = \tilde{\theta}$.   [3 marks]

   A. True   B. False

   (iv) If both $\hat{\theta}$ and $\tilde{\theta}$ are unbiased estimators of θ and $\mathrm{Var}(\hat{\theta}) > \mathrm{Var}(\tilde{\theta})$, then $\hat{\theta}$ is relatively more efficient than $\tilde{\theta}$.   [3 marks]

   A. True   B. False

   (v) If $\hat{\theta}$ is an unbiased estimator of θ, then $\hat{\theta}$ is a consistent estimator of θ.

   [3 marks]

   A. True   B. False

   [Total: 15 marks]

2. Let $X_1, X_2, \cdots, X_n$ be an independent random sample from the uniform distribution $U(0, \theta)$.

   (i) Construct a pivotal variable based on the sample maximum $X_{(n)}$. [6 marks]

   (ii) Use the result in part (a), or otherwise, to construct a $1 - \alpha$ confidence interval for θ.   [6 marks]

   [Total: 12 marks]

3. The table below records the sales of a convenience food, in thousands of dollars, for each of five regions during a month before and a month after an advertising campaign respectively. Assume that the sales before and after campaign both come from normal distributions, with the same variance but possibly different means. Note: Sample variance $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

Region	A	B	C	D	E	Sample mean	Sample variance
Sales before campaign	2.4	2.6	3.9	2.0	3.2	2.82	0.4476
Sales after campaign	3.0	2.5	4.0	4.1	4.8	3.68	0.6776

(i) Determine, at the 5% level of significance, whether an increase in the mean amount of one-month sales has occurred after the advertising campaign.

[6 marks]

(ii) Construct a 95% confidence interval (correct to a 3 decimal places) for the difference between the mean amounts of one-month sales before and after the advertising campaign respectively. [6 marks]

[Total: 12 marks]

4. Let X_1, X_2, X_3, X_4 be a random sample of STAT2601 scores, assumed to be $N(\mu_x, \sigma_x^2)$, and let Y_1, Y_2, Y_3, Y_4, Y_5 be a random sample of STAT3902 scores, assumed to be $N(\mu_y, \sigma_y^2)$. Suppose $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, Y_5$ are independent, and the following data are observed:

$$x_1 = 75 \quad x_2 = 95 \quad x_3 = 80 \quad x_4 = 65$$

$$y_1 = 80 \quad y_2 = 75 \quad y_3 = 90 \quad y_4 = 70 \quad y_5 = 70.$$

(Note: $\chi_{0.95,3} = 0.35$, $\chi_{0.05,3} = 7.82$, $\chi_{0.95,4} = 0.71$, $\chi_{0.05,4} = 9.49$, $\chi_{0.95,5} = 1.15$, and $\chi_{0.05,5} = 11.07$, where $P(\chi_n^2 \le \chi_{\alpha,n}^2) = 1 - \alpha$, in which χ_n^2 follows a χ^2 distribution with degree of freedom n.)

(Note: $F_{0.05,3,4} = 6.59$, $F_{0.05,4,3} = 9.12$, $F_{0.05,1,1} = 161.45$, $F_{0.05,2,2} = 19$, and $F_{1-\alpha,m,n} = \frac{1}{F_{\alpha,n,m}}$ where $P(F_{n_1,n_2} \le F_{\alpha,n_1,n_2}) = 1 - \alpha$, in which F_{n_1,n_2} follows a F distribution with degree of freedom (n_1, n_2).)

(i) Find a 90% confidence interval estimate for σ_x^2. [6 marks]

(ii) Find a 90% confidence interval estimate for σ_x^2/σ_y^2. [6 marks]

[Total: 12 marks]

5. Let $X_1, X_2, \cdots, X_n$ be an independent random sample from a distribution, which has the density given by

$$f(x; \theta) = \frac{\theta}{x^{\theta+1}} \quad \text{for} \quad x \geq 1,$$

where $\theta > 0$ is an unknown parameter.

(i) Write down the likelihood function of θ based on $X_1, X_2, \cdots, X_n$. [4 marks]

(ii) Find a scalar sufficient statistic for θ. [4 marks]

(iii) Find the Fisher information about θ contained in the data $X_1, X_2, \cdots, X_n$. [5 marks]

(iv) Find the Cramer-Rao Lower Bound for estimation of θ. [3 marks]

(v) (a) Show that the MLE of θ is $\hat{\theta} = n/(\sum_{i=1}^{n} \ln X_i)$. [5 marks]
    (b) State its asymptotic distribution. [4 marks]
    (c) Prove that $\hat{\theta}$ is consistent. [4 marks]

(vi) Find the rejection region of the most powerful test for the hypotheses:

$$H_0 : \theta = 2 \quad \text{versus} \quad H_1 : \theta = 3.$$

[6 marks]

(vii) Show that the generalized likelihood ratio test statistic for testing

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta \text{ unrestricted,}$$

where $\theta_0 > 0$ is fixed, is

$$-2 \ln \Lambda = 2n[\theta_0 \bar{Y} - 1 - \ln(\theta_0 \bar{Y})],$$

where Λ denotes the generalized likelihood ratio, $\bar{Y} = \frac{1}{n}(Y_1 + \cdots + Y_n)$ and $Y_i = \ln(X_i)$, for $1 \leq i \leq n$. [7 marks]

(viii) Denote $\lambda(\bar{Y}) = 2n[\theta_0 \bar{Y} - 1 - \ln(\theta_0 \bar{Y})]$. Show that for fixed θ_0, $\lambda(\bar{Y})$ monotonically decreases as $\bar{Y}$ increases from 1 to $1/\theta_0$, and monotonically increases as $\bar{Y}$ increases from $1/\theta_0$ to ∞.

Describe the rejection region based on the generalized likelihood ratio test statistic. [7 marks]

(**Note:** For the rejection regions in both (vi) and (viii), there is no need to specify the critical values at a significance level.)

[Total: 49 marks]

A LIST OF STATISTICAL FORMULAE

1. $M_X(t) = \mathrm{E}(e^{tX}).$ $\mathrm{E}(X^r) = \left(\dfrac{\mathrm{d}^r}{\mathrm{d}t^r} M_X(t)\right)\Bigg|_{t=0}.$

2. $\overline{X} = \dfrac{1}{n}\sum\limits_{i=1}^{n} X_i.$ $S^2 = \dfrac{1}{n}\sum\limits_{i=1}^{n}\left(X_i - \overline{X}\right)^2.$

3. CLT: $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \approx \mathrm{N}(0,1)$ for large n.

4. Normal population $\Longrightarrow \dfrac{nS^2}{\sigma^2} \sim \chi^2_{n-1},\ \dfrac{\overline{X} - \mu}{S/\sqrt{n-1}} \sim t_{n-1}.$

5. $\chi^2_\nu \sim \sum\limits_{i=1}^{\nu} Z_i^2.$ $t_\nu \sim \dfrac{Z}{\sqrt{\chi^2_\nu/\nu}}.$ $F_{\nu_1,\nu_2} \sim \dfrac{\chi^2_{\nu_1}/\nu_1}{\chi^2_{\nu_2}/\nu_2}.$

6. $\mathrm{Bias}(\hat{\theta}) = \mathrm{E}(\hat{\theta}) - \theta.$ $\mathrm{E}\left[(\hat{\theta} - \theta)^2\right] = \mathrm{Var}(\hat{\theta}) + \left[\mathrm{Bias}(\hat{\theta})\right]^2.$

7. $I(\theta) = \mathrm{E}\left[\left(\dfrac{\partial \log f(X;\theta)}{\partial \theta}\right)^2\right] = \mathrm{E}\left[-\dfrac{\partial^2 \log f(X;\theta)}{\partial \theta^2}\right].$ $\mathrm{Var}(\hat{\theta}) \geq \dfrac{1}{nI(\theta)}.$

8. $\mathbf{f}(x_1, x_2, \ldots, x_n; \theta) = g\left(u(x_1, x_2, \ldots, x_n), \theta\right) \cdot h(x_1, x_2, \ldots, x_n).$

9. $\dfrac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}}.$ $\dfrac{\overline{X} - \mu_0}{S/\sqrt{n-1}}.$

10. $\dfrac{\overline{X} - \overline{Y} - \delta}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}.$ $\dfrac{\overline{X} - \overline{Y} - \delta}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}},$ where $S_p^2 = \dfrac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2}.$

11. $\overline{x} \pm z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}.$ $\overline{x} \pm t_{\alpha/2,n-1}\dfrac{s}{\sqrt{n-1}}.$ $\left(\dfrac{ns^2}{\chi^2_{\alpha/2,n-1}}, \dfrac{ns^2}{\chi^2_{1-\alpha/2,n-1}}\right).$

12. $\overline{x} - \overline{y} \pm z_{\alpha/2}\sqrt{\dfrac{\sigma_x^2}{n_1} + \dfrac{\sigma_y^2}{n_2}}.$ $\overline{x} - \overline{y} \pm t_{\alpha/2,n_x+n_y-2}\sqrt{\dfrac{n_x s_x^2 + n_y s_y^2}{n_x + n_y - 2}\left(\dfrac{1}{n_x} + \dfrac{1}{n_y}\right)}.$

$\left(\dfrac{n_x(n_y - 1)s_x^2}{n_y(n_x - 1)s_y^2} \cdot \dfrac{1}{F_{\alpha/2,n_x-1,n_y-1}}, \dfrac{n_x(n_y - 1)s_x^2}{n_y(n_x - 1)s_y^2} \cdot F_{\alpha/2,n_y-1,n_x-1}\right).$

Cumulative distribution function $\Phi(x)$ for $N(0,1)$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Upper percentile for the student's t distribution t_ν

ν	60.0%	66.7%	75.0%	80.0%	87.5%	90.0%	95.0%	97.5%	99.0%	99.5%	99.9%
1	0.325	0.577	1.000	1.376	2.414	3.078	6.314	12.706	31.821	63.657	318.31
2	0.289	0.500	0.816	1.061	1.604	1.886	2.920	4.303	6.965	9.925	22.327
3	0.277	0.476	0.765	0.978	1.423	1.638	2.353	3.182	4.541	5.841	10.215
4	0.271	0.464	0.741	0.941	1.344	1.533	2.132	2.776	3.747	4.604	7.173
5	0.267	0.457	0.727	0.920	1.301	1.476	2.015	2.571	3.365	4.032	5.893
6	0.265	0.453	0.718	0.906	1.273	1.440	1.943	2.447	3.143	3.707	5.208
7	0.263	0.449	0.711	0.896	1.254	1.415	1.895	2.365	2.998	3.499	4.785
8	0.262	0.447	0.706	0.889	1.240	1.397	1.860	2.306	2.896	3.355	4.501
9	0.261	0.445	0.703	0.883	1.230	1.383	1.833	2.262	2.821	3.250	4.297
10	0.260	0.444	0.700	0.879	1.221	1.372	1.812	2.228	2.764	3.169	4.144
11	0.260	0.443	0.697	0.876	1.214	1.363	1.796	2.201	2.718	3.106	4.025
12	0.259	0.442	0.695	0.873	1.209	1.356	1.782	2.179	2.681	3.055	3.930
13	0.259	0.441	0.694	0.870	1.204	1.350	1.771	2.160	2.650	3.012	3.852
14	0.258	0.440	0.692	0.868	1.200	1.345	1.761	2.145	2.624	2.977	3.787
15	0.258	0.439	0.691	0.866	1.197	1.341	1.753	2.131	2.602	2.947	3.733
16	0.258	0.439	0.690	0.865	1.194	1.337	1.746	2.120	2.583	2.921	3.686
17	0.257	0.438	0.689	0.863	1.191	1.333	1.740	2.110	2.567	2.898	3.646
18	0.257	0.438	0.688	0.862	1.189	1.330	1.734	2.101	2.552	2.878	3.610
19	0.257	0.438	0.688	0.861	1.187	1.328	1.729	2.093	2.539	2.861	3.579
20	0.257	0.437	0.687	0.860	1.185	1.325	1.725	2.086	2.528	2.845	3.552
21	0.257	0.437	0.686	0.859	1.183	1.323	1.721	2.080	2.518	2.831	3.527
22	0.256	0.437	0.686	0.858	1.182	1.321	1.717	2.074	2.508	2.819	3.505
23	0.256	0.436	0.685	0.858	1.180	1.319	1.714	2.069	2.500	2.807	3.485
24	0.256	0.436	0.685	0.857	1.179	1.318	1.711	2.064	2.492	2.797	3.467
25	0.256	0.436	0.684	0.856	1.178	1.316	1.708	2.060	2.485	2.787	3.450
∞	0.253	0.431	0.674	0.842	1.150	1.282	1.645	1.960	2.326	2.576	3.090

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