Q1) (i) The definition of theoretical cdf of a random variable X is $F_X(x) = Pr(X \le x)$.

We can use the empirical cdf to estimate $Pr(X \le x)$, which is defined as $\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(x_i \le x)$, where x_i is the ith observation in our random sample.

(ii) $Pr(X \le 4)$ can be estimated by $\hat{F}_{10}(4) = \frac{1}{10} \sum_{i=1}^{10} I(x_i \le 4)$

$$= 0.7.$$

$Pr(3 < X < 7)$ can be estimated by $\frac{1}{10} \sum_{i=1}^{10} I(3 < x_i < 7) = 0.3$.

Note 1: In this question, we know X is a discrete r.v., and therefore, $Pr(3 < X < 7) = Pr(4 \le X \le 6) = Pr(X \le 6) - Pr(X \le 3)$.

Hence, $Pr(3 < X < 7)$ can be estimated by $\hat{F}_{10}(6) - \hat{F}_{10}(3) = 0.9 - 0.6 = 0.3$.

Note 2: If we don't know whether X is discrete or continuous, then $Pr(a < X < b)$ can be estimated by $\frac{1}{n} \sum_{i=1}^{n} I(a < x_i < b)$. Similarly, $Pr(a \le X < b)$ can be estimated by $\frac{1}{n} \sum_{i=1}^{n} I(a \le x_i < b)$, and so forth.

Q2) $Pr(X=x) = 2\left(\frac{1}{3}\right)^x$ for $x \in \mathbb{N}$.

(i) $F_X(x) = Pr(X \le x) = \sum_{i=1}^{x} Pr(X=i)$

$= \sum_{i=1}^{x} 2\left(\frac{1}{3}\right)^i = 2 \sum_{i=1}^{x} \left(\frac{1}{3}\right)^i = 2\left[\frac{\frac{1}{3} - \left(\frac{1}{3}\right)^{x+1}}{1 - \frac{1}{3}}\right] = 1 - \left(\frac{1}{3}\right)^x$ for $x \in \mathbb{N}$.

Note: Let $S_n = \sum_{i=1}^{n} P^i$. Then, $PS_n = \sum_{i=2}^{n+1} P^i = S_n - P + P^{n+1}$

$\Rightarrow S_n(P-1) = P^{n+1} - P \Rightarrow S_n = \frac{P - P^{n+1}}{1 - P}$.

In general, $F_X(x) = \begin{cases} 1 - \left(\frac{1}{3}\right)^{\lfloor x \rfloor} & \text{if } x \ge 1 \\ 0 & \text{otherwise} \end{cases}$, where $\lfloor x \rfloor$

is the largest integer s.t. $x \ge \lfloor x \rfloor$.

Note: $X \sim Geo\left(P = \frac{2}{3}\right)$.

(ii) $M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} Pr(X=x)$

$= \sum_{x=1}^{\infty} e^{tx} 2\left(\frac{1}{3}\right)^x = 2 \sum_{x=1}^{\infty} \left(e^t \frac{1}{3}\right)^x$

$= 2\left(\frac{e^t \frac{1}{3}}{1 - e^t \frac{1}{3}}\right) = \frac{\frac{2}{3} e^t}{1 - \frac{1}{3} e^t}$ for $t < \ln 3$

Note: $e^t \frac{1}{3} < 1 \Leftrightarrow e^t < 3 \Leftrightarrow t < \ln 3$

Note: $\sum_{k=1}^{\infty} P^k = \frac{P}{1-P}$ if $|P| < 1$ and $\sum_{k=0}^{\infty} P^k = \frac{1}{1-P}$ if $|P| < 1$

(iii) $\because X \sim Geo\left(P = \frac{2}{3}\right)$

$\therefore E(X) = \frac{1}{P} = \frac{3}{2}$, $Var(X) = \frac{1-P}{P^2} = \frac{3}{4}$

Q3) $M_x(t) = E(e^{tX}) = E\left(\sum\limits_{k=0}^{\infty} \frac{(tX)^k}{k!}\right)$

$= E\left[\sum\limits_{k=0}^{\infty} \frac{t^k X^k}{k!}\right] = \sum\limits_{k=0}^{\infty} \frac{t^k E(X^k)}{k!}$

$= 1 + \sum\limits_{k=1}^{\infty} \frac{t^k E(X^k)}{k!}$

$= 1 + \sum\limits_{k=1}^{\infty} \frac{t^k \, 0.8}{k!}$

$= 1 + 0.8 \sum\limits_{k=1}^{\infty} \frac{t^k}{k!}$

$= 1 + 0.8\left[\sum\limits_{k=0}^{\infty} \frac{t^k}{k!} - 1\right]$

$= 1 + 0.8\left[e^t - 1\right]$

$= 0.2 + 0.8 e^t$

$\therefore \quad X \sim Ber(p = 0.8)$

Note: $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}, \; \forall x$

Q4) $X_i \overset{iid}{\sim} P(3, \theta)$. Define $Y = \sum\limits_{i=1}^{n} X_i$

(i) $M_Y(t) = \bar{E}(e^{tY}) = \bar{E}(e^{t\sum\limits_{i=1}^{n} X_i})$

$= E(e^{tX_1} e^{tX_2} \cdots e^{tX_n}) \overset{ind}{=} \bar{E}(e^{tX_1}) \bar{E}(e^{tX_2}) \cdots E(e^{tX_n})$

$\overset{identical}{=} \prod\limits_{i=1}^{n} M_{X_i}(t) = \prod\limits_{i=1}^{n} \left(\frac{\theta}{\theta - t}\right)^3 = \left(\frac{\theta}{\theta - t}\right)^{3n}$ for $t < \theta$.

Note: $M_X(t) = \left(\frac{\theta}{\theta - t}\right)^3$ as $X \sim P(3, \theta)$.

$\therefore Y \sim P(3n, \theta)$.

(ii) $E(Y) = \frac{3n}{\theta}$

$E(cY) = c\bar{E}(Y) = c\left(\frac{3n}{\theta}\right) = \frac{1}{\theta} \Rightarrow c = \frac{1}{3n}$.

(iii) Let $W = 3\theta Y + 1$.

Then, $M_W(t) = \bar{E}(e^{tW}) = \bar{E}(e^{t(3\theta Y + 1)})$

$= E(e^{(3\theta t)Y} \cdot e^t) = e^t \bar{E}(e^{(3\theta t)Y})$

$= e^t M_Y(3\theta t) = e^t \left(\frac{\theta}{\theta - 3\theta t}\right)^{3n}$

$= e^t \left(\frac{1}{1 - 3t}\right)^{3n}$ for $t < \frac{1}{3}$.

Q5) $M_x(t) = \frac{1}{4}e^{-3t} + \frac{1}{2} + \frac{1}{4}e^t$

(i) $M_x'(t) = -\frac{3}{4}e^{-3t} + \frac{1}{4}e^t$

$M_x''(t) = \frac{9}{4}e^{-3t} + \frac{1}{4}e^t$

$\bar{E}(X) = M_x'(0) = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$

$\bar{E}(X^2) = M_x''(0) = \frac{9}{4} + \frac{1}{4} = \frac{5}{2}$

$Var(X) = \bar{E}(X^2) - [\bar{E}(X)]^2 = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$

(ii) By observation,

$$Pr(X=x) = \begin{cases} \frac{1}{4} & \text{if } x=-3 \\ \frac{1}{2} & \text{if } x=0 \\ \frac{1}{4} & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$$

$\bar{E}(X) = \sum_{\forall x \in S_x} x \, Pr(X=x) = -3\times\frac{1}{4} + 0\times\frac{1}{2} + 1\times\frac{1}{4} = -\frac{1}{2}$

$\bar{E}(X^2) = \sum_{\forall x \in S_x} x^2 \, Pr(X=x) = (-3)^2\times\frac{1}{4} + 0^2\times\frac{1}{2} + 1^2\times\frac{1}{4} = \frac{5}{2}$

$\therefore Var(X) = \frac{9}{4}$

Q6) $X \sim \text{Exp}(\mu)$, $Y \sim \text{Exp}(\lambda)$, $X \perp\!\!\!\perp Y$.
Let $W = X + Y$.   Let $U = X$

Then, $Y = W - X = W - U$.

$$\begin{cases} X = U & \text{------- ①} \\ Y = W - U & \text{------ ②} \end{cases}$$

$$J = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$\begin{aligned} f_{W,U}(w,u) &= f_{X,Y}(x,y) |J| \\ &\overset{\text{ind}}{=} f_X(x) f_Y(y) |1| \\ &= \mu e^{-\mu x} \lambda e^{-\lambda y} \\ &= \mu \lambda e^{-\mu x - \lambda y} \\ &= \mu \lambda e^{-\mu u - \lambda(w-u)} \\ &= \mu \lambda e^{(\lambda - \mu)u - \lambda w} \end{aligned}$$
, for $0 \le u \le w$.

Note: By ②,
$$\left. \begin{aligned} W &= Y + U \\ Y &\ge 0 \end{aligned} \right\} \Rightarrow W \ge U$$

$$\begin{aligned} f_W(w) &= \int_0^w f_{W,U}(w,u) \, du = \int_0^w \mu \lambda e^{(\lambda-\mu)u - \lambda w} \, du \\ &= \mu \lambda e^{-\lambda w} \int_0^w e^{(\lambda-\mu)u} \, du = \frac{\mu \lambda}{\lambda - \mu} e^{-\lambda w} \left[e^{(\lambda-\mu)u} \right]_0^w \\ &= \frac{\mu \lambda}{\lambda - \mu} e^{-\lambda w} \left[e^{(\lambda-\mu)w} - 1 \right] \\ &= \frac{\mu \lambda}{\lambda - \mu} \left(e^{-\mu w} - e^{-\lambda w} \right), \quad \text{for } w > 0. \end{aligned}$$

Q6) cont.

Note:
$$\int_0^\infty t_W(w)\,dw = \frac{\mu\lambda}{\lambda-\mu}\left[\int_0^\infty e^{-\mu w}\,dw - \int_0^\infty e^{-\lambda w}\,dw\right]$$

$$= \frac{\mu\lambda}{\lambda-\mu}\left\{\frac{[e^{-\mu w}]_0^\infty}{-\mu} - \frac{[e^{-\lambda w}]_0^\infty}{-\lambda}\right\}$$

$$= \frac{\mu\lambda}{\lambda-\mu}\left\{\frac{[-1]}{-\mu} - \frac{[-1]}{-\lambda}\right]$$

$$= \frac{\mu\lambda}{\lambda-\mu}\left\{\frac{\lambda-\mu}{\mu\lambda}\right\} = 1$$

$$M_W(t) = \bar{E}[e^{tW}] = \bar{E}[e^{tX}e^{tY}]$$

$$\stackrel{ind.}{=} E(e^{tX})\,\bar{E}(e^{tY}) = \left(\frac{\mu}{\mu-t}\right)\left(\frac{\lambda}{\lambda-t}\right) \quad \text{for } t < \min\{\lambda,\mu\}.$$

Q7) (i) The joint probability mass function of X_1 and X_2 can be presented with a 2 by 2 contingency table.

X_2 \ X_1	$X_1=0$	$X_1=1$	$X_1=2$	$Pr(X_2=x_2)$
$X_2=0$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
$X_2=1$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
$X_2=2$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
$X_2=3$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
$Pr(X_1=x_1)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Note: $Pr(X_1=x_1 \wedge X_2=x_2) = Pr(X_1=x_1) Pr(X_2=x_2)$,

$\forall x_1 \in \{0,1,2\}$, $\forall x_2 \in \{0,1,2,3\}$.

$\therefore X_1 \perp\!\!\!\perp X_2$.

(ii) Let $Y_1 = X_1 \cdot X_2$.

Then, the support of Y_1 is $S_{Y_1} = \{0,1,2,3,4,6\}$.

Let $Y_2 = \max\{X_1, X_2\}$.

Then, the support of Y_2 is $S_{Y_2} = \{0,1,2,3\}$.

Q7) (ii) cont.

Y_1 \ Y_2	$Y_2 = 0$	$Y_2 = 1$	$Y_2 = 2$	$Y_2 = 3$	$Pr(Y_1 = y_1)$
$Y_1 = 0$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{6}{12}$
$Y_1 = 1$	0	$\frac{1}{12}$	0	0	$\frac{1}{12}$
$Y_1 = 2$	0	0	$\frac{2}{12}$	0	$\frac{2}{12}$
$Y_1 = 3$	0	0	0	$\frac{1}{12}$	$\frac{1}{12}$
$Y_1 = 4$	0	0	$\frac{1}{12}$	0	$\frac{1}{12}$
$Y_1 = 6$	0	0	0	$\frac{1}{12}$	$\frac{1}{12}$
$Pr(Y_2 = y_2)$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{5}{12}$	$\frac{3}{12}$	1

$\because \quad 0 = Pr(Y_1 = 1 \cap Y_2 = 0) \neq Pr(Y_1 = 1) Pr(Y_2 = 0) = \frac{1}{12} \times \frac{1}{12} = \frac{1}{144}$

$\therefore \quad Y_1 \not\!\perp\!\!\!\perp Y_2$

Q8) $X = \xi_1 + \xi_2$, $\xi_1 \sim N(\theta, 1)$, $\xi_2 \sim N(\lambda\theta, \lambda^2)$, $\xi_1 \perp\!\!\!\perp \xi_2$
$\lambda \geq 1$.

(i) $M_X(t) = E(e^{tX}) = E\left(e^{t(\xi_1 + \xi_2)}\right)$

$= E\left(e^{t\xi_1} e^{t\xi_2}\right) \overset{ind}{=} E(e^{t\xi_1}) E(e^{t\xi_2})$

$= M_{\xi_1}(t) M_{\xi_2}(t) = \exp\left(\theta t + \frac{t^2}{2}\right)\exp\left(\lambda\theta t + \frac{\lambda^2 t^2}{2}\right)$

$= \exp\left[(\lambda+1)\theta t + \frac{(\lambda^2+1)t^2}{2}\right]$, $\forall t$.

(ii) $M'_X(t) = \exp\left[(\lambda+1)\theta t + \frac{(\lambda^2+1)t^2}{2}\right]\left[(\lambda+1)\theta + (\lambda^2+1)t\right]$

$M''_X(t) = \exp\left[(\lambda+1)\theta t + \frac{(\lambda^2+1)t^2}{2}\right]\left[(\lambda+1)\theta + (\lambda^2+1)t\right]^2$
$\qquad + \exp\left[(\lambda+1)\theta t + \frac{(\lambda^2+1)t^2}{2}\right](\lambda^2+1)$

$M_X^{(3)}(t) = \exp\left[(\lambda+1)\theta t + \frac{(\lambda^2+1)t^2}{2}\right]\left[(\lambda+1)\theta + (\lambda^2+1)t\right]^3$
$\qquad + \exp\left[(\lambda+1)\theta t + \frac{(\lambda^2+1)t^2}{2}\right]2\left[(\lambda+1)\theta + (\lambda^2+1)t\right](\lambda^2+1)$
$\qquad + \exp\left[(\lambda+1)\theta t + \frac{(\lambda^2+1)t^2}{2}\right]\left[(\lambda+1)\theta + (\lambda^2+1)t\right](\lambda^2+1)$

$\therefore E(X^3) = M_X^{(3)}(0) = \left[(\lambda+1)\theta\right]^3 + 2\left[(\lambda+1)\theta\right](\lambda^2+1) + (\lambda+1)\theta(\lambda^2+1)$

$= \theta(\lambda+1)\left[\theta^2(\lambda+1)^2 + 2(\lambda^2+1) + \lambda^2+1\right]$

$= \theta(\lambda+1)\left[\theta^2(\lambda+1)^2 + 3\lambda^2 + 3\right]$

(iii) $X \sim N\left(\theta(\lambda+1), \lambda^2+1\right)$