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Optimal Sequencing in Single-Player Games

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Abstract. An important problem in single-player video game design is how to sequence game elements within a level (or "chunk") of the game. Each element has two critical features: a *reward* (e.g., earning an item or being able to watch a cinematic) and a degree of *difficulty* (e.g., how much energy or focus is needed to interact with the game element). The latter property is a distinctive feature in video games. Unlike passive services (like a trip to the spa) or passive entertainment (like watching sports or movies), video games often require concerted effort to consume. We study how to sequence game elements to maximize overall experienced utility subject to the dynamics of adaptation to rewards and difficulty and memory decay. We find that the optimal design depends on the relationship between rewards and difficulty, leading to qualitatively different designs. For example, when the proportion of reward-to-difficulty is high, the optimal design mimics that of more passive experiences. By contrast, the optimal design of games with low reward-to-difficulty ratios resembles work-out routines with "warm-ups" and "cool-downs." Intermediate cases may follow the classical "mini-boss, end-boss" design where difficulty has two peaks. Numerical results reveal optimal designs with "waves" of reward and difficulty with multiple peaks. Level designs with multiple peaks of difficulty are ubiquitous in video games. In summary, this paper provides practical guidance to game designers on how to match the design of single-player games to the relationship between reward and difficulty inherent in their game's mechanics. Our model also has implications for other interactive services that share similarities with games, such as summer camps for children.

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Keywords: video games • level design • memory decay • adaptation

1. Introduction

Video games are big business, representing the largest and fastest-growing segment of the entertainment industry.[1] However, not all games are successful. One example of a failed game—*E.T. The Extra-Terrestrial*—is so notorious that it became the symbol of the video game crash in the early 1980s.[2] Postmortems of *E.T.*'s failure point to several causes, not the least of which its poor design. By *design*, we mean the various elements of the game experience: art, graphics, game mechanics, story, and level design. The focus of this paper is on the latter (and is defined in more detail later). In particular, we focus on *single-player* games where game designers must pay careful attention to curating the experience of their players (as opposed to multiplayer games where players can interact with one another to generate experiences). The focus on single-player games is well justified. Despite the ready connectivity of today's gaming world, single-player experiences remain one of the most popular gaming segments on consoles, PCs, and mobile devices.[3]

Single-player video games can be very long experiences, sometimes lasting dozens of hours. Accordingly, games must be broken down into "sessions" that can be consumed in one sitting. The most common form of "sessioning" in games is by the notion of levels. A *level* is a discrete unit of gameplay with a beginning, middle, and end that moves forward the game's story and often introduces new obstacles or game mechanics.[4] Level design concerns finding the right balance of game elements and sequencing them to make the level engaging and satisfying.

Some practicing game designers have proposed the use of optimization tools to assist in designing levels. Paul Tozour, an experienced game designer, wrote about the challenge in an article for Gamasutra, a leading video game design website at the time.[5] As stated in this article, a major consideration when designing a level is balancing "reward" and "difficulty" in the arrangement of game elements. To make things concrete, consider the design of a side-scrolling action game like Capcom's classic *Mega Man 2*. Each level consists of platforming sections (i.e., sections that involve skilled jumping), standard enemy encounters, and one or more "boss" (i.e., difficult) enemy encounters. Standard and boss enemy encounters test the player's strategy and reflexes while platforming sections serve as tests of dexterity and hand-eye coordination. Different types of encounters also net different rewards. Defeating standard enemies may offer much-needed boosts to health or ammunition, whereas boss fights may earn the player new weapons or unlock new areas for exploration. Although *Mega Man 2* is a classic video game from the 1980s, the challenge of level design is as relevant today as it ever was. Many popular games are still designed in the mold of classics like *Mega Man 2* (including the *Lego Star Wars* series, the *New Super Mario Bros.* series, and *Minecraft Dungeons*), whereas other large "open-world games" offer "story mission" components with a sequential level-based structure (such as *Cyberpunk 2077* and the *Assassin's Creed* series).

This paper takes up this basic level design question, as proposed by Tozour and others, to sequence a set of given game elements (obstacles, enemy encounters, puzzles, etc.) to form an enjoyable player experience. The question of designing game elements is equally as interesting but beyond the scope of our study here. The assumption that game elements are given and then assembled into levels is consistent with game design practice. Consider, for example, the classic video game *Mario Brothers 3* by Nintendo that contains multiple "worlds" that consist of themed collections of levels with similar enemies and encounter styles. The enemies and encounter types are designed at the "world" level, whereas individual levels within the world sequence these enemies and encounter types. See Tozour (2013) for further discussion.

There are by now canonical level designs in video games. An intuitive design is one of increasing difficulty and reward as the level proceeds. As the player meets earlier tests, they are more prepared to tackle later challenges. However, there is also logic for a U-shaped design where levels start difficult, become easier, then crescendo toward a difficult finish. This design was not uncommon in coin-operated video game arcades, where having a rapid succession of failed attempts could drive up revenue. Social pressure and bragging rights among arcade patrons can drive players to "overcome" the initial challenge, only to be rewarded by a section of the game that is easier to handle, leading up to a "boss" of monumental difficulty.

Another classical design for console action games (like *Megaman 2* described previously) is the "mini-boss-end-boss" structure, where levels start out easy, reach a peak of tension in the middle of the level with a "mini-boss" encounter, then easing off before another crescendo to an even more difficult "end-boss" encounter. Other level designs resemble more of a workout routine: starting easy (warm-up) and ending easy (cool down) with an intermediate peak of difficulty.

Our research question is simple: under what conditions are these qualitatively different level designs optimal? The "conditions" refer to the nature of the game elements themselves, namely their rewards and difficulties. Our analysis reveals that differences in the reward-to-difficulty ratio lead to qualitatively different optimal level designs. The notion of optimality is that of maximizing the player's experienced utility accrued up to the end of the level. This objective reflects the fact that player satisfaction is experienced dynamically throughout the level and is assessed when the player decides whether to continue the game upon the completion of a level.

To answer this research question, we develop an optimization model for deciding the sequence of a given set of game elements to optimize the experienced utility of the player taking into account three psychological factors: accomplishment adaptation, stress adaptation, and memory decay. Accomplishment adaptation refers to the process by which utilities from rewards wane as players become accustomed to them.[6] Stress adaptation refers to how disutility for expending effort diminishes as players become accustomed to certain challenges. This phenomenon is well understood by game designers. Players can adapt to difficulty quickly as they become accustomed to challenges (Kalmpourtzis 2018, Schell 2019). Memory decay refers to the psychological fact that people tend to put more emphasis on recent experiences than older experiences.[7] The limits of attention and memory capacity have been identified as a key component in understanding game design. For example, in section 27 of part 6 of Hiwiller (2015), an examination of how appropriate design needs to consider the limited memory capacity of players. Chapter 4 of Hodent (2017) describes the theory of memory loss in detail with the concept of forgetting curves.

Other authors have studied related research questions leading to optimization problems with a similar structure. The most related papers to ours are the seminal (Das Gupta et al. 2016) and related papers (Roels 2019, 2020; Li et al. 2022)) that study the optimal design of experiential services considering both memory decay and adaptation to rewards. These papers find U-shaped and so-called IU-shaped structures for service quality against time. Although our method of analysis draws much inspiration from these papers, our model and results are different. Most critically, difficulty and stress are essential characteristics of the video game experience that are not considered in these previous models.

Video games are not passive, and so it is not a surprise that models that assume a passive consumer (Das Gupta et al. 2016, Roels 2019, Li et al. 2022) do not suffice to capture the tradeoffs that interest us. Indeed, we find many level designs (like the classical mini-boss-end-boss design) that are not predicted by existing models.

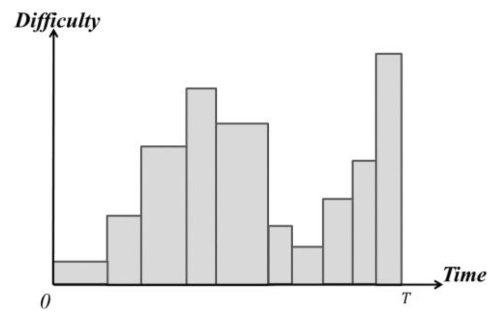
As for our findings, we analyze our proposed mathematical model of optimal level design to characterize when different qualitative designs are optimal. Our strongest analytical results are in the case when reward and difficulty are proportional; that is, easy game elements give small rewards, whereas hard game elements give large rewards. This is common in the design of individual game elements, as this is consistent with the psychological theory of "flow" championed by psychologist Csikszentmihalyi (1990), whose ideas have significant influence among video game designers[8] and academic researchers of video game design (Cowley et al. 2008).

In the theory of flow, the difficulty and reward for experiences should be balanced to help the participant achieve "optimal" experience called *flow*. If an activity requires little effort, an outsized reward feels hollow and unearned. Meanwhile, a task that is very difficult but reaps little reward leads to frustration. In the "sweet spot" of flow, the participant feels sensations of timelessness, happiness, and acute focus. Indeed, video games are often cited as an example of an experience highly adept at achieving "flow" in players, something of concern to parents, policy makers, and researchers (Kuss and Griffiths 2012).

A practical implication of Csikszentmihalyi's theory is that "flow" is best achieved when rewards and difficulty are proportional. It turns out that a key analytical driver of results in our model is precisely the reward-to-difficulty ratio. For example, when the proportion of reward-to-difficulty is high, the optimal design mimics that of more passive experiences like that studied in Das Gupta et al. (2016). This is intuitive because when difficulty is low, the gaming experience is not unlike a passive service experience. Classic games like *Dragon's Lair*—which is essentially an animated movie with very simple interactive elements separating scenes—is an example of a game with a high reward-to-difficulty ratio.

Intermediate cases give rise to the possibility of optimal mini-boss, end boss–like designs—what we call N-shaped designs because the difficulty, in this case, follows an N-shaped pattern (Figure 1). The intuition here is that a crescendo of sustained difficulty from the beginning of the level to the end builds up too much stress in the player, which can negatively impact their remembered utility given memory decay. Instead, the design starts with a crescendo of difficulty and rewards, so that the player adjusts to difficulty slowly

Figure 1. N-Shaped Game Design



and diminishes the amount of disutility accrued due to stress. Once accustomed to a certain level of difficulty at the peak of the crescendo (where the mini-boss is encountered), the remaining pattern is similar to a pure entertainment experience with a U-shaped design. The diminuendo subsequence in the middle of the level serves to reset the reference point for rewards and helps the player relax. The final crescendo sequence helps to create a grand finale experience accentuated in the memory of the player. Our analysis shows that such designs are optimal under easy-to-accept assumptions of player behavior. Our model can also pinpoint when and how to place a peak.

We also show conditions under which an *inverted* N-shaped design is optimal. This is the case when players adapt more quickly to difficulty than to rewards. Such a setting can prevail in "serious" games designed for educational and training purposes. Here, players expect the game to be challenging and are in a mood to learn and adapt to difficulty, but unlike inverted U-shaped designs where rewards are very low, rewards for difficulty are significant enough so that, at the outset of the level, high rewards get the player "going" with positive reinforcement. We see these types of designs in educational games like the mathematics-based role-playing game *Prodigy*.

To further our analysis of the structure of optimal level designs, we undertake a thorough numerical study in Section 5, which allowed us to explore other reward structures (other than proportional) and some additional structures of the optimal level designers. There, we show that N-shaped designs perform much better than naive strategies in many scenarios. We also find that the most difficult "boss" game elements are most commonly placed at the ends of levels, even under very general reward and difficulty inputs. We also show that the distance between the "boss" and the next hardest element (the mini-boss) depends on the associated rewards. The outcome follows a pattern of "separated gains" and "integrated losses," as studied in Thaler (1985) and Thaler and Johnson (1990).

In summary, this paper provides practical guidance to game designers on how to match level design to the relationship between reward and difficulty inherent in their game's mechanics. We make the following contributions:

• To our knowledge, this is the first paper to introduce a formal mathematical model for solving the sequencing problem inherent in video game level design.

• We provide mathematical justification for common level designs seen in practice, including N-shaped level designs, showing that they are optimal under certain conditions. Previous models for studying the design of experiential services (Das Gupta et al. 2016, Roels 2019, Li et al. 2022) are unable to justify the optimality of these types of designs.

• We incorporate behavioral elements into our models that are acknowledged as being significant by game designers in a way that is mathematically elegant and tractable. This includes behavioral elements not studied in the literature on the design of experiential services.

• We show that the essence of our findings is robust to generalizations that add mathematical complexity at the cost of tractability but also capture more general game design scenarios.

It is worth noting that, whereas our focus is on the design of single-player games, our analysis applies to other interactive service settings where agents must make effort in the course of receiving a service. Examples include designing trails in an outdoor adventure park, structuring the activities in a drop-in dance class, or scheduling activities at a summer camp for children. To make this concrete, consider the summer camp example. Activities of a summer camp take different amounts of effort for children to participate in and have different rewards. The overall goal is maximizing the remembered enjoyment of the campers so that they may return customers for the next summer. Designers of a plan of activities at a summer camp may use some of the insights of this study to sequence and structure these experiences in light of the accomplishment and stress processes that we identify in our study of games.

The paper is organized as follows. Section 2 summarizes related work on video games and the design of experiential services. Section 3 presents our main mathematical model of level design that is grounded in the behavioral theories of reward-seeking, difficulty aversion, and memory loss. Section 4 presents our main theoretical findings, including characterizations of when U-shaped, inverted U-shaped, N-shaped, and inverted N-shaped designs are optimal. Section 5 provides a thorough numerical exploration that provides additional insights. Section 6 concludes the main body of the paper. The online appendix has the following content. Online Appendix A contains all technical proofs of results in the main body. Online Appendix B provides tables that summarize our main analytical findings. Online Appendix C provides an integer programming formulation for our level design problem used in our numerical study. Online Appendix D gives a full specification of the parameters for an illustrative example that appears in the paper. Online Appendix E considers an extension to our setting where game elements can be repeated as a robustness check to our main insights. Online Appendix F uses real data from the game *Mario Maker 2* to illustrate how our model can be calibrated in practice. Online Appendix G provides a description of our model in the general setting of interactive services, using the design of a summer camp for children as an illustrative example.

2. Related Work

This paper is related to two burgeoning streams of research in operations management, information systems and marketing. The first is on business and design questions motivated by the video game context. Many of these papers are motivated by a similar central question—how game design relates to player engagement, retention, and monetization?—but none specifically look at the question of level design. The second stream of research in on designing services and work routines that take into consideration customer and worker behavior. These papers form the main methodological inspiration for our work.

Research primarily motivated by video games is a new and rapidly developing area in business research, crossing the disciplinary boundaries of operations management, information systems, and marketing. This includes research on the design of in-game advertising (Turner et al. 2011, Guo et al. 2019b, Sheng et al. 2022), the design of virtual currency systems (Guo et al. 2019a, Meng et al. 2021), and the sale of virtual items (Huang et al. 2020, Jiao et al. 2020, Vu et al. 2020, Chen et al. 2021, Runge et al. 2021).

We mention three papers in the video game literature that are arguably the most related to the current study. Huang et al. (2020) study how the concept of player "engagement" can be used to improve the design of games (specifically in protocols for matching players), leading to increased play and improved revenues. Sheng et al. (2022) also formalize the concept of engagement in a dynamic model for determining the optimal deployment of revenue-generating in-game advertising. (The concept of engagement in video games is also studied by Huang et al. (2019).) Ascarza et al. (2020) conduct a large-scale field experiment to draw empirical connections between game difficulty and player retention. All three studies examine the connection between game design and player motivation. Ascarza et al. (2020) relate these concepts to the notion of the difficulty of a game.

In a high-level sense, our work also relates to player motivation and progression, but with a different lens. Although the target practical audience of the previous papers might be those at game companies working on the business side of revenue generation, our focus here is to provide tactical insights to "frontline" design staff in charge of structuring game content. We consider the issue of engagement in the design unit of a "level" and ask what we can do to maximize the utility of the player (a proxy for engagement) given a set of more granular design elements. In this sense, we build on the findings of previous research—on the importance of engagement in games—and move toward tactical level-design questions.

Our work also builds on a growing literature concerned with behavioral aspects of offering experiential services based on the seminal work of Das Gupta et al. (2016). Connections between our work and this literature were already described at some length in the introduction, so we will not belabor the connection here. We would be remiss, however, not to mention important empirical work in the operations management (OM) literature on experiential service design (Dixon and Verma 2013, Dixon et al. 2017, Dixon and Victorino 2019) that has played an important complementary role to the development of optimization models like Das Gupta et al. (2016), Roels (2019), and Li et al. (2022), often providing insights that inform and enrich modeling choices. These empirical studies investigate the sequence of service elements and their relationship with behaviors like surprise and anticipation using experiments. In these experiments, the elements only have a single property (service level), whereas the elements in our model have two properties: reward and difficulty. In addition, the prescriptions of these empirical studies and our model's prescriptions differ. Based on empirical findings, Dixon and his collaborators recommend U-shaped and crescendo designs and find no empirical evidence that prescribes N-shaped designs, which can be optimal in our model. This arises from the two factors in our model and shows how our results contrast with the prescriptions found in the literature based on experiments.

A related line of inquiry into the design of experiential services is research into how worker fatigue impacts the optimal design of training and work regimens. Although fatigue is a classical topic in the study of operations management, only recently has fatigue been studied from a mathematical optimization perspective. We mention two papers that share common attributes with our current study, namely in how their analysis must tackle two "factors" impacting their objective functions analogous to our accomplishment and stress processes.

The first paper is Baucells and Zhao (2019), which looks at how fatigue impacts disutility and productivity of workers in a continuous-time framework. Although the two factors of disutility and productivity are similar to our accomplishment and stress processes, the analysis in Baucells and Zhao (2019) is different and leads to different conclusions. In particular, the optimization problem they study allows for a continuous choice of worker effort, whereas our optimization is constrained to a given set of game elements. Moreover, Baucells and Zhao (2019) find that the optimal design of effort is one of increasing or U-shaped effort profiles, whereas our models reveal the optimality of N-shaped difficulty designs seen in practice.

The second paper is Roels (2020), which studies how to optimally design a training regimen to optimize performance on some target date. Intense training contributes to two factors: fitness and fatigue. These factors are somewhat analogous to our accomplishment and stress processes, but with some important differences. First, there is a single driver (intensity) behind both fitness and fatigue in Roels (2020), whereas in our study accomplishment is driven by rewards, and stress is driven by difficulty. This leaves open the possibility that rewards and difficulty are not perfectly correlated. However, the more significant difference between the analysis of Roels (2020) and our study is in terms of the objective function. Roels (2020) aims to optimize fitness at a given point in time (a deadline), whereas we look at remembered utility accumulated throughout the time horizon. This is an important distinction that also informs our selection of related interactive service examples in the introduction. There, we mentioned summer camps and dance classes as settings where our model applies, but did not offer training programs and rehabilitation programs, which are a better fit with the model in Roels (2020). The reason is that the goal of summer camps and dance classes are more about creating a great experience for repeat customers (that fits our model), whereas training and rehabilitation programs have a "fitness" or "readiness" goal by the end of the program (that fits the model in Roels (2020)).

Concluding our comparison with Roels (2020), we note that our analysis has more in common with the "one-factor" model in Das Gupta et al. (2016) than the "two-factor" model of Roels (2020). Despite the differences mentioned above, Roels (2020) does find N-shaped and, more generally, wave-like patterns of intensity that mirror some of our results, but in this different context.

It is also worth mentioning related literature in theoretical information economics that was initiated by Ely et al. (2015), with a growing literature of applications and extensions (Renault et al. 2017, Nalbantis and Pawlowski 2019, Buraimo et al. 2020). The major distinction between this line of research and the work following Das Gupta et al. (2016) is that the former focus on "forward-looking" behavioral concepts like "suspense" and "surprise," whereas the latter tends to focus on behaviors that are backward looking. Forward-looking concepts have the added analytical complication of tracking beliefs, something we feel complicates the tradeoffs of interest in the current study.

Finally, sequencing "jobs" in an operational setting (including sequencing work on machines and assembly lines or sequencing surgeries in operating rooms) is a classical problem in the operations management literature with papers that use linear programming (LP) and integer programming (IP) techniques to solve scheduling problems that date back to the time of some of the earliest developments in LP and IP (Bowman 1959, Bakshi and Arora 1969, Emmons 1969) up to more recent developments (Meng et al. 2020, Naderi et al. 2021). This direction of research differs from ours in several ways, but the most salient aspect is the difference in objectives. The focus of the job sequencing literature is to minimize measures of run times like "tardiness" and/or fixed costs due to setups from switching between jobs. The utility of neither the worker (which is often envisioned as a machine) nor the customer are typically considered. In contrast, sequencing in level design (and service design more broadly) naturally focuses attention on maximizing customer utility.

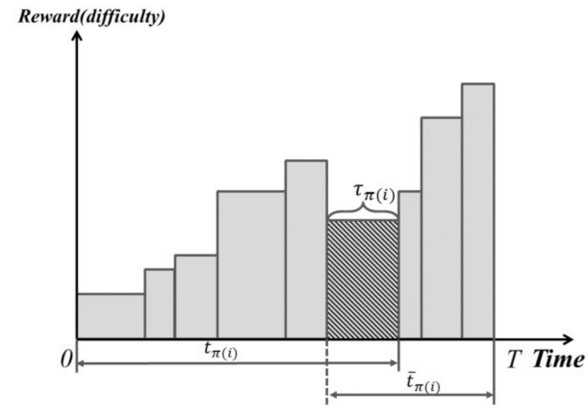
3. Model

A game designer seeks to optimally sequence a collection of n given game elements into a level that maximizes the satisfaction of a representative player. As described in the introduction, a level is a discrete "chunk" of gameplay that a player might tackle in a single "session" of play. Game elements refer to incremental units of a video game level, including enemy encounters, puzzles, or obstacles like a set of platforms to traverse. Each game element $i \in [n] \triangleq \{1, 2, \ldots, n\}$ has an associated reward r_i, a fixed duration τ_i, and a difficulty level d_i. The reward r_i can represent in-game "loot" that the player unlocks when passing game element i, or some more psychological notion of utility experienced by the player associated with the "fun" of the element or sense of accomplishment in completing it. The duration τ_i is the expected time it takes to pass game element i. The difficulty level d_i indicates how much mental and physical energy a player exhausts to pass game element i. It is important to emphasize that the game elements and their data (rewards, difficulties, and duration) are all given.

The game designer selects a permutation π of the set $[n]$ where $\boldsymbol{\pi} = (\pi(1), \ldots, \pi(n))$, where $\pi(i)$ is the ith game element in the sequence. For example, if there are three game elements indexed by the set $\{1, 2, 3\}$, then the sequence $\boldsymbol{\pi} = (2, 1, 3)$ designs the level with game element 2 first, followed by game element 1, and finally, game element 3. We assume that the elements are indexed in increasing order of rewards, that is, $r_1 \le r_2 \le \cdots \le r_{n-1} \le r_n$.

We consider a level design problem with a fixed duration $T = \sum_{i=1}^{n} \tau_i$. We denote by $t_{\pi(i)} = \sum_{j=1}^{i} \tau_{\pi(j)}$ the completion time of game element $\pi(i)$ and by $\bar{t}_{\pi(i)} = \sum_{j=i}^{n} \tau_{\pi(j)}$ the duration from the starting time of game element $\pi(i)$ until the end of the level. Observe that

Figure 2. Time Intervals for Game Element $\pi(i)$: $\tau_{\pi(i)}$, $\bar{t}_{\pi(i)}$, $t_{\pi(i)}$



$T = \bar{t}_{\pi(i)} + t_{\pi(i)} - \tau_{\pi(i)}$. Figure 2 provides a graphical representation of this notation. For simplicity of notation, we omit π in the subscripts and use $r_{(i)}$, $d_{(i)}$, $\tau_{(i)}$, t_i, and $\bar{t}_i$ to represent $r_{\pi(i)}$, $d_{\pi(i)}$, $\tau_{\pi(i)}$, $t_{\pi(i)}$ and $\bar{t}_{\pi(i)}$, respectively.

3.1. Gameplay Satisfaction

We adopt a framework similar to Das Gupta et al. (2016) and Li et al. (2022) to quantify the player's retrospective perception of a level as the remembered utility accumulated from time 0 to time T. Our expression for this remembered utility draws on three psychological concepts, namely (a) accomplishment adaptation, (b) stress adaptation, and (c) memory decay. These concepts reflect three typical behaviors in gameplay: reward seeking, difficulty aversion, and memory loss.

The accomplishment process reflects the player's passion for seeking rewards. Although people are attracted by the sensation of "winning," players also experience negative feelings during gameplay. If the game is difficult, players can come to feel anxious or frustrated, particularly when exposed to extended durations of difficulty (Chen 2007). Accordingly, we introduce a stress process that reflects the dynamics of a player's aversion to difficulty.

We introduce a memory decay process to reflect the player's memory loss due to the player's limited ability to remember what happened during the experience of a level. We must therefore examine a player's remembered utility of a level when assessing his appreciation of the design. Table 1 summarizes the relationship between the psychological process and the player behavior in the gameplay.

To formalize the accomplishment and stress processes, we follow the adaptation model of Aflaki and Popescu (2013) in a similar pattern to Das Gupta et al. (2016), where experienced utility and disutility are functions of deviations from a reference point, and this

Table 1. Psychological Process and Player Behavior

Psychological process	Player behavior	Outcome
Accomplishment process	Reward seeking	Utility
Stress process	Difficulty aversion	Disutility
Memory decay process	Memory loss	Remembered utility

reference point evolves according to a differential equation akin to Newton's law of cooling. In our model, the accomplishment process is the source of utility and the stress process is the source of disutility. Each of these processes evolves according to its own adaptive process with given parameters. These two processes are described in the next two subsections.

As the player has both positive and negative feelings from gameplay, utility from rewards and disutility from difficulty jointly affect the game experience. Following our description of the accomplishment and stress processes, we combine them to determine a (net) utility process in Section 3.1.3. At time t during the player's experience of the level, we determine an instantaneous (net) utility at time t by subtracting the instantaneous disutility from the instantaneous utility at time t. In doing so, we adopt a linear-form model similar to Roels (2020), which is based on the athletic performance model by Banister et al. (1975) with two effects: one positive (fitness) and one negative (fatigue). Although performance is not the subject of our study, we believe a linear-form model is justified here because it similarly weighs two psychological impacts on utility: one positive (reward) and one negative (difficulty). This matches with game design theory, which states that the game should balance its reward scheme and the degree of challenge (Schell 2019).

For the memory decay process, we consider a memory decay process with exponential memory decay as in Das Gupta et al. (2016) and Li et al. (2022). The memory decay process determines the relative weight of each game element and converts instantaneous utility to remembered utility. Therefore, the sequence of game elements will affect the perception of the game experience.

Combining the three psychological effects, we present the framework of our study in Figure 3.

3.1.1. Accomplishment Process. The accomplishment process reflects the psychological phenomenon of reward-seeking and adaptation to rewards. On the one hand, players prefer to receive more rewards, on the other hand, they gradually adapt to the gain and seek greater rewards (Plass et al. 2015).

To model the accomplishment process, we follow the adaptation model of Aflaki and Popescu (2013). For a given schedule $\boldsymbol{\pi}$, we denote by $f_{\boldsymbol{\pi}}(t)$ the player's reference reward at time t. For simplicity of notation, we omit $\boldsymbol{\pi}$ in the subscript and use $f(t)$. The instantaneous utility experienced at time $t \in [t_{i-1}, t_i]$ is a function of the difference between the current reward and the reference reward, which is given by

$$u_r(t) = U_r\left(r_{(i)} - f(t)\right), \tag{1}$$

where $U_r(\cdot)$ is the player's utility function for rewards. We assume that utility function $U_r(\cdot)$ is linear, such that $U_r\left(r_{(i)} - f(t)\right) = u_{r,0} + a\left(r_{(i)} - f(t)\right)$, where $u_{r,0}$ is the initial utility from the experience and a is a coefficient. We can normalize $u_{r,0}$ to zero and a to one without loss by simply rescaling utilities (recall that utilities are only defined up to affine scaling; Mas-Colell et al. 1995, chapter 1).

We assume that the rate of change of the reference reward is proportional to the instantaneous utility $u_r(t)$; that is, the rate of change of reference reward $f(t)$ at time $t \in [t_{i-1}, t_i]$ is

$$\frac{df(t)}{dt} = \alpha u_r(t) = \alpha\left(r_{(i)} - f(t)\right),$$

where we refer to $\alpha > 0$ as the *degree of reward-seeking* of the player. Parameter α depicts the speed of adaptation to rewards. The larger is the risk-seeking degree α, the faster the reference reward accumulates. Players with very large α have insatiable appetites for rewards, even as they earn rewards they require even greater rewards to stay happy.

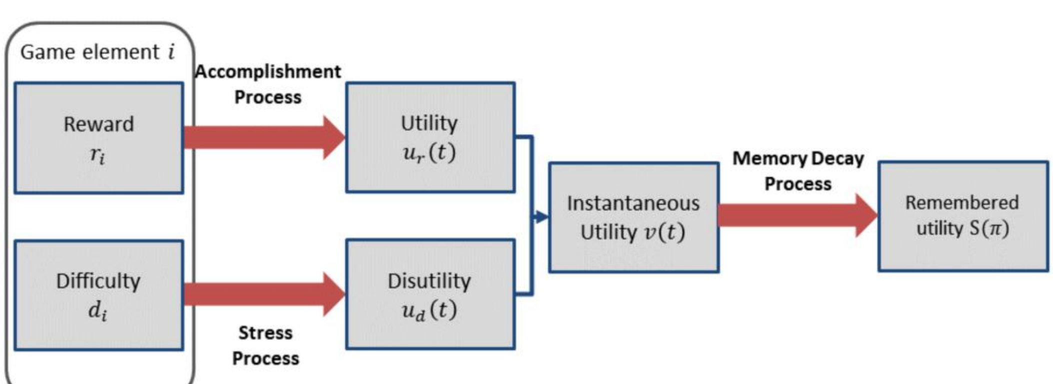


Figure 3. (Color online) Combining Effects of Accomplishment, Stress, and Memory Decay

The reference reward at time $t \in [t_{i-1}, t_i]$ for a player with risk-seeking degree α is

$$f(t) = r_{(i)} - \left(\left(r_{(1)} - f(0) \right) + \sum_{j=2}^{i} \left(r_{(j)} - r_{(j-1)} \right) e^{\alpha t_{j-1}} \right) e^{-\alpha t}. \tag{2}$$

With (1) and (2), the utility at $t \in [t_{i-1}, t_i]$ can be expressed as

$$u_r(t) = \left(\left(r_{(1)} - f(0) \right) + \sum_{j=2}^{i} \left(r_{(j)} - r_{(j-1)} \right) e^{\alpha t_{j-1}} \right) e^{-\alpha t}. \tag{3}$$

3.1.2. Stress Process. The stress process reflects the psychological phenomenon of difficulty aversion and adaptation. A game is not an unbroken sequence of rewards. Effort must be exerted to earn rewards, and this effort is proportional to the difficulty of the game element. We assume that players adapt to difficulty analogously to how they adapt to rewards. This fits the common understanding of game design, which suggests that players learn from playing and find that challenge diminishes when faced with equally difficult game elements (Kalmpourtzis 2018, Schell 2019).

As illustrated by Figure 3, the stress process governs disutility due to effort exerted in overcoming difficulty. For a given schedule $\boldsymbol{\pi}$, we denote by $g_{\boldsymbol{\pi}}(t)$ the player's reference difficulty at time t. For simplicity of notation, we often omit $\boldsymbol{\pi}$ in the subscripts and use $g(t)$ instead. The disutility at time $t \in [t_{i-1}, t_i]$ is a function of the difference between the current difficulty and the reference difficulty, which is given by

$$u_d(t) = U_d\left(d_{(i)} - g(t)\right), \tag{4}$$

where $U_d(\cdot)$ is the disutility function. As before, we assume that U_d is linear and let $U_d(d - g) = \delta(d - g)$, where δ is a given positive constant. We scale δ to one without loss for simplicity of the analysis. For completeness, we verify this assertion in Lemma EC.9 in Online Appendix A.[9]

Same as the accomplishment process, we assume the rate of change in the reference difficulty is proportional to the disutility at time t; that is, the change rate of the reference difficulty $g(t)$ at time $t \in [t_{i-1}, t_i]$ is

$$\frac{dg(t)}{dt} = \beta u_d(t) = \beta\left(d_{(i)} - g(t)\right),$$

where $\beta > 0$ (with $\beta \neq \alpha$) is the *degree of difficulty-aversion*. Parameter β depicts the speed of adaptation to difficulty. The larger is the difficulty-aversion degree β, the faster the reference difficulty accumulates.

The reference difficulty at time $t \in [t_{i-1}, t_i]$ is

$$g(t) = d_{(i)} - \left(\left(d_{(1)} - g(0) \right) + \sum_{j=2}^{i} \left(d_{(j)} - d_{(j-1)} \right) e^{\beta t_{j-1}} \right) e^{-\beta t}. \tag{5}$$

With (4) and (5), the disutility at $t \in [t_{i-1}, t_i]$ can be expressed as

$$u_d(t) = \left(\left(d_{(1)} - g(0) \right) + \sum_{j=2}^{i} \left(d_{(j)} - d_{(j-1)} \right) e^{\beta t_{j-1}} \right) e^{-\beta t}. \tag{6}$$

3.1.3. Memory Decay Process. The memory decay process reflects the psychological phenomenon of memory loss, and it converts instantaneous utility into remembered utility. This works on the (net) utility derived from the instantaneous utility from rewards and disutility from difficulty. Instantaneous utility is affected by both the accomplishment and the stress processes.

As shown in Figure 3, we assume the instantaneous utility experienced at time $t \in [t_{i-1}, t_i]$ is a function of the utility from rewards and disutility from difficulty given by

$$v(t) \triangleq V(u_r(t), u_d(t)), \tag{7}$$

where $V(\cdot)$ is an aggregate utility function over utilities u_r and disutilities u_d.

We assume that the utility function $V(\cdot)$ is linear with $V(u_r(t), u_d(t)) = v_0 + \delta_r u_r(t) - \delta_d u_d(t)$, where v_0 is the initial instantaneous utility, and $\delta_r, \delta_d > 0$ are given coefficients of the utility. Same as before, we normalize v_0 to zero and scale δ_r and δ_d to one without loss for simplicity of the analysis. This is verified in Lemma EC.9 in Online Appendix A.

We assume that the player has an exponential memory decay process with rate $\gamma > 0$ and $\gamma \neq \alpha, \beta$. Then the player's cumulative remembered utility $S(\boldsymbol{\pi})$ is given by

$$S(\boldsymbol{\pi}) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} v(t) e^{-\gamma(T-t)} dt, \tag{8}$$

where $v(t)$ is as defined in (7). Therefore, the latest game element will weigh more when players recall the game journey.

Combining (3), (6), (7), and (8) shows that the remembered utility of a level can be expressed as

$$\begin{aligned} S(\boldsymbol{\pi}) = &\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\left(r_{(1)} - f(0) \right) e^{-\alpha t} + \sum_{j=2}^{i} \left(r_{(j)} - r_{(j-1)} \right) e^{-\alpha\left(t - t_{j-1}\right)} \right) e^{-\gamma(T-t)} dt \\ &- \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\left(d_{(1)} - g(0) \right) e^{-\beta t} + \sum_{j=2}^{i} \left(d_{(j)} - d_{(j-1)} \right) e^{-\beta\left(t - t_{j-1}\right)} \right) e^{-\gamma(T-t)} dt. \end{aligned} \tag{9}$$

We assume that the player does not have any previous gameplay experience, so that $f(0) = 0$ and $g(0) = 0$. Let

$r_{(0)} = 0$ and $d_{(0)} = 0$. Continuing from (9), we have

$$
\begin{aligned}
S(\boldsymbol{\pi}) &= \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\sum_{j=1}^{i}\left(r_{(j)} - r_{(j-1)}\right)e^{-\alpha\left(t-t_{j-1}\right)}\right)e^{-\gamma(T-t)}dt\\
&\quad-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\sum_{j=1}^{i}\left(d_{(j)} - d_{(j-1)}\right)e^{-\beta\left(t-t_{j-1}\right)}\right)e^{-\gamma(T-t)}dt,\\
&= \sum_{i=1}^{n}\left(r_{(i)} - r_{(i-1)}\right)\frac{e^{-\alpha\bar{t}_i} - e^{-\gamma\bar{t}_i}}{\gamma-\alpha}\\
&\quad-\sum_{i=1}^{n}\left(d_{(i)} - d_{(i-1)}\right)\frac{e^{-\beta\bar{t}_i} - e^{-\gamma\bar{t}_i}}{\gamma-\beta},\\
&= \sum_{i=1}^{n} r_{(i)}\left(\frac{e^{-\alpha\bar{t}_i} - e^{-\gamma\bar{t}_i}}{\gamma-\alpha} - \frac{e^{-\alpha\bar{t}_{i+1}} - e^{-\gamma\bar{t}_{i+1}}}{\gamma-\alpha}\right)\\
&\quad-\sum_{i=1}^{n} d_{(i)}\left(\frac{e^{-\beta\bar{t}_i} - e^{-\gamma\bar{t}_i}}{\gamma-\beta} - \frac{e^{-\beta\bar{t}_{i+1}} - e^{-\gamma\bar{t}_{i+1}}}{\gamma-\beta}\right).
\end{aligned}\tag{10}
$$

3.2. Level Design Problem

In this section, we formulate the level design problem. To simplify the expression of game satisfaction, we first introduce the function $\Phi(t \mid \theta,\gamma)$, where θ can be either the risk-seeking degree α or the difficulty-aversion degree β. The function $\Phi(t \mid \theta,\gamma)$ is given by

$$\Phi(t \mid \theta,\gamma) = \frac{e^{-\theta t} - e^{-\gamma t}}{\gamma-\theta},\tag{11}$$

where $\theta \neq \gamma$, $\theta,\gamma > 0$, and $\Phi(t \mid \theta,\gamma) \geq 0$.

It is straightforward to see that $\Phi(t \mid \theta,\gamma)$ is continuous and twice differentiable in t. As shown in Lemma EC.1 in Online Appendix A, $\Phi(t \mid \theta,\gamma)$ is a concave-convex function with one inflection point and one stationary point. Let $T_0(\theta,\gamma)$ be the inflection point and $T_0'(\theta,\gamma)$ be the stationary point, whose formulation is shown in (EC.1) in Onlien Appendix A. These inflection and stationary points are important indicators of structure discussed later in Theorem 3 and Propositions EC.1 and EC.2.

By (10) and (11), we can formulate the level design problem as

$$
\begin{aligned}
\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} r_{(i)}\left(\Phi\left(\bar{t}_i \mid \alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} \mid \alpha,\gamma\right)\right)\\
&\quad-\sum_{i=1}^{n} d_{(i)}\left(\Phi\left(\bar{t}_i \mid \beta,\gamma\right) - \Phi\left(\bar{t}_{i+1} \mid \beta,\gamma\right)\right).
\end{aligned}\tag{12}
$$

In Online Appendix G, we demonstrate how this model can apply to interactive service problems that go beyond the video game context.

4. Optimal Structure of Game Design

We now examine the structural properties of optimal solutions to (12). This section contains two subsections. The first considers the special case where rewards and difficulties are proportional. As mentioned in the introduction, this is consistent with the concept of "flow" and is a common design principle in video games (Chen 2007). In the second section, we examine the case of the more general reward and difficulty patterns.

4.1. Sequencing Game Elements with Proportional Reward

In this section, we consider the case that the reward is proportional to the difficulty of the game element, with a uniform reward ratio $k > 0$, where $r_i = kd_i$.

Under proportional rewards, the player's remembered utility with proportional reward can be expressed by

$$
\begin{aligned}
S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} kd_{(i)}\left(\Phi\left(\bar{t}_i \mid \alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} \mid \alpha,\gamma\right)\right)\\
&\quad-\sum_{i=1}^{n} d_{(i)}\left(\Phi\left(\bar{t}_i \mid \beta,\gamma\right) - \Phi\left(\bar{t}_{i+1} \mid \beta,\gamma\right)\right),\\
&= \sum_{i=1}^{n} d_{(i)}((k\Phi(\bar{t}_i \mid \alpha,\gamma) - \Phi(\bar{t}_i \mid \beta,\gamma))\\
&\quad-(k\Phi(\bar{t}_{i+1} \mid \alpha,\gamma) - \Phi(\bar{t}_{i+1} \mid \beta,\gamma))).
\end{aligned}
$$

To simplify the expression, we define the function

$$\Psi(t \mid \alpha,\beta,\gamma,k) = k\Phi(t \mid \alpha,\gamma) - \Phi(t \mid \beta,\gamma).$$

Using this notation, we rewrite the level design problem with proportional reward (LDPP) as

$$\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) = \sum_{i=1}^{n} d_{(i)}\left(\Psi\left(\bar{t}_i \mid \alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+1} \mid \alpha,\beta,\gamma,k\right)\right).\tag{LDPP}$$

It is straightforward to see that $\Psi(t \mid \alpha,\beta,\gamma,k)$ is continuous and twice differentiable in t. As shown in Lemmas EC.2 and EC.3 in Online Appendix A, we prove that $\Psi(t \mid \alpha,\beta,\gamma,k)$ can have one or two of inflection point(s) and one or two stationary point(s).

Let $T_1(\alpha,\beta,\gamma,k)$ and $T_2(\alpha,\beta,\gamma,k)$ be the inflection points when there are two points, and $T_2(\alpha,\beta,\gamma,k)$ be the unique inflection point when there is only one inflection point. For simplicity, we will drop the arguments and use T_1 and T_2 when there is no possibility for confusion. For example, in Theorems 1 and 2, we find that the player prefers either a crescendo or diminuendo subsequence within $[0,(T-T_2)^+]$, $[(T-T_2)^+,(T-T_1)^+]$, and $[(T-T_1)^+,T]$.

Our analysis suggests that the player's tastes are influenced by the joint effects of the parameters α, β, γ, and k. To express the structural property of the optimal solution, we define the two thresholds:

$$\underline{k} \triangleq \begin{cases} \dfrac{\beta+\gamma}{\alpha+\gamma}, & \text{if } \alpha > \beta, \\ \dfrac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha < \beta \text{ and } \alpha > \gamma, \\ 0, & \text{if } \alpha < \beta \text{ and } \alpha < \gamma \end{cases} \quad \text{and}$$

$$\overline{k} \triangleq \begin{cases} \dfrac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha > \beta \text{ and } \beta > \gamma, \\ +\infty, & \text{if } \alpha > \beta \text{ and } \beta < \gamma, \\ \dfrac{\beta+\gamma}{\alpha+\gamma}, & \text{if } \alpha < \beta. \end{cases}$$

First, we describe properties of the optimal sequence when the game's duration is sufficiently long (i.e., $T > T_2$).

Theorem 1. *When the game duration is sufficiently long (i.e., $T > T_2$), in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPP, the elements' rewards (difficulties) are in the following structure.*

(i) *When $k \le \underline{k}$, the optimal structure is an inverted U-shaped sequence.*

(ii) *When $\underline{k} < k \le \overline{k}$, there are two cases of the optimal structure.*

(iia) *If $\alpha > \beta$, the optimal structure is a N-shaped sequence.*

(iib) *If $\alpha < \beta$, the optimal structure is an inverted N-shaped sequence.*

(iii) *When $k > \overline{k}$, the optimal structure is a U-shaped sequence.*

We present the optimal structures in Figure 4 and summarize the mathematical expressions of the optimal structure in Table EC.1 in Online Appendix B. Theorem 1 is arguably the central result of the paper, and so we deliberate on its meaning in the next few paragraphs.

In case (i), the rewards are so low in proportion to difficulty that utilities from the stress adaptation process dominate the accomplishment process. Here, we need to think about managing the stress of the player, so a huge jump in difficulty will cause a lot of disutility, so we have a warm-up and cool-down to avoid jumps. In other words, when the reward ratio is low (i.e., $k < \underline{k}$), the problem will become a workout design problem, whose optimal structure is an inverted U-shaped sequence regardless of the values of α and β. This is easy to understand. When you do a workout, the player gets tired very easily. We see this game design in genres that require intensive body movement like the arcade classics *Dance Dance Revolution* or *Whack-A-Mole*.

Conversely, when the reward ratio is high (i.e., $k \ge \overline{k}$), the problem essentially becomes the design of an entertainment or service experience as studied in Das Gupta et al. (2016). Accordingly, the optimal structure follows the U-shaped pattern identified in Das Gupta et al. (2016). This optimal structure is well suited to games in which the plot is the most important issue (e.g., interactive fiction like the classic arcade game *Dragon's Lair*).

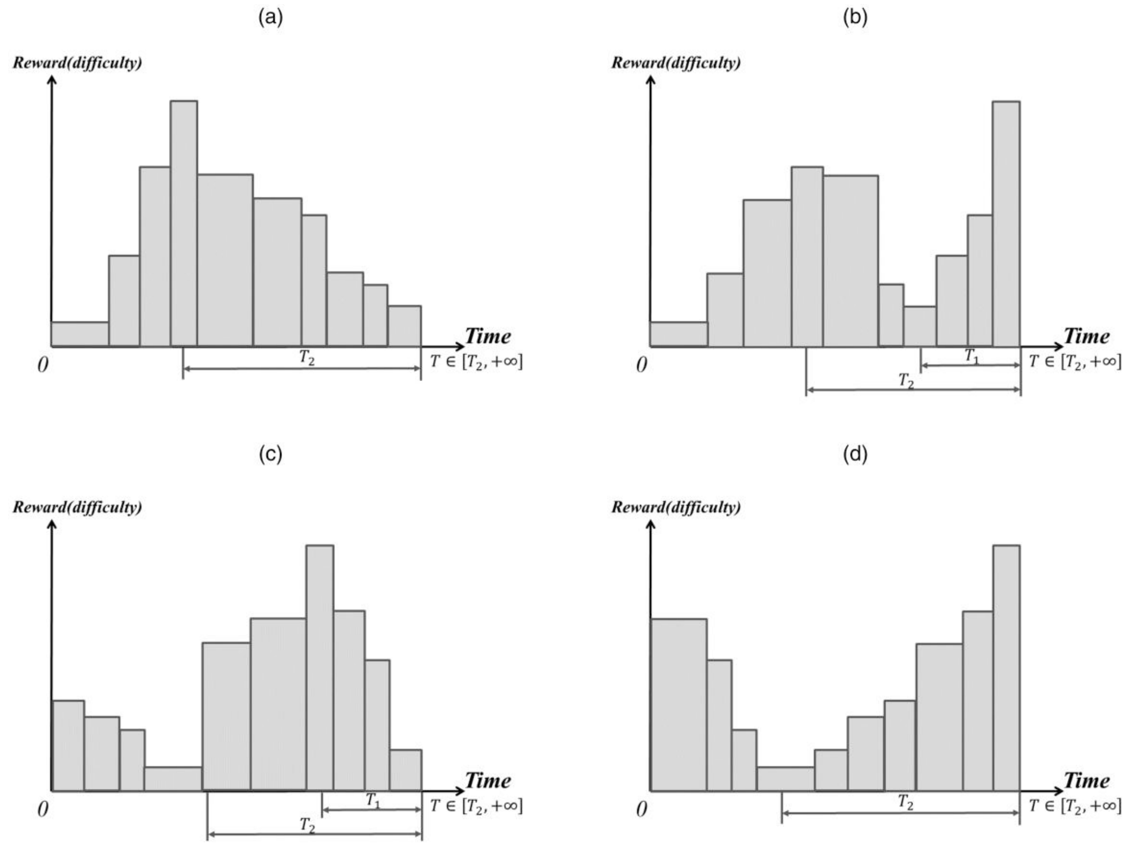
These two extreme cases are well-covered by previous literature, whereas the intermediate case (ii) yields fresh insights. In case (ii), rewards and difficulties are roughly even in weight (with k between $\underline{k}$ and $\overline{k}$) and so the degrees of reward-seeking α and difficulty aversion β start to play a pivotal role. Because this "second-order effect" has bite, we no longer see the "extreme" cases of U-shaped and inverted U-shape. (We will see the even more extreme designs of pure crescendo and diminuendo arise in short duration levels in Theorem 2.) Indeed, case (iia) yields an N-shaped design that starts with a preliminary crescendo of difficulty followed by a U-shaped finish. Case (iib) has an inverted N-shaped design that starts with a diminuendo of difficulty followed by an inverted U-shaped finish. In both of these cases, we see a more even distribution of hard and easy elements, common to many popular video games. Let's examine these two subcases in turn.

Case (iia) is distinguished by game designs with similar rewards and difficulties, but where the degree of reward-seeking outstrips the degree of difficulty aversion; that is, $\alpha > \beta$. It is our contention that $\alpha > \beta$ is a common case for players who play games largely as a form of entertainment. The player adjusts quickly to rewards and so demands increasing rewards to maintain a given level of utility. On the other hand, the players adapt slowly to difficulty and so suffer a lot of disutility if there is a sudden spike in challenge.

This is reflected in the optimal N-shaped design. The design starts with a crescendo of difficulty and rewards so that the player adjusts to difficulty slowly and diminishes the amount of disutility accrued. Once accustomed to a certain level of difficulty at the peak of the crescendo, the remaining pattern is similar to a pure entertainment experience with a U-shaped design. The diminuendo subsequence in the middle of the level serves to reset the reference point for rewards and helps the player relax. The final crescendo sequence helps create a grand ending experience accentuated in the memory of the player who otherwise adapts quickly to rewards.

By contrast, case (iib) is distinguished by game designs with similar rewards and difficulties, but now where the degree of difficulty aversion outstrips the degree of reward seeking, that is, $\alpha < \beta$. We believe this scenario is common in "serious" games that are played not purely for entertainment but for educational, training, and adherence purposes (e.g., Plass et al. (2015) and Kalmpourtzis (2018) for discussions of study games, Sardi et al. (2017) for medical programs, and Seaborn and Fels (2015) for workplace incentive programs). The online game *Prodigy* is designed for school-age children to learn mathematics in a role-playing game (RPG) style environment. In a game like *Prodigy*, players are in a

Figure 4. Optimal Structures of the LDPP



Notes. (a) Inverted U-shape when $k \leq \underline{k}$. (b) N-shape when $\underline{k} < k \leq \overline{k}$ and $\alpha > \beta$. (c) Inverted N-shape when $\underline{k} < k \leq \overline{k}$ and $\alpha < \beta$. (d) U-shape when $k > \overline{k}$.

learning mode (no one mistakes Prodigy for a pure entertainment game) so they can adjust quickly to difficulty, whereas they are pleasantly surprised to be getting rewards while learning math and so adjust slowly in their expectations of rewards.

The inverted N-shaped design is intuitive under these conditions. The initial diminuendo subsequence at the beginning provides the player with a spike in initial rewards, which translates into a spike of utility because adaptation to rewards is slow. On the other hand, an initial spike in difficulty that slowly diminishes is expected in an educational game whose goal is to teach a difficult topic like mathematics. Players quickly adjust to these expectations as they figure out the types of questions or problems they are being presented with. As the player moves to the later part of the level, the inverted U-shaped subsequence is reminiscent of case (i). Players have experienced enough rewards to undertake an ascending peak of rewards and difficulties, followed by a cool down. The decrescendo at the end takes advantage of a steady decline in disutility as the game elements become easier.

It is important to appreciate the differences between case (iia) and case (iib). In case (iia), ending the level with a U-shaped subsequence will create high utilities, but we need a crescendo subsequence in the beginning to let the player adapt to difficulty first. In case (iib), difficulty plays a more important role. This time, ending the level with an inverted U-shaped sequence will create high utilities, but we need a diminuendo subsequence in the beginning to give the player an initial sense of accomplishment at the outset. This design takes advantage of their fresh mind at the outset to get some of the difficult tasks under their belt, then reset their nerves for a final inverted U-shaped push.

From both the theoretical results and the practical use, we can tell that the game designer will have to understand the difficulty of the game to make a better design. If he is designing a low-difficulty game, then he can create an experience similar to pure entertainment. If he is designing a high-difficulty game, then he can forge an experience like a workout. When the designer is designing a game with medium difficulty, then the distribution of easy and hard elements should be more balanced and follow the characteristics of the players. For games with a greater emphasis on entertainment, an N-shaped design with a mini-boss-end-boss structure is optimal. For games designed to educate or train, an inverted N-shaped design should be considered.

We complete the analysis initiated in Theorem 1 by investigating the case of short levels (i.e., when $T < T_2$).

Theorem 2. *When the game duration is short (i.e., $T < T_2$), the optimal schedule $\boldsymbol{\pi}^*$ of the LDPP exhibits the following structure:*

(i) *When $k \le \overline{k}$, the optimal structure degenerates to a diminuendo sequence if $T < T_2$.*

(ii) *When $\underline{k} < k \le \overline{k}$, there are two cases of the optimal structure.*

(iia) *When $\alpha > \beta$, the optimal structure degenerates to a U-shaped sequence if $T_1 < T < T_2$, and a crescendo sequence if $T < T_1$.*

(iib) *When $\beta > \alpha$, the optimal structure degenerates to an inverted U-shaped sequence if $T_1 < T < T_2$, and a diminuendo sequence if $T < T_1$.*

(iii) *When $k > \underline{k}$, the optimal structure degenerates to a crescendo sequence if $T < T_2$.*

The previous proposition suggests that game duration is another key issue. If the game is designed with a compact duration, then you can only form part of the optimal sequence. This echoes the findings in Das Gupta et al. (2016) and Li et al. (2022) that the optimal structure may degenerate when the duration is not long enough. Crescendo and diminuendo designs are also common in games. Mobile games in the "endless runner" genre (like the popular *Jetpack Joyride*) start easy and quickly build toward greater and greater difficulty, reflecting a crescendo design. By contrast, many of the original arcade games, like *Donkey Kong*, start punishingly difficult. This reflects the different types of players that the games were designed to attract. In the arcades of the late 1970s and early 1980s, video gaming had a public and competitive feel (captured, for example, in the 2007 documentary *King of Kong: A Fistful of Quarters*). Games that presented a stern challenge were favored by players as a way to "rank" the gaming abilities of those in the arcades.

Table 2 summarizes the results in Theorems 1 and 2 on the optimal structure of levels based on our model.

We can see that the value-to-reward ratio k, parameters α, β, and game duration T can jointly affect the optimal structure. When $T < T_2$, the optimal structure starts to degenerate. N-shaped and inverted N-shaped sequences can be optimal only when the reward ratio is in the medium level $\underline{k} < k \le \overline{k}$.

Finally, we consider the special case where reward equals difficulty (i.e., $k = 1$). In this case, we can interpret that the player accumulates a sense of accomplishment purely by the challenge of the game elements. The following corollary gives a very compact breakdown of how all six possible game designs (crescendo, diminuendo, inverted U-shape, U-shape, N-shape, inverted N-shape) are possible as the remaining parameters (besides k) change.

Corollary 1. *When the reward equals the difficulty of each game element, in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPP, the elements' rewards (difficulties) are in the following structure.*

(i) *When $\alpha > \beta$, the optimal structure is an N-shaped sequence if $T > T_2$, a U-shaped sequence if $T_1 < T < T_2$, and a crescendo sequence if $T < T_1$.*

(ii) *When $\beta > \alpha$, the optimal structure is an inverted N-shaped sequence if $T > T_2$, an inverted U-shaped sequence if $T_1 < T < T_2$, and a diminuendo sequence if $T < T_1$.*

One takeaway here is how pivotal a role is played by the parameters α and β. As we discussed earlier, one can associate $\alpha > \beta$ with audiences that are looking for more entertainment experiences, whereas $\alpha < \beta$ is associated with more learning or training experiences. Corollary 1 highlights that these two orientations support fundamentally different optimal level designs. This is a nontrivial design insight for developers of educational games who might otherwise benchmark their level design against entertainment-focused games.

4.2. Game Design with General-Reward Scheme

In this section, we consider the general case where rewards and difficulties are no longer proportional. In this case, we refer to (12) as the level design problem with general reward (LDPG). When there is no proportional relationship, we were, for the most part, only able to analyze (12) numerically as an integer optimization problem. See Online Appendix C for a description of the IP formulation that we worked with and see Section 5 for our numerical findings.

One special case we were able to analyze was the case where all elements share a common reward (alternatively, a common difficulty). When the elements share a

Table 2. Optimal Structures of the LDPP

k	Reward ratio	α, β	T	Duration	Optimal structure
$k \le \underline{k}$	Low	$\alpha,\beta > 0$	$T > T_2$	Long	Inverted U-shape
			$0 < T \le T_2$	Short	Diminuendo
$\underline{k} < k \le \overline{k}$	Medium	$0 < \beta < \alpha$	$T > T_2$	Long	N-shape
			$T_1 < T \le T_2$	Medium	U-shape
			$0 < T \le T_1$	Short	Crescendo
		$0 < \alpha < \beta$	$T > T_2$	Long	Inverted N-shape
			$T_1 < T \le T_2$	Medium	Inverted U-shape
			$0 < T \le T_1$	Short	Diminuendo
$k > \overline{k}$	High	$\alpha,\beta > 0$	$T > T_2$	Long	U-shape
			$0 < T \le T_2$	Short	Crescendo

common difficulty $d_i = d$ for all $i \in [n]$, the level design problem with general reward and fixed difficulty (LDPGFD) can be expressed by

$$\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) = \sum_{i=1}^{n} r_{(i)}\left(\Phi\left(\bar{t}_i \mid \alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} \mid \alpha,\gamma\right)\right) - \sum_{i=1}^{n} d\left(\Phi\left(\bar{t}_i \mid \beta,\gamma\right) - \Phi\left(\bar{t}_{i+1} \mid \beta,\gamma\right)\right),$$
$$= \sum_{i=1}^{n} r_{(i)}\left(\Phi\left(\bar{t}_i \mid \alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} \mid \alpha,\gamma\right)\right) - d\Phi\left(T \mid \beta,\gamma\right). \tag{13}$$

When the elements share a fixed reward $r_i = r$ for all $i \in [n]$, the level design problem with general reward and fixed reward (LDPGFR) can be expressed by

$$\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) = \sum_{i=1}^{n} r\left(\Phi\left(\bar{t}_i \mid \alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} \mid \alpha,\gamma\right)\right) - \sum_{i=1}^{n} d_{(i)}\left(\Phi\left(\bar{t}_i \mid \beta,\gamma\right) - \Phi\left(\bar{t}_{i+1} \mid \beta,\gamma\right)\right),$$
$$= r\Phi\left(T \mid \alpha,\gamma\right) + \sum_{i=1}^{n} d_{(i)}\left(\Phi\left(\bar{t}_{i+1} \mid \beta,\gamma\right) - \Phi\left(\bar{t}_i \mid \beta,\gamma\right)\right). \tag{14}$$

In these two settings, we were able to show the following structural result. We should remark that this result can be shown via similar arguments to that found in Das Gupta et al. (2016) because once rewards or difficulties are fixed, the model effectively becomes a "single factor" model like that studied in Das Gupta et al. (2016). To be self-contained, we include a detailed proof in the online appendix but are careful to point out the similarities between our argument and those found in Das Gupta et al. (2016).

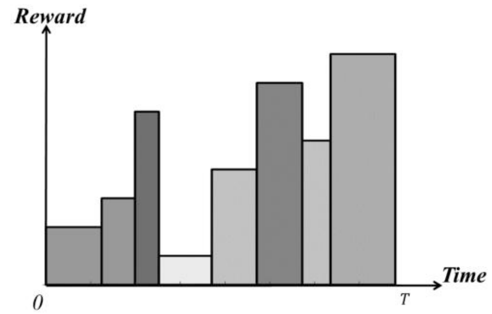
Theorem 3. *Recall that $T_0(\alpha,\gamma)$ and $T_0(\beta,\gamma)$ are the unique inflection point of function $\Phi(\alpha,\gamma)$ and $\Phi(\beta,\gamma)$, respectively.*

(i) *When the elements share a fixed difficulty in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPGFD, the rewards are in a U-shaped sequence if $T > T_0(\alpha,\gamma)$, and a crescendo sequence if $T < T_0(\alpha,\gamma)$.*

(ii) *When the elements share a fixed reward in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPGFR, the difficulties are in an inverted U-shaped sequence if $T > T_0(\beta,\gamma)$, and a diminuendo sequence if $T < T_0(\beta,\gamma)$.*

Although we cannot prove the properties of optimal structure for the general level design problem (12), we found some interesting results that are commonly observed in video game level designs seen in practice. We mention here one numerical instance with an optimal sequence, as illustrated in Figure 5. The parameters defining the instance can be found in Table EC.3 in Online Appendix D.

Figure 5. Optimal Sequence of the LDPG



Note. The heights of the shaded bars mark the rewards of the game elements, and the grayscale shadings represent the difficulty of the game elements.

In Figure 5, the heights of the shaded bars mark the rewards of the game elements and the grayscale shadings represent the difficulty of the game elements. The height of the bars indicates the size of the reward, while darker shades correspond to more difficulty.

In this example, the optimal schedule exhibits a wave-like structure. Difficulty increases for a certain period (e.g., the first three elements in Figure 5), then the game turns easy in a short time, followed by another crescendo subsequence of difficulty (e.g., the middle three elements in Figure 5). To match the changes in difficulty, the reward sequence also follows a wave-like structure. This design pattern matches recommendations by game designers like Hiwiller (2015) and Hodent (2017), and service designers like Hormeß and Lawrence (2012), which indicate that a structure with multiple peaks and drops is preferred by the players and customers.

We may develop an intuition for how the wave structure arises via the discussion that follows Theorem 1. Peaks in difficulty are followed by a "cooling" period to slow the stress process. Rewards also work in patterns of crescendos and diminuendos so that players do not become "numb" to high rewards by adjusting their expectations. Players look for a challenging and rewarding experience, making intermittent crescendos of difficulty and reward attractive, but an ever-increasing crescendo makes players increasingly stressed at the same time of becoming inured to the rewards. An example of a popular game with this "wave-like" pattern of difficulty and rewards is the *Plants vs Zombies* series of mobile games. In the games in this series, the player makes defenses using plants to ward off waves of attacking zombies. Zombies come in waves of varying difficulties.

Table 3 summarizes the optimal structures and the conditions, and we summarize the mathematical expressions of the optimal structure in Table EC.2 in Online Appendix B.

As a final note, we have included the description of a related model in an appendix of the paper (see Online Appendix E). One of the distinguishing features of

Table 3. Optimal Structures of the LDPG

Problem	Situation	T	Duration	Optimal structure
LDPGFD	Fixed difficulty	$T > T_0$	Long	U-shape
		$0 < T \le T_0$	Short	Crescendo
LDPGFR	Fixed reward	$T > T_0$	Long	Inverted U-shape
		$0 < T \le T_0$	Short	Diminuendo
LDPG	General	$T > 0$	Any duration	Wave-like

games is that elements are virtual, meaning that they can be reproduced costlessly multiple times within a level. This is in contrast with service design problems, like those studied in Das Gupta et al. (2016), where repeating a service element may be costly or not possible.

Here the decision space is extended to allow the level designer to choose the number of each game element to deploy (within a given time limit) as well as how to sequence these elements. This is related to, but different than, the challenge of choosing the duration of elements studied in Das Gupta et al. (2016). In Online Appendix E, we study the optimal structure of the final sequence of game elements (allowing for repeats) in both the proportional reward and general reward settings. Our results in this setting are consistent with the findings in the base model we study in this section. Accordingly, these results should be viewed as a robustness check for our main conclusions.

5. Numerical Study

We use numerical approaches to gain further insight into the structure of optimal level designs. We began this analysis with an illustrative result in Figure 5 in the previous section but take a more systematic approach. We are interested in questions of the prevalence of the different optimal level designs (U-shaped, N-shaped, crescendo, etc.) across many instantiations. In the first two subsections, we look at structured reward and difficulty data (either proportional or other structured protocols). These results show that N-shaped optimal designs are not uncommon (e.g., Table 4) and are often fairly well approximated by U-shaped designs (Table 5) in remembered utility. Diminuendo and the inverted U- and N-shaped designs are much less common in our experiments. Optimal U-shaped designs are the most common.

In a third section, we examine optimal level structure when rewards and difficulties of elements are unstructured. As might be predictable, the optimal design is wave-like with a high probability, mimicking what we see in Figure 5. N-shaped designs are also more prevalent than U-shaped designs. In this section, we also investigate the question of where the boss (i.e., the most difficult element) is typically positioned and the average distance between "peaks" of waves.

In our experiments, we consider level design problems with eight elements. We generate 150 instances of rewards, difficulties, durations, and parameters α, β, and γ. For each instance in Section 5.1, we randomly generate difficulties $\boldsymbol{d}$ and reward ratios k from independent uniform distribution Uniform$(0, 10)$, and we set the reward vector as $\boldsymbol{r} = k\boldsymbol{d}$. For each instance in Section 5.2, we use the protocol-based reward as presented in Table 6 that look at structured but nonproportional reward structures and generate random data appropriately. For the instances in Section 5.3, the rewards, difficulties, and durations are randomly generated from independent uniform distribution Uniform$(0, 10)$. As Das Gupta et al. (2016), parameters α, β, and γ are drawn from independent Gamma distribution Gamma(k_G, θ_G), with shape k_G and scale θ_G. This leads to unstructured rewards and difficulties. We conduct the numerical study under parameters $k_G \in \{1, 2, 3\}$ and $\theta_G \in \{0.125, 0.25, 0.375\}$. For all experiments, we compute the optimal sequence by solving the IP in Online Appendix C.

Table 4. Percentage of the Optimal Structures Under the Proportional Reward Scheme (%)

(k_G, θ_G)	Crescendo	Diminuendo	Inverted U-shape	U-shape	Inverted N-shape	N-shape	N-shape in theory
(1, 0.125)	22.67	4.67	10.00	58.67	0.67	3.33	44.00
(1, 0.25)	6.67	0.67	12.00	61.33	2.67	16.67	44.67
(1, 0.375)	6.67	0	12.00	64.00	4.67	12.67	34.67
(2, 0.125)	2.67	0.67	10.00	68.67	2.67	15.33	38.67
(2, 0.25)	0	0	19.33	56.67	1.33	22.67	41.33
(2, 0.375)	0.67	0	18.67	64.67	2.67	13.33	34.67
(3, 0.125)	0	0	8.67	66.67	6.67	18.00	40.67
(3, 0.25)	0	0	18.00	60.00	2.67	19.33	36.00
(3, 0.375)	0	0	22.67	62.00	2.00	13.33	34.67

Table 5. Average Gap of Sequences in Different Structures When N-Shaped Sequence Is Optimal (%)

(k_G, θ_G)	Crescendo	Diminuendo	Inverted U-shape	U-shape	Inverted N-shape	N-shape
$(1, 0.125)$	112.39	122.24	108.58	5.54	89.93	0
$(1, 0.25)$	170.17	104.91	89.95	7.33	80.23	0
$(2, 0.125)$	108.90	99.76	84.55	8.01	72.60	0
$(2, 0.25)$	97.55	107.68	89.82	8.94	87.08	0

5.1. Sequencing Under Proportional Reward Scheme

In this section, we analyze the optimal structure under a proportional reward scheme. We start by analyzing the distribution of optimal structures. We record the percentage of different optimal structures across 150 instances and present the results in Table 4.

We can see from the table, as k_G increases, less crescendo and diminuendo sequences are optimal. An explanation is that there are fewer degenerate cases as k_G grows. Crescendo and diminuendo sequences are degenerate cases, as we summarized in Table 2. The percentage of inverted and U-shaped optimal sequences is stable under different situations. There are more N-shaped optimal sequences when θ_G is of medium value, and there are only a few instances where the inverted N-shaped sequence is optimal. The last column of the table, labeled "N-shape in theory" records the percentage of instances where the parameters are such that an N-shape design is optimal if the level was sufficiently long (i.e., $T > T_2$). Around 40% of instances were found to be in this category. However, because we conducted our numerical study with only eight elements, this may not be long enough to show the whole N-shaped structure in some of these instances. In these cases, the N-shaped sequence degenerates into a crescendo or U-shaped sequence, even if the parameters allow for an N-shaped structure.

Furthermore, we study the differences in remembered utility among different sequences and compare them when N-shaped sequence is optimal. Let $\boldsymbol{\pi}_i$ stand for the schedule in structure i, where $i \in \{1, \ldots, 6\}$ stands for the crescendo, diminuendo, inverted U-shape, U-shape, inverted N-shape, and N-shape, respectively. Let the sequence in structure $i \in \{1, \ldots, 6\}$ that has the largest remembered utility be $\boldsymbol{\pi}_i^*$, and the optimal sequence be $\boldsymbol{\pi}^*$. Then, the optimal gap of the best solution in structure i is given by $p_i = |S(\boldsymbol{\pi}^*) - S(\boldsymbol{\pi}_i^*)| / S(\boldsymbol{\pi}^*) \cdot 100\%$.

Next, we report on the optimality gap of all structures for the instances where an N-shaped sequence is optimal in Table 5. We can see from the table that the optimal gap is stable with different (k_G, θ_G) for many structures. U-shaped sequence has the least gap to the N-shaped sequence, and the gap increases as k_G grows. The optimal U-shaped sequence thus acts as a reasonable heuristic, but the average gap is still considerable, often larger than 5%.

5.2. Sequencing Under Protocol-Based Reward Scheme

Of course, we would like to go beyond proportional rewards. Instead of going immediately to completely general reward structures (which we take up in the next section), here we restrict to nonproportional but structured protocols for rewards and difficulties. Our main question is whether N-shaped designs remain a prevalent design in these more general scenarios. These protocols can be thought of as broad strategies to enhance the engagement of players at the game element design stage. Table 6 illustrates an example of the protocols we consider.

In protocol 1, the reward increases as the difficulty grows, but it follows a nonlinear relationship $r_i = d_i^2/4$ rather than a linear relationship with the difficulty as in the proportional case. In protocol 2, the reward decreases as the difficulty grows. The rewards are in a U-shaped sequence in protocol 3 and an inverted U-shaped sequence in protocol 4.

We investigate the structure of the optimal sequence under the same environment as Section 5.1. However, because we consider a nonproportional reward scheme, we cannot identify the structure of the sequence based on either the reward or the difficulty. Instead, we analyze the reward ratios of a sequence of the elements. Let the reward ratio of element i be q_i given by $q_i = r_i/d_i$.

We now present the percentage of different reward ratio sequences in Table 7. Protocol 1 is the only protocol that supports inverted U-shaped sequences. Protocol 1 is closest to the proportional reward scheme, and this is the reason why it supports most of the optimal structures proposed in Section 4.1. Protocol 2 only supports the crescendo, U-shaped, and N-shaped sequence, but not the wave-like sequence. One explanation is that the highest-difficulty elements have the least rewards, and this reduces the number of wave-like sequences because reward and difficulty are negatively correlated. Protocol 3 supports more U-shaped sequences and less

Table 6. Reward Protocols

No.	Reward	Difficulty
1	$\boldsymbol{r} = (1/4, 1, 9/4, 4, 25/4, 9, 49/4, 16)^{\mathrm{T}}$	$\boldsymbol{d} = (1, 2, 3, 4, 5, 6, 7, 8)^{\mathrm{T}}$
2	$\boldsymbol{r} = (16, 49/4, 9, 25/4, 4, 9/4, 1, 1/4)^{\mathrm{T}}$	
3	$\boldsymbol{r} = (16, 9, 4, 1, 1/4, 9/4, 25/4, 49/4)^{\mathrm{T}}$	
4	$\boldsymbol{r} = (1/4, 9/4, 25/4, 49/4, 16, 9, 4, 1)^{\mathrm{T}}$	

Table 7. Optimal Reward Ratio Sequences Under Different Protocols (%)

(k_G, θ_G)	Protocol	Crescendo	Diminuendo	Inverted U-shape	U-shape	Inverted N-shape	N-shape	Wave-like
$(1, 0.125)$	1	27.33	0.67	3.33	61.33	0	6.00	1.33
	2	32.00	0	0	68.00	0	0	0
	3	3.33	0	0	30.00	0	15.33	51.33
	4	1.33	0	0	11.33	0	31.33	56.00
$(1, 0.25)$	1	10.00	2.00	14.00	57.33	0	14.00	2.67
	2	12.67	0	0	87.33	0	0	0
	3	0	0	0	26.00	0	10.00	64.00
	4	0	0	0	9.33	0	26.67	64.00
$(2, 0.125)$	1	4.67	0	6.00	66.00	0	21.33	2.00
	2	2.67	0	0	97.33	0	0	0
	3	0	0	0	20.67	0	1.33	78.00
	4	0	0	0	13.33	0	27.33	59.33
$(2, 0.25)$	1	1.33	0	31.33	45.33	0	13.33	8.67
	2	0	0	0	100.00	0	0	0
	3	0	0	0	18.67	0	0	81.33
	4	0	0	0	4.00	0	37.33	58.67

N-shaped sequences than protocol 4. The relative proportion of the optimal sequence is stable as θ_G changes, but there are less crescendo and diminuendo sequences when k_G is large. Similar to Section 5.1, there may be fewer degenerate instances, and hence, there are fewer crescendo and diminuendo sequences. All of these demonstrate how patterns in the rewards and difficulties leads to related structures in the optimal level design.

5.3. Sequencing Under General Reward Scheme

In this section, we analyze the optimal sequence under a general reward scheme. Following the definition of reward ratio $q_i = r_i/d_i$, we start by presenting the distribution of optimal reward ratio structures in Table 8.

From the table, we see that most of the instances have wave-like structure in the reward ratios. There are more inverted N-shaped and N-shaped sequences than inverted U-shaped and U-shaped sequences. No crescendo or diminuendo sequences are found. There are more wave-like sequences when k_G is moderate, whereas θ_G does not appear to have any systematic impact.

Because of the prevalence of wave-like results in the simulations, the rest of our numerical investigations explore some of the salient features in the "waves" that we see. One obvious feature is the location of the largest "peak." This corresponds to asking about the location of the "boss" of the level. The most common design in practice is to find the "boss" at the end of the level, and so we explore how prevalent this is in our optimal level designs. Another natural question is when "peaks" (bosses and mini-bosses) are batched close together in the "waves" or spaced farther apart. This tells us something about the tempo of difficulty in the optimal level design.

To investigate this, we simplify things by isolating attention to one boss and one mini-boss in our simulations. Based on this, we consider four configurations. Let element $n-1$ be the mini-boss and element n be the boss. The setup of the rewards and difficulties of the bosses are presented in Table 9, where the parameters x and y in the reward columns vary in $\{1, \ldots, 10\}$. We randomly generate 150 instances of the difficulty and reward of the rest nonboss elements with Uniform$(0, 10)$, and we consider an identical duration with $\tau = 5$ for all the elements, which facilitates our analysis of the optimal positions of the bosses.

In the first study, we investigate the optimal position of the boss (i.e., element n). To better present the trend of the change in the optimal position of the boss, we follow configuration 1 in Table 9. The distribution of the optimal positions of the boss is presented in Figure 6. From the figure, we see that the distribution of the optimal boss position changes as x increases. The optimal positions are evenly distributed when x is small. As x increases, there are more instances where the boss is placed in the final slot. Because the reward increases as x grows, the result suggests that the boss with higher reward is optimal to be scheduled at the end of the game. The benefit is that a boss with higher reward scheduled at the end can provide higher remembered utility for the players. Placing the most influencing element at the end is a common phenomenon studied in the literature (Kahneman et al. 1993, Das Gupta et al. 2016).

Table 8. Percentage of the Optimal Reward Rate Structures Under the General Reward Scheme (%)

(k_G, θ_G)	Crescendo	Diminuendo	Inverted U-shape	U-shape	Inverted N-shape	N-shape	Wave-like
$(1, 0.125)$	0	0	0	0.67	3.33	5.33	90.67
$(1, 0.25)$	0	0	0.67	0	2.0	3.33	94.00
$(2, 0.125)$	0	0	0	0	3.33	1.33	95.33
$(2, 0.25)$	0	0	0.67	0.67	0.67	2.67	95.33

Table 9. Configurations of the Numerical Study

	Boss		Mini-boss	
Configuration	Reward	Difficulty	Reward	Difficulty
1	$r_n = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$	$d_n = 15$	$r_{n-1} = 13$	$d_{n-1} = 13$
2a	$r_n = 15$		$r_{n-1} = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_{n-1}$	
2b	$r_n = 60$		$r_{n-1} = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_{n-1}$	
3	$r_n = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$		$r_{n-1} = (0.8 + 0.2 \cdot e^{y-5}) \cdot d_{n-1}$	

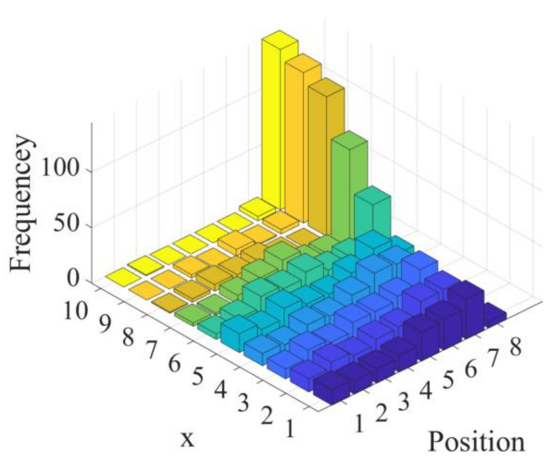
We now study the distance between the boss and mini-boss. We consider two cases listed as configurations 2a and 2b in Table 9: (i) when $r_n = 15$ and (ii) when $r_n = 60$. We record the distance between the bosses under these two cases and present them in Figure 7.

We can see from the figure that the distribution of the distance is different when $r_n = 15$ and when $r_n = 60$. When $r_n = 15$ and x is small, there are more instances with small distances. When $r_n = 60$ and x is large, there are more instances with larger distances. This result echoes the study of Thaler (1985) and Thaler and Johnson (1990), which reveal that people prefer separate gains and integrated losses. The intuition behind the result is that it is better to have wonderful moments separated to enjoy all of them and to integrate unhappy moments to minimize pain.

Based on the previous results, we extend the investigation to the average distance between the bosses. We follow configuration 3 in Table 9 and conduct the numerical study with different values of r_{n-1} and r_n. The distribution of average distances is shown in Figure 8.

From the figure, we can see that the average distance increases as x (r_n) and y (r_{n-1}) increase. The case with $x, y = 10$ has largest average distance. The result suggests that it is optimal to separate the bosses when the reward rates of the bosses are high. The separate-boss solution provides more remembered utility because it takes advantage of the accomplishment, stress, and memory decay processes. Same as before, the results are consistent with the study on the spread effect (Loewenstein and Prelec 1993, Dixon and Verma 2013).

Figure 6. (Color online) Optimal Position of the Boss as r_n Changes, Where $r_n = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$



In this section, we performed numerical simulations to get further insight into optimal level design structure. There is also a natural question about how a game designer can calibrate the parameters of our model (α, β, and γ) based on available game data. We describe how to perform this calibration using real data from the game *Mario Maker 2* in Online Appendix F.

6. Conclusion

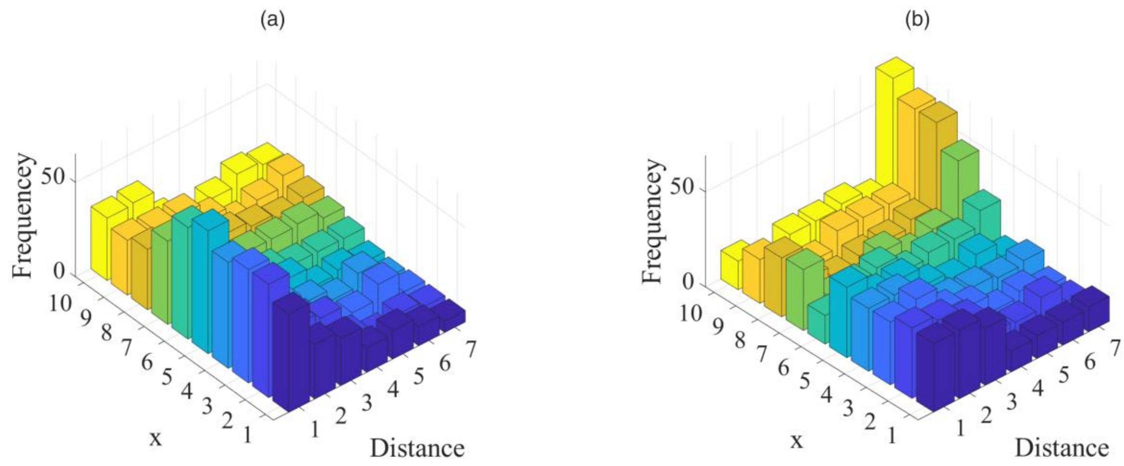
In this paper, we presented a mathematical model to analyze the problem of designing video game levels for players who are reward-seeking, difficulty averse, and suffer from memory decay. Our analysis shows that the relative strengths of these factors and the properties of the game elements used to sequence a level give rise to a variety of different level designs. Online Appendix B summarizes these findings in two convenient tables.

We believe that future research into level design can further explore some of the complexities that we see in practice but are beyond the scope of the current model. First, in this model, we have assumed that players assess utility in a backward-looking manner at the end of the level. This is consistent with the experiential services literature initiated by Das Gupta et al. (2016), but alternative "forward-looking" models (like those found in Ely et al. (2015)) offer other modeling opportunities to examine the optimal structure of video game levels. It would be interesting to see if these alternative theoretical foundations could provide additional insight into why certain level designs are prevalent in practice.

Our model assumes that players "stick around" until the end of the level before deciding whether to continue playing the game. We did this for tractability purposes, because otherwise we would need to track some "forwarding looking" information about what the player thinks will happen later when deciding if to quit a level midstream. We believe an extension that incorporates quitting behavior would be a major contribution because retention of players is a core concern of game design, particularly in free-to-play games.

Some of the results we have may have promise for understanding the design of games in the "endless runner" genre, typified by the high revenue-generating *Jetpack Joyride* on mobile platforms. In endless runners,

Figure 7. (Color online) Distance Between the Bosses as r_{n-1} Changes, Where $r_{n-1} = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$

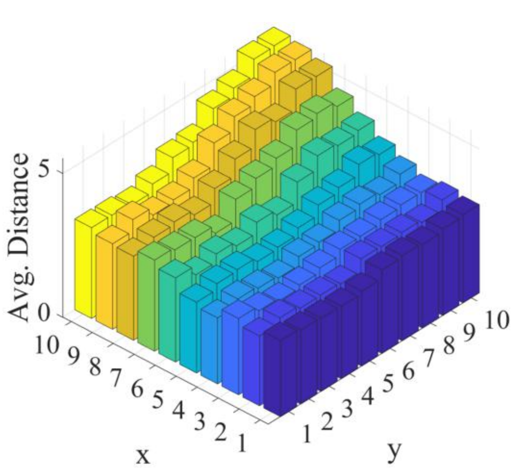


Notes. (a) When $r_n = 15$. (b) When $r_n = 60$.

levels are procedurally generated (meaning generated randomly as they are encountered) and, in principle, have no end (hence the adjective "endless"). An infinite horizon dynamic model would be needed to study this problem, but we believe many of the insights we have developed here would be applicable in this setting, particularly the notion of how "peaks" and "valleys" of difficulty manage reward-seeking and difficulty-aversion behaviors of players.

Finally, there are applications of game design that extend beyond the classical entertainment setting studied here. The concept of gamification—using games to help people learn or comply with medical regimes, for example—is a growing area of application (Plass et al. 2015, Seaborn and Fels 2015, Sardi et al. 2017, Kalmpourtzis 2018). We expect this trend to continue as games become more widely accepted as a form of meaningful interaction in society.

Figure 8. (Color online) Average Distance Between the Bosses as r_n and r_{n-1} Change, Where $r_n = (0.8 + 0.2 \cdot e^{x-5}) \cdot d_n$ and $r_{n-1} = (0.8 + 0.2 \cdot e^{y-5}) \cdot d_{n-1}$



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Endnotes

[1] See https://www.reuters.com/article/esports-business-gaming-revenues-idUSFLM8jkJMl (accessed December 30, 2022).

[2] See https://www.npr.org/2017/05/31/530235165/total-failure-the-worlds-worst-video-game.

[3] In a recent report, Sony revealed that single-player experiences are more popular than multiplayer experiences on the Sony PlayStation platform (https://www.vice.com/en/article/5dp34k/internal-sony-docs-explain-how-activities-became-a-cornerstone-for-ps5). Large game developer EA also reports robust sales of single-player games in 2021 across all platforms (https://www.pcgamer.com/singleplayer-games-live-service/). The slate of best-selling games during 2020 on Steam (the predominant delivery platform for games on PC) features numerous single-player games (Assasin's Creed Odyssey, Uncharted 4, Horizon: Zero Dawn, Cyberpunk 2077, etc.) (https://store.steampowered.com/sale/BestOf2020). A recent market research report shows that single-player mobile games remain the most popular single segment of the video game industry (https://edg.io/resources/blog/state-of-online-gaming-2021/, accessed December 30, 2022).

[4] The use of the word level here should not be confused with the notion of experience level or skill level of players. Level here exclusively refers to a discrete chunk of gameplay.

[5] See https://www.gamedeveloper.com/design/decision-modeling-and-optimization-in-game-design-part-9-modular-level-design (accessed December 30, 2022).

[6] Studies from the psychology literature that examine and measure the adaptation process are referenced in detail in Das Gupta et al. (2016) and Li et al. (2022).

[7] Studies from the psychology literature that discuss memory decay are also explored at length in Das Gupta et al. (2016) and Li et al. (2022).

[8] See for instance this article on Gamasutra on the concept of flow: https://www.gamedeveloper.com/design/cognitive-flow-the-psychology-of-great-game-design (accessed December 30, 2022).

[9] Arguing for a simple normalization of utility without loss does not suffice here because we have already executed a normalization of the utilities for rewards in the previous section. This is why we introduce a secondary argument for why we may assume $\delta = 1$ without loss found in Lemma EC.9.

References

Aflaki S, Popescu I (2013) Managing retention in service relationships. *Management Sci.* 60(2):415–433.

Ascarza E, Netzer O, Runge J (2020) The twofold effect of customer retention in freemium settings. Working paper, Harvard Business School, Cambridge, MA.

Bakshi MS, Arora SR (1969) The sequencing problem. *Management Sci.* 16(4):B–247.

Banister EW, Calvert TW, Savage MV, Bach T (1975) A systems model of training for athletic performance. *Australian J Sports Medicine* 7(3):57–61.

Baucells M, Zhao L (2019) It is time to get some rest. *Management Sci.* 65(4):1717–1734.

Bowman EH (1959) The schedule-sequencing problem. *Oper. Res.* 7(5):621–624.

Buraimo B, Forrest D, McHale IG, Tena J (2020) Unscripted drama: Soccer audience response to suspense, surprise, and shock. *Econom. Inquiry* 58(2):881–896.

Chen J (2007) Flow in games (and everything else). *Comm. ACM* 50(4):31–34.

Chen N, Elmachtoub AN, Hamilton ML, Lei X (2021) Loot box pricing and design. *Management Sci.* 67(8):4809–4825.

Cowley B, Charles D, Black M, Hickey R (2008) Toward an understanding of flow in video games. *Comput. Entertainment* 6(2):1–27.

Csikszentmihalyi M (1990) *Flow: The Psychology of Optimal Experience* (Harper & Row, New York).

Das Gupta A, Karmarkar US, Roels G (2016) The design of experiential services with acclimation and memory decay: Optimal sequence and duration. *Management Sci.* 62(5):1278–1296.

Dixon M, Verma R (2013) Sequence effects in service bundles: Implications for service design and scheduling. *J. Oper. Management* 31(3):138–152.

Dixon MJ, Victorino L (2019) The sequence of service: An affect perspective to service scheduling. *Handbook of Service Science,* vol. II (Springer, Berlin), 49–76.

Dixon MJ, Victorino L, Kwortnik RJ, Verma R (2017) Surprise, anticipation, and sequence effects in the design of experiential services. *Production Oper. Management* 26(5):945–960.

Ely J, Frankel A, Kamenica E (2015) Suspense and surprise. *J. Political Econom.* 123(1):215–260.

Emmons H (1969) One-machine sequencing to minimize certain functions of job tardiness. *Oper. Res.* 17(4):701–715.

Guo H, Hao L, Mukhopadhyay T, Sun D (2019a) Selling virtual currency in digital games: Implications for gameplay and social welfare. *Inform. Systems Res.* 30(2):430–446.

Guo H, Zhao X, Hao L, Liu D (2019b) Economic analysis of reward advertising. *Production Oper. Management* 28(10): 2413–2430.

Hiwiller Z (2015) *Players Making Decisions: Game Design Essentials and the Art of Understanding Your Players* (New Riders).

Hodent C (2017) *The Gamer's Brain: How Neuroscience and UX Can Impact Video Game Design* (CRC Press, Boca Raton, FL).

Hormeß M, Lawrence A (2012) Boom! Wow. Wow! WOW! BOOOOM!!!. *Touchpoint* 4(2):24–27.

Huang Y, Jasin S, Manchanda P (2019) Level Up": Leveraging skill and engagement to maximize player game-play in online video games. *Inform. Systems Res.* 30(3):927–947.

Huang Y, Lim KH, Lin Z (2020) Leveraging the numerosity effect to influence perceived expensiveness of virtual items. *Inform. Systems Res.* 32(1):93–114.

Jiao Y, Tang CS, Wang J (2020) Opaque selling in player-vs-player games. Preprint, submitted April 10, https://dx.doi.org/10.2139/ssrn.3558774.

Kahneman D, Fredrickson BL, Schreiber CA, Redelmeier DA (1993) When more pain is preferred to less: Adding a better end. *Psych. Sci.* 4(6):401–405.

Kalmpourtzis G (2018) *Educational Game Design Fundamentals: A Journey to Creating Intrinsically Motivating Learning Experiences* (CRC Press, Boca Raton, FL).

Kuss DJ, Griffiths MD (2012) Online gaming addiction in children and adolescents: A review of empirical research. *J. Behav. Addiction* 1(1):3–22.

Li Y, Dai T, Qi X (2022) A theory of interior peaks: Activity sequencing and selection for service design. *Manufacturing Service Oper. Management* 24(2):993–1001.

Loewenstein GF, Prelec D (1993) Preferences for sequences of outcomes. *Psych. Rev.* 100(1):91.

Mas-Colell A, Whinston MD, Green JR (1995) *Microeconomic Theory* (Oxford University Press, New York).

Meng L, Zhang C, Shao X, Zhang B, Ren Y, Lin W (2020) More MILP models for hybrid flow shop scheduling problem and its extended problems. *Internat. J. Production Res.* 58(13): 3905–3930.

Meng Z, Hao L, Tan Y (2021) Freemium pricing in digital games with virtual currency. *Inform. Systems Res.* 32(2):481–496.

Naderi B, Roshanaei V, Begen MA, Aleman DM, Urbach DR (2021) Increased surgical capacity without additional resources: Generalized operating room planning and scheduling. *Production Oper. Management* 30(8):2608–2635.

Nalbantis G, Pawlowski T (2019) US demand for European soccer telecasts: A between-country test of the uncertainty of outcome hypothesis. *J. Sports Econom.* 20(6):797–818.

Plass JL, Homer BD, Kinzer CK (2015) Foundations of game-based learning. *Edu. Psych.* 50(4):258–283.

Renault J, Solan E, Vieille N (2017) Optimal dynamic information provision. *Games Econom. Behav.* 104:329–349.

Roels G (2019) Optimal structure of experiential services: Review and extensions. *Handbook of Service Science*, vol. II (Springer, Berlin), 105–146.

Roels G (2020) High-performance practice processes. *Management Sci.* 66(4):1509–1526.

Runge J, Nair H, Levav J (2021) Price promotions for "freemium" app monetization. Preprint, submitted March 19, https://dx.doi.org/10.2139/ssrn.3357275.

Sardi L, Idri A, Fernández-Alemán JL (2017) A systematic review of gamification in e-health. *J. Biomedical Inform.* 71:31–48.

Schell J (2019) *The Art of Game Design: A Book of Lenses* (CRC Press, Boca Raton, FL).

Seaborn K, Fels DI (2015) Gamification in theory and action: A survey. *Internat. J. Humum Comput. Stud.* 74:14–31.

Sheng L, Ryan CT, Nagarajan M, Cheng Y, Tong C (2022) Incentivized actions in freemium games. *Manufacturing Service Oper. Management* 24(1):275–284.

Thaler R (1985) Mental accounting and consumer choice. *Marketing Sci.* 4(3):199–214.

Thaler RH, Johnson EJ (1990) Gambling with the house money and trying to break even: The effects of prior outcomes on risky choice. *Management Sci.* 36(6):643–660.

Tozour P (2013) Decision modeling and optimization in game design, part 9: Modular level design. Accessed December 29, 2022, https://www.gamedeveloper.com/design/decision-modeling-and-optimization-in-game-design-part-9-modular-level-design.

Turner J, Scheller-Wolf A, Tayur S (2011) Scheduling of dynamic in-game advertising. *Oper. Res.* 59(1):1–16.

Vu D, Zhao X, Stecke K (2020) Pay-to-win in video games: Microtransactions and fairness concerns. Preprint, submitted August 20, https://dx.doi.org/10.2139/ssrn.3658537.