

THE UNIVERSITY OF HONG KONG  
DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

STAT2602 PROBABILITY AND STATISTICS II

May 11, 2021

Time: 2:30 p.m. - 4:30 p.m.

*Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.*

Answer ALL FIVE questions. Marks are shown in square brackets.

**S&AS: STAT2602 Probability and Statistics II**

1. Let  $X_1, X_2, \dots, X_n$  be an independent random sample from  $N(2\theta, 5\theta^2)$  with  $\theta \neq 0$ . Define  $\hat{\theta} = \frac{A}{n} \sum_{i=1}^n X_i$ , and let  $\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$  be its mean squared error.

- (i) Choose the value of  $A$  such that  $\hat{\theta}$  is an unbiased estimator of  $\theta$ . [3 marks]
- (ii) Choose the value of  $A$  such that  $\text{MSE}(\hat{\theta})$  is minimized. [5 marks]
- (iii) Show that  $T_1 = \left( \frac{1}{2n} \sum_{i=1}^n X_i \right)^2$  is a consistent estimator of  $\theta^2$ . [3 marks]
- (iv) Show that  $T_1$  is an asymptotically unbiased estimator of  $\theta^2$ . [3 marks]
- (v) Calculate  $EX_1^2$ , and then construct  $T_2$ , a method of moments estimator of  $\theta^2$ . [3 marks]
- (vi) Show that  $T_2$  is an unbiased estimator of  $\theta^2$ . [3 marks]

[Total: 20 marks]

2. Let  $X_1, X_2, \dots, X_n$  be an independent random sample from  $N(\mu, \sigma^2)$ .

- (i) Find the MLEs of  $\mu$  and  $\sigma^2$  on the space

$$\Omega_1 = \{(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}.$$

[5 marks]

- (ii) Find the MLEs of  $\mu$  and  $\sigma^2$  on the space

$$\Omega_2 = \{(\mu, \sigma^2) : \mu_0 \leq \mu \leq \mu_1, \sigma^2 > 0\},$$

where  $\mu_0$  and  $\mu_1$  are two given finite constants with  $\mu_0 < \mu_1$ . [5 marks]

- (iii) Will the MLE of  $\mu$  in part (ii) be the UMVUE of  $\mu$ ? Explain it. [5 marks]
- (iv) Suppose that  $\mu = 0$  and  $n = 20$ . Find a rejection region of size 0.05 for testing hypotheses

$$H_0 : \sigma^2 = 1 \quad \text{versus} \quad H_1 : \sigma^2 = 2.$$

[5 marks]

[Total: 20 marks]

3. If  $X_1, X_2, \dots, X_n$  are independent and uniformly distributed on the interval  $(0, 1)$ . Define  $X_{(n)} = \max_{1 \leq i \leq n} X_i$ .

- (i) As  $n \rightarrow +\infty$ , will  $X_{(n)}$  converge in probability to any limit? If yes, identify the limit and provide a proof; if no, justify your claim.

[10 marks]

**S&AS: STAT2602 Probability and Statistics II**

- (ii) Find the asymptotic distribution of  $n(1 - X_{(n)})$  and provide a proof. [10 marks]  
**[Total: 20 marks]**

4. Let  $X_1, X_2, X_3, X_4$  be a random sample of STAT2601 scores, assumed to be  $N(\mu_x, \sigma_x^2)$ , and let  $Y_1, Y_2, Y_3, Y_4, Y_5$  be a random sample of STAT2602 scores, assumed to be  $N(\mu_y, \sigma_y^2)$ . Suppose  $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, Y_5$  are independent, and the following data are observed:

$$x_1 = 75 \quad x_2 = 95 \quad x_3 = 80 \quad x_4 = 65$$

$$y_1 = 80 \quad y_2 = 75 \quad y_3 = 90 \quad y_4 = 70 \quad y_5 = 70$$

- (i) Find a 90% confidence interval estimate for  $\sigma_x^2$ . [5 marks]  
(ii) Find a 90% confidence interval estimate for  $\sigma_x^2/\sigma_y^2$ . [5 marks]  
(iii) Find the distribution of  $\frac{(X_1 - X_2)^2/\sigma_x^2}{(Y_1 - Y_2)^2/\sigma_y^2}$ . [5 marks]  
(iv) Based on the result in (iii), construct an alternative 90% confidence interval estimate for  $\sigma_x^2/\sigma_y^2$ . [3 marks]  
(v) Is the confidence interval estimate in (ii) better than the one in (iv)? Explain your answer. [2 marks]

**[Total: 20 marks]**

5. Suppose that the score of STAT2602 follows the distribution  $N(\mu, \sigma^2)$ . The teacher wants to see whether the mean score of STAT2602 is 80 or less than 80 by using hypothesis testing. So, he collects an independent random sample consisting of the scores of 100 students, and finds that the mean of this random sample is 77 and the variance of this random sample is 9.

- (i) State the null and alternative hypotheses for the teacher. [5 marks]  
(ii) Suppose that  $\sigma^2 = 16$ , and the teacher uses  $\bar{X}$  as the test statistic and  $\{\bar{X} < 79.5\}$  as the rejection region. Write down the power function  $\pi(\mu)$  in terms of the cumulative distribution function of  $N(0, 1)$ . [5 marks]  
(iii) Does this test have the significance level  $\alpha = 0.05$ ? [2 marks]  
(v) Find the value of  $\lim_{\mu \rightarrow -\infty} \pi(\mu)$ . [3 marks]  
(vi) If the value of  $\sigma^2$  is unknown, should the teacher reject the null hypothesis at the significance level  $\alpha = 0.05$  based on the generalized likelihood ratio test? [5 marks]

**[Total: 20 marks]**

A LIST OF STATISTICAL FORMULAE

1. CLT:  $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \approx N(0, 1)$  for large  $n$ .
2. Normal population  $\implies \frac{nS^2}{\sigma^2} \sim \chi_{n-1}^2$ ,  $\frac{\bar{X} - \mu}{S/\sqrt{n-1}} \sim t_{n-1}$ .
3.  $\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$ .  $E[(\hat{\theta} - \theta)^2] = \text{Var}(\hat{\theta}) + [\text{Bias}(\hat{\theta})]^2$ .
4.  $I(\theta) = E \left[ \left( \frac{\partial \ln f(X; \theta)}{\partial \theta} \right)^2 \right] = E \left[ -\frac{\partial^2 \ln f(X; \theta)}{\partial \theta^2} \right]$ .  $\text{Var}(\hat{\theta}) \geq \frac{1}{nI(\theta)}$ .
5.  $\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ .  $\bar{x} \pm t_{\alpha/2, n-1} \frac{s}{\sqrt{n-1}}$ .  $\left( \frac{ns^2}{\chi_{\alpha/2, n-1}^2}, \frac{ns^2}{\chi_{1-\alpha/2, n-1}^2} \right)$ .
6.  $\bar{x} - \bar{y} \pm z_{\alpha/2} \sqrt{\frac{\sigma_x^2}{n_1} + \frac{\sigma_y^2}{n_2}}$ .  $\bar{x} - \bar{y} \pm t_{\alpha/2, n_x+n_y-2} \sqrt{\frac{n_x s_x^2 + n_y s_y^2}{n_x + n_y - 2} \left( \frac{1}{n_x} + \frac{1}{n_y} \right)}$ .  
 $\left( \frac{n_x(n_y-1)s_x^2}{n_y(n_x-1)s_y^2} \cdot \frac{1}{F_{\alpha/2, n_x-1, n_y-1}}, \frac{n_x(n_y-1)s_x^2}{n_y(n_x-1)s_y^2} \cdot F_{\alpha/2, n_y-1, n_x-1} \right)$ .
7.  $z_{0.9} = -1.28$ ,  $z_{0.2} = 0.85$ ,  $t_{0.025, 10} = 2.228$ ,  $t_{0.05, 99} = 1.65$
8.  $\chi_{0.05, 20}^2 = 31.41$ ,  $\chi_{0.05, 3}^2 = 7.815$ ,  $\chi_{0.95, 3}^2 = 0.352$ .
9.  $F_{0.05, 3, 4} = 6.59$ ,  $F_{0.05, 4, 3} = 9.12$ ,  $F_{0.05, 1, 1} = 161.45$ .

\*\*\*\*\* END OF PAPER \*\*\*\*\*