<u>THE UNIVERSITY OF HONG KONG</u>

<u>DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE</u>

STAT2602 PROBABILITY AND STATISTICS II

December 20, 2019 **Time: 9:30 a.m. – 11:30 a.m.**

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer ALL SIX questions. Marks are shown in square brackets.

1. Suppose that a random variable X has a (probability) density function given by

$$f(x) = \begin{cases} 2e^{-2x}, & \text{for } x > 0; \\ 0, & \text{otherwise,} \end{cases}$$

(i) Calculate the moment generating function of X. **[6 marks]**

(ii) Calculate $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$. **[6 marks]**

(iii) Calculate $\mathrm{E}(e^{X/2})$, $\mathrm{E}(e^{X})$ and $\mathrm{E}(e^{3X})$, if they exist. **[3 marks]**

(iv) Based on an independent random sample $\mathbf{X} = \{X_1, X_2, ..., X_n\}$ from the distribution of X, provide a consistent estimator for $\theta = \mathrm{E}(e^{\sin(X)})$, where $\sin(\cdot)$ is the sine function. **[5 marks]**

[Total: 20 marks]

2. Suppose that a person takes two steps to cook a dish. At step one, the cooking time (in hour) T_1 follows a distribution with mean 0.1 and variance 0.1. At step two, the cooking time (in hour) T_2 follows another distribution with mean 0.3 and variance 0.3. Suppose that T_1 and T_2 are independent. What is the approximate probability that this person can cook 50 dishes within 19 hours?

[Total: 15 marks]

3. Let $\mathbf{X} = \{X_1, X_2, ..., X_n\}$ be an independent random sample from the Poisson distribution with parameter $\theta > 0$, where a random variable X follows the Poisson distribution having the mean θ, variance θ, and (probability) density function:

$$\mathrm{P}(X = x) = \frac{\theta^x e^{-\theta}}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

(i) Find $\hat{\theta}_1$, the MLE of θ on the space $\Omega_1 = \{\theta : \theta > 0\}$. **[5 marks]**

(ii) Show that $\hat{\theta}_1$ is an unbiased estimator of θ. **[2 marks]**

(iii) Show that $\hat{\theta}_1$ is a consistent estimator of θ. **[3 marks]**

(iv) Will $\hat{\theta}_1$ be the UMVUE of θ? Explain. **[5 marks]**

(v) Find $\hat{\theta}_2$, the MLE of θ on the space

$$\Omega_2 = \{\theta : 0 < \theta < \theta_*\},$$

where θ_* is a given finite positive constant. **[10 marks]**

(vi) Let $\delta = \mathrm{P}(X > 3)$. Find an MME of δ. **[5 marks]**

[Total: 30 marks]

4. Let $\mathbf{X} = \{X_1, X_2, X_3, X_4, X_5\}$ be an independent random sample from $\mathrm{N}(\mu, \sigma^2)$.

   (i) Determine the distribution of $W_1 = \dfrac{(X_2 - X_3)^2 + (X_4 - X_5)^2}{2\sigma^2}$. **[5 marks]**

   (ii) Determine the distribution of $W_2 = \dfrac{2(X_1 - \mu)}{\sqrt{(X_2 - X_3)^2 + (X_4 - X_5)^2}}$. **[5 marks]**

   (iii) Construct a $(1-\alpha)$ confidence interval of μ by using W_2. **[5 marks]**

   [Total: 15 marks]

5. Suppose that 9 persons are randomly selected, and their height (in cm) is measured in the morning and evening. Let X denote the height measured in the morning, and Y denote the height measured in the evening. The observed values for this sample are given in the following table:

Number (i)	1	2	3	4	5	6	7	8	9
Height in the morning (x_i)	170	175	160	161	165	172	170	180	174
Height in the evening (y_i)	170	174	158	160	163	172	168	177	172

   Suppose that $X \sim \mathrm{N}(\mu_x, \sigma_x^2)$, $Y \sim \mathrm{N}(\mu_y, \sigma_y^2)$, and $D = X - Y \sim \mathrm{N}(\mu_d, \sigma_d^2)$.

   (i) Find a 95% confidence interval of μ_x. **[4 marks]**

   (ii) Find a 95% confidence interval of μ_d. **[4 marks]**

   (iii) Is it reasonable to conclude that the height in the morning is higher than that in the evening? State your reason(s). **[2 marks]**

   [Total: 10 marks]

6. Let $\mathbf{X} = \{X_1, X_2, ..., X_n\}$ be an independent random sample from the Poisson distribution with parameter $\theta > 0$.

   (i) Find the rejection region of the most powerful test for hypotheses:

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2.$$

   [5 marks]

   (ii) Find the critical value such that this test has an exact size 0.05. **[5 marks]**

   [Total: 10 marks]

A LIST OF STATISTICAL FORMULAE

1. $M_X(t) = \mathrm{E}(e^{tX})$. $\quad \mathrm{E}(X^r) = \left(\dfrac{\mathrm{d}^r}{\mathrm{d}t^r} M_X(t)\right)\Bigg|_{t=0}$.

2. $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$. $\quad S^2 = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$.

3. CLT: $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \approx N(0,1)$ for large n.

4. Normal population $\Longrightarrow \dfrac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$, $\dfrac{\overline{X} - \mu}{S/\sqrt{n-1}} \sim t_{n-1}$.

5. $\mathrm{Bias}(\hat{\theta}) = \mathrm{E}(\hat{\theta}) - \theta$. $\quad \mathrm{E}\left[(\hat{\theta} - \theta)^2\right] = \mathrm{Var}(\hat{\theta}) + \left[\mathrm{Bias}(\hat{\theta})\right]^2$.

6. $I(\theta) = \mathrm{E}\left[\left(\dfrac{\partial \ln f(X;\theta)}{\partial \theta}\right)^2\right] = \mathrm{E}\left[-\dfrac{\partial^2 \ln f(X;\theta)}{\partial \theta^2}\right]$. $\quad \mathrm{Var}(\hat{\theta}) \geq \dfrac{1}{nI(\theta)}$.

7. $\mathbf{f}(x_1, x_2, \ldots, x_n; \theta) = g\left(u(x_1, x_2, \ldots, x_n), \theta\right) \cdot h(x_1, x_2, \ldots, x_n)$.

8. Expontial family: $f(x;\theta) = h(x)c(\theta)\exp\left(\sum_{i=1}^{s} p_i(\theta)t_i(x)\right)$.

9. $\overline{x} \pm z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}$. $\quad \overline{x} \pm t_{\alpha/2,n-1}\dfrac{s}{\sqrt{n-1}}$. $\quad \left(\dfrac{ns^2}{\chi^2_{\alpha/2,n-1}}, \dfrac{ns^2}{\chi^2_{1-\alpha/2,n-1}}\right)$.

10. $\overline{x} - \overline{y} \pm z_{\alpha/2}\sqrt{\dfrac{\sigma_x^2}{n_1} + \dfrac{\sigma_y^2}{n_2}}$. $\quad \overline{x} - \overline{y} \pm t_{\alpha/2,n_x+n_y-2}\sqrt{\dfrac{n_x s_x^2 + n_y s_y^2}{n_x + n_y - 2}\left(\dfrac{1}{n_x} + \dfrac{1}{n_y}\right)}$.

Cumulative distribution function $\Phi(x)$ for $N(0,1)$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Upper percentile for the student's t distribution t_ν

ν	60.0%	66.7%	75.0%	80.0%	87.5%	90.0%	95.0%	97.5%	99.0%	99.5%	99.9%
1	0.325	0.577	1.000	1.376	2.414	3.078	6.314	12.706	31.821	63.657	318.31
2	0.289	0.500	0.816	1.061	1.604	1.886	2.920	4.303	6.965	9.925	22.327
3	0.277	0.476	0.765	0.978	1.423	1.638	2.353	3.182	4.541	5.841	10.215
4	0.271	0.464	0.741	0.941	1.344	1.533	2.132	2.776	3.747	4.604	7.173
5	0.267	0.457	0.727	0.920	1.301	1.476	2.015	2.571	3.365	4.032	5.893
6	0.265	0.453	0.718	0.906	1.273	1.440	1.943	2.447	3.143	3.707	5.208
7	0.263	0.449	0.711	0.896	1.254	1.415	1.895	2.365	2.998	3.499	4.785
8	0.262	0.447	0.706	0.889	1.240	1.397	1.860	2.306	2.896	3.355	4.501
9	0.261	0.445	0.703	0.883	1.230	1.383	1.833	2.262	2.821	3.250	4.297
10	0.260	0.444	0.700	0.879	1.221	1.372	1.812	2.228	2.764	3.169	4.144
11	0.260	0.443	0.697	0.876	1.214	1.363	1.796	2.201	2.718	3.106	4.025
12	0.259	0.442	0.695	0.873	1.209	1.356	1.782	2.179	2.681	3.055	3.930
13	0.259	0.441	0.694	0.870	1.204	1.350	1.771	2.160	2.650	3.012	3.852
14	0.258	0.440	0.692	0.868	1.200	1.345	1.761	2.145	2.624	2.977	3.787
15	0.258	0.439	0.691	0.866	1.197	1.341	1.753	2.131	2.602	2.947	3.733
16	0.258	0.439	0.690	0.865	1.194	1.337	1.746	2.120	2.583	2.921	3.686
17	0.257	0.438	0.689	0.863	1.191	1.333	1.740	2.110	2.567	2.898	3.646
18	0.257	0.438	0.688	0.862	1.189	1.330	1.734	2.101	2.552	2.878	3.610
19	0.257	0.438	0.688	0.861	1.187	1.328	1.729	2.093	2.539	2.861	3.579
20	0.257	0.437	0.687	0.860	1.185	1.325	1.725	2.086	2.528	2.845	3.552
21	0.257	0.437	0.686	0.859	1.183	1.323	1.721	2.080	2.518	2.831	3.527
22	0.256	0.437	0.686	0.858	1.182	1.321	1.717	2.074	2.508	2.819	3.505
23	0.256	0.436	0.685	0.858	1.180	1.319	1.714	2.069	2.500	2.807	3.485
24	0.256	0.436	0.685	0.857	1.179	1.318	1.711	2.064	2.492	2.797	3.467
25	0.256	0.436	0.684	0.856	1.178	1.316	1.708	2.060	2.485	2.787	3.450
∞	0.253	0.431	0.674	0.842	1.150	1.282	1.645	1.960	2.326	2.576	3.090

********** END OF PAPER **********