EE 382V: Social Computing	**Fall 2018**

Lecture 1: August 24

Lecturer: Vijay Garg *Scribe: Patrick Sigourney*

1.1 Introduction

Scribe: Students are required to act as scribe and take notes during one class section. Link to Google Docs sheet for scribe duty signups was emailed on Aug 24. The notes need to be formatted using LaTeX (overleaf.com is an online tool for this). The generated PDF will be sent to the class TA for review and revisions, after the student has made all necessary corrections and submitted the final .TEX and .PDF files to the TA, the notes will be posted for student use:

https://github.com/Sunny666/EE382V_Social_Computing

TA: The class TA is Sunny Bhagia (sunnysb@utexas.edu). Office hours are TBA and available either in-person or via Skype.

Assignment Teams: All assignments will be done in self-selected teams of 2, with the exception of the textbook chapter presentation, which will be done in assigned teams of 2. Links to Google Docs sheets for both assignment and presentation teams was emailed on Aug 24.

Exams: There will be 2 midterm exams and a final exam. For each exam you are allowed one 8.5"x11" page of notes, handwritten, both sides.

Textbook Assignments: Each month, 5-6 sections of the textbook will be covered by assigned student presentations. In addition to these presentations, all students will need to select 2 of these sections each month, read them, and write a half-page summary for each section. This will be handed in at the start of the first class session each month.

1.1.1 Topics for the Class

1. Matching: Buyers and sellers, pairing, stable matches.
2. Auctions: Buyer's valuation of item
3. Games: Two-player games, payoff matrix with row player & column player; prisoner's dilemma.
4. Voting: One-to-one and priority list; Arrow's Theorem.
5. Experts Algorithm: Accuracy of past predictions impacts weight applied to future predictions
6. Security: RSA public key algorithm

1.2 Matching

Bipartite graph: A graph $G = (V, E)$ is bipartite if V can be partitioned into A and B such that there

are no edges between vertices in the same partition. ie: Elements in A can only be connected to elements of B and elements in B can only be connected to elements of A.

Matching: A matching M is a subset of edges E ($M \subseteq E$) such that no two edges in M share any vertex. ie: one-to-one relationship; a pairing.

Perfect Matching: Perfect matching occurs when the vertex partitions A and B are of the same cardinality (number of elements) ($|A| == |B|$), each element of A is connected to exactly one element of B, and each element of B is connected to exactly one element of A.

Let $G = (A, B, E)$ be a bipartite graph such that $|A| == |B|$.
$M \subseteq E$ is a perfect matching if $|M| == |A|$

Maximal Matching: A matching M is maximal if it cannot be extended to include more matches.

Maximum Matching: A matching M is maximum in a graph if it has the largest cardinality possible. A maximum matching is always a maximal matching, whereas maximal matching is not always maximum matching and may have scope for improvement.

1.2.1 Hall's Marriage Theorem - Conditions for Perfect Matching (PM)

For a vertex a in set A, we define the neighboring set of a as:
$N(a) = \{b \in B | (a, b) \in E\}$
(B is a neighboring set of a which contains all b where there is an edge between a and b)

Let s be a subset of vertex set A; Then the neighboring set of subset s is defined as:
$N(s) = \bigcup_{a \in s} N(a)$
(Neighbor of subset s is the union of all neighboring sets $N(a)$ of each element a in s)

Constricted Set: A constricted set is one where there are more elements in the set s than there are elements in the associated neighbor set $N(s)$.
$\exists s : |s| > |N(s)|$
(There exists a subset s where the number of elements in s is greater than the number of elements in the neighbor set $N(s)$)

Perfect Matching is not possible with a constricted set.

Perfect Matching is possible if: $\forall s \subseteq A : |s| <= |N(s)|$
(For all subsets s in A, the cardinality of s is less than or equal to the cardinality of the neighbor set)

EE 382V: Social Computing **Fall 2018**

Lecture 1: August 24

Lecturer: Vijay Garg *Scribe: Jesus Valenzuela Ochoa*

1.1 Proof that no constricted set $\Rightarrow$ a Perfect Match (PM)

Use induction on n $|A| = |B| = n$
Base Case $n = 1$

$$a_1 \odot \leftrightarrow \odot b_1, \text{ There is } PM$$

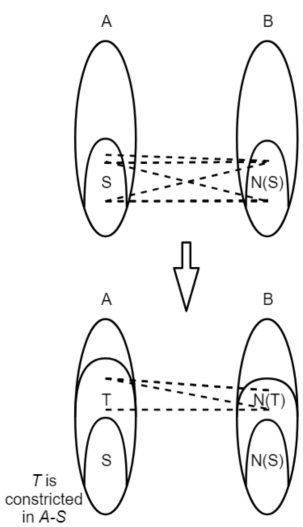
Induction Case
Assume that the theorem is true for all values above $|A| < n$

Case 1
$\exists S$ so that $|S| \equiv |N(S)|$

Case 2
$\forall S : |S| < |N(S)|$

1.1.1 Case 1.

For a selected subset S of vertices from set A, the size of a subset created with the neighbors of the elements of S, i.e. $N(S)$, equals the size of subset S.



It is needed to confirm that for the selected subsets $(S, N(S), E_s)$, the remaining subset $A - S$ does not have any constricted set when matched to subset $B - N(S)$.

This allows to iteratively break the problem into smaller parts, checking that remainder is not constricted after removing any given subset of A and its their neighbors.

If T is constricted in A-S(as shown in the lower section of Figure 1.1)
$\Rightarrow S \cup T$ is constricted in A $\Rightarrow$ **contradiction**

Figure 1.1: Splitting sets.

1.1.2 Case 2.

For a selected subset S of vertices from set A, the size of a subset created with the neighbors of the elements of S, i.e. $N(S)$, is less than the size of subset S.

Pick any vertex a in A and match to any vertex b in $N(a)$, remove both a and b from the consideration.

- Verify that there is no constriction after removal.
- Now use induction as the set size has decreased.

In implementation, you want your algorithm to make matching maxim*um* by the time it is maxim*al*.

1.2 Alternating Path

An alternating path is one that alternates between matched and non-matched edges.

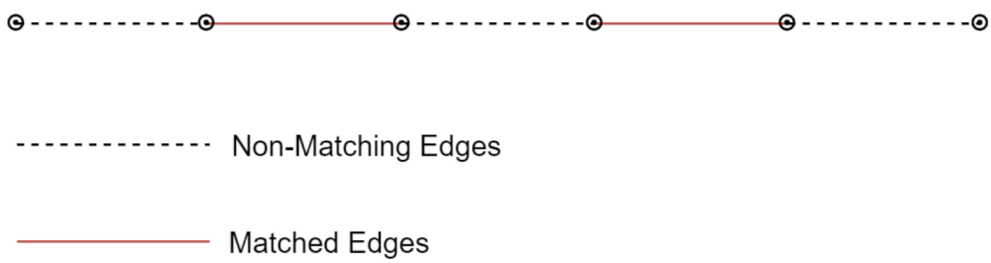


Figure 1.2: Alternating path.

1.2.1 Augmenting Path

An *augmenting path* is an alternating path starts with an exposed vertex in A and ends with and exposed vertex in B. An **exposed vertex** is one that has only non-matching edges attached to it.

If an augmenting path is found, the size of the matching can be augmented (incremented by 1) by exchanging the matched edges for the non matched edges.

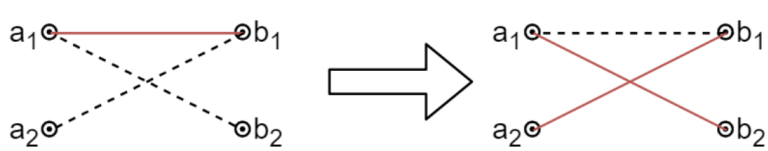


Figure 1.3: Augmenting path edge exchange.

All-edges start as unmatched. Consider using the B.F.S. algorithm to find a match. Always try to find an augmenting path.

B.F.S. Tree(with a twist)

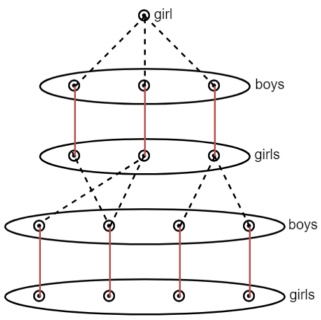


Figure 1.4: Example of a B.F.S.-Tree

This particular example has a constricted set, since there is one more girl than boys.

With B.F.S.-Tree construction, either an augmenting path is found or there is a constricted set. (By B.F.S.-Tree property).

TIP
If there is no constricted set, then you know there is an augmenting path.

Theorem 1.1 (Theorem) *A matching M is maxim*um *if and only if there are no augmenting paths with respect to M.*

Proof:

- The existence of augmenting path indicates matching is not maxim*um*, since matched and non-matching edges can be inverted, increasing matching size by 1.

- To prove that M is not maximum $\Rightarrow \exists$ an augmenting path:

Let M' be a maximum matching,

- Compute $Q = M \triangle M' \leftarrow symmetrical$ ($\triangle \equiv xor$ of M and M')

$$M \triangle M' = (M' - M) \bigcup (M - M')$$

- Since $|M'| > |M|$, Q has more edges from M' than from M.

- Each vertex is adjacent to at most one edge in $M \cap Q$ and at most one edges in $M' \cap Q$

- Q is composed of cycles(even number of edges) and paths(odd number of edges) that alternate between edges from M and M'. Cycles will always be constructed by an even number of edges since we are working with bipartite graphs and a cycle with an odd number of edges would only be achievable by matching elements of the same set. Paths found must have more edges from M' since its size is larger than that of M, hence representing the existence of an alternating augmenting path with reference to M, such as the one shown in Figure 1.5.

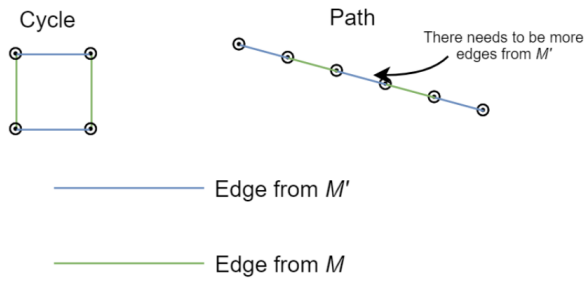


Figure 1.5: Cycle vs path construct

■

1.3 Weak Duality

Maximum matching problem with bipartite graphs:

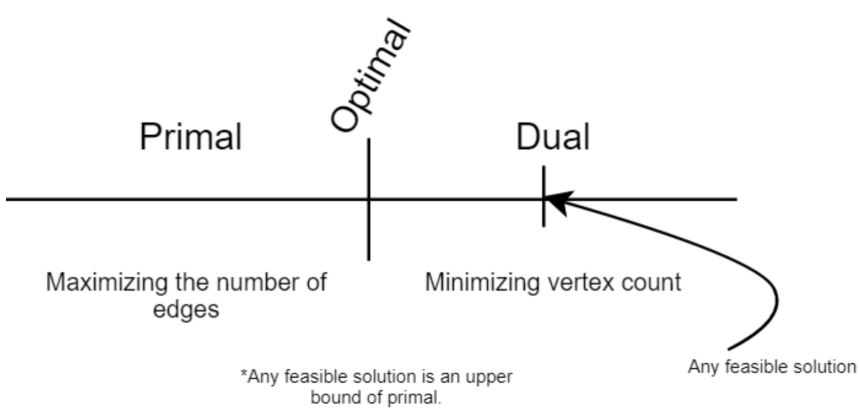


Figure 1.6: Weak duality in matching.

Let G be a bipartite graph, described as $G = (A, B, E)$. $X \subseteq A \cup B$ is a vertex cover of $G = (A, B, E)$ if every edge is adjacent to at least one vertex in X.

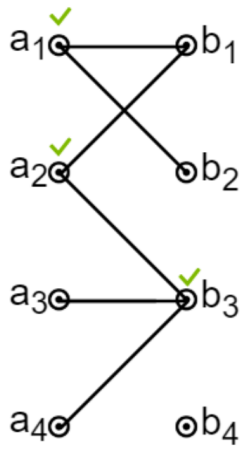


Figure 1.7: Example of a vertex cover. Analyzing what vertices combined are attached to all edges, we can establish $X = (a_1, a_2, b_3)$ for a cover of size 3.

Theorem 1.2 *Given any vertex cover X, and any matching M, $|M| \leq |X|$*

Proof: Each element of X can only have one match. All its other edges are removed once it matches. ■

Theorem 1.3 (König, Egerváry Theorem) *For any bipartite graph, the maximum size of a matching equals the size of the minimum vertex cover.*

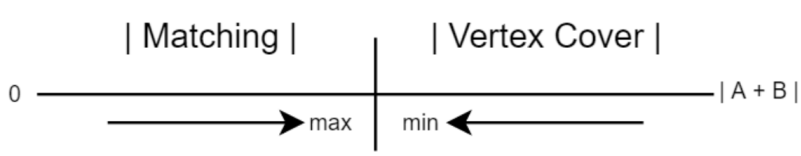


Figure 1.8: Min-Max theorem applied to weak duality in matching.

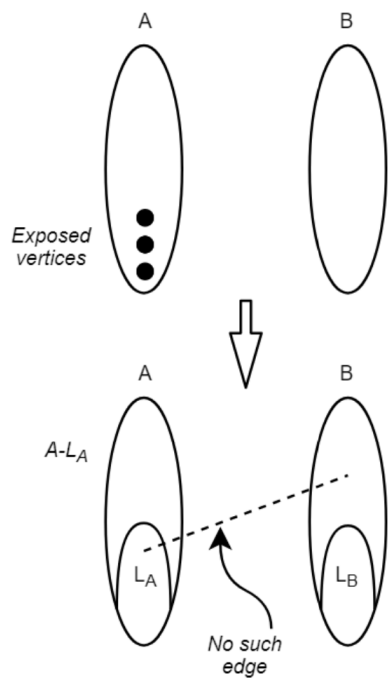


Figure 1.9: Abstraction of sets into L. There may not be any edges between L_A and $B - L$ as all the vertices in L_A are either unmatched or have categorized edges they are connected to in B into L_B, per L's definition.

Let
L: Vertices that can be reached by B.F.S.-tree from an exposed vertex.

M' : maximum matching of a bipartite graph.

$C' = (A - L_A) \cup L_B$

$L = L_A \cup L_B$
$L_A = A \cap L$
$L_B = B \cap L$

Claim

- C' is a vertex cover.
  Every edge in a bipartite graph belongs to either an alternating path, which places its b_x endpoint in $B \cap L$, or it has an a_x endpoint in $A - L_A$. If said edge is matched but not in an alternating path, then its a_x endpoint cannot be in an alternating path, since said path must have included the edge and thus belongs to $A - L_A$. If the edge is unmatched but not in an alternating path, then its a_x endpoint cannot be in an alternating path, for such a path could be extended by adding the edge to it. In consequence, C' forms a vertex cover.

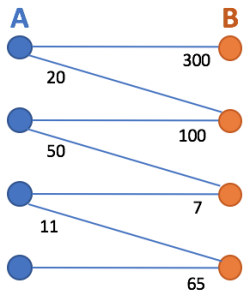
- $|C'| = |M'|$
  Every vertex in C' is an endpoint of a matched edge: every vertex in $A - L_A$ is matched because L includes all unmatched A vertices, and every vertex in $B \cap L$ must also be matched, for if there existed an alternating path to an unmatched vertex then changing the matching by removing the matched edges from this path and adding the unmatched edges in their place would increase the size of the matching. However, no matched edge can have both of its endpoints in C'. Thus, C' is a vertex cover of size equal to M', and must be a minimum vertex cover.

Scribe Notes: Introduction to Weighted Bipartite Matching
August 24th 11:25am-12:00pm

Kelsey Sandlin

1 Introduction to Weighted Bipartite Matching

A **weighted bipartite graph** is a bipartite graph in which the edges between A and B have weights assigned to them. Example:



Weighted Bipartite Graphs turn out to be more useful than bipartite graphs. The goal is to find a *perfect matching** that maximizes the weight:

$$max(\sum_{e \in M} w(e))$$

1.1 A note on *perfect matching** in weighted bipartite graphs

There was a handful of student questions regarding selecting an *imperfect* match to get a larger maximum of matched edge weights.

In a weighted bipartite graph, one can always add zero-weight edges between points in A and points in B. Therefore if the solution is not a perfect matching, we can make it one at no additional cost by matching any un-matched pairs with zero-weight edges. For the examples today, you can assume |A| = |B|.

1.2 Realistic Examples

Weighted bipartite graphs make are more useful for solving realistic problems than non-weighted bipartite graphs – you can think of set A as real estate sellers and set B as potential buyers where the edge weights are the prices of the houses. The sellers want to maximize profits, so the goal is to maximize matched edge weights:

$$max(\sum_{e \in M} w(e))$$

You could also use set A to represent Jobs and set B to represent Workers – with the assumption that each worker can only perform one job. The goal is to finish all the jobs (therefore perfect matching is required) while minimizing the amount paid to the workers (matched edge weights):

$$min(\sum_{e \in M} w(e))$$

Aside: Are the maximization and minimization problems the same problem?

Yes – you can convert from maximization to minimization (or vice-versa) by multiplying the edge weights by -1:

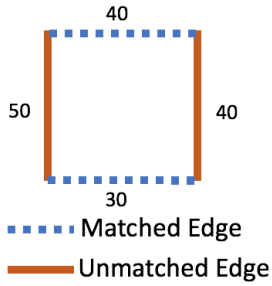
$$max(\sum_{e \in M} w(e)) = -min(\sum_{e \in M} -w(e))$$

1.3 Introduction to the Algorithm

How can we solve this problem to find the optimal perfect matching that maximizes edge weights?

- We will use some ideas about augmenting paths from the previous lecture
- Apply the Alternating Path Algorithm, but only flip the path if the weight of the flipped version is greater than the current weight

This algorithm introduces the concept of a negative cycle. A **Negative Cycle** is an alternating cycle that when flipped improves the weight. In the below example a negative cycle, you can see that if you flip, you'll get an increased total weight.



Either there is a negative cycle in the graph, or you are at an optimal solution.

Unweighted Bipartite Matching Algorithm is O(mn) where m is number of edges and n is number of vertices.

Introduction to Kuhn-Munkres Algorithm: There might exist a more efficient algorithm based on Duality. This dual algorithm will deal with minimizing labels assigned to each vertex.

EE 382V: Social Computing **Fall 2018**

Lecture 2, Session 1: August 25

Lecturer: Vijay Garg *Scribe: Matt Wey*

2.1 Introduction

In this section we will be solving the Assignment Problem specifically using the *Kuhn-Munkres* (Hungarian) Algorithm.

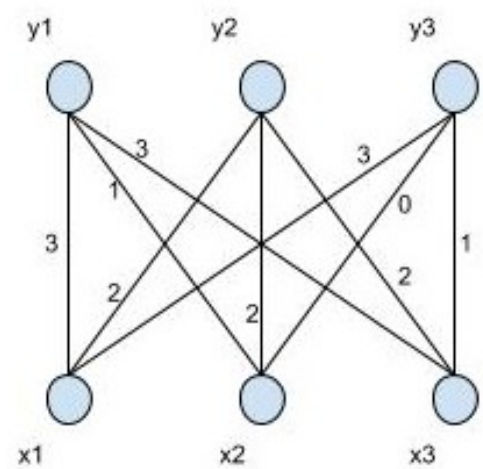


Figure 2.1: For the Assignment Problem we will consider this of bipartite graph.

2.2 Assignment Problem

The goal for the Assignment Problem is to find perfect matching with the maximum weight or the minimum cost. For our example we will be considering the bipartite graph shown in Figure 2.1. The key idea to solving this problem is duality.

2.2.1 Feasibility Constraints

Similar to assigning a weight to an edge, to add a feasibility constraint to the problem we give a label to each vertex. To be considered a feasibility constraint it must meet the following condition.

$$l(x_i) + l(y_j) \geq w(x_i, y_j); \qquad i = 1, 2, \ldots, n; j = 1, 2, \ldots, n \tag{2.1}$$

Where $l(x_i)$ is the label at vertex x_i, and $w(x_i, y_j)$ is the weight from vertex x_i to vertex y_j. The feasibility constraint is considered to create a "tight" edge when

$$l(x_i) + l(y_j) = w(x_i, y_j) \tag{2.2}$$

2.2.2 Equality Graph

When all edges in a given labeling l are tight we get an Equality Graph for that labeling, E_l, which is a subgraph of the original graph (Figure 2.1), an example of which is shown in Figure 2.2.

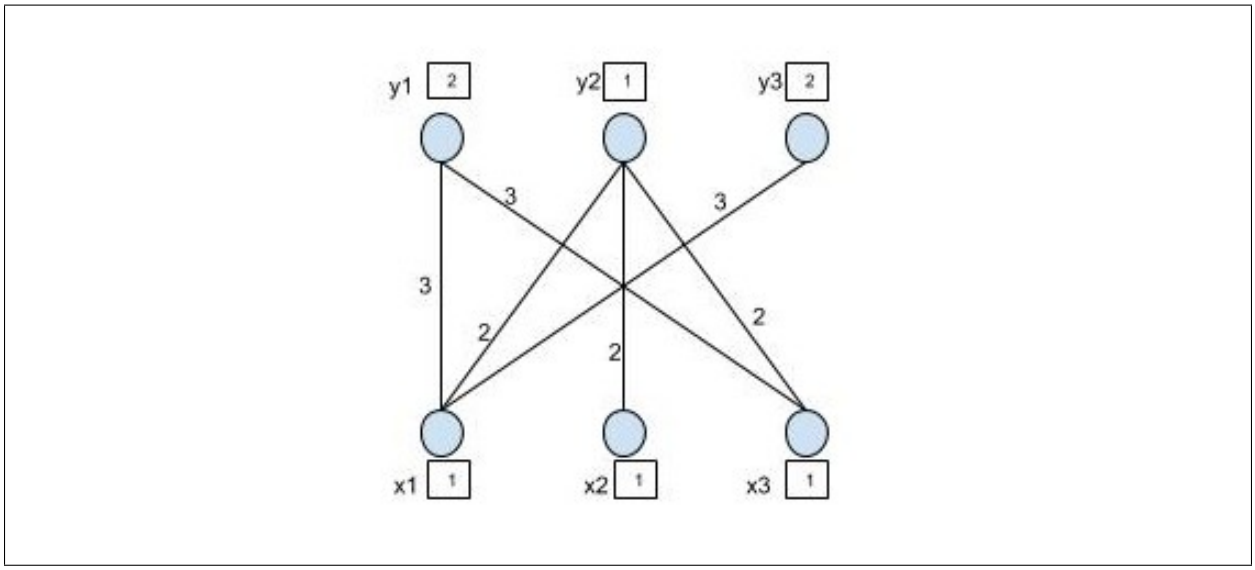


Figure 2.2: An equality graph. This particular equality graph is a subgraph of Figure 2.1.

2.2.3 Kuhn-Munkres Algorithm

As a method to solving the assignment problem we will explore the Kuhn-Munkres (Hungarian) Algorithm.

Theorem 2.1 *If l is feasible and M is a perfect matching in E_l, then M is a max-weight matching.*

Note that this method uses Equality Graphs which require tight edges so for each graph formed in the process the edges should remain tight.

Proof: Let M' be any perfect matching (not necessarily in E_l)

$$w(M') = \sum_{e \in E} w(e) \leq \sum_{e \in M'} l(e_x) + l(e_y) = \sum_{v \in V} l(v)$$

M is a perfect matching in E_l

$$w(M) = \sum_{e \in M} w(e) = \sum_{v \in V} l(v)$$

and finally

$$w(M') \leq w(M)$$

■

We can use Theorem 2.1 in creating the algorithm that will solve the Assignment Problem. It is important to note that the input for our algorithm is a bipartite graph with the connected edges and the weights for the edges. Any non-existent edges can also be considered edges zero weight.

Invariants

l is feasible;

M is some matching in E_l, where E_l is an equality graph with labeling l

The following pseudocode explains how our algorithm will solve the Assignment Problem.

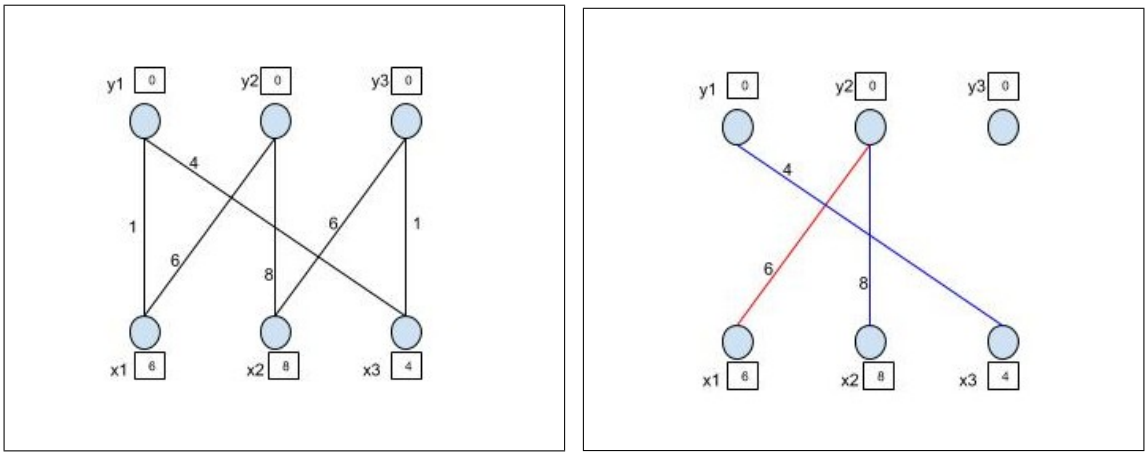
```
while(|M| < n) {
  - if there is an augmenting path in E_l
    - use the augmenting path to augment and increase the size of M
  - else
    - improve l to l' such that E_l exists in E_l'
}
```

Notice each iteration through the while loop will each increase the size of M OR change E_l to include an additional augmenting path. Note that $N_l(S)$ represents the neighbors of set S in labeling the l. The steps of the algorithm are as follows:

1. Initialization: l is initial labeling, $M = \{\}$
2. if M is a perfect matching stop. Otherwise pick a free vertex u such that $u \in X$.
   Assign $S = \{u\}, T = \emptyset$
3. if $N_l(S) = T$
$$\alpha_l = M_{x \in S, y \notin T}(l(x) + l(y) - w(x, y))$$
$$l'(v) = \begin{cases} l(v) - \alpha_l; & v \in S \\ l(v) + \alpha_l; & v \in T \\ l(v); & otherwise \end{cases}$$
4. if $N_l(S) \neq T$ pick $y \in (N_l(S) - T)$
   if y is free then u to y is an augmenting path.
   Augment path and go back to 2.
   if y is match to z
   $S = S \cup \{z\}, T = T \cup \{y\}$. Go to 3.

2.2.4 Hungarian Algorithm Example

- Given a weighted graph, create a basic initialization labeling all the vertices with the max weight of its connected edges as shown in Figure 2.3 (a).
$$\forall y_j : \quad l(y_j) = 0, \quad l(x_i) = \max(w(x_i, y_j))$$
- Select an initial matching as shown in Figure 2.3 (b).



(a) Initialized labeled graph with trivial labeling

(b) Initial equality graph E_l

Figure 2.3

- Assign $S = \{x_1\}, T = \emptyset$. Since $N_l(S) \neq T$ to go step 4.

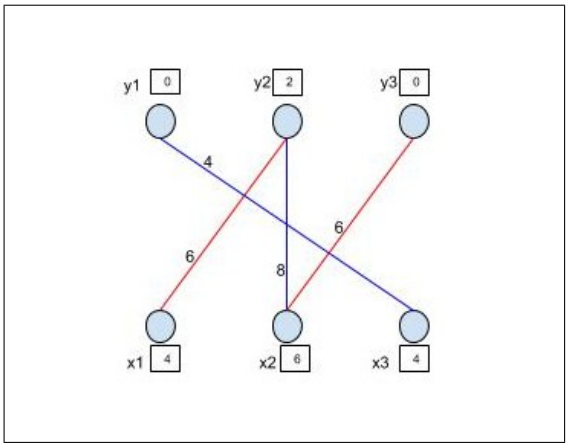
- Assign $y \in (N_l(S) - T)$. We choose y_2, which is matched, and increase the graph by assigning $S = \{x_1\} \cup \{x_2\}, T = \emptyset \cup \{y_2\}$. Go to 3.

- Now $S = \{x_1, x_2\}, T = \{y_2\}, N_l(S) = T$. Calculate α_l

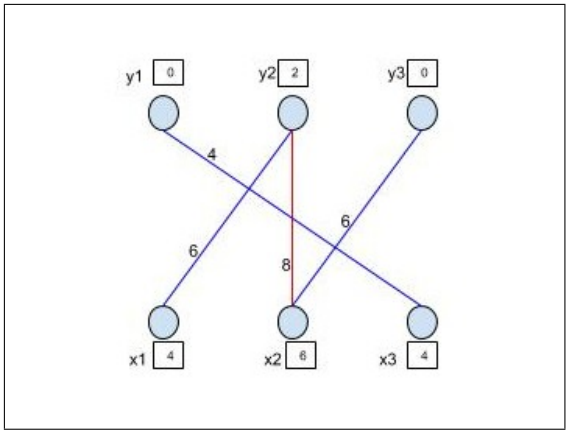
$$\alpha_l = min \begin{cases} (x_1, y_1) & 6 + 0 - 1 \\ (x_1, y_3) & 6 + 0 - 0 \\ (x_2, y_1) & 8 + 0 - 0 \\ (x_2, y_3) & 8 + 0 - 6 \end{cases} = 2$$

- Update Labels by reducing labels of S by 2 and increasing labels of T by 2 and add edge (x_2, y_3) to the equality graph. See Figure 2.4 (a).

- We now have a path starting and ending with unselected edges, so we augment the path and add the updated equality graph to the matching M as shown in Figure 2.4(b).



(a) Equality graph updated to include augmented path.



(b) Graph showing the final optimal matching with total weight 16.

Figure 2.4

- Now that we have complete vertex coverage (perfect matching), by Theorem 2.1 we know we have an optimal (max-weight) solution and can calculate the final value by summing the selected edges or summing the vertex labels.

EE 382V: Social Computing	**Fall 2018**

Lecture 6: August 25

Lecturer: Vijay Garg — *Scribe: Ari Bruck*

6.1 Demange, Gale, and Sotomayor aka `DGS/Auction` Algorithm

After the discussion of the KuhnMunkres algorithm, the professor introduced a new algorithm. The algorithm will provide the best possible assignment of goods and bidders such that the prices are maximized.

Input :`Bipartite graph with non-negative integer weights on the edges`,
`B` : set of Bidders,
`G` : set of Goods,
$\mathtt{W}_{i,j}$: weight between bidder i and good j
Output :`Maximum weight matching`

Q :Queue of Bidders
P_j :Price of Good j
$Owner_j$:Current winning bidder of Good j
n_m :Size of the matching (In the example in class $n_g \leq n_b$ [more bidders than goods] so $n_m = n_g$)
δ :The incremental price increase in auction price as a result of a matching

Algorithm 1 DGS/Auction

```
for all goods j do
    P_j ← 0                                  ▷ Each good's price starts at 0
    Owner_j ← NULL                           ▷ bidders do not own any good
Q ← B                                        ▷ all bidders are in the queue
n_m ← min(n_g, n_b)
δ ← 1/(n_m + 1)

while Q ≠ ∅ do
    i ← Q.dequeue()
    find j that maximizes W_{i,j} − P_j      ▷ Find the good that has best "effective" payoff
    if W_{i,j} − P_j ≥ 0 then                ▷ If the good adds to the overall weight matching
        Q.enqueue(Owner_j)                   ▷ Replace current owner
        Owner_j ← i                          ▷ with new one
        P_j ← P_j + δ                        ▷ Increase the auction price for that good
return (j, Owner_j)∀j.  ▷ maximum weight matching has been found, return all goods and their owners
```

<u>Correctness:</u> The proof of correctness is based on showing that the algorithm gets into an equilibrium, a situation where all bidders "are happy".

Definition: Bidder i is δ-happy with respect to P if $\exists$ good j s.t. **either**:

$Owner_j = i$
AND
$\forall$ goods j' : $\delta + W_{i,j} - P_j \geq W_{i,j'} - P_{j'}$

OR

$Owner_j \neq i$
AND
$\forall$ goods j : $W_{i,j} \leq P_j$

Loop Invariant: $\forall$ bidders $\not\subset Q$ are δ-happy

Initially: `TRUE`
Q is initialized to all bidders

At Runtime: `TRUE`
For the bidder i dequeued in an iteration, the loop exactly chooses the j that makes him happy, if such j exists, and the δ-error is due to the final increase in P_j.
Therefore this iteration cannot hurt the invariant for any other i': any increase in P_j for j that is not owned by i' does not hurt the inequality while an increase for the j that was owned by i' immediately enqueues i'.

The running time analysis above implies that the algorithm terminates, at which point Q must be empty and thus all bidders must be δ-happy.

■

6.1.1 δ-Happy Bidders

Claim: If all bidders are δ-happy then for every matching M'

$$n\delta + \sum_{i=owner_j} W_{i,j} \geq \sum_{(i,j)\in M'} W_{i,j}$$

Proof: Fix a bidder i and assume i receives item j by this algorithm such that:

$\delta + W_{i,j} - P_j \geq W_{i,j'} - P_{j'}$ where j' is item received in M'

$$\text{Sum over all } i = \sum_{i=owner_j} (\delta + W_{i,j} - P_j) \geq \sum_{i,j'} (W_{i,j'} - P_{j'})$$
$$= n\delta + \sum_{i=owner_j} (W_{i,j} - P_j) \geq \sum_{i,j'} (W_{i,j'} - P_{j'})$$

Since both the algorithm and M' give matchings, each j appears at most once on the left hand side and at most once on the right hand side.
Moreover, if some j does not appear on the left hand side then it was never picked by the algorithm and thus $P_j = 0$. Thus when we subtract $\sum_j P_j$ from both sides of the inequality, the LHS becomes the LHS

of the inequality in the lemma and the RHS becomes at most the RHS of the inequality in the lemma such that

$$= n\delta + \sum_{i=owner_j} W_{i,j} \geq \sum_{i,j'} W_{i,j'}$$

■

Some things to note: In the algorithm, either some bidder gets out of Q or the price of good j goes up by δ. Since all weights (prices) $\in \mathbb{Z}$ (rational prices can be scaled to integers), then $n\delta < 1$.

No P_j can ever increase once its value is above $C = max_{i,j} W_{i,j}$. It follows that the total number of iterations of the main loop is at most $Cn/\delta = O(Cn^2)$ where n is the total number of vertices (goods+bidders).
Each loop can be trivially implemented in $O(n)$ time, giving total running time of $O(Cn^3)$, which for the unweighted case, $C = 1$, matches the running time of the basic alternating paths algorithm on dense graphs.

References

[] Noam Nisan, Auction Algorithm for Bipartite Matching, *Turing's Invisible Hand: Computation, Economics, and Game Theory* (July 2009), https://agtb.wordpress.com/2009/07/13/auction-algorithm-for-bipartite-matching/

EE 382V: Social Computing	**Fall 2018**

Lecture 2: August 25

Lecturer: Vijay Garg — *Scribe: Javier Palomares*

2.1 Linear Programming

Linear Programming involves finding the maximum possible value of some objective subject to some constraints. For example, we might want to maximize the harvest value of some plot of land where we can plant different crops. In linear programming models these type of problems are modeled as a set of decision variables
$\{x_1, x_2, \ldots, x_n\}$
and an objective function
$c_1x_1 + c_2x_2 + \ldots + c_nx_n$
subject to some constraints
$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$
$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$
$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$
$x_1, x_2, \ldots, x_n \geq 0$
Linear programming only deals with problems that are linear with respect to the decision variables in the objective function and constraints.

2.1.1 Example of Linear Programming

Maximize the objective function
$x_1 + x_2$
subject to
$2x_1 + x_2 \leq 1$
$x_1 + 2x_2 \leq 1$
$x_1, x_2 \geq 0$

To solve the problem we need to find the point that maximizes the objective function within the union of the spaces defined by each of the constraints. This point is called the optimal solution, and the value the objective function takes at this space is called the optimal value. The space defined by the set of constraints is called the feasible space.

Figure 2.1 shows the feasible space defined by the constraints. Possible values of the objective function correspond to the level sets of $x_1 + x_2$. Since the objective function grows as x_1 and x_2 grow, level sets up and to the right corresponding to a bigger value in the objective function. It turns out that the corner of the feasible space at $(x_1, x_2) = (1/3, 1/3)$ is the optimal point.

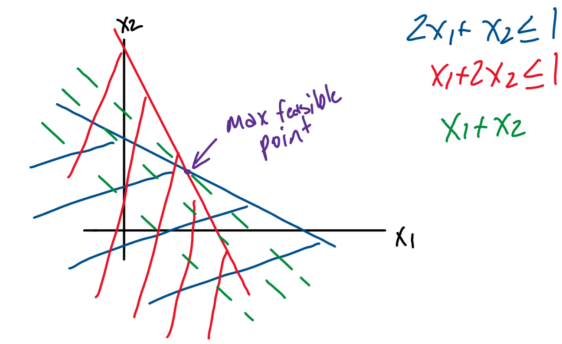


Figure 2.1: Finding optimal solution in the feasible space

2.1.2 Solution at corner points

A lot of Linear Programming was formulated by George Dantzig who found that optimal points are always at the corners of the feasible space. He formulated this into the Simplex algorithm.

2.2 Duality

It turns out that every linear problem has an associated problem called the simplex. If the linear problem (called the primal) is a maximization problem, the dual is a minimization problem, and solutions to the dual problem are an upper limit on the optimal value to the primal problem. If the primal is a minimization problem, then the dual is a maximization problem. Moreover, duality also states that the optimal value of the dual problem equals the optimal value of the primal.

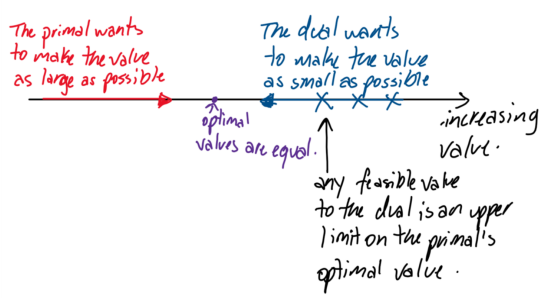


Figure 2.2: The dual places an upper limit on the primal's optimal value

2.2.1 Duality in practice

If we can find a vertex cover for a bipartite graph of size n, then by duality, we can prove that there then there cannot exist a matching of size bigger than n. The minimization problem (finding a vertex cover) places an upper bound on the maximization problem (cardinality of the matching).

2.2.2 Duality example

minimize
$7x_1 + x_2 + 5x_3$
subject to
$x_1 - x_2 + 3x_3 \geq 10$
$5x_1 + 2x_2 - x_3 \geq 6$
$x_1, x_2, x_3 \geq 0$

The dual problem is

maximize
$10p_1 + 6p_2$
subject to
$p_1 + 5p_2 \leq 7$
$-p_1 + 2p_2 \leq 1$
$3p_1 - p_2 \leq 5$
$p_1, p_2, p_3 \geq 0$

Duality tells us that if we find *any* solution to the dual problem, even if it's not the optimal solution, with a value V, then the optimal value of the primal cannot be less than V.

2.2.3 Max-Flow/Min-Cut

In a max flow, the goal is to maximize the flow from a source node to a destination node subject to the constrained capacity of edges between intermediate nodes. Additionally, flow must be conserved, so the flow into an intermediate node is constrained to equal the flow out of a node.

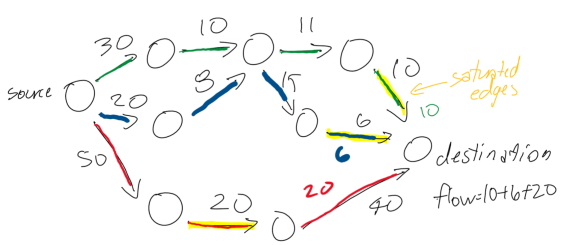


Figure 2.3: Network flow from a source to a destination along edges with capacity

In this example, we see that the edges with capacity 10, 6, and 20 become saturated and constraint the flow from the source to the destination. It turns out that these edges form a min s-t cut according to the min max theorem. In a flow network, a s-t cut is a set of edges that when removed isolates the source and destination. The min s-t cut has a minimum value of weight associated with the set of edges. So if we find the min s-t cut, the weight of the edges is equal to the max flow from the source to the destination.

2.2.3.1 Min-Max Theorem

The max-flow min-cut theorem states that in a flow network, the amount of maximum flow is equal to capacity of the minimum cut.

It turns out that matching problems can be modeled as a max flow problem by adding dummy source and destination nodes, and edges with capacity 1 from the source to the girls and from the boys to the destination, and of infinite capacity between boys and girls.

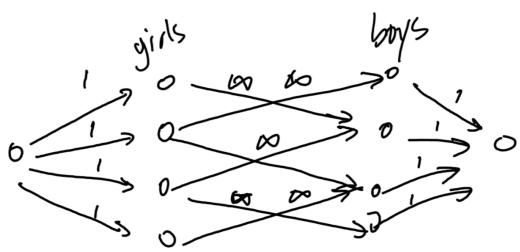


Figure 2.4: Matching Problem as a network flow

2.3 Appendix

In this section, we'll write the primal using vectors and arrays. Note that this was not covered in lecture, and is here just for extra information

We can rewrite the objective function as
$\mathbf{c}'\mathbf{x}$
where $\mathbf{x}$ is the vector of decision variables and $\mathbf{c}$ is the vector of coefficients in the objective function.
Bold face indicates vectors, and $\mathbf{a}'\mathbf{b}$ is the transpose of $\mathbf{a}$ times $\mathbf{b}$.
Then the coefficients of the constraints form the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

with rows $\mathbf{a}_i$ and columns $\mathbf{A}_j$
So the primal can be rewritten into the following structure
minimize
$\mathbf{c}'\mathbf{x}$
subject to
$\mathbf{a}_i'\mathbf{x} \geq b_i \in M_1$
$\mathbf{a}_i'\mathbf{x} \leq b_i, i \in M_2,$
$\mathbf{a}_i'\mathbf{x} = b_i, i \in M_3,$
$x_j \geq 0, j \in N_1$
$x_j \leq 0, j \in N_2$
x_j free, $j \in N_3$

The dual then has the structure
maximize
$\mathbf{p}'\mathbf{b}$
subject to
$p_i \geq 0, i \in M_1$
$p_i \leq 0, i \in M_2$
p_i free, $i \in M_3$
$\mathbf{p}'\mathbf{A}_j \leq c_j, j \in N_1$
$\mathbf{p}'\mathbf{A}_j \geq c_j, j \in M_2,$
$\mathbf{p}'\mathbf{A}_j = c_j, j \in N_3,$

EE 382V: Social Computing	**Fall 2018**

Lecture 2: August 25

Lecturer: Vijay Garg — *Scribe: Swapna M Mukrappilly*

2.1 Posets (Partially Ordered Sets)

A partially ordered set is a set P equipped with a binary relation $\leq$ which is a partial order on X.

$$P = (X, \leq)$$

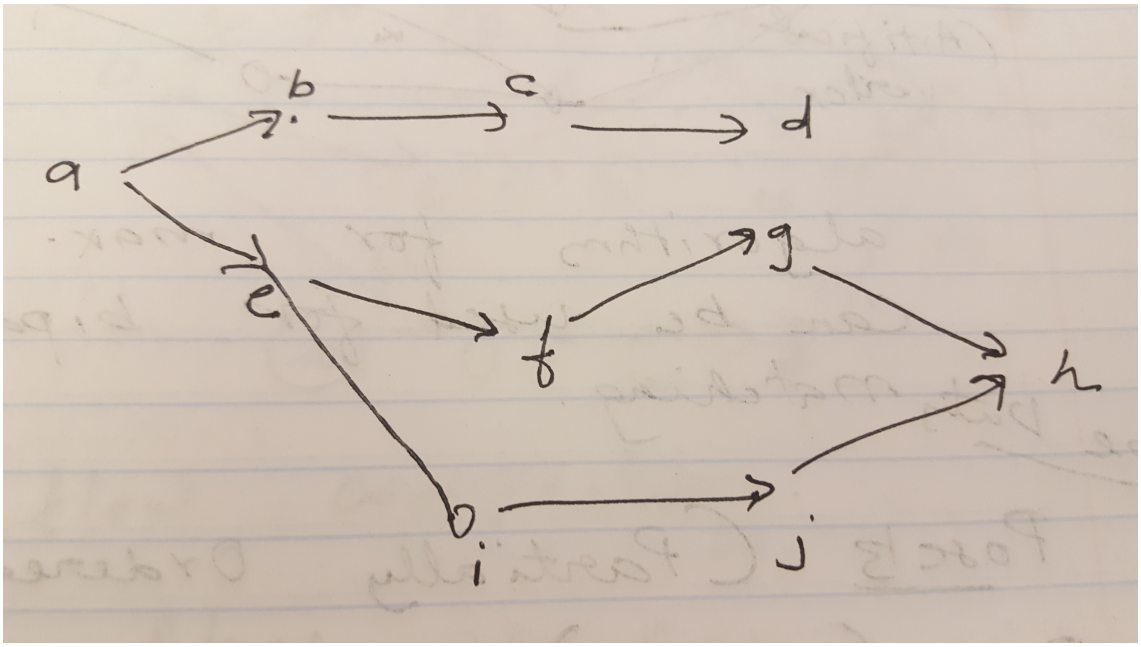
X: Any set.

The relation $\leq$ is **reflexive**, **transitive**, **antisymmetric** relation.

For all a,b,c $\in \mathbb{X}$ it must satisfy,

- a $\leq$ a (reflexivity: every element is related to itself)
- If a $\leq$ b and b $\leq$ a, then a = b (antisymmetry: two distinct elements cannot be related in both directions)
- If a $\leq$ b and b $\leq$ c, then a $\leq$ c (transitivity: if a first element is related to second element and that element is related to a third element, then the first element is related to the third element)

2.2 Chains and Antichains



$C \subseteq X$ is a chain if,

$$\forall x, y \in C, \quad (x \leq y) \quad or \quad (y \leq x)$$

In the above figure, {a,b,c,d} is a chain.

$A \subseteq X$ is an antichain if,

$$\forall x, y \in A, \quad (x \not\leq y) \wedge (y \not\leq x)$$

In the above figure, {c,f,i} is an antichain.

2.3 Dilworth's Theorem

Dilworth's theorem, also called min-max theorem states that,

Minimum number of chains required to cover the poset = maximum sized antichain.

2.4 Answers to problems on Matching

Exercise 1-1

There exists an augmenting path in G with respect to M iff there exists a directed path in D between an exposed vertex in A and an exposed vertex in B

Prove the above claim

Answer:

Proof:-

Let us consider an a maximal matching M and a maximum matching M*. The symmetric difference D is a subgraph between M and M∗, $D = M \bigoplus M*$

As both M and M∗ are matchings, each vertex in D can have degree not greater than 2, because it can be adjacent to at most one edge from each matching. If a vertex has degree equal to 2, then it is adjacent to one edge from M and one edge from M∗. Therefore, all the cycles and paths are alternating but cycles can only be even-length.

As |M∗| > |M| and $D = M * \bigoplus M = (M * \cup M) \backslash (M * \cap M)$ it is also true that $|D \cap M *| > |D \cap M|$, meaning that there are more edges from M* than M in D. Since cycles can only be even length, the augmenting path can only be odd-length paths that start and end with edges from M∗. Hence an augmenting path has to be a directed path and not a cycle.

If the vertices in A and B were not exposed, there would already been an edge between A and B resulting in a non-disjoint set.

Thus we can prove that an augmenting path can exist if and only if there is a directed path between an exposed vertex in A and an exposed vertex in B

Exercise 1-2

An edge cover of a graph G = (V, E) is a subset of R of E such that every vertex of V is incident to at least one edge in R. Let G be a bipartite graph with no isolated vertex. Show that the cardinality of the minimum edge cover R∗ of G is equal to |V | minus the cardinality of the maximum matching M∗ of G. Is this true also for non-bipartite graphs?

Answer:

Let $\nu(G)$ be the size of the maximum matching M*.

Let n =|V| be the number of vertices in the graph G.

We will construct an edge cover as follows,

A maximum matching M* covers $2\nu(G)$ vertices.Since there are no isolated vertices, the vertices that are left uncovered $= n - 2\nu(G)$.

If we use one edge to cover one vertex, the total size of the edge cover |R*|, $\rho(G)$

$$\leq \nu(G) + n - 2\nu(G)$$
$$\leq n - \nu(G)$$
$$\rho(G) + \nu(G) \leq n \qquad .....(1)$$

Let R* be the minimum edge cover. Since removing any edge from R* yields an uncovered vertex, each edge has an endpoint of degree 1 in (V, R*)

This implies that (V, R*) consists of disjoint sets. The number of components in (V, R*)

$$= |V| - |R*|$$
$$= n - \rho(G)$$

Now we construct a matching M of size $n - \rho(G)$ by choosing one edge from each disjoint set in (V, R*)

Maximum matching $|M*| \geq n - \rho(G)$

$$\nu(G) \geq n - \rho(G)$$
$$\nu(G) + \rho(G) \geq n \qquad .....(2)$$

From (1) and (2) we get,

$$\nu(G) + \rho(G) = n$$

The result is true for non-bipartite graphs too.

Exercise 1-3

Show that in any graph G= (V,E) (not necessarily bipartite), the size of any maximal matching is at least half the size of a maximum matching M*

Answer:

Consider a maximum matching M* of size $\alpha(G)$. For every edge in M*, at least one endpoint of the edge must be present in any maximal matching(or the edge can be added to the matching, and hence it won't be maximal). Hence, the number of vertices covered by any maximal matching must at least be $|M*| = \alpha(G)$. Since an edge in the matching can cover at most two vertices, the number of edges in the maximal matching must be $\alpha(G)/2$.

Session 1

EE 382V: Social Computing	**Fall 2018**

Lecture 3 Session 1: September 28

Lecturer: Vijay Garg *Scribe: Bhargavi Karthik*

3.1 LP-Duality

In Kuhn-Munkres (KM) algorithm, we find matching using the concept of labeling the vertices such that it satisfies the feasibility constraints. Using LP-Duality, we can come up with similar ideas for solving problems in a more intuitive way.

3.1.1 Example for LP Duality

There is an objective function which we are trying to minimize,

$7x_1 + x_2 + 5x_3$

subject to constraints,

$x_1 - x_2 + 3x_3 \geq 10$

$5x_1 + 2x_2 - x_3 \geq 6$

where x_1, x_2, x_3 are the decision variables and assumption is that,

$x_1, x_2, x_3 \geq 0$

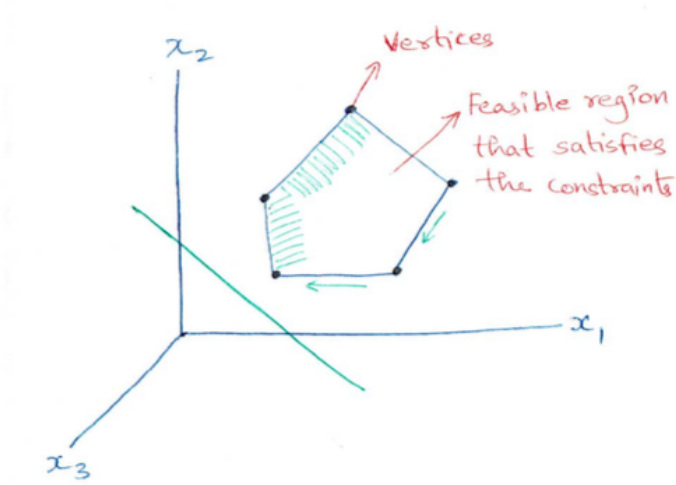


Figure 3.1: Finding optimal solution in the feasible space

Every constraint is in some sense dividing up the feasible region into two half spaces. The goal here is to find the optimal solution. If we can draw level sets for the given objective function, which means we are trying to optimize along some dimension. In this case we are trying to find the minimum. While trying to optimize along some direction, the point one always hits will be a corner. One of the important properties as per the Simplex algorithm is - "The extremal point is always a vertex".

The approach is to figure out one feasible point and start from there. If it is not a corner, keep moving until we find a corner and keep optimizing. However, if we are at a corner and if there are n other corners to go to, then go in the direction that has the most advantage.

Suppose we have solution, x = (2, 1, 3). Lets substitute in the objective function,

$7(2) + 1 + 5(3) = 14 + 1 + 15 = 30$

this solution is also feasible as it satisfies the constraints,

$2 - 1 + 3(3) = 10$

$5(2) + 2(1) - 3 = 9$

Trying to find a bound on primal:

$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10$ by comparing coefficients point-wise and $x_i \geq 0$

solution has to be ≥ 10 but we are trying to get a better or greater lower bound

Add up the constraints equations,

$6x_1 + x_2 + 2x_3 \geq 16$

The objective function is still greater than the above equation. Hence new lower bound is 16.

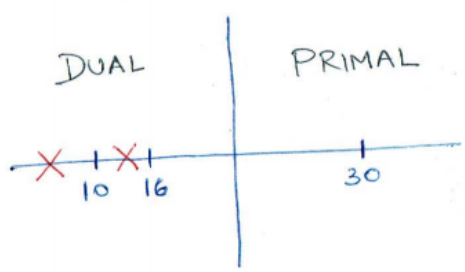


Figure 3.2: Representation of the example as duality

The goal of primal was to minimize and so now the goal of dual will be to maximize. We have to figure out what combination of each constraint should we add such that it gives the tightest bound on the objective function.

We have n decision variables and m constraints in our primal. So dual will have m variables (one variable for each constraint in primal) and n constraints (since n decision variables in primal).

Objective function is $10y_1 + 6y_2$ subject to constraints,

$y_1 + 5y_2 \leq 7$

$-y_1 + 2y_2 \leq 1$

$3y_1$ - $y_2 \leq 5$

In a generalized representation of the primal, objective function can be written as

$$\sum_{i=1}^{n} c_i x_i$$

constraints can be written as

$$\sum_{j} A_{ij} x_j \geq b_j$$

where A is a matrix of dimension m*n, c and b are vectors of size n and m respectively.

For dual, b is translated to c, c is translated to b and matrix A becomes A^T.

3.1.2 Strong Duality

Theorem 3.1 *The primal program has finite optimum if and only if its dual has finite optimum. Moreover, if $x*$ and $y*$ are optimal solutions for (P) and (D) then,*

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Note: This translates into all the min-max theorems such as max flow = min cut, max matching = min vertex cover and min number of chains to cover a poset = max sized anti-chain

3.1.3 Weak Duality

Theorem 3.2 *If x and y are feasible solutions for (P) and (D) then,*

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

Proof:

Claim 3.3

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j$$

This claim is true because y is feasible and $x \geq 0$

Claim 3.4

$$\sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \geq \sum_{i=1}^{m} b_i y_i$$

This claim is true because x is feasible and $y \geq 0$

Equating claim 3.3 to claim 3.4

Hence proved,

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

■

3.1.4 Complementary Slackness Conditions

Theorem 3.5 *Given that x,y are (P) and (D) feasible solutions. Then x and y are both optimal if and only if the following complementary slackness conditions are satisfied:*

Claim 3.6

$$\forall j \quad either(x_j = 0) \quad or \quad \sum_{i=1}^{m} a_{ij} y_i = c_j$$

Claim 3.7

$$\forall i \quad either(y_i = 0) \quad or \quad \sum_{j=1}^{n} a_{ij} x_j = b_i$$

3.2 LP Duality and Maximum Matching

For the Kuhn-Munkres Algorithm, we started with x variables and came up with labels through initial feasible labeling. These labels are the dual variables. And we tried to get complementary slackness.

3.2.1 Maximum Matching Problem As Linear Program

Primal:

$$max \sum_{i,j} W_{ij} x_{ij}$$

subject to:

$$\sum_{j} x_{ij} \leq 1$$

and

$$\sum_{i} x_{ij} \leq 1$$

where x_{ij} is an edge between A and B, i is an index from A and j is an index from B. For each edge, there is a weight W_{ij}. We need to pick the edges such that the sum of their weights is maximum.

Replacing integral constraint by linear constraint,

$$x_{ij} \in \{0, 1\} \Rightarrow x_{ij} \geq 0$$

The above transformation is called LP-Relaxation. This can be used only when we can prove that the optimal solution is always going to be integral.

When $|A| = |B| = n$

Number of decision variables = n^2

Number of constraints = 2n

Dual:

Number of decision variables = 2n

Number of constraints = n^2

Objective function (minimizing vertex labels):

$$min \sum u_i + \sum v_j$$

subject to: $u_i + v_j \geq w_{ij}$

$u_i \geq 0 \quad and \quad v_j \geq 0$

where u_i is constraints for side A and v_j is constraints for side B

EE 382V: Social Computing **Fall 2018**

Lecture 3 Session 2 : September 28

Lecturer: Vijay Garg *Scribe: Carmina Francia*

3.1 Stable Matching/Marriage

Matching has a lot of real world applications. We see it in markets, buyers/sellers, computer processes, residencies, and so on. So far we've used matching in the context of girls likes boys and boys like girls. These matchings can sometimes have *preferences*.

This is called **Stable matching/marriage**. It is finding a matching between two equal sets each following an ordered preference list.

For example, let's create two preference lists:

Table 3.1: Men Preference List

M1	W4	W1	W2	W3
M2	W2	W3	W1	W4
M3	W2	W4	W3	W1
M4	W3	W1	W4	W2

Table 3.2: Women Preference List

W1	M4	M1	M3	M2
W2	M1	M3	M2	M4
W3	M1	M2	M3	M4
W4	M4	M1	M3	M2

This is still a bipartite and a good matching would be when we don't have one woman assigned to two men and vice versa.

Blocking Pair

What kind of marriage would not last long? Let's say we have the following matchings: *(m,w)* and *(m', w')*. If *m* and *w'* are not engaged, but m prefers *w'* to w and *w'* prefers *m* to *m'* , then there's an incentive for them to break. This is called a **blocking pair**.

The goal of stable matching is to find a matching that has *no* blocking pair.

3.2 Gale and Shapley Algorithm

The idea is similar to DGS algorithm: people bidding and going for the most valuable item. We will have men proposing (bidders). The difference is the other set, in this case women, has a say. Women can reject men.

Data: men preference list and women preference list
Result: (m,w) matchings
Assuming number of men is equal to number of women;
Put all men in a list (initially everyone if free);
while *free list of men is not empty* **do**
  pick any free man;
  man proposes to most preferred woman that he has not proposed to;
  if *woman is free* **then**
    men is engaged with that woman;
  else
    woman rejects the less desirable man of the two men based on her preference list;
  end
  rejected man go back to free list (does not matter where in the list)
end

Key notes:

- *engaged* means man and woman are not married. Matchings can change as algorithm progresses.
- Women rejects based on their preference list
- if you're a *free* woman, you cannot say no. Only way to reject is if a better man comes.
- Once a woman is engaged, she cannot be free.
- Algorithm **ALWAYS** terminates and everybody is married.

This example is *man-optimal*. It is seen as worse from women's perspective. However, as the algorithm progresses, men decrease on their preference while women are safe or their preference increases.

3.3 Stable Matching Example

Let's walk through a man-optimal example...

Table 3.3: Men Preference List

M1	**W4**	W1	W2	W3
M2	*W2*	**W3**	W1	W4
M3	**W2**	W4	W3	W1
M4	*W3*	**W1**	W4	W2

Table 3.4: Women Preference List

W1	**M4**	M1	M3	M2
W2	*M1*	**M3**	M2	M4
W3	*M1*	**M2**	M3	M4
W4	*M4*	**M1**	M3	M2

1. Let's begin with *M1*. *W4* is free so *M1* and *W4* becomes engaged.
2. We move on to *M2*. *W2* is free and becomes engaged with *M2*.
3. We go to *M3*. *W2* is already engaged, so we check the woman's preference list. *W2* prefers *M3* over *M2*. So now, *M3* and *W2* are engaged. *M2* goes back to the free list.
4. We go to *M4*. *W3* is free and becomes engaged with *M4*.

5. We pick *M2*. The next woman on his preference list that he has not proposed to is *W3*. *W3* is engaged to *M4*. We check the *W3*'s preference list. *W3* prefers *M2* over *M4*. So *M2* and *W3* becomes engaged and *M4* goes back to the list.
6. We pick *M4*. *W1* is free so *M4* and *W1* becomes engaged.

The list is now empty and returns the matchings: *(M1, W4)*, *(M2, W3)*, *(M3, W2)*, *(M4, W1)*. Since we don't see any blocking pairs, this is a stable matching.

Now let's say it's a woman-optimal. Same steps apply except we would initialize with all women in the free list and pick a woman to start with. Once the list is empty and everybody is married/matched, you will notice that we come up with the same matchings:*(W1, M4)*, *(W2, M3)*, *(W3, M2)*, *(W4, M1)*.

Note: If the man-optimal and woman-optimal are *equal*, this is the **only stable matching**.

Session 1

EE 382V: Introduction to Social Computing **Fall 2018**

Lecture 4 Session 1: September 29
Stable Marriage

Lecturer: Vijay Garg *Scribe: Tiffany Tillett*

4.1 Review

Recall the algorithm to find a stable marriage from last class.

4.1.1 Inputs

1. mPref: data structure containing preferences for each man
2. wPref: data structure containing preferences for each woman
3. rank: where rank[i][j] indicates the rank of man j for woman i; higher preference correlates to lower rank

4.1.2 Stable Marriage Algorithm

1. $freelist$ = list of men who are free (initially has all men)
2. while ($freelist$ is not empty):
    3. choose some man m from $freelist$
    4. m proposes to his most preferred woman w that he has not already proposed to
    5. if w is free:
        6. w accepts proposal (assign m to w)
    7. else: // w is already engaged to m'
        8. if $rank[w][m] > rank[w][m']$: // w prefers m'
            9. add m back to $freelist$ // reject m; w still engaged to m'
        10. else:
            11. add m' back to the $freelist$ // reject m'
            12. w gets engaged to m
13. print assignment

4.2 Stable Marriage Algorithm Correctness

Theorem 4.1 *The Algorithm terminates with a matching.*

Proof:

Claim 4.2 *The last woman in a man's preference list must be free*

Invariant: Once a woman is engaged, she stays engaged.

In order for a man to make it to his last preference, the previous $n-1$ women must already be engaged since the only reason they can reject his proposal is if they are engaged to someone that they prefer more. Since the number of men and women is equal, the last woman in the list can never be engaged.

Since the last woman in a man's preference list will always be free, every man is guaranteed that he will match with one woman, so a matching will exist.

■

Theorem 4.3 *The algorithm (if it terminates) ends with a stable matching.*

Proof: Assume that $(m, w')(m', w)$ is a blocking pair where m is married to w' and w is married to m'.

In order for $(m, w')(m', w)$ to be a blocking pair, this means that m must prefer w to w' and that w must prefer m to m'.

Since m prefers w to w', he would have proposed to w first and gotten rejected.

Invariant: A woman's assignment can only improve with execution of the algorithm.

Therefore, w can only have rejected m if she had a proposal from someone she preferred more. If that person was m', we know that she must prefer m' to m. If that person was not m', we know that she prefers that person to m and that she prefers m' to that person, so by extension, she prefers m' to m. Generally, w will always prefer her current fiance to all men she rejected, so she must prefer m' to m.

The requirement for (m, w) to be a blocking pair was that m must prefer w to w' and that w must prefer m to m'. These two requirements cannot co-exist, so by contradiction, we know that $(m, w')(m', w)$ cannot be a blocking pair. ■

4.3 Posets: Partially Ordered Sets

4.3.1 Definition

A poset is defined as $(X, \leq)$ where X is a set and $\leq$ is a binary relation on that set.

A reflexive poset has the following properties:

1. reflexive - for all vertices v in V: $v \leq v$
2. anti-symmetric - for all pairs of vertices u, v in V: if $v \leq u$ and $u \leq v$ then $u == v$

3. transitive - for all groups of vertices u, v, w in V: $u \leq v \leq w$ implies that $u \leq w$

Consider all following posets to be reflexive posets.

4.3.2 Example 1

X: $\mathbb{R}^n$

$\leq$: $U \leq V \equiv$ for all i: $u[i] \leq v[i]$

Claim 4.4 *$(X, \leq)$ is a poset.*

4.3.2.1 Numerical Example 1

$M1 = (2, 5, 3)$; $M2 = (6, 8, 5)$

$2 \leq 6$, $5 \leq 8$, $3 \leq 5$ says that $M1 \leq M2$.

4.3.2.2 Numerical Example 2

$M1 = (1, 3, 1)$; $M2 = (2, 1, 2)$

$1 \leq 2$, $3 > 1$. What does this mean?

Definition 4.5 *$M1$ is incomparable to $M2$ if $(M1 \nleq M2)$ and $(M2 \nleq M1)$.*

4.3.3 Example 2

$z = \{a, b, c\}$

X: { {}, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c} }

$\leq$: $\subseteq$

$\{a, b\} \leq \{a, b, c\}$

$\{a, b\}$ is incomparable to $\{b, c\}$

4.3.4 Dealing with Incomparable Items

If $M1$ and $M2$ are incomparable, try to find $M3$ which is better than both $M1$ and $M2$.

$M1 = (1, 1, 2, 3)$

$M2 = (2, 2, 1, 2)$

Note: the numbers here indicate preference where 1 indicates first choice, 2 indicates second choice, etc

find $M3 = (1, 1, 1, 2)$ such that $M3 \leq M1$ and $M3 \leq M2$.

4.3.5 Hasse Diagrams: Pictoral Representations of Posets

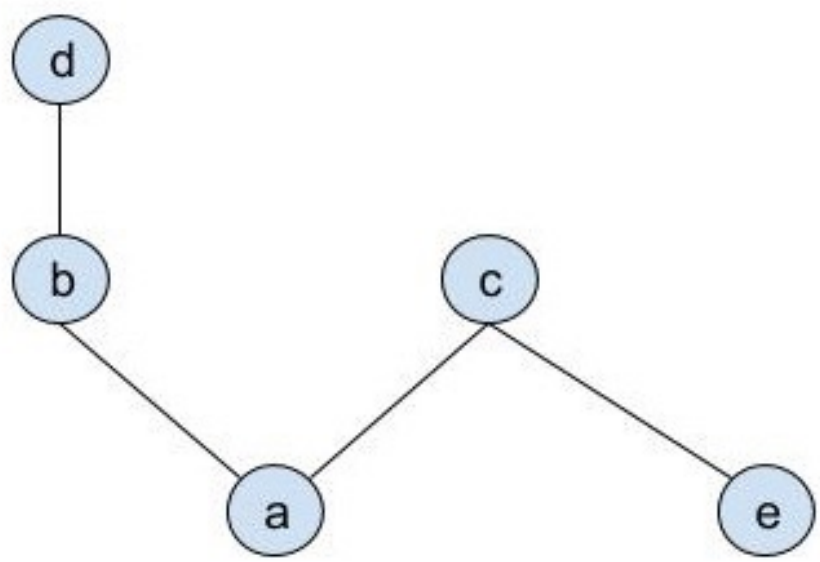


Figure 4.1: Hasse Diagram example

This diagram shows that:

$a \leq b$

$a \leq c$

$b \leq d$

$e \leq c$

c and d are incomparable

a and e are incomparable

4.3.6 Operations on Posets

Given a poset $(X, \leq)$:

Definition 4.6 *Let $x, y \in X$. $z \in X$ is meet of x and y if:*

1. $z \leq x$ *and* $z \leq y$

   AND

2. *for all* z'*:* $z' \leq x$ *and* $z' \leq y \implies z' \leq z$

Where meet is also known as the greatest lower bound. The first condition is the lower bound, and the second condition is that the lower bound must be greater than all other lower bounds.

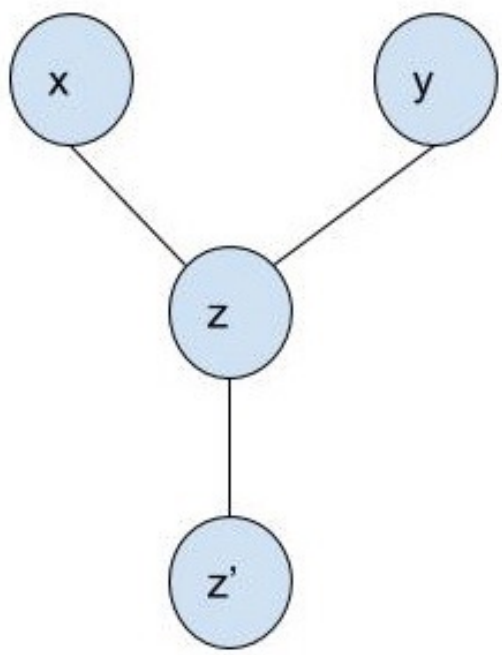


Figure 4.2: Meet Example; z is the meet of x and y

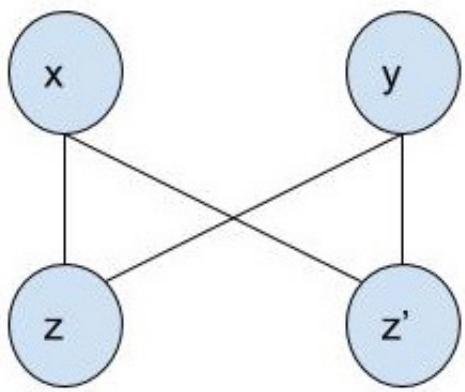


Figure 4.3: z is not a meet; z and z' are incomparable

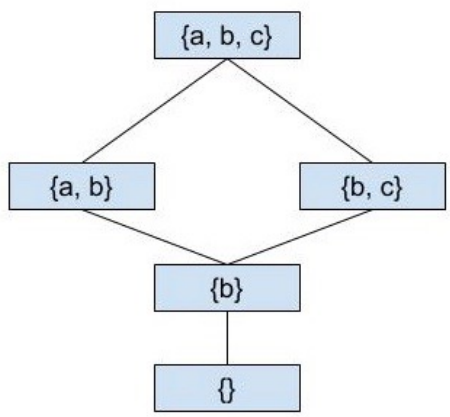


Figure 4.4: meet corresponds to intersection of sets

4.3.7 Lattice Posets

Definition 4.7 *A poset $(X, \leq)$ is lattice (nice) if for all $a, b \in X$, $meet(a, b)$ exists and $join(a, b)$ exists.*

Definition 4.8 *$join(a, b)$ is the least upper bound (reverse of meet).*

$(a \vee b) = join(a, b)$

$(a \wedge b) = meet(a, b)$

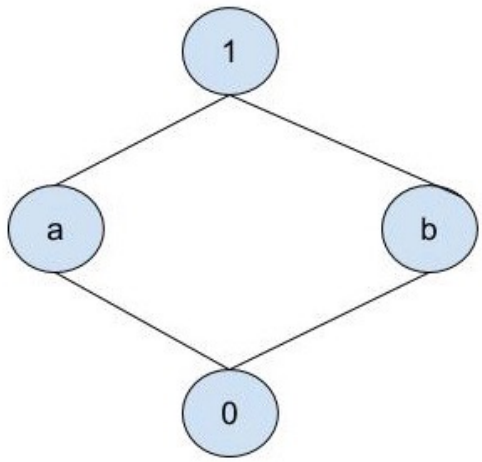


Figure 4.5: meet(a,1) = a

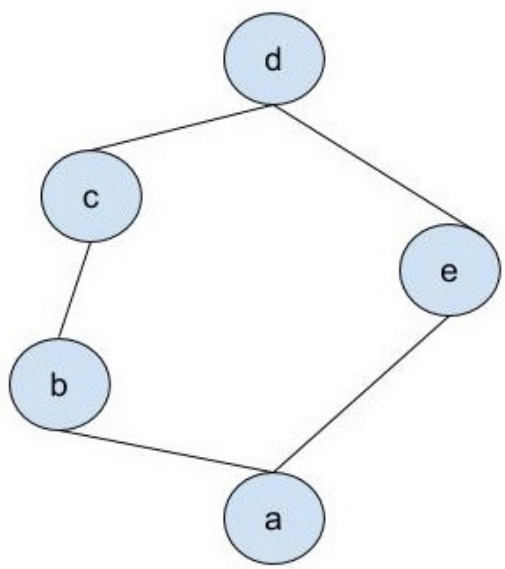


Figure 4.6: meet(c,e) = a; join(c,e) = d; meet(b,e) = a; join(b,e) = d

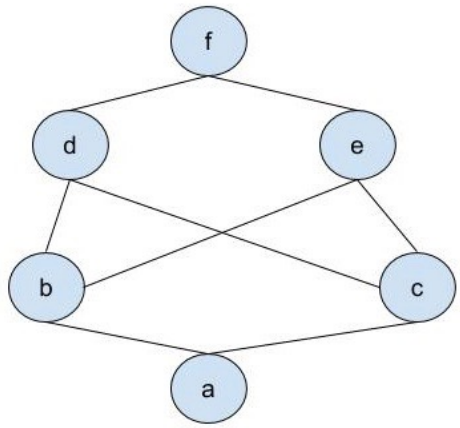


Figure 4.7: meet(b,c) = a; There is no meet of d and e since b and c are incomparable. a cannot be the meet since b and c are bigger than a.

The definition of a lattice can be extended to apply to all finite subsets of X.

Definition 4.9 *A lattice $L = (X, \leq$ is distributive if for all $x, y, z \in X$:*

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
   OR
2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Referencing Figure 4.6:

$e \wedge (b \vee c) = e \wedge c = a$

$(e \wedge b) \vee (e \wedge c) = a \vee a = a$

This works out. However, we need to show this for all x, y, z.

$c \wedge (b \vee e) = c \wedge d = c$

$(c \wedge b) \vee (c \wedge e) = b \vee a = b.$

This does not work. This lattice is not distributive.

There is a theorem which states that lattices of certain shapes cannot be distributive.

EE 382V: Social Computing **Fall 2018**

Lecture 4, Session 4: September 29

Lecturer: Vijay Garg *Scribe: Zeid Ayssa*

4.1 Stable Marriage Optimality

After we introduced $Gale-Shapley$ stable matching algorithm, we need to prove that the resulting matching is optimal i.e. there is no better matching for men.

Note: throughout this section, we will assume men optimal matching. Furthermore, lower ranking numbers mean higher preference i.e.1 means it is the man's/woman's 1st choice and 4 means that it is the man's/woman's 4th choice.

Assuming men optimal assignment, Below is the Hasse diagram of all possible assignments. Note that this is not necessarily matching assignments, but rather all possible men assignments to women. This is not necessarily a matching due to the fact that if all men were matched to their first woman preference, we could have multiple men preferring the same women and we cannot match a women to more than one man.

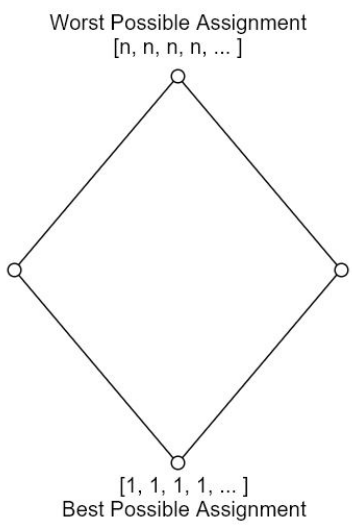


Figure 4.1: Hasse diagram of all possible men assignments

Example: if we have two matchings $M_1 = [1, 2, 1, 3, 1]$ and $M_2 = [2, 3, 2, 3, 1]$, if $M_1 \leq M_2$, where $M_1 \leq M_2$ is defined as $\forall i : M_1[i] \leq M_2[i]$, then we know that every man in M_1 is either happier or as happy compared to M_2 matching.

Let u and v be vectors of size n representing two different men assignments in figure 4.1 where $u \leq v$ is defined as $\forall i : u[i] \leq v[i]$

We know from a previous section that figure 4.1 is a Partially Ordered Set (POSET) because it is Reflexive, Anti-Symmetric, and Transitive.

Claim 4.1 *$(X, \leq)$ is a lattice poset where X is the set of all possible assignments shown in figure 4.1*

Definition 4.2 *A poset is lattice if every two elements in the poset have a unique upper bound or join and a unique lower bound or meet. Minimum upper bound or join is denoted by $\sqcup$ and maximum lower bound or meet is denoted by $\sqcap$.*

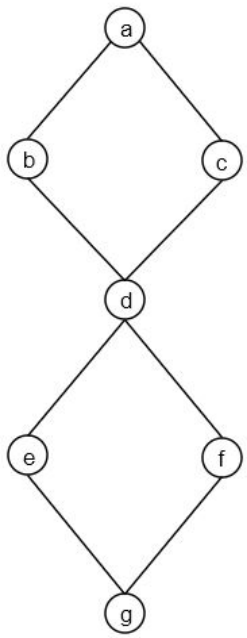


Figure 4.2: Hasse diagram of a lattice poset

Figure 4.2 is an example of a lattice poset i.e. for every two elements there is a unique upper and lower bounds. Let us check:

$b \sqcap c = d$ and $b \sqcup c = a$

$e \sqcap c = g$ and $e \sqcup c = a$

Proof: Let u and v be two different assignment sets with size 4

$u[1, 3, 2, 1]$

$v[2, 1, 3, 1]$

$w = u \sqcap v = [1, 1, 2, 1]$ i.e. pairwise min operation.

We know that $w \leq u$ and $w \leq v$, so w is a more preferable matching to men than u or v. This also means that w will always be greater than or equal to any other assignment set that is less than or equal to u or v. For example, if there is another set $g = [1, 1, 1, 1]$ then $g < u$ and $g < v \implies g \neq w \implies w > g$ since w is already defined as the meet between sets u and v. This proves that the poset is a lattice poset. ■

Claim 4.3 *$(X, \leq)$ is a distributed lattice poset*

We need to verify that poset $(X \leq)$ is a distributive lattice.

Let $x = [a_1, a_2, a_3, \ldots]$, $y = [b_1, b_2, b_3, \ldots]$, and $z = [c_1, c_2, c_3, \ldots]$ be different sets in the poset. For a poset to be a distributed lattice we need to verify that $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$ which means we need to verify that:

$min(a_x, max(b_x, c_x) = max(min(a_x, b_x), min(a_x, c_x))$

Proof: Let $x[1, 3, 2, 1]$, $y[2, 2, 3, 1]$, and $z[3, 1, 2, 3]$ be three different assignment sets with size 4. Now, let us see if this is a distributed lattice.

<u>1st *Element*</u>

$min(1, max(2, 3)) = min(1, 3) = 1$

$max(min(1, 2), min(1, 3)) = max(1, 1) = 1$

<u>2nd *Element*</u>

$min(3, max(2, 1)) = min(3, 2) = 2$

$max(min(3, 2), min(3, 1)) = max(2, 1) = 2$

<u>3rd *Element*</u>

$min(2, max(3, 2)) = min(2, 3) = 2$

$max(min(2, 3), min(2, 2)) = max(2, 2) = 2$

<u>4th *Element*</u>

$min(1, max(1, 3)) = min(1, 3) = 1$

$max(min(1, 1), min(1, 3)) = max(1, 1) = 1$

Given that for all elements in the assignment sets x, y, and z the result of $min(a_x, max(b_x, c_x)$ was equal to the result of $max(min(a_x, b_x), min(a_x, c_x))$, then this is a distributive assignment set. Since this should hold true for all possible assignment sets in our poset $(X, \leq)$ then we can say that this is a distributed lattice. ■

Why do we care if it is a distributed lattice or not? If a poset is a distributive lattice then it makes it similar to Boolean algebra since union and intersection distribute over each other. This property will allow us define a Boolean predicate later on to help us prove optimal stable matching.

4.1.1 Hasse Diagram of All Assignments and Stable Matching Region

Figure 4.3 shows the Hasse diagram of all possible assignments. It also shows the men optimal and women optimal stable matching. There is a region that has different sets of stable matching i.e. no blocking pair, but there is only one optimal stable matching for men and women.

A men optimal stable matching is a matching where there is no other stable matching set that leads to happier men i.e. there is no other stable set that is less than the current stable matching. The same applies to women where a women optimal stable matching is a matching where there is no other stable matching set that leads to happier women i.e. there is no other stable set that is less than the current stable matching.

Starting from the initial set $[1, 1, 1, \ldots]$ and taking any path when using $Gale-Shapley$ algorithm will always give us the optimal men or women stable matching.

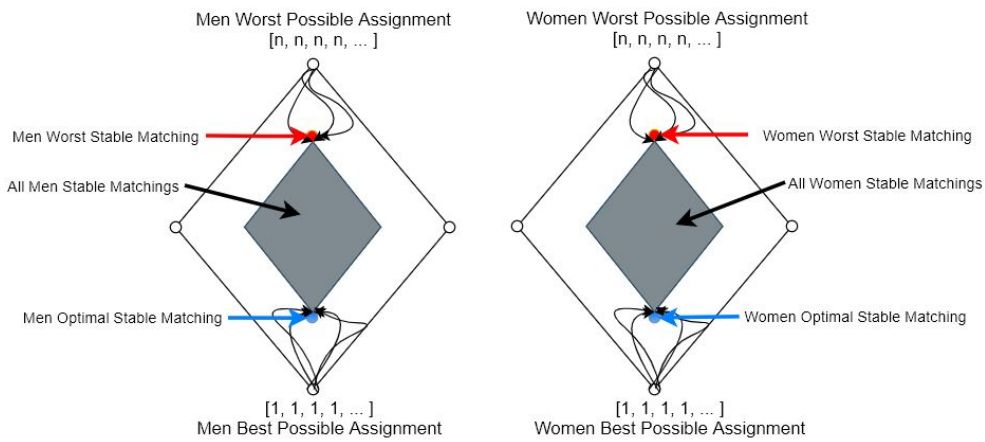


Figure 4.3: Hasse diagram of all possible assignments along with the stable matching region

4.1.2 Finding Stable Matching Assignments

If we want to write an algorithm to find the optimal stable matching, we need to figure out some sort of logic to determine if the current man and woman matching is the best we can do before moving on. Trying to go over all the possible matchings would take a lot of time since the total number of possible assignments is n^n because for every man there are n women i.e. $m = [\ldots n] \implies [n][n]$ array.

Figure 4.4 shows the preference for men and women where we can formally represent them as:

<u>Men Preferences:</u>

$m_1 = [2, 3, 1, 4]$

$m_2 = [2, 1, 3, 4]$

$m_3 = [1, 3, 4, 2]$

$m_4 = [3, 2, 4, 1]$

<u>Women Preferences:</u>

$w_1 = [3, 2, 1, 4]$

$w_2 = [1, 2, 4, 3]$

$w_3 = [4, 1, 2, 3]$

$w_4 = [4, 3, 2, 1]$

Figure 4.4 also shows one possible stable matching $\{\{m_1, w_2\}, \{m_2, w_1\}, \{m_3, w_3\}, \{m_4, w_4\}\}$.

4.1.2.1 Forbidden State

Figure 4.5 shows a matching that has a forbidden state. We can see from figure 4.5 that w prefers m' more than m, we can also see that m' prefers w more than w', therefore this a blocking pair which means that this is not a stable matching.

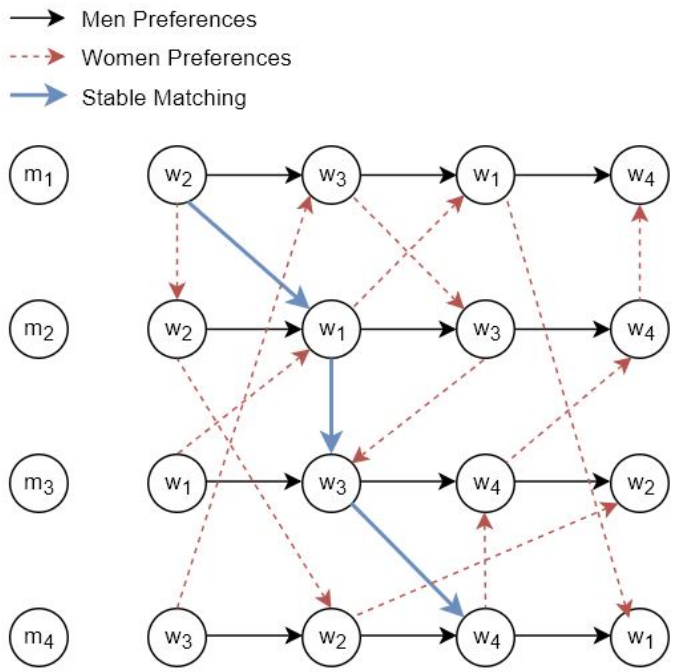


Figure 4.4: Possible men and women matching

From figure 4.5 we can say that a matching is not stable if there is a preference arrow that points to a node before the current matching region since it will represent a blocking pair. Similarly, we can say that a matching is stable if there are no preference arrows pointing to a node that is before the current matching region. If our algorithm makes sure that there are no forbidden states, then we know for sure that if a women rejects a man then there is no way there was going to be a stable marriage that would include this pair. If a matching does not include a forbidden state, then this also proves the claim that this is a distributed lattice.

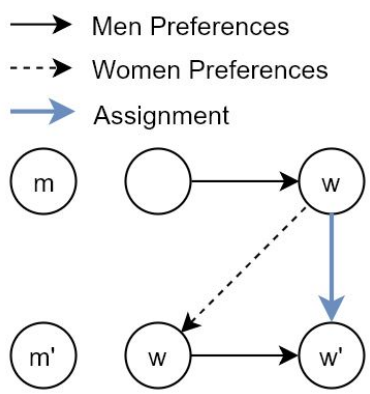


Figure 4.5: Forbidden state

Figure 4.6 shows a matching that has a blocking pair or a forbidden state. A matching with a forbidden state means that the matching is not stable.

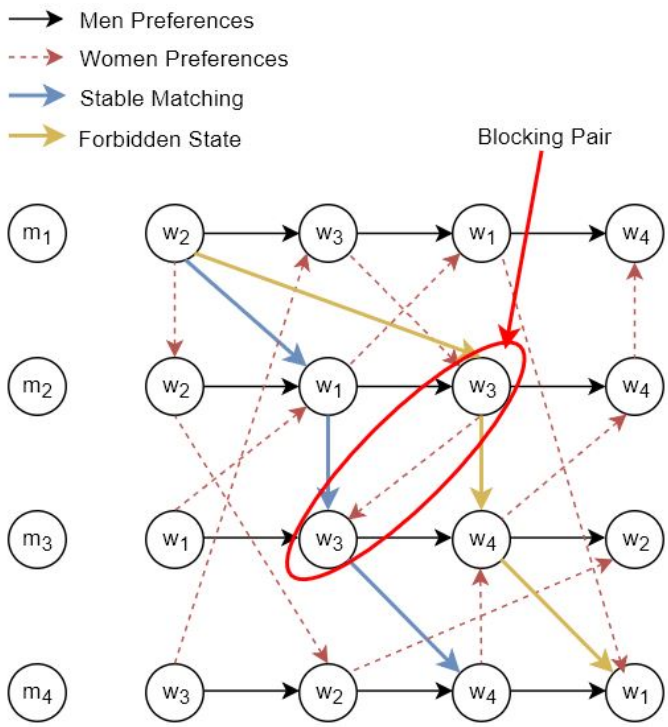


Figure 4.6: Possible men and women matching

In the next class we will go over a formal definition of the forbidden state and how it will help us prove that $Gale - Shapley$ algorithm will give us an optimal stable matching. We will define an algorithm that uses (G, i, B) where:

G : Assignment

i : men

B : Boolean Predicate, where B:

- Is a matching
- Has no blocking pair

EE 382V: Social Computing	**Fall 2018**

Scribe Assignment 1: September 29

Lecturer: Vijay Garg *Scribe: Zach Southwell*

1.1 The LP formulation of Max-Flow Min-Cut problem

1.1.1 Problem Domain

We are given a connected graph which is directed from an origin node s to a destination node t, where each edge in the graph has some non-negative capacity which represents the maximum flow across the edge.

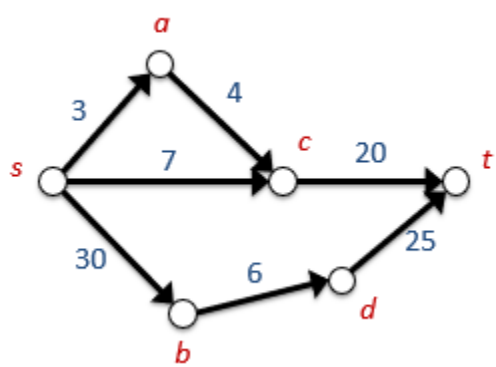


Figure 1.1: sample flow

1.1.2 Primal Problem

We want to calculate the *maximum flow* which can be sent from s to t. Another way to state this is that we want to maximize the flow through each edge from s subject to these constraints:

- Conservation constraints: The size of a the incoming flow for a vertex must be the same as the outgoing flow
- Capacity constraints: No edge may carry more than its maximum capacity. Define $c(u, v)$ as the maximum capacity for an edge $(u, v) \in E$
- The flow through each edge must be non-negative

This can be stated as the following linear program:

Maximize:

$$\sum_{v:(s,v)\in E} f(s,v)$$

Subject to:

$$\sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w)$$
$$f(u,v) \leq c(u,v) \qquad \forall (u,v) \in E$$
$$f(u,v) \geq 0 \qquad \forall (u,v) \in E$$

1.1.3 Dual Problem

The dual problem is to *cut* the graph into two components with one containing s and the other containing t by removing the edges with the minimum total capacity. Let S be the component containing s and T be the component containing t. Every node in the graph (except for s and t) has a dual variable $y(u)$ which is 1 if $u \in S$ and 0 otherwise, and every edge in the graph has a dual variable $y(u,v)$ which is 1 if $u \in S$ and $v \in T$ and 0 otherwise. The dual problem can be stated as:

Minimize:

$$\sum_{(u,v)\in E} c(u,v) * y(u,v)$$

Subject to:

$$y(v) + y(s,v) \geq 1 \qquad \forall v : (s,v) \in E$$
$$y(v) - y(u) + y(u,v) \geq 0 \qquad \forall (u,v) \in E (s \neq u, t \neq v)$$
$$-y(u) + y(u,t) \geq 0 \qquad \forall u : (u,t) \in E$$

1.1.4 Primal Example

For figure 1.1, the primal problem can be stated as:

Maximize:

$$f(s,a) + f(s,c) + f(s,b)$$

Subject to:

$$\begin{gathered}
f(s,a) - f(a,c) = 0 \\
f(s,b) - f(b,d) = 0 \\
f(a,c) + f(s,c) - f(c,t) = 0 \\
f(b,d) - f(d,t) = 0 \\
f(s,a) \leq 3 \\
f(s,c) \leq 7 \\
f(s,b) \leq 30 \\
f(a,c) \leq 4 \\
f(b,d) \leq 6 \\
f(c,t) \leq 20 \\
f(d,t) \leq 25 \\
f(u,v) \geq 0 \qquad \forall (u,v) \in E
\end{gathered}$$

1.1.5 Dual Example

Minimize:

$$c(s,a)*y(s,a)+c(s,b)*y(s,b)+c(s,c)*y(s,c)+c(a,c)*y(a,c)+c(b,d)*y(b,d)+c(c,t)*y(c,t)+c(d,t)*y(d,t)$$

Subject to:

$$\begin{gathered}
y(a) + y(s,a) \geq 1 \\
y(b) + y(s,b) \geq 1 \\
y(c) + y(s,c) \geq 1 \\
y(c) - y(a) + y(a,c) \geq 0 \\
y(d) - y(b) + y(b,d) \geq 0 \\
-y(c) + y(c,t) \geq 0 \\
-y(d) + y(d,t) \geq 0
\end{gathered}$$

1.1.6 Solution

The following mapping satisfies the primal with a value of 16:

$$f(s,a) = 3; f(s,c) = 7; f(s,b) = 6; f(a,c) = 3; f(c,t) = 10; f(b,d) = 6; f(d,t) = 6$$

And the following 'cut' satisfies the dual with a value of 16:

$$y(s,a) = 1; y(s,c) = 1; y(s,b) = 0; y(a,c) = 0; y(c,t) = 0; y(b,d) = 1; y(d,t) = 0$$

Since the primal and dual solutions have the same value, we know that these solutions are optimal.

A value of 1 for $y(s,a), y(s,c), y(b,c)$ in the dual problem indicates that the edge belongs to the 'cut' set. Finding these constraining edges is the intuitive and natural approach to solving the maximization problem. This is interesting because even without formal knowledge of LP duality in this case, you would have a tendency to construct the dual in order to solve the primal problem.

EE 382V: Social Computing **Fall 2018**

Lecture : LP Formulation of the Stable Marriage Problem

Lecturer: Vijay Garg *Scribe: Eric Addison*

0.1 Introduction

In this note, we will work out a linear programming (LP) formulation of the Stable Marriage problem. The problem will be briefly reviewed, followed by introduction of relevant notation. Characteristics of the problem will be translated into linear algebraic constraints, which will allow us to write down the LP statement of the problem, along with its dual (DLP). Finally, we will work through a small example.

The formulation and notation in this note is based primarily on that found in [1], however several other articles can easily be found that provide similar treatments.

0.2 Review of the Stable Marriage Problem

Consider two sets of agents M and W (e.g. men and women), with $|M| = |W|$ for simplicity. Each agent provides a complete list of preferences of the members from the opposite set, which implicitly defines a ranking operator ($\geq_m$ for each man m and $\geq_w$ for each woman w) such that $(M, \geq_w)$ and $(W, \geq_m)$ are totally ordered sets. We say that a woman w prefers man m to another man m' if $m \geq_w m'$.

The goal of the stable marriage problem is to find a *stable matching* between the two sets M and W. Define a *matching* as a mapping $\mu : (M \cup W) \to (M \cup W)$ where:

- $\mu(w) = m \Leftrightarrow \mu(m) = w$, i.e. m is matched to w
- if $\mu(w) \notin M$, then $\mu(w) = w$ and w is not matched
- if $\mu(m) \notin W$, then $\mu(m) = m$ and m is not matched

A matching is *stable* if $\nexists (m, w)$ such that $m >_w \mu(w)$ and $w >_m \mu(m)$ (where $>_m$ is the irreflexive version of $\geq_m$). That is, there is no pair of matched man and matched woman who both prefer each other to their current match. Such a pair is know as a *blocking pair*.

0.3 LP Formulation

We begin by defining the *incidence vector* of a matching μ as a vector $x \in \{0, 1\}^{|M| \times |W|}$, indexed such that $x_{m,w} = 1 \Rightarrow \mu(m) = w$, else $x_{m,w} = 0$. The following linear constraints are necessary and sufficient to

classify an arbitrary x as a matching:

$$\sum_{j \in W} x_{m,j} \leq 1 \quad \forall m \in M \tag{A}$$

$$\sum_{i \in M} x_{i,w} \leq 1 \quad \forall w \in W \tag{B}$$

$$x_{m,w} \geq 0 \quad \forall (m,w) \in M \times W \tag{C}$$

Constraints (A) and (B) together define the requirement that a man (or woman) can be either unmatched or matched to a single other person: no double matchings. Constraint (C) is satisfied by the definition of x, but explicitly listing it will complete the linear program.

If the nature of the incidence vector is unclear, imagine representing is as a matrix X, where each entry X_{ij} is a boolean value that determines whether man m_i is matched to woman w_j. Clearly to form a valid matching, we can have at most a single 1 in each row and column, or else there would be a man or woman who is matched to more than one partner. A perfect matching is achieved when the matrix X is a permutation matrix.

0.3.1 The Assignment Problem

Before describing the linear program associated with the marriage problem (MP), we can first do so for the simpler assignment problem (AP), which contains only one sided preferences. In this problem, imagine that the set M is a set of buyers, and W is a set of sellers. Define a *fractional matching* as a vector $x \in \mathbb{R}^{|M| \times |W|}$ that satisfies constraints (A-C). The linear program AP can then be written as

$$\text{Maximize} \sum_{(i,j) \in M \times W} \alpha_{ij} x_{ij}$$

$$\text{subject to constraints A-C}$$

where α_{ij} is the total value to both parties of a matching between i and j. We know from linear optimization theory that the optimal solution(s) x^* will lie at corners of the feasible region, which given these constraints will result in a solution in the form of the previously defined incidence vector, which takes values $x^* \in \{0,1\}^{|M| \times |W|}$.

Unfortunately, this linear program is not in the standard form of

$$\begin{aligned} \text{Maximize } & \mathbf{c}^T \mathbf{x} \\ \text{s.t. } & A\mathbf{x} \leq \mathbf{b} \\ & x_i \geq 0 \end{aligned}$$

from which we could easily write down the dual problem as:

$$\begin{aligned} \text{Minimize } & \mathbf{b}^T \mathbf{y} \\ \text{s.t. } & A^T \mathbf{y} \geq \mathbf{c} \\ & y_i \geq 0 \end{aligned}$$

Where we have switched from index notation to vector notation. How can we reform the constraints (A) and (B) to look like standard form? Start by referring to the incidence vector $\mathbf{x}$ by a single index, i.e. $x_{i,j} \to x_k$, where $k = (i-1)N + j$. We can pick out the vector $\mathbf{x}^i$, where $\mathbf{x}^i_j = x_{i,j}$ for a particular man m_i by multiplying $\mathbf{x}$ by the following matrix:

$$A = [\underbrace{\mathbf{e}_1, \mathbf{e}_1, ..., \mathbf{e}_1}_{N}, \underbrace{\mathbf{e}_2, \mathbf{e}_2, ..., \mathbf{e}_2}_{N}, ..., \underbrace{\mathbf{e}_N, \mathbf{e}_N, ..., \mathbf{e}_N}_{N},]$$

Where the basis vector $\mathbf{e}_i \in \mathbb{R}^N$ has all zeros except for the ith row, and $A \in \mathbb{R}^{N \times N^2}$. To be more explicit, the matrix A looks like:

$$A = \begin{bmatrix} 1 & 1 & ... & 1 & 0 & 0 & ... & 0 & 0 & ... \\ 0 & 0 & ... & 0 & 1 & 1 & ... & 1 & 0 & ... \\ 0 & 0 & ... & 0 & 0 & 0 & ... & 0 & 1 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where each block of 1's is N elements long.

We can then rewrite constraint (A) as:

$$A\mathbf{x} \leq \mathbf{1}_N$$

where the vector of ones $\mathbf{1}_N = [1, 1, ..., 1] \in \mathbb{R}^N$.

Similarly, we can rewrite constraint (B) using a matrix that picks out the woman vector $x_{i,w}$:

$$B = [\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N, \mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N, ...] = [\underbrace{I_N, I_N, ..., I_N}_{N}]$$

Where I_N is the $N \times N$ identity matrix. Now constraint (B) becomes:

$$B\mathbf{x} \leq \mathbf{1}_N$$

We can combine these into a single constraint by stacking the matrices:

$$\begin{bmatrix} A \\ B \end{bmatrix} \mathbf{x} = D\mathbf{x} \leq \mathbf{1}_{2N}$$

Now we can write down the dual of the assignment problem as:

$$\begin{aligned} \text{Minimize } & \mathbf{1}_{2N}^T \mathbf{y} \\ \text{s.t. } & D^T \mathbf{y} \geq \alpha \\ & y_i \geq 0 \end{aligned}$$

We can choose at this point to rewrite the vector $\mathbf{y}$ as

$$\mathbf{y} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

which then changes the dual problem statement to

$$\begin{aligned} \text{Minimize } & \mathbf{1}_N^T \mathbf{u} + \mathbf{1}_N^T \mathbf{v} \\ \text{s.t. } & A^T \mathbf{u} + B^T \mathbf{v} \geq \alpha \\ & u_i, v_j \geq 0 \end{aligned}$$

or, moving back to index notation,

$$\begin{aligned} \text{Minimize } & \sum_{i \in M} u_i + \sum_{j \in W} v_j \\ \text{s.t. } & u_i + v_j \geq \alpha_{ij} \quad \forall (i,j) \in M \times W \\ & u_i, v_j \geq 0 \end{aligned}$$

As discussed previously in class, the dual variables u_i and v_j can be interpreted as node labels in a bipartite graph.

0.3.2 The Marriage Problem

The marriage problem, having two-sided preferences, requires an additional constraint to avoid blocking pairs:

$$\sum_{j>_m w} x_{m,j} + \sum_{i>_w m} x_{i,w} + x_{m,w} \geq 1 \quad \forall (m,w) \in M \times W \tag{D}$$

where the notation $j >_m w \equiv \{j \in W : j >_m w\}$ and $i >_w m \equiv \{i \in M : i >_w m\}$, for example, $j >_m w$ is the set of all women that are more preferable to m than is w. Constraint (D) demands that at least one of three conditions are met:

- m is matched to a woman more desirable than w
- w is matched to a man more desirable than m
- m is matched to w

How does constraint (D) ensure there are no blocking pairs? Consider two matches (m, w') and (m', w), where $w >_m w'$ and $m >_w m'$, so that (m, w) is a blocking pair. Then the pair (m, w) violates constraint (D):

$$\sum_{j>_m w} x_{m,j} + \sum_{i>_w m} x_{i,w} + x_{m,w} = 0 + 0 + 0 = 0$$

because m is not matched to a woman more desirable than w, w is not matched to a man more desirable than m, and m is not matched to w. Therefore, $\exists$ a blocking pair $(m, w) \Rightarrow$ constraint (D) is violated, so

$$\text{Constraint (D) is satisfied } \Rightarrow \nexists \text{ any blocking pairs.}$$

The linear program for the marriage problem (MP) can then be written as

$$\text{Maximize} \sum_{(i,j) \in M \times W} x_{ij}$$

$$\text{subject to constraints A-D}$$

But once again, the constraint (D) is not in standard form. We can go through a similar exercise as before, creating a large single matrix that allows us to combine all constraints into a single constraint.

For instance, consider the first sum in constraint (D):

$$\sum_{j>_m w} x_{m,j}$$

For each man m_i, define a relative ranking indicator matrix $R^i \in \mathbb{R}^{N \times N}$ such that $R^i_{k,l} = 1 \Rightarrow w_k \geq_m w_l$, else $R^i_{k,l} = 0$. That is, $R^i_{k,l}$ indicates whether man m_i prefers woman w_k over w_l. Then form a block matrix $R = \text{diag}(R^1, R^2, ..., R^N) \in \mathbb{R}^{N^2 \times N^2}$, which allows us to write:

$$\sum_{j>_m w} x_{m,j} \geq z, \; \forall (m,w) \rightarrow R\mathbf{x} \geq \mathbf{z}$$

For brevity, and because the remainder of the reduction to standard form is rather tedious, we'll cut right to the chase at this point.

Note that whereas constraints (A) and (B) each introduced N decision variables into the dual assignment problem (which we called u_i and v_j), constraint (D) will introduce an *additional* N^2 decision variables! Further, since the inequality on constraint (D) is $\geq$, we can flip this and carry on by multiplying the inequality through by -1, which will have the effect of bringing in the new decision variables as subtractions in the dual marriage problem.

This results in the following dual problem, with a third decision variable γ introduced to represent the constraints from (D):

$$\begin{aligned}
\text{Minimize } & \sum_{i \in M} \alpha_i + \sum_{j \in W} \beta_j - \sum_{(i,j) \in M \times W} \gamma_{i,j} \\
\text{s.t. } & \alpha_m + \beta_w - \sum_{j <_m w} \gamma_{m,j} - \sum_{i \leq_w m} \gamma_{i,w} \geq 1 \quad \forall (m,w) \in M \times W \\
& u_i, v_j, \gamma_{i,j} \geq 0
\end{aligned}$$

Note that the third term in constraint (D) has been absorbed into the second summation in the dual problem constraint by replacing $<_w$ with $\leq_w$.

0.4 Example

We can work through the stable marriage problem in the linear programming setting by following the example, presented in the spirit of [2].

Consider an $N = 3$ instance of the stable marriage problem. Let the men's P_M and women's P_W preference lists be given by:

$$P_M = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad P_W = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

The constraints can be written exhaustively as follows:

$$x_{1,1} + x_{1,2} + x_{1,3} \leq 1 \tag{A1}$$
$$x_{2,1} + x_{2,2} + x_{2,3} \leq 1 \tag{A2}$$
$$x_{3,1} + x_{3,2} + x_{3,3} \leq 1 \tag{A3}$$
$$x_{1,1} + x_{2,1} + x_{3,1} \leq 1 \tag{B1}$$
$$x_{1,2} + x_{2,2} + x_{3,2} \leq 1 \tag{B2}$$
$$x_{1,3} + x_{2,3} + x_{3,3} \leq 1 \tag{B3}$$
$$(x_{1,3}) + (x_{3,1}) + x_{1,1} \geq 1 \tag{D[1,1]}$$
$$(x_{1,1} + x_{1,3}) + (x_{2,2} + x_{3,2}) + x_{1,2} \geq 1 \tag{D[1,2]}$$
$$(0) + (x_{2,3} + x_{3,3}) + x_{1,2} \geq 1 \tag{D[1,3]}$$
$$(0) + (x_{1,1} + x_{3,1}) + x_{2,1} \geq 1 \tag{D[2,1]}$$
$$(x_{2,1} + x_{2,3}) + (0) + x_{2,2} \geq 1 \tag{D[2,2]}$$
$$(x_{2,1}) + (x_{3,3}) + x_{2,3} \geq 1 \tag{D[2,3]}$$
$$(x_{3,2} + x_{3,3}) + (0) + x_{3,1} \geq 1 \tag{D[3,1]}$$
$$(x_{3,3}) + (x_{2,2}) + x_{3,2} \geq 1 \tag{D[3,2]}$$
$$(0) + (0) + x_{3,3} \geq 1 \tag{D[3,3]}$$

And subject to those, we wish to maximize the objective function:

$$\text{Maximize: } p = x_{1,1} + x_{1,2} + x_{1,3} + x_{2,1} + x_{2,2} + x_{2,3} + x_{3,1} + x_{3,2} + x_{3,3}$$

We can follow our nose through a brute-force solution to this problem by first noticing the rather convenient fact that, since man m_3 and woman w_3 both prefer each other to any others, they must be matched according to constraint (D[3,3])! That is, we know $x_{3,3} \geq 1$. Given the non-negativity constraint and constraints (A3) and (B3), we can also immediately identify that $x_{3,1} = x_{3,2} = x_{1,3} = x_{2,3} = 0$, and that $x_{3,3} = 1$. At this point our unused constraints now look like:

$$x_{1,1} + x_{1,2} \leq 1 \tag{A'1}$$
$$x_{2,1} + x_{2,2} \leq 1 \tag{A'2}$$
$$x_{1,1} + x_{2,1} \leq 1 \tag{B'1}$$
$$x_{1,2} + x_{2,2} \leq 1 \tag{B'2}$$
$$x_{1,1} \geq 1 \tag{D'[1,1]}$$
$$(x_{1,1}) + (x_{2,2}) + x_{1,2} \geq 1 \tag{D'[1,2]}$$
$$(0) + (1) + x_{1,2} \geq 1 \tag{D'[1,3]}$$
$$(0) + (x_{1,1}) + x_{2,1} \geq 1 \tag{D'[2,1]}$$
$$(x_{2,1}) + (0) + x_{2,2} \geq 1 \tag{D'[2,2]}$$
$$(x_{2,1}) + (1) \geq 1 \tag{D'[2,3]}$$
$$(1) + (0) \geq 1 \tag{D'[3,1]}$$
$$(0) + (x_{2,2}) \geq 1 \tag{D'[3,2]}$$

From here, everything quickly collapses to find $x_{1,1} = 1$, $x_{2,2} = 1$, and all others equal 0.

We are then left with an incidence vector $x =$[1 0 0 0 1 0 0 0 1], or a matching of $(1, 1)$, $(2, 2)$, and $(3, 3)$. It can be easily verified that this matching is stable, and in fact we know that it *must* be stable since it was constructed using the constraints that guarantee a stable matching.

From the other direction, we can examine the dual for this problem. The dual is not so easy to solve by inspection as the primal was for this contrived case, so here we only display the dual objective and constraints for completeness. We wish to minimize the dual objective function

$$\begin{aligned}\text{Minimize: } d = {} & \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 \\ & - \gamma_{1,1} - \gamma_{1,2} - \gamma_{1,3} - \gamma_{2,1} - \gamma_{2,2} - \gamma_{2,3} - \gamma_{3,1} - \gamma_{3,2} - \gamma_{3,3}\end{aligned}$$

subject to:

$$\begin{aligned}\alpha_1 + \beta_1 - (\gamma_{1,2}) - (\gamma_{1,1} + \gamma_{2,1}) &\geq 1 \\ \alpha_1 + \beta_2 - (0) - (\gamma_{1,2}) &\geq 1 \\ \alpha_1 + \beta_3 - (\gamma_{1,1} + \gamma_{1,2}) - (\gamma_{1,3}) &\geq 1 \\ \alpha_2 + \beta_1 - (\gamma_{2,2} + \gamma_{2,3}) - (\gamma_{2,1}) &\geq 1 \\ \alpha_2 + \beta_2 - (0) - (\gamma_{1,2} + \gamma_{2,2} + \gamma_{3,2}) &\geq 1 \\ \alpha_2 + \beta_3 - (\gamma_{2,2}) - (\gamma_{1,3} + \gamma_{2,3}) &\geq 1 \\ \alpha_3 + \beta_1 - (0) - (\gamma_{1,1} + \gamma_{2,1} + \gamma_{3,1}) &\geq 1 \\ \alpha_3 + \beta_2 - (\gamma_{3,1}) - (\gamma_{1,2} + \gamma_{3,2}) &\geq 1 \\ \alpha_3 + \beta_3 - (\gamma_{3,1} + \gamma_{3,2}) - (\gamma_{1,3} + \gamma_{2,3} + \gamma_{3,3}) &\geq 1\end{aligned}$$

References

[1] Alvin E. Roth, Uriel G. Rothblum, and John H. Vande Vate. Stable matchings, optimal assignments, and linear programming. *Math. Oper. Res.*, 18(4):803–828, 1993.

[2] V. J. Sethuraman and C.P. Teo. Linear programming brings marital bliss. *Mathematical Medley*, 25(2):89–91, 1998.

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Lecture 15: October 26

Lecturer: Vijay Garg *Scribe: Jose Carlos Martinez Garcia-Vaso*

15.1 Game Theory

15.1.1 Introduction

Everything in life is like a a game. We make choices for everything we do, and the consequences of those choices can reward or penalize us. For example, we might be faced with the choice to take Mo-Pac or i35 in our morning commute. Choosing the best route will make for a faster commute. These choices are not made in isolation, but they are often based on the action of other players in the game. Back to the morning commute example, we might choose to take Mo-Pac over i35 because less people are choosing that option, and there is less traffic. This idea is at the core of game theory.

John von Newman stated the field of game theory by studying the zero-sum games. Later on, game theory was generalized by John Nash, with his contributions on finite games. Nash was awarded the Nobel Prize for his contributions to game theory.

15.1.2 Ingredients of Game Theory

1. **Players**: $N = \{1, 2, \ldots, n\}$

   The players are the participants in the game.

2. **Actions/strategies**:

   Strategies are the options or choices the players can make during the game. We will assume there is a finite number of strategies that a player can use.

3. **Utility function**: $U_i = (S_1, S_2, \ldots, S_n)$

   The utility function quantifies the utility of the strategies to the players with respect to the other players choice of strategies. We assume that we know the utility of all the actions available to all the players, so that we know the utility of our actions depending on the other players' actions. This is necessary for the analysis.

15.1.3 The Two Player Game

Consider the following game. Two students are faced with the option of spending most of their time preparing for a presentation or studying for an exam. If a student chooses to spend most of his time studying for the exam he will get a grade of 92 in the exam. However, if he prepares for the presentation, he will get an 80 in the exam instead. Both students are paired together to work on the presentation, and their presentation grade depends on the work that both of them put into it. If both students choose to spend most of their time working in the presentation, they will both get a 100 in the presentation. If only one student spends

most of her time in the presentation, both students will get a 92. Finally, if none of the students chooses to work on the presentation, they will get an 84.

The utility function for the previously explained strategies can be summarized in bi-matrix (normal) form as shown in *Table 15.1*. In the utility function we consider the final total grade of the students, which is obtained from the average of the grades for the exam and the presentation.

Table 15.1: Presentation-Exam game utility function in bi-matrix (normal) form.

		Student 2	
		Presentation	**Exam**
Student 1	**Presentation**	90, 90	86, 92
	Exam	92, 86	88, 88

Below are the all possible combination of strategies the students may pick:

- **(Presentation, Presentation)**: Both students work on their presentations, then, they both get a 100 in the presentation and an 80 in the exam. Their final grade is 90.
- **(Presentation, Exam) / (Exam, Presentation)**: Both students get a 92 in the presentation, because one of them worked on the presentation. The student that worked on the exam will get a 92 in the exam, and the other an 80. The final grades are 86 for the student that worked in the presentation, and 92 for the student that worked in the exam.
- **(Exam, Exam)**: Both students worked on the exam, then, they both get a 92 in the exam and an 84 in the presentation. Their final grade is 88.

For the analysis of the game, assume the following:

1. **Rational selfish players**: The players will use the strategy that provides them with the highest utility (payoff), regardless of how their strategies affect the other players.
2. **Common knowledge of the game structure**: Every player knows the utility function. That is they know how every action they take will compare to every possible action by the other players. Moreover, players know the other players are also rational and selfish.

The first thing to look when analyzing a game is if there exist a dominant strategy.

Definition 15.1 *A **dominant strategy** is a strategy that is better (the utility for that strategy is greater or equal) than all the other strategies regardless of the strategies the other players choose.*

In the previous example, Exam is a dominant strategy. The utility for a player choosing the Exam strategy is greater or equal than any other strategy the other player can choose. Moreover, if a strategy yields a utility strictly greater than any strategy the other player can choose, that strategy is strictly dominant. In this case, Exam is also strictly dominant.

Definition 15.2 *A **strictly dominant strategy** is a strategy that is strictly better (the utility for that strategy is strictly greater) than all the other strategies regardless of the strategies the other players choose.*

Different utility functions yield different dominant strategies. For example, let's alter the previous game's utility function to that shown in *Table 15.2*. In this case, Presentation will be the strictly dominant strategy.

Table 15.2: Modified Presentation-Exam game utility function.

		Student 2	
		Presentation	**Exam**
Student 1	**Presentation**	98, 98	94, 96
	Exam	96, 94	92, 92

15.1.4 The Prisoner's Dilemma

Consider the following scenario. Two subjects are arrested and imprisoned for the same crime. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the prisoners on the principal charge, but they have enough to convict both on a lesser charge, such as resisting arrest. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to confess that the other committed the crime, or to not confess and cooperate with the other prisoner by remaining silent. The offer gives the utility function in *Table 15.3*.

Table 15.3: Prisoner's dilemma utility function.

		Prisoner 2	
		Not Confess	**Confess**
Prisoner 1	**Not Confess**	-1, -1	-10, 0
	Confess	0, -10	-4, -4

The prisoner's dilemma game has the strictly dominant strategy of Confess for both players. Therefore, we expect both prisoners to confess and serve 4 years in prison. However, they would be better off if both of them choose not to confess. In that case they would only serve 1 year. A dominant strategy is not necessarily socially optimal.

EE 382V: Social Computing **Fall 2018**

Lecture 16: October 26

Lecturer: Vijay Garg *Scribe: Michael Romer*

16.1 Introduction

For purposes of discussion, the following will be described from the perspective of two-player games. However, many of these concepts generalize to n-player games as well.

To put the following in the context of a concrete game, we first recap the two-player game where two students partnered together can each choose to either prepare for their joint presentation (strategy P) or prepare for their exam (strategy E). The payoffs of each student are given by the following bimatrix:

		Player 2	
		P	E
Player 1	P	$(98, 98)$	$(94, 96)$
	E	$(96, 94)$	$(92, 92)$

16.2 Best Response

First we define the notion of a *best response*, which is informally the best strategy a player can pick assuming they already know the strategy picked by the other player.

Definition 16.1 (Best Response) *Let $\mu_i(s,t)$ be the payoff for player i for a strategy profile (s,t). A strategy s for player 1 is a* ***best response (BR)*** *to a strategy t for player 2 if $\mu_1(s,t) \geq \mu_1(s',t)\ \forall s' \neq s$. If $\mu_1(s,t) > \mu_1(s',t)\ \forall s' \neq s$, then s is a* ***strict best response*** *to t.*

In the game above, if either student knows their partner plans to prepare for the exam, then that student's best response would be to prepare for the presentation since their payoff would be 94 versus 92.

16.3 Equilibrium

Using the idea of best response, we are able to find certain equilibrium states within a given game. One common form of equilibrium named after the mathematician John Nash is the *Nash equilibrium*.

Definition 16.2 (Nash Equilibrium) *A strategy profile (s,t) is a* ***Nash equilibrium*** *if strategy s is a best response to strategy t and if strategy t is a best response to strategy s.*

In a Nash equilibrium, neither player has any motivation to deviate from their selected strategy. When extended to n-player games, a strategy profile $s = (s_1, s_2, \ldots, s_n)$ is a Nash equilibrium if every s_i is a best response to s_j for all $j \neq i$.

16.3.1 Finding Nash Equilibria

Consider the same game as described in section 16.1 except with the payoffs now given by the following bimatrix:

		Player 2	
		P	E
Player 1	P	$(90, 90)$	$(86, 92)$
	E	$(92, 86)$	$(88, 88)$

To determine which of the strategy profiles is a Nash equilibrium, consider each profile and determine if any player can increase their payoff by changing only their own strategy.

- For strategy profile (P, P), player 1 can increase their payoff from 90 to 92 by changing their strategy from P to E. The same is similarly true for player 2, thus (P, P) is *not* a Nash equilibrium.
- For strategy profile (P, E), player 1 can increase their payoff from 86 to 88 by changing their strategy from P to E. Similarly for strategy profile (E, P), player 2 can increase their payoff from 86 to 88 by changing their strategy from P to E. Thus, neither (P, E) nor (E, P) is a Nash equilibrium.
- For strategy profile (E, E), neither player 1 nor player 2 can increase their payoff by changing their strategy alone. Therefore, (E, E) is the *only* Nash equilibrium in this game.

16.3.2 Trivial Game

It is possible for all strategy profiles in a game to be Nash equilibria. One such example is the trivial game in which all payoffs are the same regardless of the action any player takes.

		Player 2	
		A	B
Player 1	A	$(1, 1)$	$(1, 1)$
	B	$(1, 1)$	$(1, 1)$

16.4 Mixed Strategies

In the games considered up to this point, each player has selected as their strategy one action from a finite set of possible discrete actions. These strategies are known as *pure strategies*. In some games, using a pure strategy is the worst possible strategy to use.

16.4.1 Rock-Paper-Scissors

Consider the game Rock-Paper-Scissors with the following payoff matrix:

		Player 2		
		R	P	S
	R	$(0,0)$	$(-1,1)$	$(1,-1)$
Player 1	P	$(1,-1)$	$(0,0)$	$(-1,1)$
	S	$(-1,1)$	$(1,-1)$	$(0,0)$

Suppose the game was setup such that, e.g., player 1 had to act first followed by player 2. If player 1 used a pure strategy, such as always picking rock, then player 2 would be able to win every time. In general, it is not possible for a player to play a *deterministic pure strategy* such as this and *not* have player order influence the game's outcome.

By considering only pure strategies, it is also possible to show that there exist no Nash equilibria in the Rock-Paper-Scissors game above. However, by allowing for randomized strategies where each player can choose an action probabilistically, it is possible for players to effectively act simultaneously. When players pick these so-called *mixed strategies*, the game's outcome becomes independent of player order.

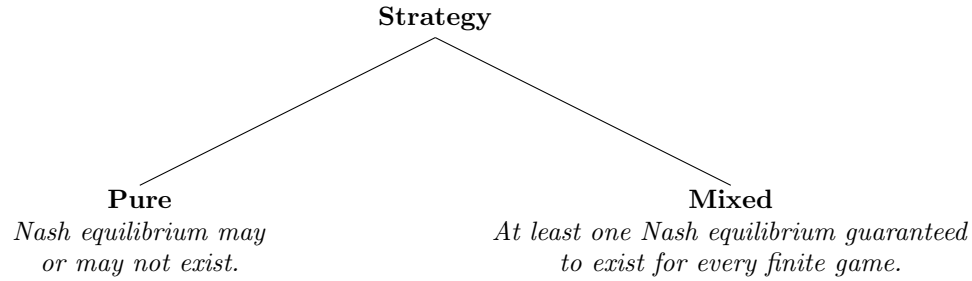
Definition 16.3 (Mixed Strategy) *Let $S = \{s_1, s_2, \ldots, s_n\}$ be the finite set of pure strategies available to each player for a given game. A probability vector $\mathbf{s} = (p_{s_1}, p_{s_2}, \ldots, p_{s_n})$ is a* ***mixed strategy*** *for a player where p_{s_i} is the probability of that player selecting the pure strategy s_i.*

Mixed strategies allow a player to pick their strategy at random rather than selecting a single pure strategy. In the game above, one possible mixed strategy is to select either rock, paper, or scissors with equal probability. The mixed strategy in this case corresponds with the probability vector $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. The Nash equilibrium for the Rock-Paper-Scissors game is the strategy profile $\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$ where both players pick with equal likelihood either rock, paper, or scissors.

16.4.2 Existence of Nash Equilibrium

In his 1950 paper, John Nash showed that for every *finite game*, that is any game with a finite number of players and finite number of strategies, at least one mixed strategy is guaranteed to exist that is also a Nash equilibrium. On the other hand, as was demonstrated in the Rock-Paper-Scissors game above, a Nash equilibrium is *not* guaranteed to exist when only considering the set of possible pure strategies.

The following tree summarizes the two main classes of strategies discussed so far along with their respective results:



16.4.3 Payoffs in Mixed Games

To analyze any game, you need the bimatrix of payoffs. In a game with mixed strategies, the bimatrix lists the *expected payoffs* rather than the absolute payoffs for each player. In this case, the mixed strategy profile corresponding to a Nash equilibrium will maximize each player's expected payoff given the strategies selected by all other players.

16.5 A Tale of Two Firms

In each of the following two-player games that follow, each player may only play a pure strategy. Mixed strategies are not allowed.

16.5.1 Manufacturing Items

Suppose two firms, Firm 1 and Firm 2, can employ only one of the following pure strategies:

- Make low-priced items (strategy L)
- Make high-priced items (strategy H)

Additionally, we know the following information about the firm's target markets:

- 60% of the population will purchase exclusively low-priced items.
- If Firm 1 and Firm 2 compete with each other, then Firm 1 will win 80% of the target market.
- If the two firms don't compete, they each get 100% of their respective target markets.

This results in the following bimatrix of payoffs:

		Firm 2	
		L	H
Firm 1	L	$(.48, .12)$	$(.60, .40)$
	H	$(.40, .60)$	$(.32, .08)$

NOTE: This is an *asymmetric* game.

16.5.1.1 Analysis

Given the information above, the question is what will each firm do? When analyzing player strategies in any game, ask yourself the following questions:

- Are there any strict dominant strategies?
- Are there any Nash equilibria?

If each player has a strict dominant strategy, then the optimal strategy profile that yields the greatest payoff for each player is easily determined. However, in this game only Firm 1 has a strict dominant strategy, which is to make low-priced items.

Even though Firm 2 has no strict dominant strategy of its own, it is able to see from the payoff matrix the strict dominant strategy that Firm 1 will follow. Given Firm 1's strategy will be to produce low-priced items, Firm 2's best response will be to produce high-priced items as this will yield a payoff of .40 as opposed to the .12 they would receive if they competed with Firm 1 and produced low-priced items.

Therefore, in this game, Firm 1 will produce low-priced items, and Firm 2 will produce high-priced items.

16.5.2 Servicing Clients

Now instead of manufacturing items, suppose Firm 1 and Firm 2 can each offer services to one of the three following clients: A, B, or C. Additionally, we know the following information:

- If both firms pursue the same client, they split the business from that client evenly in half. The amount of business available from a given client depends on the size of that client.
- Firm 1 is too small to get business on its own from any client. If Firm 1 pursues a different client from Firm 2, then Firm 1's payoff will be zero.
  - This means that in order to have a nonzero payoff, Firm 1 needs to coordinate with Firm 2.
- Firm 2 can get the full business of either client B or client C on its own; however, client A is so big that if only one firm approaches it, then that firm's payoff will be zero.

The following bimatrix describes the above scenario along with giving the payoffs for each firm:

		Firm 2		
		A	B	C
	A	$(4,4)$	$(0,2)$	$(0,2)$
Firm 1	B	$(0,0)$	$(1,1)$	$(0,2)$
	C	$(0,0)$	$(0,2)$	$(1,1)$

16.5.2.1 Analysis

By looking at the bimatrix of payoffs, we can readily tell that no player in this game has a strict dominant strategy.

If no player has a strict dominant strategy, we then wish to identify any of the Nash equilibria in the game. In the absence of a strict dominant strategy, strategic players will pick strategies that form a Nash equilibrium because that would be the point at which none of the players have a motivation to deviate from their selected strategy.

In the game above, one can show that a Nash equilibrium exists at the strategy profile (A, A). This also happens to be the game's *social optimum*, which is the strategy profile that maximizes the *social value*, i.e., the sum of the payoffs for that profile. For this particular game, the optimal social value is 8.

EE 382V: Social Computing **Fall 2018**

Lecture 17: October 26

Lecturer: Vijay Garg *Scribe: Jeffrey D Nelson*

17.1 Games with Multiple Nash Equilibria

There are certain games in which more than one Nash Equilibrium exists. The following figure depicts a game in which you and a partner are preparing slides for a presentation. Each of you can choose to prepare the slides with either Microsoft PowerPoint or Apple Keynote.

	ppt	keynote
ppt	1,1	0,0
keynote	0,0	1,1

Table 17.1: Combining Presentation Slides

If you choose a different presentation tool than your partner, you will both get a payoff of 0, as you will be unable to combine your slides. Your best response is to choose the same tool as your partner, regardless of the tool chosen. For this reason, (ppt, ppt) and (keynote, keynote) are both Nash Equilibrium.

17.1.1 Battle Of The Sexes

Another game with multiple Nash Equilibria is the Battle of the Sexes, where you and your significant other go to see a movie together and must choose between Action and Romance.

	romance	action
romance	2,1	0,0
action	0,0	1,2

Table 17.2: Battle of the Sexes

In this game, both (romance, romance) and (action, action) are Nash Equilibrium, but one player will receive a higher payoff.

17.2 Schelling's Focal Point

For games in which there are multiple Nash Equilibria, there may be one NE that is more likely to be chosen because of information available outside of the context of the game. For example, in the game combining presentation slides, (ppt, ppt) may be more likely due to PowerPoint being more popular than Keynote. We call this point the **Schelling Focal Point** and define it as the point most likely to be chosen in the absence of communication.

17.3 Games with No Nash Equilibrium

Consider the following game, in which you and a friend flip two pennies. If the pennies match, you receive a payoff of 1 and your friend receives a payoff of -1. Otherwise, you receive a payoff of -1 and your friend receives a payoff of 1.

	heads	tails
heads	1,-1	-1,1
tails	-1,1	1,-1

Table 17.3: Matching Pennies

This game is also called a **Zero-Sum Game** as the overall payoff is always zero. It should be clear that there is no Nash Equilibrium, as there is no pair of strategies that are best responses to each other.

17.3.1 Mixed Strategy for Matching Pennies

As we learned in the previous lecture, there exists at least one Nash Equilibrium in a finite game where each player is using a mixed strategy. To derive the NE for Matching Pennies, we define the following:

- Player2 chooses heads with probability q and tails with probability $1 - q$.
- Player1 chooses heads with probability p and tails with probability $1 - p$.

Payoffs for player1:

- Payoff for choosing heads:

$$(q)(1) + (1 - q)(-1) \rightarrow 2q - 1 \tag{17.1}$$

- Payoff for choosing tails:

$$(q)(-1) + (1 - q)(1) \rightarrow 1 - 2q \tag{17.2}$$

Player1 must be "indifferent" about his responses, so we solve:

$$1 - 2q = 2q - 1 \rightarrow q = \frac{1}{2} \tag{17.3}$$

In solving for player2's expected payoff w/ respect to p, we see that $p = 1/2$ from the symmetric principle. Therefore, the Nash Equilibrium for Matching Pennies is:

$$[(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})] \tag{17.4}$$

In other words, player1 chooses heads with probability 0.5 and tails with probability 0.5, as does player2.

17.3.2 Football Plays

Consider the following game of football. An offense can choose to pass or run the ball, while a defense can choose to defend against the pass or the run. The following matrix defines the payoff for each strategy.

defense

offense

	defend pass	defend run
pass	0,0	10,-10
run	5,-5	0,0

Table 17.4: Football Plays

Our goal is to compute a Nash Equilibrium vector for each team's mixed strategy, and so we define the following:

- Offense passes with probability p
- Defense defends the pass with probability q

Expected payoff for offense when choosing to pass:

$$(q)(0) + (1-q)(10) \rightarrow 10 - 10q \tag{17.5}$$

Expected payoff for offense when choosing to run:

$$(q)(5) + (1-q)(-5) \rightarrow 5q \tag{17.6}$$

By the indifference principle:

$$10 - 10q = 5q \rightarrow q = 2/3 \tag{17.7}$$

By the symmetric principle, we could solve for the defense's expected payoff and get $p = \frac{1}{3}$. Therefore, the Nash Equilibrium vector for this game is:

$$[(pass = \frac{1}{3}, run = \frac{2}{3}), (defendPass = \frac{2}{3}, defendRun = \frac{1}{3})] \tag{17.8}$$

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Lecture 18: October 27

Lecturer: Vijay Garg *Scribe: Lydia Guarino*

18.1 Pareto Dominant Strategies

In a multi-participant game, one strategy profile is *Pareto dominant* compared to another if it leaves at least one participant better off and no-one worse off.

18.1.1 Definition

Strategy profile $(s_1, s_2, ...s_n)$ *Pareto dominates* strategy profile $(s_1^{'}, s_2^{'}, ...s_n^{'})$ iff:

$\forall i \in N : U_i(s_1, s_2, ...s_n) \geq U_i(s_1^{'}, s_2^{'}, ...s_n^{'})$

and

$\exists j \in N : U_j(s_1, s_2, ...s_n) > U_j(s_1^{'}, s_2^{'}, ...s_n^{'})$

18.1.2 Example

In the game represented below, strategy pair (A_1, A_1) Pareto dominates (A_1, A_2), because both players receive a higher payoff. (A_2, A_1) Pareto dominates (A_2, A_2) because Player 2 receives the same payoff as in (A_2, A_2), but Player 1 receives a higher payoff. (A_1, A_2) Pareto dominates (A_2, A_2) because Player 1 receives the same payoff, but Player 2 receives a higher payoff. (A_2, A_2) is Pareto dominated by all other strategies.

x	A_1	A_2
A_1	3, 5	2, 4
A_2	7, 2	2, 2

18.2 Pareto Optimal Strategies

Strategy profile $(s_1, s_2, ...s_n)$ is *Pareto optimal* if there does not exist any strategy profile that *Pareto dominates* $(s_1, s_2, ...s_n)$.

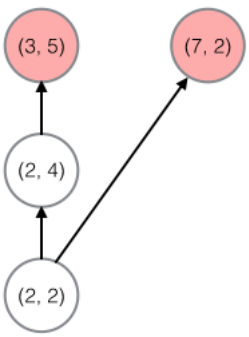
18.2.1 Example

In our same game, strategy pairs (A_1, A_1) and (A_2, A_1) are Pareto optimal because no other pair Pareto dominates them.

x	A_1	A_2
A_1	3, 5	2, 4
A_2	7, 2	2, 2

18.2.2 Poset example

This same game can be represented as a poset, where it is immediately evident that these strategy pairs represent the maximal elements in the poset. Note that representative payoffs (3, 5) and (7, 2) are incomparable, which allows this game to have multiple Pareto optimal strategy profiles.



18.3 Pareto Optimality, Nash Equilibrium and Strictly Dominant Strategies

Some interesting relationships to note for Pareto optimality are that strategy profiles that are Pareto optimal, may not be Nash Equilibrium or Strictly Dominant Strategies. In our example game of the Prisoner's Dilemma, you can see that the Nash Equilibrium and Strictly Dominant Strategy are for both suspects to confess, however, all strategy profiles *except* a double confession are considered Pareto optimal.

x	don't confess	confess
don't confess	-1, -1	-10, 0
confess	0, -10	-4, -4

18.4 Social Optimality

Strategy profile $(s_1, s_2, ...s_n)$ is *Socially optimal* if the sum of the payoffs in the strategy are maximized for the choice.

Lemma 18.1 *Social optimality implies Pareto optimality.*

x	A_1	A_2
A_1	$3 + 5 = 8$	$2 + 4 = 6$
A_2	$7 + 2 = 9$	$2 + 2 = 4$

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Lecture 19: October 27

Lecturer: Vijay Garg *Scribe: Marina Thomas*

19.1 Auctions

Auctions have been part of the history of world economy for a very long time. Today, this kind of economic activity is very prominent even in the internet world. For example, various search companies like Google,Yahoo! and Microsoft uses the auction format to sell slots for sponsored content. The very high demand for these slots have caused it to be priced very high. In this lecture we will study the behavior of buyers and sellers during auctions.

19.1.1 Types of Auctions

A primary concept when modelling auctions is that each bidder has an intrinsic value for the item being auctioned i.e a price up to which he/she is willing to pay for the the item, but not any higher. This price is also referred to as bidders' true value.

Consider the case of a seller auctioning a painting to buyers. Say that the seller has a valuation for the painting as, 100,000 and consider buyers' valuation in dollars as below:

- 200,000
- 250,000
- 98,000
- 300,000

In the above example as the seller's valuation is $100,000 and the maximum valuation by a buyer/bidder is $300,000, we say that there is a surplus of $200,000 that can be generated by the sale of this item, that will go from the buyer to the seller.

Knowledge of the system:

Consider a system in which the seller and buyers know each others' true values.

Case 1: Seller gets to decide the price/commitment mechanism.
If the seller knows the above declared true values assigned to the item by the bidders, and if he could set the price, he will simply set it to just below $300,000, and declare that he will not accept any lower price. The buyer with value $300,000 will buy the item, and the full value of the surplus, i.e. $ 200,000 will go to the seller. In this case, the seller has no need for an auction: he gets as much as he could reasonably expect just by announcing the right price.

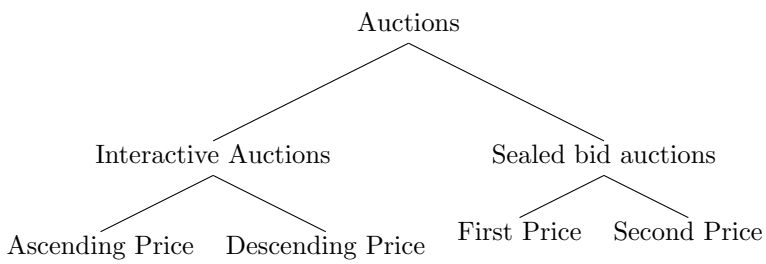
Case 2: Bidder to decide the price/commitment mechanism.
The bidder with true valuation $ 300,000 could announce that she is willing to purchase the item for a

price just above the larger of $ 100,000 and the values held by all other buyers(i.e $ 250,000). With this announcement, the seller would still be willing to sell since the price would be above $ 100,000. In this case seller gets $150,000 surplus and $ 50,000 of the surplus will go to the buyer. As with the seller's commitment, this commitment by the buyer also requires knowledge of everyone else's true values.

Is it possible to develop an algorithm where each player need not reveal their valuation?

A mechanism for that could be auctions.

Most of the revenue for Google comes from selling the slots through auctions. Below tree shows the different kinds of auctions we will explore as part of this lecture.



Ascending price auctions are also called English auctions. Similarly descending price auctions are also called Dutch auctions.

For ascending price auctions, the auction starts at a low value and is incremented in small deltas. As the price increases, bidders drop out when the amount crosses their true value. Finally the bidder with the second highest bid also will drop out when the amount is greater than his true value and only one bidder will remain, the one with the highest value bid and he will win the auction at the currently incremented price.

For descending price auctions, the bid starts at a very high value, say 1 million and the amount is brought down in small deltas. When the amount reaches the true value of the highest bidder, he wins the auction and will pay his true value.

In first price auctions, the bidder with the highest bid wins the auction and will pay his true value.

In second price auctions, the the highest bidder will win the object and will pay the value of the second highest bid. These auctions are also called Vickery auctions.

For now we can assume that there are no ties.

For sealed bid auctions, the participants reveal the valuation to a central agency. There is only one chance for each bidder and the winner is the one with highest price. Based on the format of the auction the bidder either pays second price or first price.

For the previously mentioned example, consider the scenario where buyers lie for second price auctions:

- 200,000 - consider bidder lied and put value as 275,000 - Scenario (a)
- 250,000 - consider bidder lied and put value as 301,000 - Scenario (b)
- 98,000
- 300,000 - consider bidder lied and put value as 248,000 - Scenario (c)

The impact of the above actions

- (a) - Bidder lied and added 275,000 as value. In this case, the highest bidder will now have to pay 275,000 and so his surplus will be reduced to $25,000. For the seller, this will be an additional gain of $25,000. Bidder A did not benefit by lying. His payoff is still zero and hence it was not beneficial to lie.
- (b) - Bidder B lied and added 301,000 as the value. The highest bid now belongs to this bidder, and he will win the auctioned item. There will be a loss of 50,000 for this bidder. Hence lying was disadvantageous for the bidder. As in the previous case this will be an advantage for the seller.
- (c) The bidder lied and put value as 248,000. The bidder will loose the bid and will loose $52,000 as he his true value was $300,000. Lying has become disadvantageous to the bidder in this case too.

Based on the above, it can be inferred that for second price auctions, dominant strategy is to tell the truth. Here DSIC or Dominant Strategy Incentive Compatible Mechanism is being truthful

Another example for DSIC is stable matching. For example, suppose that a man(m) lied or was careless while creating the ordered list of preferences of women and chose only the first 2 preferences(w1,w2) correctly hoping that he will be assigned to one of his first 2 preferences. If there are other men(m1,m2) who also prefers women(w1,w2) and if w1,w2 also prefers m1 and m2 to m, m will not be paired to either of them. Since m lied and/or was careless and did not have the rest of his preferences in their true order, he might end up being paired with a least preferred partner. Had he been truthful, he could have got a better choice.

When we design algorithm/mechanism for auctions, it will be nice if it has nice properties like DSIC and social welfare etc.

Unlike second price auctions, first price auction does not have a dominant strategy. The seller starts the bid at a very high value and no bidder says anything until someone actually accepts the bid. Bidders learn nothing during the auction other than that no one has yet accepted the bid. Each bidder i, has a first price pi at which she'll be willing to break the silence and accept the item at price pi. The item goes to the bidder with the highest bid value; and this bidder pays the value of her bid in exchange for the item.

Why might a seller opt for second price auction? For first price auctions, no one might turn up or, people might under value the item.

eBay uses second price auction.

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Lecture 21: October 27

Lecturer: Vijay Garg — *Scribe: Robert Pate*

21.1 Background on Lattice Linear Predicates

We proved some things about stable marriage, such as stable matching always exists, but with Lattice Linear Predicates we can prove it more easily and apply the same approach to the biggest problems such as housing allocation, market clearing, etc.

We had algorithms already for all of these problems but this gives us extra insight into them.

21.2 Consistent Global State

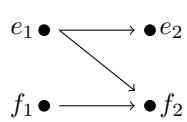
Definition 21.1 *Lattice*

A poset where a join and meet exist for all pairs of elements

Definition 21.2 *Distributive Lattice*

Lattice where joins and meets distribute over each other.

For example we have an example poset below where there are 4 elements, the events are proposals, and the orderings indicate the order the proposals were done.

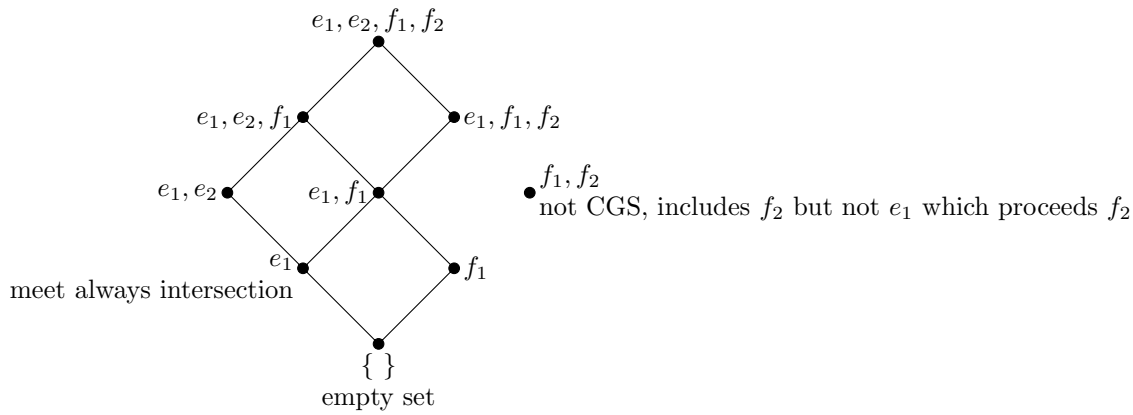


From this poset we define a 2nd poset $X(e_1, e_2, f_1, f_2)$ which has the same 4 elements. There are 16 possible subsets, but we don't want all of them. We want just the "consistent global states." This is also used in distributive systems by treating this as the order of events in a system.

Definition 21.3 *Consistent Global State*

G, a subset of X, is a CGS if
$\forall\ e, f \in X\ :\ (f \in G) \wedge (e < f) \Rightarrow e \in G$

Graph of X drawn upward:



We started with the smallest at the bottom which is always the empty set, and draw each of the consistent global states. This also draws a natural ordering relationship between them because our $<$ relation is defined by being a subset of.

Claim 21.4 *A set of CGS is a distributive lattice*

Proof: *Definitions:*
Elements: all are CGS due to our claim
Lattice: our relation must be poset ("subset of" is reflexive, anti-symmetric, and transitive)
Distributive: must be closed under meets and joins (a meet of multiple CGS is an intersection of them)

So we must show:
G *is a CGS* $\wedge$ H *is a CGS* $\Rightarrow G \cap H$ *is a CGS*

Meet is greatest lower bound. $G \cap H$ *is lower than* G *because any intersection is smaller. It's also the greatest lower bound because we can have elements lower than* $G \cap H$.

1. *Pick any two events where:*
   $e, f \in X$
   $f \in G \cap H$
   $e < f$

Now we prove $e \in G \cap H$*:*

2. $f \in G \cap H$ *implies* $(f \in G) \wedge (f \in H)$.

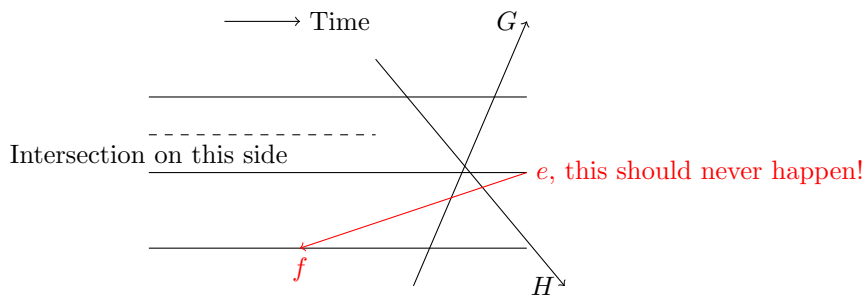
3. $(e < f) \wedge (G$ *is CGS)* $\Rightarrow e \in G$.

4. $(f \in H) \wedge (e < f) \Rightarrow e \in H$

5. $(e \in H) \wedge (e \in G) \Rightarrow e \in G \cap H$

■

21.2.1 Drawing a poset on a time line



There cannot be arrows going backwards (such as e, f as pictured above) in a CGS since it would no longer be consistent. Consistent means for every included event, all prior events are also included. In the above picture, if we have $e < f$ but e is not in G nor H, they would not be consistent.

21.2.2 Unions

We've shown this works for meets, a similar argument can be made to show it's also closed under joins where the relation is unions instead of intersections.

21.3 Lattice Linear Predicates

Back to our example, these events could be proposals from men to women or proposals from buyers to sellers. And the subsets tells us what has happened so far and tells us the state of the system.

It depends on the problem you're looking at, but in the example of the Stable Matching Problem, a state is "good" if the current state of proposals forms a stable matching.

Our goal would be to find the states that are good for us. We'll define a boolean predicate that's much easier to analyze called a Lattice Linear Predicate.

Definition 21.5 *Lattice Linear Predicate*

Given a distributive lattice L and given a boolean predicate B, B is a Lattice Linear Predicate if:
$\forall$ *CGSs* G*:* $\neg B(G) \Rightarrow \exists$ *i:forbidden*(G, i, B)

LLP Example

If you're happy, you're done.

If not, there must be some process, such as a man in SMP, that if it doesn't move then you cannot reach a good state (such as a stable marriage). If we can find him, then we can advance him to proceed toward stable marriage.

Back to the lattice, start with the lowest set (always empty set) then proceed to e_1 or f_1. The problem is that the lattice could be huge! If some predicate is LLP it saves us from having to explore the entire lattice. It lets us know a given combination is bad so we can throw it away and continue. Any point that we find is forbidden we know all older points are also forbidden since we're dealing with Consistent Global States. For the same reason, and any future states where we haven't advanced the problem process are also forbidden.

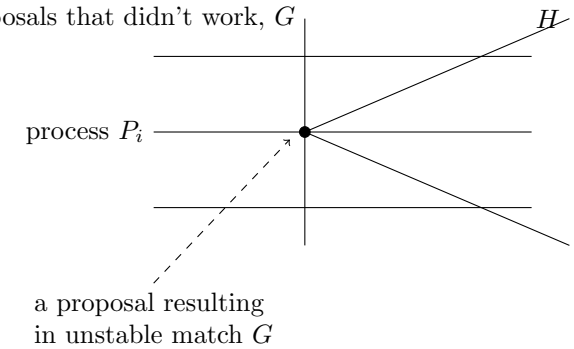
Definition 21.6 *Berkoff's Theorem*

In any CGS, if my predicate is not true, unless I advance, it can never be true.

Definition 21.7 *Forbidden*

Given any distributive lattice L and predicate B
forbidden $(G, i, B) \equiv$
$\forall H \in L : G \leq H :$
$(G[i] = H[i]) \Rightarrow \neg B(H)$

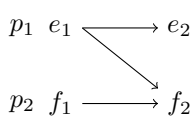
One set of proposals that didn't work, G



Consider the point pictured above. Think of it as a proposal that results in an unstable matching G. There are at least one of these such that if we continue around it we'll never succeed. In any state in the future from here (such as H) it will not be good because of this blocking pair.

This is valuable in helping us prune our search space to speed up our algorithms, if we can find it.

Simple Example Forbidden Process:



Given a predicate $B(G) \equiv$ "any state which contains at least one event from p_2." Unless we execute one event from p_2, we'll never make the predicate true. So p_2 is the forbidden process, and $\{\}, \{e_1\}, \{e_1, e_2\}$ are all bad states.

Claim 21.8 *B is a lattice linear if and only if B is "closed under meets."*
B is true in $G \wedge B$ is true in $H \Rightarrow B(G \cap H)$.

If true in 2 places, find meet. If false then not lattice linear. So if not true in e_1, should be at least 1 dimension that if I don't progress on it I can never make it true. So if I can advance something else to make B true then e_1 is not forbidden.

21.4 Lattice Linear Algorithm

21.4.1 Recap

1. Defined notion of predicate on a lattice

2. Defined which predicates are good
3. Defined closure of meets

Some predicates are nice from both meet and join perspectives, such as SMP, but not all. Payoff: if algorithm is lattice linear then it's efficient

21.4.2 LLP Algorithm to find the minimum feasible vector

- One variable: where are you in the lattice, as a global vector G
- init G as the least CGS in the lattice
- return G as the least CGS that is feasible, e.g. minimum market clearing price
- B feasible predicate: an assignment is feasible if for example it's a stable marriage or market clearing price
- Set an upper bound T that is the top most element, such as the number of women. Top is the end, bottom is the start.
- Note this is a generic algorithm, but "forbidden" will differ for each type of problem
- "Successor" is defined in respect to chain in poset. Go to next event on the same process.
- Never define the whole lattice, it's huge. break poset into chains (however you want) and call them processes, so long as lattice linear.
- Max time: total count of events $O(n^2)$

```
 vector function getLeastFeasible (T: vector, B: predicate)
   var G: the least CGS in the lattice
   while (∃ j : forbidden (G, j, B) do
        for all j in forbidden(G, j, B):
            if (G[j] = T[j]) then return null // reached the top, no feasible vector
            else G[j] := successor of G_j
end while
return G
```

$\forall j$: forbidden (G, j, B) is equivalent to $\neg B(G)$ because B is lattice linear. If this is not true it implies there must exist something that is forbidden. So if your predicate is not true, find one that is forbidden and advance that.

21.5 Proving lattice-linearity of predicates

Lemma 21.9 *Lattice Linearity has the property:*
if B_1 is LL $\wedge$ B_2 is LL $\Rightarrow$ $B_1 \wedge B_2$ is LL

Proof:

1. Pick any global state and if it's not true here, we can find a forbidden

2. Find the forbidden: if $B_1 \wedge B_2$ is not true, then either B_1 is not true or B_2 is not true.
3. Say B_1 is not true, and we know B_1 is LL then there exists at least one place we can advance to make it true
4. You'll find at least one process on which you must advance

■

21.6 Conclusion

Goal for next session is be able to show:

- SMP predicate is lattice linear
- Market clearing price is lattice linear
- Housing allocation problem is lattice linear
- And that the same algorithm works for all 3

EE 382V: Social Computing **Fall 2018**

Lecture 7, Session 1: November 9

Lecturer: Vijay Garg *Scribe: Joshua Musick*

7.1 Voting

Voting is the aggregation of beliefs / rankings. Examples include gathering Concensus, or Search Engines.

If we consider a group of people on an island, and they all vote given two choices, it's easy to determine what is preferred by the majority. But, when you add a third choice, it makes things complicated. In particular if Alice gives the following preference for choices A, B, and C.

$$\text{We have a Problem!} \begin{cases} A > B \\ B > C \\ C > A \end{cases}$$

Definition 7.1 (Plurality) *When chosing order, we want the following properties to produce a ranking.*

1. *Transitivity* - $\forall x, y, z : (x > y) \wedge (y > z) \implies (x > z)$
2. *Completeness* - $\forall x, y : (x > y) \vee (y > x)$

7.1.1 Sample Voting Ballot with Three Choices

Each voter is represented by V_i, and the columns represent their preferences with their first choice on top, second below their first etc.

V_1	V_2	V_3	Final Ranking
A	A	A	A
B	B	C	B
C	C	B	C

Table 7.1: Simple Case

V_1	V_2	V_3	
A	B	C	(A > B) : 2
B	C	A	(B > C) : 2
C	A	B	(C > A) : 2

Table 7.2: Problematic Case

This lead to the discovery of **Condorcet's Paradox (1700's)**.

College	National Rank	Class size	Money Offered
X	4	40	3K
Y	8	18	1K
Z	12	24	8K

Table 7.3: Best College Example

Voting in context of plurality is related to comparing 2 choices. In Table 7.3, how do you decide which college is the best?

7.2 Voting Systems

7.2.1 Axioms

1. **Unanimity** (U)
   If all voters prefer A over B, then in the final ranking, A must be placed over B.

2. **Independence of Irrelevant Alternative** (IIA)
   For each pair of alternatives x, y the group ranking should depend only on how each individual voter ranks x and y reltative to each other.

3. **Non-Dictatorial** (ND)
   Voting system is not a Dictatorial system.

Critic	Unanimous Vote	IIA Vote
C_1	CK > GF	CK > GF > PF
C_2	CK > GF	CK > GF > PF
C_3	CK > GF	CK > GF > PF
C_4	GF > CK	GF > PF > CK
C_5	GF > CK	GF > PF > CK

Table 7.4: Movie Preferences for Unanimous and IAA Votes

In the first vote, (unanimous vote), CK is the obvious winner, 3 to 2. In the second case, it is not so obvious. If first choice gets 2 points, second 1 point, and third 0 points, the totals are as follows:

Movie	Vote Totals
CK	6
GF	7
PF	2

Table 7.5: Vote totals for IAA Example

Once we add a third option, now the GF becomes the winner. Furthermore, PF is also now an Irrelevant Alternative.

7.2.2 Systems

1. **Majority Rule** [U ✓] [IIA ✗] [ND ✓]
   A system where the option with the most votes or highest ranking is adopted. This system can lead to the Condorcet Paradox

2. **Positional Voting** [U ✓] [IIA ✗] [ND ✓]
   This system uses Bordat Counting, to assign a number to each choice and the choice with the largest sum wins. This does not solve the Condorcet Paradox, if selections are symmetric.

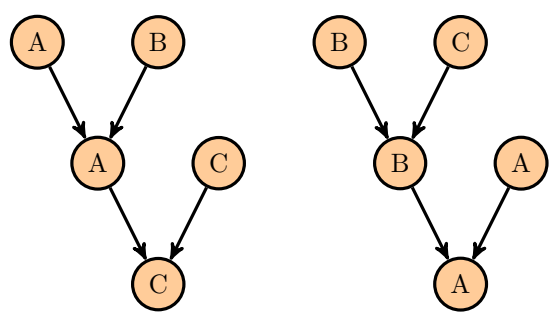
V_1	V_2	V_3	Option Totals
A (2)	A (2)	C (2)	A : 5 (winner)
B (1)	C (1)	A (1)	B : 1
C (0)	B (0)	B (0)	C : 3

Table 7.6: Sample Positional Voting with Assigned Values for each choice

V_1	V_2	V_3
A	B	C
B	C	A
C	A	B

Table 7.7: Symmetric Vote Choices

As you reconsider Table 7.7, if we instead of summing point values (Bordat Counting), only compare two choices at a time, we create a new problem. A problem of order matters, see below:



In the depiction above, which is comparing two elements at a time, and then comparing the winner to the next element, demonstrates the problem. Outcomes will vary simply based on what order you perform your comparison.

Definition 7.2 (Social Choice) *The top most choice.*

Definition 7.3 (Social Welfare) *The complete ranking of all choices.*

Note: Each definition above can be deduced from the other...

3. **Dictatorial** [U ✓] [IIA ✓] [ND ✗]
   All voters agree with the dictator.

7.3 Arrow's Theorem

Theorem 7.4 (Arrow's Theorem) *There is no voting system that satisfies Unanimity, Independence of Irrelevant Alternatives, and Non-Dictatorial. - Kenneth Arrow (1950s)*

Definition 7.5 (Quorum) *S is a quorum if all the members unanimously prefer A over B and all members not in S prefer B over A, then the final ranking places A over B.*

Lemma 7.6 *If S is a quorum and T is a quorum, then $S \cap T$ is a a quorum.*

Given options $A, B,$ and C

S: All voters prefer A to B and All voters in S' prefer B to A.

T: All voters prefer B to C and all voters in T' prefer C to B.

$$A > B$$

$$B > C$$

By transitivity, $A > C$ in final ranking.

Now focus on $S \cap T$:

$$(S \cap T)^c = S^c \cap T^c$$

$$v = w, x, y, z$$

$$s_w = x, y, z$$

$$s_x = w, y, z$$

$$s_y = w, x, z$$

$$s_z = w, x, y$$

The intersection of all of these sets is the null set.

$\therefore$ All the axioms can not be met by any voting system.

EE 382V: Social Computing	**Fall 2018**

Lecture 7: November 9

Lecturer: Vijay Garg *Scribe: Cong Wang*

7.1 Single-peaked preference

A standard example of cycling, in which there are three people, and three alternatives, A, B, C, and preferences are

$$X \succ_1 Y \succ_1 Z$$
$$Y \succ_2 Z \succ_2 X$$
$$Z \succ_3 X \succ_3 Y$$

Suppose that X, Y and Z correspond to amounts of money to spend on education, with X corresponding to a small amount, Y to a medium amount, and Z to a large amount. Then the preferences of voter 1 make sense: she is happiest with the smallest amount, and second-happiest with a medium amount. The preferences of voter 2 also make sense: he is happiest with a medium amount, but if not medium, then he prefers a large amount. The preferences of voter 3, on the other hand, are harder to justify in a simple way: he prefers a large amount, but his second choice is a small amount, with medium coming last. In other words, the first two voters have preferences that can be explained by proximity to a fixed number: each of them has an "ideal" amount that they'd like, and they evaluate the alternatives by how close they come to this ideal. The third voter's preferences can't be explained this way: there's no "ideal" quantity such that both large and small are close to it, but medium isn't. This is not to say that a person couldn't hold these preferences, but they're more unusual.

It turns out that cycling will not be a problem, if people's preferences are aligned in a particular way : if preferences are single-peaked. Let's assume that the k alternatives are named X_1, X_2,..., X_k, and that voters all perceive them as being arranged in this order. We say that a voter has a **single-peaked preference**, if there is no alternative X_s for which both neighboring alternative X_{s-1} and X_{s+1} are ranked above X_s. In other words, a voter never prefers two options that lie on opposite sides of a middle option.

Such preferences are called single-peaked because the condition we impose is equivalent to the following one : each voter i has a top-ranked option X_t, and here preferences fall off on both sides of X_t :

$$X_t \succ_i X_{t+1} \succ_i X_{t+2} \succ_i ...$$

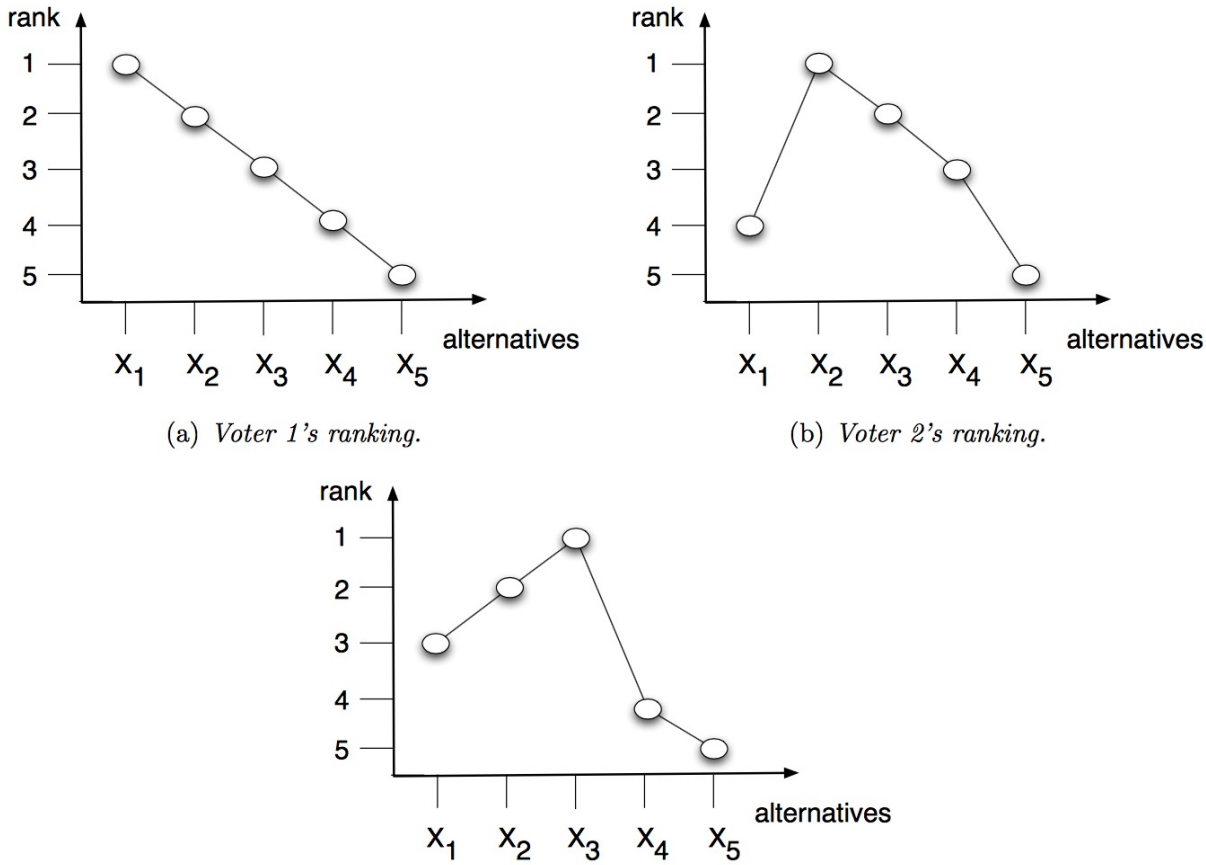
and

$$X_t \succ_i X_{t-1} \succ_i X_{t-2} \succ_i ...$$

Pictorially, this can be represented as in figure 7.1. The example shown there has three voters with preferences

$$X_1 \succ_1 X_2 \succ_1 X_3 \succ_1 X_4 \succ_1 X_5$$
$$X_2 \succ_2 X_3 \succ_2 X_4 \succ_2 X_1 \succ_2 X_5$$
$$X_3 \succ_3 X_2 \succ_3 X_1 \succ_3 X_4 \succ_3 X_5$$

and each of the three plots shows one of these sets of individual preferences: In the plots, there is an oval for each alternative, and its height corresponds to its position in the list. As drawn, the single peak in an individual's ranking emerges visually as a peak in the plot.



(a) *Voter 1's ranking.*

(b) *Voter 2's ranking.*

(c) *Voter 3's ranking.*

Figure 7.1: With single-peaked preferences, each voter's ranking of alternatives decreases on both sides of a "peak" corresponding to her favorite choice.

7.2 Median Voter Theorem

Preferences are single-peaked whenever there is some way of arranging the alternatives along a line so that the graphs of each voter's preferences have just one local maximum.

***Claim**: If all individual rankings are single-peaked, then majority rule applied to all pairs of alternatives produces a group preference relation that is complete and transitive.*

Let's consider the top-ranked alternative for each voter, and sort this set of individual favorites from left to right, along our linear order. Notice that if several voters have the same alternative as their respective individual favorite, then this alternative will appear multiple times in the sorted list: it is fine for the list

to have repetitions. Now consider the individual favorite that forms the median of this list - that is, the individual favorite that lies exactly at the halfway point in the sorted order. The median individual favorite is a natural idea to consider as a potential group favorite, since it naturally "compromises" between more extreme individual favorites on either side.

***The Median Voter Theorem**: With single-peaked rankings, the median individual favorite defeats every other alternative in a pairwise majority vote.*

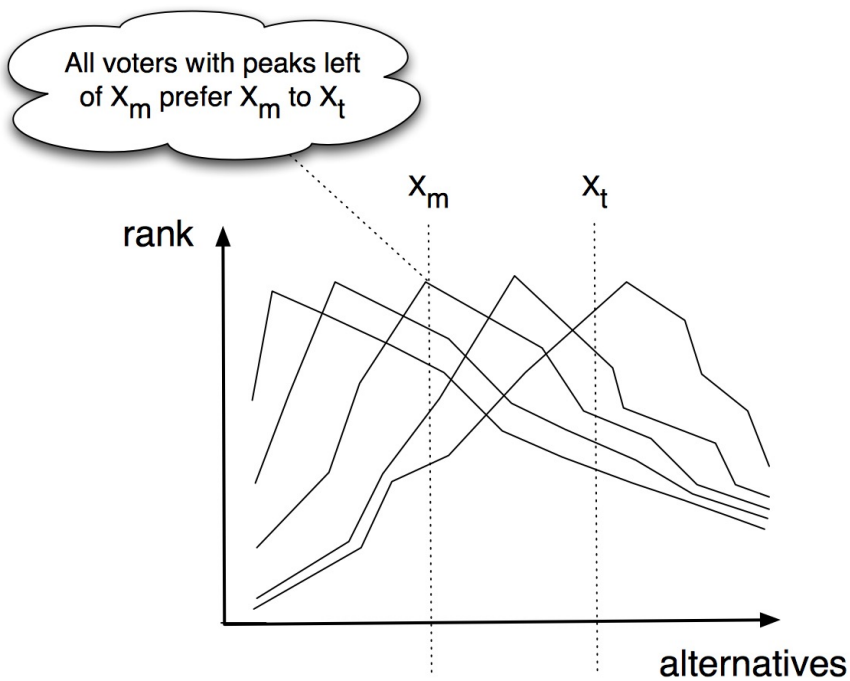


Figure 7.2: The proof that the median individual favorite X_m defeats every other alternative X_t in a pairwise majority vote: if X_t is to the right of X_m, the X_m is preferred by all voters whose peak is on X_m or to its left. (The symmetric argument applied when X_t is to the left of X_m)

Proof:
Let X_m be the median individual favorite, and let X_t be any other alternative. Let's suppose that X_t lies to the right of X_m - that is, t > m. (The case in which it lies to the left has a completely symmetric argument.) Let's also order the voters in the sorted order of their individual favorites. The argument is now depicted schematically in Figure 7.2. The number of voters k is odd, and we know that X_m is in position (k+1)/2 of the sorted list of individual favorites. This means that for everyone in the first (k+1)/2 positions, X_m is either their favorite, or their favorite lies to the left of X_m. For each voter in this latter group, X_m and X_t are both on the right-hand "down-slope" of this voter's preferences, but X_m is closer to the peak than X_t is, so X_m is preferred to X_t. It follows that everyone in the first (k+1)/2 positions prefers X_m to X_t. But this is a strict majority of the voters, and so X_m defeats X_t in a pairwise majority vote. The median individual favorite X_m can always count on gathering a majority of support against any other alternative X_t, because for more than half the voters, X_m lies between X_t and each of their respective favorites.

7.3 Housing Allocation Problem

In the "House Allocation Problem" problem, there are n agents and n houses. Each agent A_i initially owns a house H_i and has a completely ranked list of houses. This list is given by *pref*[i][k] which specifies the kth preference of agent i. Thus, *pref*[i][1] = j means that A_i prefers H_j as his top choice. The goal is to come up with an optimal house allocation defined as an allocation such that each agent has a house and no subset of agents can improve the satisfaction of agents in this subset by exchanging houses within the subset.

7.3.1 Top Trading Cycles Algorithm

We will give a simple and intuitive algorithm that will achieve these goals, the "Top Trading Cycles" algorithm.

input: n agents with strict preferences ordering over all the houses ($\succ_1$,...,$\succ_n$);
Let S_1 be the set of all agents. Set a counter t = 1;
while *size of* $S_t > 0$ **do**
  Construct a graph $G_t = (V_t, E_t)$ where $V_t = S_t$ and for each $(i, j) \epsilon E_t$ if and only if $h_j \succ_i h_k$ for all other $k \epsilon V_t$. This is the graph that results when every agent points to their most preferred house. And each agent is in at most one cycle ;
  Find any cycle C_t in G_t and for every directed edge $(i, j) \epsilon C_t$, we assign the entire cycle their most preferred houses, put (i, j) in map m, which means remove the agent from the market with her assigned house ;
  Set $S_{t+1} = S_t$ and for each $i : (i, j) \epsilon C_t$, set $S_{t+1} = S_{t+1}$ - i. Increment t (t = t+1) ;
end
Result: m

Algorithm 1: Top Trading Cycles Algorithm

Lemma *In each graph G_t constructed by the algorithm, there is at least one cycle C_t, and every agent is part of at most one cycle.*

Proof

This follows simply because by construction, G_t is a directed graph in which every vertex has out-degree exactly one. So by starting at any vertex and following edges forward, we must find a cycle, we cannot get stuck, since there is always an outgoing edge, and hence must repeatedly visit some vertex.

Claim *The top trading cycle algorithm terminates with a stable allocation.*

Proof

Suppose there is some trading coalition S with respect to the final allocation. Let N_1 be the set of people removed in the first stage. No one in N_1 can be part of S, because they all received their favorite items, so they cannot improve by any swap. Let N_2 be the set of people removed in the second stage. Each of these people gets their favorite item not owned by anyone in N_1. So the only way they could improve is by trading with someone in N_1, but since none of those people belong to S, they cannot improve by trading with anyone in S. This can be repeated for people in N_3, N_4... to show that no one can belong to S. Thus the assumption that there is a trading coalition is false, which means the allocation is stable.

7.3.2 Example

Consider 4 agents with the following preference ordering over each other's houses:

$$\succ_1: 2 \succ 3 \succ 1 \succ 4$$

$$\succ_2: 1 \succ 3 \succ 2 \succ 4$$

$$\succ_3: 1 \succ 2 \succ 3 \succ 4$$

$$\succ_4: 2 \succ 3 \succ 4 \succ 1$$

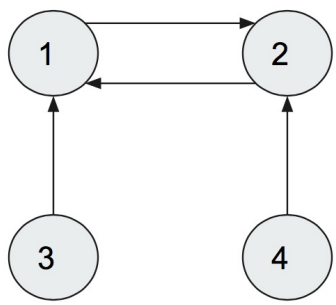


Figure 7.3: remove (1,2) and (2,1)

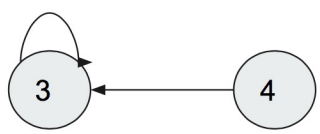


Figure 7.4: remove (3,3)

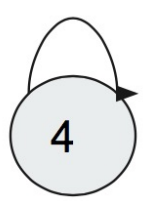


Figure 7.5: remove (4,4)

So the result is (A_1, H_2), (A_2, H_1), (A_3, H_3), (A_4, H_4)

References

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[2] lecture slides from UMass. *https://people.cs.umass.edu/ sheldon/teaching/mhc/cs312/2014sp/Slides/top-trading-cycles.pdf* (2014).

[3] lecture slides from UPenn. *https://www.cis.upenn.edu/ aaroth/courses/slides/agt17/lect11.pdf*

EE 382V: Social Computing **Fall 2018**

Lecture 8: November 10

Lecturer: Vijay Garg *Scribe: Shiyang Cheng*

8.1 Ceaser Cipher

In cryptography, *Caeser Cipher* is one of the simplest encryption techniques. It exploit substitution cipher in which each letter in the plaintext is replaced by a letter some fixed number of postions down the alphabet.

For example:

Cipher: $HPPE\ NPSOJOH$

Plain: *good morning*

In which, H maps to g, and so on.

8.2 RSA - Asymmetric cryptography

RSA stands for Rivest-Shamir-Adleman.

8.2.1 History

Cryptanalysis of the Enigma ciphering system enabled the western Allies in World War II to read substantial amounts of Morse-coded radio communications of the Axis powers that had been enciphered using Enigma machines. This yielded military intelligence which, along with that from other decrypted Axis radio and teleprinter transmissions, was given the codename Ultra. This was considered by western Supreme Allied Commander Dwight D. Eisenhower to have been "decisive" to the Allied victory.

The Enigma machines were a family of portable cipher machines with rotor scramblers. Good operating procedures, properly enforced, would have made the plugboard Enigma machine unbreakable. However, most of the German military forces, secret services and civilian agencies that used Enigma employed poor operating procedures, and it was these poor procedures that allowed the Enigma machines to be reverse-engineered and the ciphers to be read.

The German plugboard-equipped Enigma became Nazi Germany's principal crypto-system. It was broken by the Polish General Staff's Cipher Bureau in December 1932, with the aid of French-supplied intelligence material obtained from a German spy. A month before the outbreak of World War II, at a conference held near Warsaw, the Polish Cipher Bureau shared its Enigma-breaking techniques and technology with the French and British. During the German invasion of Poland, core Polish Cipher Bureau personnel were evacuated, via Romania, to France where they established the PC Bruno signals intelligence station with French facilities support. Successful cooperation among the Poles, the French, and the British at Bletchley Park continued until June 1940, when France surrendered to the Germans.

From this beginning, the British Government Code and Cypher School (GC&CS) at Bletchley Park built up

an extensive cryptanalytic capability. Initially, the decryption was mainly of Luftwaffe (German air force) and a few Heer (German army) messages, as the Kriegsmarine (German navy) employed much more secure procedures for using Enigma. Alan Turing, a Cambridge University mathematician and logician, provided much of the original thinking that led to the design of the cryptanalytical bombe machines that were instrumental in eventually breaking the naval Enigma. However, the Kriegsmarine introduced an Enigma version with a fourth rotor for its U-boats, resulting in a prolonged period when these messages could not be decrypted. With the capture of relevant cipher keys and the use of much faster US Navy bombes, regular, rapid reading of U-boat messages resumed[WikiEnigma].

In 1970, *James H. Ellis*, a British cryptographer at the UK Government Communications Headquarters (GCHQ), conceived of the possibility of "non-secret encryption", (now called public key cryptography), but could see no way to implement it. In 1973, his colleague *Clifford Cocks* implemented what has become known as the RSA encryption algorithm, giving a practical method of "non-secret encryption", and in 1974, another GCHQ mathematician and cryptographer, *Malcolm J. Williamson*, developed what is now known as *Diffie-Hellman* key exchange[WikiRSA].

8.2.2 RSA Use Case

There are three individuals: *Alice*, *Bob*, *Eve*. *Alice* want to communicate with *Bob* as figure-8.1. *Eve* is a passive attacker, who intercepts all the message between *Alice* and *Bob*. *Eve* also could send message to each other pretend to be one of them. *Alice* and *Bob* want to keep the message secrecy and authentication at the same time.

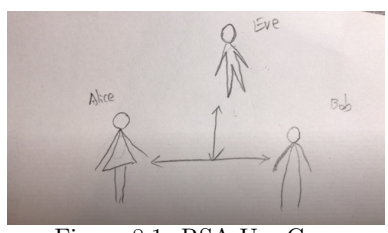

Figure 8.1: RSA Use Case

We use the properties:

1. Private and public keys can invert each other.

If we use private key to encrypt a message, we can use public key to decrypt the message. Otherwise, if we use public key to encrypt a message, we can use private key to decrypt the message.

2. If we have public key, we cannot find private key (because of the fact that a one way function)

Solution:

Let's define:

e_A: *Alice*'s public key

d_A: *Alice*'s private key

e_B: *Bob*'s public key

d_B: *Bob*'s private key

E: encryption function

D: decryption function

Assume that *Alice* gets *Bob*'s public key e_A, and *Bob* gets *Alice*'s public key. Nobody shares private key to anyone.

Alice want to send message M. She applies the encryption function to M, using encryption function E with *Bob*'s public key e_B as $E(M, e_B)$. Then encrypt the encrypted message again using her own private key d_A, which is $E(E(M, e_B), d_A)$.

Bob could use *Alice*'s public key e_A to decrypt the message into $E(M, e_B)$. If and only if the message comes from *Alice*, *Bob* could get the message $E(M, e_B)$, and decrypt the message again using his own private key d_B into message M.

In this way, *Bob* use *Alice*'s public key e_A to authenticate *Alice*. And use his own private d_B to decrypt the message which provide the secrecy at the same time.

8.2.3 RSA Algorithms

8.2.3.1 Generate Key Pairs

Algorithm 1 generateKeyPair()

1. Take two big prime numbers p and q, let $n = pq$
2. Compute *Euler*'s totient function $\Phi(n) = (p-1) * (q-1)$
3. Choose any number e that relatively prime to $\Phi(n)$, which means $gcd(e, \Phi(n)) = 1$
4. Find d such that $d * e \equiv 1 \; mod \; \Phi(n)$ (take mod both side)
5. throw away p, q, $\Phi(n)$
6. return (n, e) as public key, (n, d) as private key

This section gives the algorithms is meant to generate key pair which is (e, d) as algorithm-1. The function to encrypt message M as C:

$C = M^e \; mod \; n$

The function to decrypt encrypted message C back to M:

$M = C^d \; mod \; n$

8.2.3.2 Modulo Properties

Modulo operation finds the remainder after division of one number by another, which represent as *mod*.

Modulo operation has properties:

1. $(a + b) \; mod \; n = ((a \; mod \; n) + (b \; mod \; n))$
2. $(a * b) \; mod \; n = ((a \; mod \; n) * (b \; mod \; n))$

References

[WikiRSA] WIKIPEDIA, Public-key cryptography *wikipedia.org* (2018),

[WikiEnigma] WIKIPEDIA, Cryptanalysis of the Enigma *wikipedia.org* (2018),

EE 382V: Social Computing	**Fall 2018**

Lecture 2: November 10

Lecturer: Vijay Garg *Scribe: Porter Perry*

2.1 Introduction

This lecture focuses on the underlining number theory, algorithms, and and mathematical utilities that form the basis for the RSA encryption and decryption system (crypto) for security and authentication.

2.2 Fermat's Little Theorem

First we explore *Fermat's Little Theorem* (FLT), which was stated in a letter dated in 1640 by the French mathematician Pierre de Fermat. This theorem is highly useful for simplifying the computation of exponents in modular arithmetic.

Theorem 2.1 (Fermat's Little Theorem) *For any prime number p, and any integer number a where $0 < a < p - 1$, then $a^{(p-1)} \equiv 1 \pmod{p}$.*

Note: As explained during the lecture, the equivalence relational notation above is commonly used in academic mathematical writings. The modular equivalent relations are the same as applying the modulus operation on both sides of the equivalent equation. The above equivalence relation can be re-written as $a^{(p-1)} \pmod{p} = 1 \pmod{p}$.

2.2.1 FLT Examples

Ex 1. Let $p = 5$, and $a = 3$.

$$\begin{aligned} a^{(p-1)}(\text{mod } p) &= 3^{(5-1)}(\text{mod } 5) \\ &= 3^4(\text{mod } 5) \\ &= 81(\text{mod } 5) \\ &= 1 \end{aligned}$$

Ex 2. Let $p = 7$, and $a = 4$.

$$\begin{aligned} a^{(p-1)}(\text{mod } p) &= 4^{(7-1)}(\text{mod } 7) \\ &= 4^6(\text{mod } 7) \\ &= 4096(\text{mod } 7) \\ &= 1 \end{aligned}$$

2.3 Euler's Theorem

We now explore *Euler's Theorem*, written by the Swiss mathematician Leonhard Euler in 1736. This theorem is a generalization of Fermat's little theorem.

Theorem 2.2 (Euler's Theorem) *Let n be any number.*
Let a be relatively prime to n, where $gcd(a, n) = 1$ (the greatest common divisor between them is 1), then

$a^{\phi(n)} \equiv 1(mod\ n)$

where $\phi(n)$ is Euler's totient function.

Definition 2.3 (Euler's totient function) $\phi(n) = |\{i_{1 \leq i \leq n} | gcd(i, n) = 1\}|$
Euler's totient function is the size of the set of positive integers, up to n, that are relatively prime to n. Said another way, it is the number of integers i in the range $1 \leq i \leq n$ for which the greatest common divisor $gcd(n, i)$ is equal to 1.

2.3.1 Euler's Totient Function Examples

Ex 1. Let $n = 5$.
$\phi(5) = |\{1, 2, 3, 4\}| = 4$

Ex 2. Let $n = 6$.
$\phi(6) = |\{1, 5\}| = 2$

Ex 3. Let $n = 7$.
$\phi(7) = |\{1, 2, 3, 4, 5, 6\}| = 6$

Ex 4. Let $n = 12$.
$\phi(12) = |\{1, 5, 7, 11\}| = 4$

Ex 5. Let $n = 13$.
$\phi(13) = |\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}| = 12$

2.3.2 Euler's Totient Function Properties

A couple interesting and useful properties of Euler's totient function that play a key role in the RSA cryptosystem are:

- Euler's totient function is a multiplicative, meaning that if two numbers are m and n are relatively prime, then $\phi(mn) = \phi(m)\phi(n)$.
- If a number p is prime, then $\phi(p) = (p - 1)$.

Claim 2.4 *Let p and q both be prime numbers, then $\phi(pq) = (p-1)(q-1)$*

2.3.3 Examples

Ex 1. Let $p = 2$ and $q = 3$.

$$\begin{aligned}\phi(2 \times 3) = \phi(6) &= (2-1)(3-1) \\ &= (1)(2) \\ &= 2\end{aligned}$$

This matches the result of $\phi(6)$ of Euler's Totient Function Ex. 2 in the previous section.

Ex 2. Let $p = 5$ and $q = 3$.

$$\begin{aligned}\phi(5 \times 3) = \phi(15) &= (5-1)(3-1) \\ &= (4)(2) \\ &= 8 \\ &= |\{1, 2, 4, 7, 8, 11, 13, 14\}| = 8\end{aligned}$$

2.4 Greatest Common Divisor

[From Wikipedia]
Euclid's algorithm is an efficient method for computing the greatest common divisor (GCD) of two numbers, the largest number that divides both of them without leaving a remainder.

The Euclidean algorithm is based on the principle that the greatest common divisor of two numbers does not change if the larger number is replaced by its difference with the smaller number. For example, 21 is the GCD of 252 and 105 (as $252 = 21 \times 12$ and $105 = 21 \times 5$), and the same number 21 is also the GCD of 105 and $252105 = 147$. Since this replacement reduces the larger of the two numbers, repeating this process gives successively smaller pairs of numbers until the two numbers become equal. When that occurs, they are the GCD of the original two numbers. By reversing the steps, the GCD can be expressed as a sum of the two original numbers each multiplied by a positive or negative integer, e.g., $21 = 5 \times 105 + (2) \times 252$.

The *extended Euclidean algorithm* is an extension to the Euclidean algorithm, and computes, in addition to the greatest common divisor of integers a and b, also the coefficients of Bzout's identity, which are integers x and y such that

$ax + by =$gcd(a, b)

The extended Euclidean algorithm is particularly useful when a and b are coprime. With that provision, x is the modular multiplicative inverse of a modulo b, and y is the modular multiplicative inverse of b modulo a. Similarly, the polynomial extended Euclidean algorithm allows one to compute the multiplicative inverse in algebraic field extensions and, in particular in finite fields of non prime order.

2.5 Large Prime Numbers

The generation of prime numbers is foundational to the RSA cryptosystem. It would be nice to have a fast algorithm to do this. This can be accomplished using by an algorithms that checks if a randomly generated number is prime or not.

In 2002, Agrawal, Kayal, and Saxena published the paper titled "PRIMES is in P" in which they presented a deterministic algorithm that determines if any given general number is prime or composite. While this algorithm is of huge academic importance, it is not typical used in practice in favor of other faster options.

The *Prime Number Theorem* describes limiting distribution of of prime numbers, which become less common as they become larger, and formally quantifies the rate at which they occur.

Theorem 2.5 (Prime Number Theorem) *Let n be any positive number.*
Let $\pi(n)$ be the prime counting function that gives the number of primes less than or equal to n. The function $n/\ln(n)$ is a good approximation for $\pi(n)$ in that the limit of the ratio of the two functions is equal to one as n increases without bound:

$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln(n)} = 1$

This means that for a large value of n, the probability that a randomly generated number not greater than n is prime is very close to $1/\ln(n)$.

The above probabilistic statement of the prime number theorem is used to determine approximately how many random numbers will need to be generated and tested before a prime number will likely be found. For example, to find a 64-digit random prime number, a total of $\ln 10^{64} \approx 148$ randomly generated numbers will be tested before a prime number is found. The quantity of numbers tested can be reduced by half if only odd random numbers are tested, which for this example would be $(\ln 10^{64})/2 \approx 74$.

2.6 Fast Computation of Modulus Exponents

Consider the example of computing $7^{1024} \bmod 5$. The direct method involves first calculating 7^{1024} by multiplying 7 by itself 1024 times, then taking the result modulo 5. A more efficient way to solve the above is to take advantage of the following modular multiplication rule:

$a^2 \bmod n = (a \times a) \bmod n = ((a \bmod n) \times (a \bmod n)) \bmod n$

We can use the method of repeated squaring to calculate the answer more efficiently:

$7 \bmod 5 = 2$
$7^2 \bmod 5 = (2 \times 2) \bmod 5 = 4$
$7^4 \bmod 5 = (4 \times 4) \bmod 5 = 16 \bmod 5 = 1$
$7^8 \bmod 5 = (1 \times 1) \bmod 5 = 1$
$7^{16} \bmod 5 = (1 \times 1) \bmod 5 = 1$
...
$7^{512} \bmod 5 = (1 \times 1) \bmod 5 = 1$
$7^{1024} \bmod 5 = (1 \times 1) \bmod 5 = 1$

The above example happened to have an exponent that was conveniently a power of 2. What if it was desired to solve 7^{1025}? Combining the repeated squaring result with the modular multiplication rule, we get the following:

$7^{1025} \bmod 5 = ((7^{1024} \times 7^1) \bmod 5 = (7^{1024} \bmod 5) \times (7^1 \bmod 5)) \bmod 5 = (1 \times 2) \bmod 5 = 2$

2.6.1 Repeated Squaring and Multiplication Algorithm

The exponential $a^e \pmod{n}$ can be solved using the following iterative algorithm:

```
Step 1:  Let e_k e_{k-1} ... e_1 e_0 be the binary representation of e.
Step 2:  Set the variable c to 1
Step 3:  Repeat steps 3a and 3b for i=k, k-1, ..., 1, 0
      Step 3a:  c := c^2 mod n
      Step 3b:  if (e_i) = 1, then c := (a * c) mod n
```

Alternatively, this can also be implemented as a recursive algorithm:

```
Function exp(a, e, n)
    if (e is 1) return a
    else
        if (e is even)
            return (exp(a, e/2, n)^2 mod n)
        else
            return (a * exp(a, e/2, n)^2 mod n)
```

Session 3

EE 382V: Social Computing	**Fall 2018**

Lecture 8 Session 3: November 10

Lecturer: Vijay Garg *Scribe: Haleigh Walker*

8.1 RSA Underlying Mathematics

Proof: We want to demonstrate the correctness of our RSA algorithm and Euler's Theorem for any integer M which is relatively prime to n.

$M^{\Phi(n)} \equiv 1 \; mod \; n$

Here $\Phi(n)$ is the Euler totient function giving numbers of positive integers less than n which are relatively prime to n. For prime numbers p:

$\Phi(p)$ = p - 1

Therefore:

$\Phi(n) = \Phi(p) \cdot \Phi(q)$
= (p - 1) $\cdot$ (q - 1)
= n - $(p + q)$ + 1

Since d is relatively prime to $\Phi(n)$, its inverse e is in the ring of integers modulo $\Phi(n)$:

e $\cdot$ d $\equiv 1 \; mod \; \Phi(n)$

We now want to prove that our equations for encryption and decryption hold true when e and d are chosen correctly.

D(E(M)) $\equiv (E(M))^d \equiv (M^e)^d$ mod $n = M^{ed}$ mod n
$E(D(M)) \equiv (D(M))^e \equiv (M^d)^e$ mod $n = M^{ed}$ mod n
$M^{ed} \equiv$M$^{k\Phi(n)+1}$ mod n (for some integer k)

For all M such that p does not divide M

M$^{p-1} \equiv 1$ mod p

We also know that:

$\Phi(n) = (p-1) \cdot (q-1)$
$k\Phi(n) = k(p-1) \cdot (q-1)$

Therefore:

$M^{k\Phi(n)+1} = (M^{p-1})^{k(q-1)} \cdot M = 1 \cdot M$

Since (p - 1) divides $\Phi(n)$, from here we can conclude:

$M^{k\Phi(n)+1} \equiv M \; mod \; p$

Similarly
$M^{k\Phi(n)+1} \equiv M \bmod q$

Since $n = pq$, it is therefore implied that for all M
$\mathrm{M}^{ed} \equiv \mathrm{M}^{k\Phi(n)+1} \equiv M \bmod n$

Therefore E and D are inverse permutations. ■

8.2 Finite Distributive Lattice (FDL)

The lattice is used to model the search space of the combinatorial optimization problem. We want a parallel algorithm to solve problems with n processes. The goal of the combinatorial optimization problem is to find the smallest element in the lattice which satisfies the predicate (feasible solutions). We will assume the bollean predicate B is lattice linear.

Algorithm: getLeastFeasible
    if predicate is true
        you are done
    else
        there is at least one place you must advance to make the predicate true (at least 1 forbidden)

We will model the current state of the system with vector G, the global state, where G[i], the state of the process, is maintained at process i.

Definition 8.1 *Forbidden State: Given a predicate B, and a vector $G \in L$, a state G[i] is forbidden if for any vector $H \in L$, where $G \leq H$, if H[i] equals G[i], then B is false for H.*

forbidden(G,i,B) $\forall H \in L$: $G \leq H$: $(G[i] = H[i]) \Longrightarrow \neg B(H)$.

Definition 8.2 *Lattice Linear Predicates: We define a predicate B to be lattice-linear with respect to a lattice L if for any global state G, B is false in G implies that G contains a forbidden state. A boolean predicate B Is lattice-linear with respect to a lattice L iff*

$\forall G \in L : \neg B(G) \Longrightarrow \exists\ i : forbidden(G, i, B)$

Theorem 8.3 *B_1 is lattice linear $\bigwedge$ B_2 is lattice linear $\Longrightarrow$ B_1 $\bigwedge$ B_2 is lattice linear*

Proof: Global state G where B_1 or B_2 is false (If B_2 can never be true, then $B_1 \bigwedge B_2$ can never be true)

We need to show $\neg$B(G) $\Longrightarrow \exists i$: forbidden(G, i, B)

Suppose $\neg$($\mathrm{B}_1 \bigwedge B_2$). This implies that one of the predicates is false and therefore from lattice-linearity, a forbidden state exists. ■

Ex: Smallest possible cut = optimal answer

The predicate is true in the blue circles
If the lattice is linear, you can advance to find the optimal solution
We need to make our cut so that we have P1, P2, and P3 true. See Figure 3.1 below.

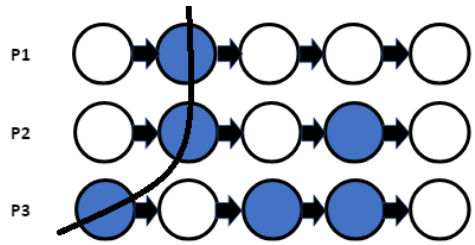

Figure 8.1: Optimal Solution for Lattice Linear Example

8.2.1 Stable Marriage

We now illustrate that the lattice-linear predicate detection technique is applicable when the search space is any distributive lattice by applying predicate detection to the stable marriage problem.

The vertices are a set of proposals E that can be made by men to women. We refer to these proposals as events which are executed by n processes P1, P2 P_n corresponding to n men. When there are no constraints, we impose a partial order $\rightarrow p$ on E with n disjoint chains to model the order in which these proposals can be made. Similar to the Gale-Shapely algorithm, each process makes proposals to women in decreasing order of male preferences. In Figure 3.2, black arrows indicate male preference in decreasing order, while red arrows indicate female preferences in increasing order.

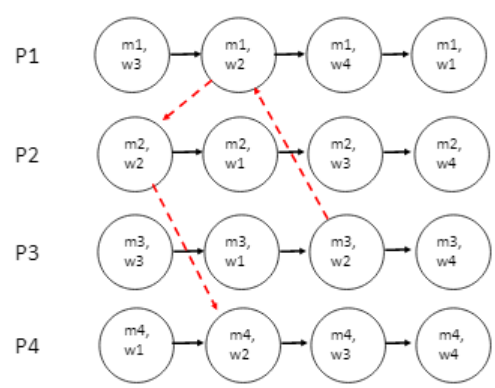

Figure 8.2: Male and Female Preferences

w2 preferences in order: m4, m2, m1, m3

Feasible Predicate: Assignment that gives a stable marriage.

E: set of proposals that can be made by men to women
$\rightarrow p$: partial order imposed

Definition 8.4 *A global state $G \subseteq E$ is consistent if*
$\forall e, f \in E : (e \rightarrow p\ f\ ) \bigwedge (f \in G) \Longrightarrow (e \in G)$.

Looking for male optimal solution that satisfies the feasible solution.

G = Global State
$B_{stablemarriage} \equiv B_{sin}(G)$
$\equiv$ frontier of consistent global state G
$\equiv$ last state in G
$\equiv$ G satisfies stable marriage predicate iff there is no red arrow pointing from frontier of G to any state in G AND G is consistent

Note: Red arrows pointing backwards from G indicate a blocking pair. A red arrow pointing within G also violates matching because it indicates the same woman for two men.

8.2.2 Prove Lattice Linear Predicate

Claim 8.5 *Backwards red arrows are forbidden.*

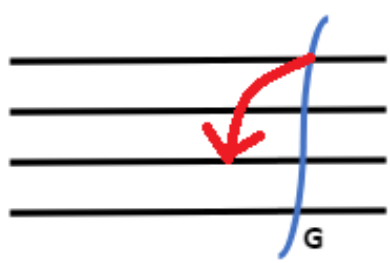

Figure 8.3: Backwards Red Arrows

The predicate can only be true if you advance on this process. You must advance at forbidden states to find the optimal solution.

This algorithm is more parallel than Gale-Shapely because multiple forbidden states can advance at the same time.

8.2.3 Constrained Stable Marriage

For the constrained stable marriage problem, we have arrows that relate proposals of different processes. In the diagram below, the black arrows indicate the order of proposal. If the process proposes to a circle an arrow is pointing to, it must also propose to the circle from which the arrow originated (See Figures 3.4 and 3.5 below).

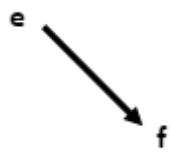

Figure 8.4: If You Include f, You Must Include e

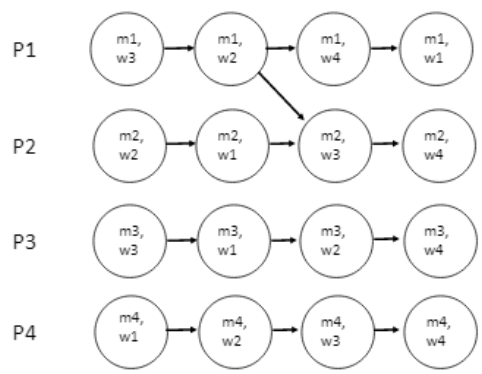

Figure 8.5: Constrained Stable Marriage

Backwards black arrows indicate that G is not consistent (future to past). See Figure 3.6 below.

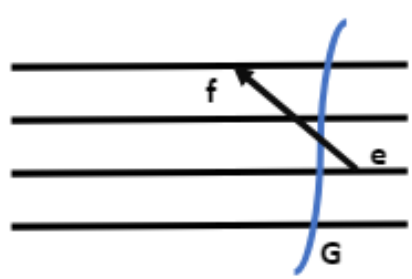

Figure 8.6: G is not Consistent

8.2.4 Algorithm Applications

The same algorithm can be applied to multiple problems, such as Market Clearing Price (VCG) and Housing Allocation problems, but the predicate continues to change.

8.2.4.1 Market Clearing Price

For VCG, if you are given a price vector, prices must increase, therefore decreasing prices are forbidden. You can then construct a preferred seller graph (bipartite graph).
Predicate: Market clearing at perfect matching.
We know that without perfect matching, a constricted set exists.
In the minimal set of constricted items, all items are forbidden.

8.2.4.2 Housing Allocation

In the Housing Allocation problem, subset matchings create cycles. Not being part of a cycle is forbidden.