THE UNIVERSITY OF HONG KONG
DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

STAT2602 PROBABILITY AND STATISTICS II

May 16, 2022 Time: 9:30 a.m. - 11:30 a.m.

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer ALL FIVE questions. Marks are shown in square brackets.

1. Let $X_1, X_2, \cdots, X_n$ be an independent random sample from $N\left(\frac{n}{n+1}\mu, \sigma^2\right)$.

   (i) Show that $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is an asymptotically unbiased estimator of μ.

   [5 marks]

   (ii) Show that

   $$\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}.$$

   Hence show that $\overline{X}$ is a consistent estimator of μ. **[5 marks]**

   (iii) If $\frac{n}{n+1}\mu = \mu_0$ (a constant), show that $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is a consistent estimator of σ^2 by using the weak law of large numbers. **[5 marks]**

   (iv) If $\mathrm{E}(X^4) = s^4 + 6s^2\sigma^2 + 3\sigma^2$ when $X \sim N(s, \sigma^2)$, show that $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is a consistent estimator of σ^2. **[5 marks]**

   [Total: 20 marks]

2. Let $X_1, X_2, \cdots, X_n$ be an independent random sample from a Laplace(θ) distribution, which has the density given by

   $$f(x;\theta) = \frac{1}{2\theta}\exp\left(-\frac{|x|}{\theta}\right) \quad \text{for} \quad -\infty < x < \infty.$$

   Here, $\theta > 0$ is a parameter. Note that if $X \sim$ Laplace(θ), $\mathrm{E}X = 0$, $\mathrm{E}|X| = \theta$ and $\mathrm{E}(X^2) = 2\theta^2$.

   (i) Find the MLE of θ on $\Omega_1 = \{\theta : \theta > 0\}$. **[5 marks]**

   (ii) Calculate the CRLB with respect to θ. **[5 marks]**

   (iii) Is the MLE in (i) the UMVUE of θ? **[5 marks]**

   (iv) Find the rejection region of the most powerful test for hypotheses:

   $$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2.$$

   [5 marks]

   [Total: 20 marks]

3. Let $X_1, X_2, \cdots, X_n$ be an independent random sample from Uniform$[\theta, 0]$ for $\theta < 0$. Let $X_{(1)}$ be the smallest value of the sample $\{X_i\}_{i=1}^{n}$.

(i) Show that $T = \dfrac{X_{(1)}}{\theta}$ is a pivotal quantity. **[10 marks]**

(ii) Give a $1-\alpha$ equal-tailed confidence interval for θ. **[10 marks]**

[Total: 20 marks]

4. Let X_1, X_2, X_3, X_4 be a random sample from $N(\mu_x, \sigma_x^2)$, and let Y_1, Y_2, Y_3, Y_4, Y_5 be a random sample from $N(\mu_y, \sigma_y^2)$. Suppose $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, Y_5$ are independent, and the following data are observed:

$$x_1 = 75 \quad x_2 = 95 \quad x_3 = 80 \quad x_4 = 65$$

$$y_1 = 80 \quad y_2 = 75 \quad y_3 = 90 \quad y_4 = 70 \quad y_5 = 70$$

(i) Find a 90% confidence interval estimate for σ_x^2. **[5 marks]**

(ii) Find a 90% confidence interval estimate for σ_x^2/σ_y^2. **[5 marks]**

(iii) Find the distribution of $\dfrac{(X_1 - X_2)^2/\sigma_x^2}{(Y_1 - Y_2)^2/\sigma_y^2}$. **[5 marks]**

(iv) Based on the result in (iii), construct an alternative 90% confidence interval estimate for σ_x^2/σ_y^2. **[3 marks]**

(v) Is the confidence interval estimate in (ii) better than the one in (iv)? Explain your answer. **[2 marks]**

[Total: 20 marks]

5. Let $\mathbf{X} = \{X_1, X_2, ..., X_n\}$ be an independent random sample from the Poisson distribution with parameter $\theta > 0$.

(i) Find the rejection region of the most powerful test for hypotheses:

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 3.$$

[5 marks]

(ii) Find the critical value such that this test has an exact size 0.05. **[15 marks]**

[Total: 20 marks]

A LIST OF STATISTICAL FORMULAE

1. CLT: $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \approx N(0,1)$ for large n.

2. Normal population $\Longrightarrow \dfrac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$, $\dfrac{\overline{X}-\mu}{S/\sqrt{n-1}} \sim t_{n-1}$.

3. $\text{Bias}(\hat{\theta}) = \text{E}(\hat{\theta}) - \theta$. $\quad \text{E}\left[(\hat{\theta}-\theta)^2\right] = \text{Var}(\hat{\theta}) + \left[\text{Bias}(\hat{\theta})\right]^2$.

4. $I(\theta) = \text{E}\left[\left(\dfrac{\partial \ln f(X;\theta)}{\partial \theta}\right)^2\right] = \text{E}\left[-\dfrac{\partial^2 \ln f(X;\theta)}{\partial \theta^2}\right]$. $\quad \text{Var}(\hat{\theta}) \geq \dfrac{1}{nI(\theta)}$.

************ END OF PAPER ************