THE UNIVERSITY OF HONG KONG
DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

STAT2602 PROBABILITY AND STATISTICS II

December 12, 2016 Time: 9:30 a.m. – 11:30 a.m.

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer ALL FIVE questions. Marks are shown in square brackets.

1. (True-false questions)

(a) If $Z \sim N(0,1)$, $V \sim \chi^2_\nu$, and Z and V are independent, then $\dfrac{Z}{\sqrt{V/\nu}}$ is t_ν.

[3 marks]

A. True B. False

(b) If $\hat{\theta}$ is an unbiased estimator of θ, then $\hat{\theta}$ is a consistent estimator of θ.

[3 marks]

A. True B. False

(c) If $\hat{\theta}$ is the MLE of θ, then $\hat{\theta}^2$ is the MLE of θ^2. **[3 marks]**

A. True B. False

(d) If both $\hat{\theta}$ and $\tilde{\theta}$ are UMVUEs of θ, then $\hat{\theta} = \tilde{\theta}$. **[3 marks]**

A. True B. False

(e) If both $\hat{\theta}$ and $\tilde{\theta}$ are unbiased estimators of θ and $\text{Var}(\hat{\theta}) > \text{Var}(\tilde{\theta})$, then $\hat{\theta}$ is relatively more efficient than $\tilde{\theta}$. **[3 marks]**

A. True B. False

[Total: 15 marks]

2. Let $X_1, X_2, \cdots, X_n$ be an independent random sample drawn from $N\left(\dfrac{n}{n+1}\mu, \sigma^2\right)$.

(a) Show that $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ is an asymptotically unbiased estimator of μ.

[5 marks]

(b) Show that

$$\text{Var}(\overline{X}) = \frac{\sigma^2}{n}.$$

Hence show that $\overline{X}$ is a consistent estimator of μ. **[6 marks]**

(c) If $\dfrac{n}{n+1}\mu = \mu_0$ (a constant), show that $S^2 = \dfrac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is a consistent estimator of σ^2 by using the weak law of large numbers. **[6 marks]**

(d) Given that if $Y \sim N(s, \sigma^2)$, $\text{E}(Y^4) = s^4 + 6s^2\sigma^2 + 3\sigma^2$. Show that S^2 is a consistent estimator of σ^2. **[6 marks]**

[Total: 23 marks]

3. Let $X_1, X_2, \cdots, X_n$ be an independent random sample drawn from $N\left(\mu, \sigma^2\right)$.

   (a) Write down the log-likelihood function $l(\mu, \sigma^2)$. **[5 marks]**

   (b) Show that $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ are the MLEs of μ and σ^2, respectively. **[10 marks]**

   (c) Calculate the CRLB with respect to μ. **[5 marks]**

   (d) Will $\overline{X}$ be the UMVUE of μ? Explain. **[2 marks]**

[Total: 22 marks]

4. Let $X_1, X_2, \cdots, X_n$ be an independent random sample drawn from $N(\mu_X, \sigma_X^2)$, and $Y_1, Y_2, \cdots, Y_m$ be an independent random sample drawn from $N(\mu_Y, \sigma_Y^2)$. Assume that these two random samples are independent. Define $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\overline{Y} = \frac{1}{m}\sum_{i=1}^{m} Y_i$, $S_X^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$, and $S_Y^2 = \frac{1}{m}\sum_{i=1}^{m}\left(Y_i - \overline{Y}\right)^2$.

   (a) Show that
   $$U = \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0,1),$$
   by using the moment generating function. **[5 marks]**

   (b) If $V_1 = \frac{nS_X^2}{\sigma_X^2} \sim \chi^2_{n-1}$, $V_2 = \frac{mS_Y^2}{\sigma_Y^2} \sim \chi^2_{m-1}$, and V_1 and V_2 are independent, show that
   $$V = V_1 + V_2 \sim \chi^2_{n+m-2},$$
   by using the moment generating function. **[5 marks]**

   (c) If U and V are independent, $\sigma_X^2/\sigma_Y^2 = d$, and d is a known constant, from U and V, construct a $1-\alpha$ confidence interval for $\mu_X - \mu_Y$. **[8 marks]**

   (d) If $d = 2$, $n = m = 6$, and we observed $\overline{X} = 2$, $\overline{Y} = 1$, $S_X^2 = 1/3$, and $S_Y^2 = 2/3$, determine whether the 95% confidence interval estimate contains 0. **[2 marks]**

[Total: 20 marks]

5. Suppose that the score of STAT2602 follows the distribution $N(\mu, \sigma^2)$. The teacher wants to see whether the mean score of STAT2602 is 80 or less than 80 by using hypothesis testing. So, he collects a random sample consisting of the scores of 100 students, and finds that the mean of this random sample is 77 and the variance of this random sample is 9.

   (a) State the null and alternative hypotheses for the teacher. **[5 marks]**

   (b) Suppose that $\sigma^2 = 16$, and the teacher uses $\overline{X}$ as the test statistic and $\{\overline{X} < 79.5\}$ as the rejection region. Write down the power function $\pi(\mu)$ in terms of the cumulative distribution function of $N(0, 1)$. **[5 marks]**

   (c) Does this test have the significance level $\alpha = 0.05$? **[2 marks]**

   (d) Find the value of $\lim_{\mu \to -\infty} \pi(\mu)$. **[3 marks]**

   (e) If the value of σ^2 is unknown, should the teacher reject the null hypothesis at the significance level $\alpha = 0.05$ based on the generalized likelihood ratio test? **[5 marks]**

[Total: 20 marks]

A LIST OF STATISTICAL FORMULAE

1. $M_X(t) = \mathrm{E}(e^{tX})$.

2. If $X \sim N(\mu, \sigma^2)$, the p.d.f. is $\dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$, and

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right), \quad \frac{nS^2}{\sigma^2} \sim \chi^2_{n-1}, \quad \frac{\overline{X} - \mu}{S/\sqrt{n-1}} \sim t_{n-1}.$$

3. If $X \sim \chi^2_\nu$,

$$M_X(t) = \frac{1}{(1-2t)^{\frac{\nu}{2}}} \quad \text{for } t < \frac{1}{2}.$$

4. $\mathrm{Bias}(\hat{\theta}) = \mathrm{E}(\hat{\theta}) - \theta$.

5. $I(\theta) = \mathrm{E}\left[\left(\dfrac{\partial \ln f(X;\theta)}{\partial \theta}\right)^2\right] = \mathrm{E}\left[-\dfrac{\partial^2 \ln f(X;\theta)}{\partial \theta^2}\right]$. $\quad \mathrm{Var}(\hat{\theta}) \geq \dfrac{1}{nI(\theta)}$.

6. If $X_1, X_2, \cdots, X_n$ are independent,

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = a_i^2 \sum_{i=1}^{n} \mathrm{Var}\left(X_i\right).$$

7. $t_{0.025,10} = 2.228$ and $t_{0.05,\infty} = 1.65$

8. For $X \sim N(0,1)$, $\mathrm{P}\left(X < -1.25\right) = 0.1$

********** END OF PAPER **********