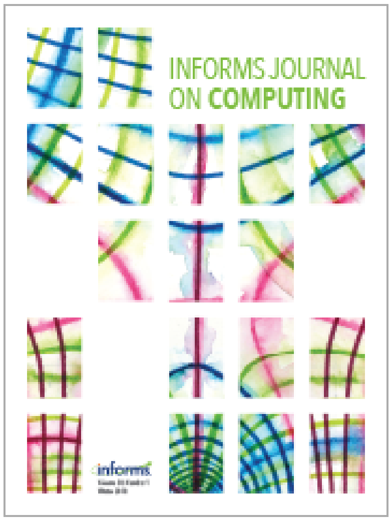

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Stabilizing Grand Cooperation via Cost Adjustment: An Inverse Optimization Approach

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Abstract. For an unbalanced cooperative game, its grand coalition can be stabilized by some instruments, such as subsidization and penalization, that impose new cost terms to certain coalitions. In this paper, we study an alternative instrument, referred to as cost adjustment, that does not need to impose any new coalition-specific cost terms. Specifically, our approach is to adjust existing cost coefficients of the game under which (i) the game becomes balanced so that the grand coalition becomes stable, (ii) a desired way of cooperation is optimal for the grand coalition to adopt, and (iii) the total cost to be shared by the grand coalition is within a prescribed range. Focusing on a broad class of cooperative games, known as integer minimization games, we formulate the problem on how to optimize the cost adjustment as a constrained inverse optimization problem. We prove $\mathcal{NP}$-hardness and derive easy-to-check feasibility conditions for the problem. Based on two linear programming reformulations, we develop two solution algorithms. One is a cutting-plane algorithm, which runs in polynomial time when the corresponding separation problem is polynomial time solvable. The other needs to explicitly derive all the inequalities of a linear program, which runs in polynomial time when the linear program contains only a polynomial number of inequalities. We apply our models and solution algorithms to two typical unbalanced games, including a weighted matching game and an uncapacitated facility location game, showing that their optimal cost adjustments can be obtained in polynomial time.

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Keywords: cooperative game • grand coalition stability • cost adjustment • inverse optimization • integer minimization game • weighted matching game • uncapacitated facility location game

1. Introduction

Cooperative game theory is concerned with the collaborative decision of multiple decision makers. Ideally, if all decision makers collaborate to form a grand coalition (i.e., grand cooperation is achieved), this will lead to a centralized optimal decision with the best payoff to all decision makers, which is the minimum total cost in the context of cost minimization. To implement this centralized optimal decision, the cost incurred by the grand coalition needs to be reasonably allocated to each individual decision maker. In the cooperative game theory (Shapley 1953), this issue is addressed by the concept of the *core* (Gillies 1959). Roughly speaking, the core is the set of cost allocations under which no individual or subcoalitions of players will be better off by leaving the grand coalition. If the core is nonempty, then the grand coalition will be regarded as being stable (Bondareva 1963).

However, many cooperative games are known to have an empty core, so that their grand coalitions are not stable. Such games are also termed as unbalanced (Shapley and Shubik 1969, Owen 1975). See, for example, facility location games (Goemans and Skutella 2004, Mallozzi 2011, Liu et al. 2016), machine scheduling games (Schulz and Uhan 2010, Liu et al. 2018), vehicle routing games (Göthe-Lundgren et al. 1996, Caprara and Letchford 2010), carrier alliance games (Agarwal and Ergun 2010), and vaccine allocation games (Westerink-Duijzer et al. 2020), to mention just a few.

Because the grand coalition usually achieves the social optimum, it is of great practical value to stabilize the grand coalition for an unbalanced game, for which instruments derived in the existing literature are inspired by the following two intuitive ideas. One idea is referred to as *subsidization*, through which an outside authority injects certain outside resources to reduce the grand coalition cost so that the players only need to share a portion of the incurred cost. The other idea is referred to as *penalization*, through which an outside authority can stabilize the grand coalition by enforcing a penalty to any subcoalition that deviates from the grand coalition. For studies on subsidization, an instrument based on the γ*-core* is investigated (Bachrach et al. 2009, Caprara and Letchford 2010, Liu et al. 2016), where γ represents the maximum portion of grand coalition cost that can be allocated to ensure a grand cooperation among all players. In this context, the value of $1-\gamma$ represents the minimum portion of the grand coalition cost that needs to be covered by the outside authority. For the study on penalization, an instrument based on the *least core value* has been introduced (Maschler et al. 1979, Kern and Paulusma 2003, Schulz and Uhan 2010), where the least core value equals the minimum penalty that needs to be enforced to stabilize the grand coalition. Recently, Liu et al. (2018) reveal the complementary roles of subsidization and penalization and apply them simultaneously to develop a new instrument for stabilizing the grand coalition.

For cooperative games whose coalition cost can be written as a function of multiple cost terms, the above instruments can be viewed as *cost impositions*, as they directly impose new cost terms to the incurred cost of certain coalitions. For example, the instrument based on the *least core value* imposes new cost terms to all subcoalitions such that their deviation cost is lifted, while the instrument based on the γ*-core* imposes a new cost term to the grand coalition cost such that the total cost to allocate is reduced.

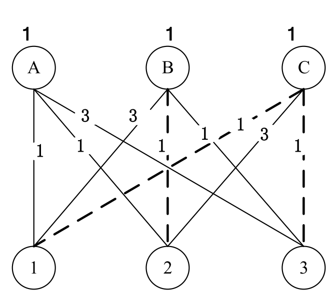
In many situations, each cost term in the definition of a cooperative game can be justified by certain operational or business meaning. For this reason, we may regard such cost terms as being understandable and acceptable to the game players. In such a context, imposing a new coalition-specific cost term may cause some additional issues such as requiring external subsidies for the grand coalition and causing loss of goodwill among subcoalitions. Hence, it raises a question of the possibility to stabilize the grand coalition of an unbalanced game without cost impositions. Our paper provides an affirmative answer to this question. We propose an alternative instrument, referred to as cost adjustment, to stabilize the grand coalition. Instead of directly imposing new cost terms to the incurred coalition cost, the instrument of cost adjustment changes only the existing cost terms of an unbalanced game. Although the idea of cost adjustment is general, in this paper, we focus on the so-called integer minimization game, that is, the game has a corresponding optimization problem that can be formulated by an integer linear program. Many optimization problems in operations research belong to such a category (Caprara and Letchford 2010; Liu et al. 2016, 2018). It is worth noting that a related yet distinct notion named the integer programming game (IP game) exists within the domain of noncooperative games. In an IP game, individual players' action sets consist of selections of integer points originating from separate polytopes. For further insights, one can explore references such as Köppe et al. (2011) and Carvalho et al. (2022).

We will begin by presenting an example in Section 1.1 that demonstrates the motivation and basic idea behind the instrument of cost adjustment, and compare it with the instruments of subsidization and penalization. Following that, we will describe several real-world applications where the idea of cost adjustment is applicable. We then explain in Section 1.2 the connection between our cost adjustment and the inverse optimization. In Section 1.3, we provide a formal summary of the comparisons between the instrument of cost adjustment and other existing instruments. Finally, we describe the contributions, results, and organization of this paper in Section 1.4.

1.1. Motivation Example and Some Practical Incentives

Consider an uncapacitated facility location (UFL) game depicted in Figure 1. In this game, a facility service provider, acting as an outside authority, determines and charges facility opening costs and facility service costs. The

Figure 1. Uncapacitated Facility Location Game



Coalition	Coalition Cost
{1}	2
{2}	2
{3}	2
{1,2}	3
{1,3}	3
{2,3}	3
{1,2,3}	5

game consists of three players, labeled as 1, 2, and 3, and three potential facilities labeled as A, B, and C. Each facility has an opening cost of one and can serve each player j at a service cost r_{ij}, where $i \in \{A, B, C\}$ and $j \in \{1, 2, 3\}$. The service costs for each player-facility pair are shown in Figure 1, near the lines connecting the facilities and the players. Each coalition of this game desires to open some facilities to serve its members, with the total cost, including the facility opening costs and the service costs, minimized.

In summary, one socially optimal solution for the grand coalition is to open facilities B and C, and to serve the players through the dashed lines in Figure 1. Thus, the minimum total cost for the grand coalition is $1+1+1+1+1=5$. In addition, the minimum total cost for each subcoalition of two members, that is, {1, 2}, {2, 3}, and {1, 3}, is $1+1+1=3$, and that for each subcoalition of one member, that is, {1}, {2}, and {3}, is $1+1=2$. Consider any cost allocation $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1+\alpha_2+\alpha_3=5$, which covers the total cost incurred by the grand coalition, where each α_j indicates the cost allocated to player j with $j \in \{1, 2, 3\}$. To ensure that no two-member subcoalition deviates from the grand coalition, it is required that $\alpha_1+\alpha_2 \le 3$, $\alpha_2+\alpha_3 \le 3$, and $\alpha_1+\alpha_3 \le 3$. Thus, $\alpha_1+\alpha_2+\alpha_3 \le (3+3+3)/2 = 4.5$, contradicting $\alpha_1+\alpha_2+\alpha_3=5$. Hence, there must exist a subcoalition of two players who will be better off by leaving the grand coalition. The grand coalition is not stable, or in other words, this game is unbalanced and has an empty core.

To make the grand coalition stable, the facility service provider can do as follows via cost imposition, that is, imposing new cost terms to the incurred cost of certain coalitions, according to the existing instrument based on the idea of either subsidization or penalization:

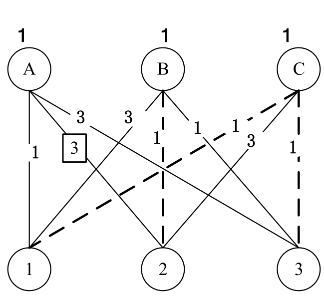
- *Subsidization-based instrument*, for example, providing a subsidy of 0.5, that is, imposing a new cost term of -0.5, to the grand coalition cost. In this case, it is sufficient to stabilize the cooperation among players {1, 2, 3} with a total shared cost reduced from 5 to 4.5 because of the new cost term imposed to the grand coalition.
- *Penalization-based instrument*, for example, charging a penalty of 1/3, that is, imposing a new cost term of 1/3 to each coalition that deviates from the grand coalition. In this case, the grand coalition is stabilized and the total shared cost among players {1, 2, 3} is still 5, while the costs of other coalitions are increased by 1/3 due to the new cost term imposed to the subcoalitions.

As explained in the previous example, either the subsidization-based or the penalization-based instrument can be used to stabilize the grand coalition. They both impose new cost terms on the incurred cost of certain coalitions, that is, providing an additional subsidy of 0.5 to the grand coalition or charging each deviating subcoalition an additional penalty of 1/3. However, as mentioned above, the consequences of subsidization and penalization, such as providing external subsidies and causing loss of goodwill among subcoalitions, are not always preferred by the central authority.

This motivates us to try the instrument defined upon cost adjustment. Specifically, to stabilize the grand coalition, the facility service provider can adjust the existing cost term r_{A2}, that is, the service cost for facility A to serve player 2, from one to three, as shown in Figure 2. After such an adjustment to the existing service cost, the minimum total cost for the grand coalition is still five, and the original optimal solution to the grand coalition is still optimal, but the minimum total cost for subcoalition {1, 2} has increased from three to four. It can be verified that under the cost allocation $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1=\alpha_2=2$ and $\alpha_3=1$, no subcoalition can be better off by leaving the grand coalition. Thus, the game becomes balanced and has a nonempty core. The grand coalition thus becomes stable. Indeed, since cost adjustment does not impose any new coalition-specific cost terms, the outside authority can charge each player directly using the existing practice.

The example provided illustrates that the instrument of cost adjustment can serve as an effective means of stabilizing the grand coalition in certain unbalanced games. Specifically, in the context of the uncapacitated facility location game presented, increasing the service cost is shown to stabilize the grand coalition without compromising the

Figure 2. Cost Adjustment for the Uncapacitated Facility Location Game



Coalition	Coalition Cost
{1}	2
{2}	2
{3}	2
{1,2}	**4**
{1,3}	3
{2,3}	3
{1,2,3}	5

optimality or total cost of the original socially optimal solution. The key difference between the cost adjustment-based instrument and the cost imposition-based instruments is that the former only adjusts existing cost terms (e.g., adjusting r_{A2} from one to three in the example) rather than imposing new cost terms (e.g., providing an additional subsidy of 0.5 or charging an additional penalty of 1/3 in the example).

The above UFL game can find its practical applications, for example, warehouse and logistics sharing operated by service platforms such as RiRiShun Logistics, where cost adjustment can be used to strengthen grand cooperation and increase social welfare. To be specific, RiRiShun is a company specializing in providing warehouse and logistics services for home appliances of China operated by Haier Group. As the Haier subsidiaries are dispersed across the country and have a need for remote storage and delivery services, RiRiShun can establish or lease warehouses in various regions and allocate transportation resources to provide services to the subsidiaries on a global scale. By doing so, RiRiShun can operate warehouses and transportation capacities to optimize operational costs while meeting the demands of the subsidiaries.

In this application, the players are subsidiaries with self-interested storage and delivery needs. The central authority, represented by RiRiShun or Haier, strives to efficiently manage warehouses and transportation capacities, offering storage and delivery services to subsidiaries while minimizing the total operational cost on a global scale. The facility opening costs include setup expenses such as labor cost, equipment operation cost, and warehouse rent, whereas the facility service costs represent the delivery expenses from warehouses to subsidiaries. Each coalition aims to use warehouses and transportation capacities to satisfy their demands while minimizing the total cost. To encourage grand cooperation among subsidiaries and attain global optimization, Haier can adjust the unit delivery cost for each pair of warehouse and subsidiary nodes.

The instrument of cost adjustment has potential for other practical real-world applications. For example, it can be applied to cooperative games occurring in transportation markets, where carriers form alliances to design collaborative service networks with the goal of maximizing their profits. To incentivize grand cooperation, a carrier alliance acting as the central authority can implement the concept of cost adjustment. This involves adjusting existing cost terms, such as creating side-payment agreements for exchanging service capacities, with the aim of guiding individual players toward centralized optimality. (See Chan et al. (2021) for a comprehensive introduction to cooperative games in transportation markets.) By using such side-payment agreements, one can apply the instrument of cost adjustment in various other cooperative applications with pricing mechanisms involved, such as those for communication services, power system services, and warehouse services, where a market operator or alliance has an incentive to coordinate different service providers for global optimization.

1.2. Cost Adjustment and Inverse Optimization

Usually in an unbalanced game, there can be more than one way to adjust costs for stabilizing the grand coalition. Consider the unbalanced UFL game introduced in Section 1.1. In addition to the cost adjustment described earlier, there is a trivial cost adjustment for the facility service provider to make, in which all the facility opening costs and service costs are changed to zero. As a result, each player is allocated zero cost, forming a cost allocation that is in the core of the game after the cost adjustment, so that the grand allocation is stabilized in the new game. Although the original socially optimal solution remains optimal for the grand coalition in the new game, such a trivial cost adjustment is unlikely to be preferred. This is because the total cost shared among the grand coalition after the adjustment becomes zero, which is hardly meaningful in practice.

Therefore, the cost adjustment for stabilizing the grand coalition needs to be optimized in such a way that the total adjustment of existing cost terms is minimized, which ensures the minimum adjustment effort for the outside authority. The cost adjustment also requires regulation to ensure that the grand coalition follows a desirable mode of cooperation as the social optimum. Additionally, the cost shared among the grand coalition, which is equivalent to the total cost of the desirable cooperation under the adjusted cost terms, should fall within a desirable range.

The optimization problem for the instrument of cost adjustment falls under the class of *inverse optimization problems*. Unlike traditional *forward optimization problems*, where we seek an optimal solution to a given objective function, inverse optimization problems involve having a solution or objective function value of a forward optimization problem provided beforehand. The task is then to determine the parameters of the forward optimization problem that yield the given solution or objective function value as optimal.

In the context of the instrument of cost adjustment, the inverse optimization problem aims to minimize the overall effort required to adjust certain parameters (i.e., cost coefficients) shared by a set of optimization problems. These optimization problems correspond to all possible coalitions of players, and the goal is to ensure that the optimal solutions to these problems satisfy certain desirable properties after the cost adjustment. These desirable properties include the common requirement found in inverse optimization, which ensures that the optimal solution and its total cost align with the desired values in the optimization problem of the grand coalition. In addition, the cooperative game defined by the multiple optimization problems must also ensure a nonempty core.

Inverse optimization problems find applications in various fields, such as network optimization (Agarwal and Ergun 2010, Xu et al. 2018), mechanism design (Beil and Wein 2003), electricity markets (Birge et al. 2017), data-driven research (Esfahani et al. 2018, Shahmoradi and Lee 2021, Chan et al. 2022, Gupta and Zhang 2022), and others (Bertsimas et al. 2012, Polydorides et al. 2012, Gorissen et al. 2013, Aswani et al. 2018).

There are two main classes of inverse optimization models studied in the existing literature. In the first class of models, we have a forward optimization problem and, in addition, a given solution to the forward optimization problem, which may not be optimal. We need to adjust some parameters of the forward optimization problem so that the given solution becomes optimal to the new problem. Naturally, we need to minimize the total effort of adjustment on the parameters. The new problem may have a different optimal objective function value from the original problem (Zhang and Liu 1996, 1999; Sokkalingam et al. 1999; Lasserre 2013; Bodur et al. 2022).

In the second class of models, we also have a forward optimization problem, but, in addition, we are given a prescribed range of the optimal objective function value rather than a specific solution to the forward optimization problem. We need to adjust some parameters of the forward optimization problem so that the optimal objective function value of the new problem is within the prescribed range (Zhang et al. 2000, Heuberger 2004, Lv et al. 2010). Another variant of such inverse optimization models (Ahmed and Guan 2005) aims to ensure that the optimal objective value of the new problem is as close as possible to a given desired value.

We refer to the above two classes of problems as the *inverse optimal solution problem* and the *inverse optimal value problem*, respectively. There are also some aliases for the inverse optimal solution problem (e.g., the inverse optimization problem mentioned in Ahuja and Orlin (2001)) and the inverse optimal value problem (e.g., the reverse problem mentioned in Heuberger (2004)).

For inverse optimization with a forward optimization problem that is polynomially solvable, extensive computational complexity results and solution algorithms have been studied (see Ahuja and Orlin (2001) and Heuberger (2004) for detailed introductions). Less is known when the forward problem is formulated as an integer program. In general, such an integer programming version of inverse optimization is hard to solve. To this end, Schaefer (2009) proposes two algorithmic approaches for problems having a property of superadditivity. Wang (2009) develops a cutting-plane algorithm for the inverse mixed integer linear programming problem, and Bodur et al. (2022) extend the results and designs a new cutting-plane solution algorithm by establishing a general characterization of optimality conditions.

The most relevant literature regarding encouraging collaboration via cost adjustment might be mechanism design for transportation alliance. As summarized by Chan et al. (2021), several existing studies have investigated cooperative transportation games in transportation markets with multiple carriers (players). In these games, each carrier has demands to transport people or goods across a transportation network and uses its transportation resources to make profits by satisfying its transportation demands. If a centralized operator is involved, a cooperatively optimal solution (optimal solution for the grand coalition) may be favored, but this can violate individual or coalitional rationality. To achieve global optimality, the operator can make a cost-per-arc side-payment agreement among players, in which carriers are rewarded for sharing resources along arcs and penalized for using resources to satisfy their own demands. The parameters of this cost-per-arc side-payment agreement can be computed using inverse optimization to ensure that no subset (or only subsets containing a few number of carriers) finds it more profitable to form a subcoalition. In some of these existing studies, the characteristic function of the transportation game can be formulated as a linear program with only continuous decision variables, making the corresponding inverse optimization problem computationally tractable (Agarwal and Ergun 2008). Although in other studies where the characteristic function can only be formulated as an integer program (with integer decisions), the corresponding inverse optimization problem may be computationally intractable. In these cases, only solutions to the relaxation of the inverse optimization problem are computed (Agarwal and Ergun 2010, Houghtalen et al. 2011, Zheng et al. 2015), which, however, may still violate individual or coalitional rationality.

Our paper differs from the existing studies on mechanism design for transportation alliance in several ways. First, we investigate more general cooperative games with characteristic functions that can be formulated as integer programs. Second, we allow adjustments on various types of cost coefficients in addition to those in the cost-per-arc side-payment agreement. Third, we take into account the full dimension of individual and coalitional rationalities. Fourth, our study goes beyond ensuring the optimality of the prescribed cooperative solution, as it also considers the need to ensure a desirable range of cost sharing among players. Moreover, our study focuses on deriving theoretical results on computational tractability and feasibility conditions for the corresponding inverse optimization problems, as well as on developing efficient algorithms if the problems are tractable and feasible.

Table 1. Comparisons of Instruments Defined on Cost Adjustment and Cost Imposition

Comparisons (action)	Cost adjustment (adjusting cost coefficients)	Subsidization (providing subsidy)	Penalization (charging penalty)
Imposing new cost term	No	Yes	Yes
Adjustable total shared cost	Yes	—	—
Controllable cooperation scheme	Yes	—	—
External resources required	No	Yes	No
Loss of goodwill for subcoalitions	Not all	—	All
Applicable to non-OPT games	No	Yes	Yes

1.3. Cost Adjustment vs. Cost Imposition

Although both cost adjustment and cost imposition can be used to stabilize the grand coalition in cooperative games, there are some differences in their implementation. In Table 1, we formally summarize some comparisons between the two methods.

As shown in Table 1, the primary distinction of cost adjustment from cost imposition is that it focuses on modifying existing cost terms instead of introducing new coalition-specific cost terms. As a result, the cost adjustment approach differs from the subsidization-based and the penalization-based instruments in the following critical aspects:

- Cost adjustment is applicable to games that have explicit optimization formulations (OPT Game), including the class of games known as IM games (formally defined in Section 2.1). In these games, such as UFL games, the characteristic functions can be formulated as explicit integer linear programs, and the cost adjustment instrument can be applied to adjust the cost coefficients. On the other hand, cost imposition (such as subsidization and penalization) can be applied to other games with no explicit optimization formulations as long as the characteristic function values are known. This is a limitation of the instrument of cost adjustment.
- Cost adjustment differs from existing subsidy and penalty-based instruments in that it does not impose costs on players that form specific coalitions. Instead, it imposes costs on players without targeting any particular coalition. This approach to stabilizing the grand coalition may be more acceptable to all players involved, making it a promising alternative to existing methods. This motivation underlies our study of cost adjustment.
- Cost adjustment can impose a desirable cooperation scheme and a desirable range of sharing cost among the players, as it takes advantages of the information of the integer-linear-program formulations for the characteristic functions. This is beyond the common aims considered in those existing instruments for stabilizing the grand coalition. It can achieve more targeted and efficient cooperation among players.

The previous implications highlight that each instrument for stabilizing grand cooperation has its own strengths and limitations, which are also associated with different tradeoffs. The instrument of subsidization, for example, involves an opportunity cost for the central authority, while the instrument of penalization may result in a loss of goodwill for subcoalitions. The instrument of cost adjustment, on the other hand, is at the efforts of adjusting cost coefficients of the game itself. Generally speaking, the instrument of cost adjustment is preferred when the central authority seeks to stabilize a grand coalition without introducing new coalition-specific cost terms. However, it is important to understand the strengths and limitations of each instrument and their potential tradeoffs to choose the most appropriate approach for different cooperative game applications.

1.4. Contributions, Results, and Organization

We now summarize the major contributions and results of our paper as follows.

First, we provide the first study on the instrument of cost adjustment to stabilize the grand coalition in a wide range of cooperative games known as *integer minimization games*. In these games, the cost of each coalition, including both the grand coalition and all subcoalitions, is determined by solving an integer program. Notably, all these integer programs share a common set of parameters, including the cost coefficients defining their objective values. Unlike other existing instruments, which are based on cost impositions, this alternative instrument is based on cost adjustment and does not create any new coalition-specific cost terms. Its ability to adjust existing cost terms and impose a desirable cooperation scheme and range of cost sharing, make it an attractive option for central authority looking to ensure stability in cooperative games.

Second, our study on this alternative instrument based on cost adjustment also contributes to the literature on inverse optimization. The cost adjustment problem for stabilizing the grand coalition is essentially an inverse optimization problem. However, when dealing with unbalanced integer minimization games that consider the full dimension of rationality, one faces a new type of inverse optimization that involves an exponential number of optimization problems that define the cooperative game. Such inverse optimization is further complicated by imposing a desirable cooperation scheme and range of cost sharing, and it has never been investigated in any existing studies on inverse optimization. As a result, our study has introduced a new class of applications of inverse optimization, one that bridges the two areas of inverse optimization and grand coalition stabilization in unbalanced cooperative games.

Third, we investigate how to optimize the cost adjustment for stabilizing the grand coalition of an integer minimization game. We formulate the problem as a constrained inverse optimization problem, prove its $\mathcal{NP}$-hardness, and derive its easy-to-check feasibility conditions. To solve such a constrained inverse optimization problem, we develop two solution algorithms based on novel linear programming reformulations. One is a cutting-plane algorithm, which runs in polynomial time when the corresponding separation problem is polynomial time solvable and the ellipsoid method is applied. The other needs to explicitly derive all the inequalities of a linear program, which runs in polynomial time when the linear program contains only a polynomial number of inequalities. These algorithms can also be adapted to produce lower or upper bounds in polynomial time on the objective value achieved by the optimal cost adjustment. Accordingly, our study enriches the limited literature on the integer programming version of inverse optimization.

Finally, we apply our models and solution algorithms to two typical unbalanced cooperative games, namely, the weighted matching game and the UFL game. We show that their optimal cost adjustment can be obtained in polynomial time, and the computational results show that their grand coalitions can be stabilized effectively by only some minor cost adjustments. This not only demonstrates the applicability of our analytical results, but also provides alternatives to stabilizing grand coalitions for these two unbalanced games, as well as reveals some interesting new properties that these two games have.

The paper unfolds as follows. In Section 2, we introduce some preliminaries, formally define the instrument of cost adjustment, and formulate the resulting optimization problem as a constrained inverse optimization problem. In Section 3, we prove the $\mathcal{NP}$-hardness of the constrained inverse optimization problem, develop the solution algorithms, and derive the feasibility conditions. In Section 4, we demonstrate the applications of the proposed instrument and solution algorithms, and in Section 5, we conclude with a discussion on directions for future research. All relevant proofs are provided in the online appendix of this paper.

2. Inverse Optimization Formulation for Instrument of Cost Adjustment

In this section, we present our inverse optimization formulation for the optimization of cost adjustments to stabilize the grand cooperation, for which we need to first introduce some preliminaries about the cooperative game theory and the concept of an integer minimization game.

2.1. Preliminaries: Cooperative Game and Integer Minimization Game

Let $N = \{1, 2, \ldots, n\}$ denote a set of n players. A *coalition* is defined as a nonempty subset of players in N, and N is also referred to as the grand coalition. Let $\mathcal{S} \subseteq 2^N \setminus \{\emptyset\}$ denote the set of potential coalitions (usually, $\mathcal{S} = 2^N \setminus \{\emptyset\}$). A *cooperative game with transferable utilities* can be represented by a pair (N, π), where $\pi : \mathcal{S} \to \mathbb{R}$ denotes a *characteristic function*, which, in the context of cost minimization, maps each coalition $S \in \mathcal{S}$ to a value $\pi(S)$ that denotes the minimum total cost for players in S to accomplish certain work through cooperation. To share costs among the players, a *cost allocation*, denoted by a vector $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n] \in \mathbb{R}^n$, is required, where each α_j for $j \in N$ indicates the cost allocated to player j. By slightly abusing the notation for convenience, we use $\alpha(S) = \sum_{j \in S} \alpha_j$ to denote the total cost allocated to a coalition S for $S \in \mathcal{S}$.

The task of the cooperation game is to share costs among all the players in a fair way. One of the most important concepts for fair cost allocations is the *core*. For a cooperative game (N, π), the core of (N, π) is defined as

$$\text{Core}(N, \pi) = \{\alpha \in \mathbb{R}^n : \ \alpha(N) = \pi(N), \text{ and } \ \alpha(S) \le \pi(S) \text{ for all } S \in \mathcal{S}\}, \tag{1}$$

which is the set of cost allocations $\alpha \in \mathbb{R}^n$ that satisfies a budget balance constraint, $\alpha(N) = \pi(N)$, as well as a coalition stability constraint, $\alpha(S) \le \pi(S)$, for each $S \in \mathcal{S}$.

A cooperative game (N, π) with a nonempty core is also known as balanced (Osborne and Rubinstein 1994), where, formally, (N, π) is called balanced if inequality $\sum_{S \in \mathcal{S}} \lambda_S \pi(S) \le \pi(N)$ holds for every balanced collection of weights $\lambda_S \in [0, 1]$ for $S \in \mathcal{S}$ with $\sum_{S \ni j} \lambda_S = 1$ for all $j \in N$. It is clear that if game (N, π) is balanced, then there is no incentive for any coalition $S \in \mathcal{S}$ to deviate from the grand coalition N. However, as mentioned in Section 1, many cooperative games have an empty core; that is, they are unbalanced, implying that their grand coalitions are not stable (e.g., see the instance of the UFL game illustrated earlier in Figure 1 of Section 1.1).

As illustrated by the example in Section 1.1, for some unbalanced cooperative games, where the characteristic function can be formulated as the optimal objective value of a minimization problem with a vector of costs belonging to the problem's parameters, one can adjust the cost vector to make the games become balanced and thus stabilize the grand coalition. Accordingly, we focus our study on a broad class of such cooperative games, known as the integer minimization (IM) games (Caprara and Letchford 2010). By Definition 1, in an IM game, the value of the characteristic function for each coalition is equal to the optimal objective value of an integer linear program whose cost coefficients represent the problem's cost vector.

Definition 1. A cooperative game with player set $N = \{1, 2, \ldots, n\}$ is called an *integer minimization* game if there exist positive integers $p \in \mathbb{Z}_+$ and $q \in \mathbb{Z}_+$, a matrix $A \in \mathbb{Z}^{p\times q}$, a matrix $B \in \mathbb{Z}^{p\times n}$, a right-hand side vector $E \in \mathbb{Z}^p$, and a cost vector $c \in \mathbb{R}^q$, which consists of the so called existing cost terms, such that for each coalition $S \in \mathcal{S}$, the value of the characteristic function, denoted by $\pi(S;c)$, depends on the cost vector c and is equal to the optimal objective value of the following integer linear program (ILP):

$$\pi(S;c) = \min\{c^T x : Ax \geq By(S) + E,\ x \in \mathbb{Z}^q\}, \tag{2}$$

where $y(S)$ denotes the incidence vector of coalition S with $y_j(S) = 1$ if $j \in S$, and with $y_j(S) = 0$ otherwise. As in Caprara and Letchford (2010), we assume that the ILP in (2) has a nonempty bounded feasible region for all $S \in \mathcal{S}$.

According to Caprara and Letchford (2010) and Liu et al. (2016, 2018), the class of IM games includes many typical unbalanced cooperative games, such as facility location games (e.g., see the example shown in Figure 1, Goemans and Skutella (2004), and Devanur et al. (2005)), machine scheduling games (Schulz and Uhan 2010, 2013), and traveling salesman games (Kimms and Kozeletskyi 2016).

Consider the UFL game introduced in the example shown in Figure 1. Let $M = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ denote the set of facilities, and $N = \{1, 2, 3\}$ denote the set of players. Let f_i indicate the opening cost of facility i for $i \in M$, and let r_{ij} indicate the service cost for facility $i \in M$ to serve player $j \in N$. This UFL game is an IM game with $\mathcal{S} = 2^N \setminus \{\emptyset\}$, because for each coalition $S \in \mathcal{S}$, the value of the characteristic function $\pi_{\mathrm{UFL}}(S;(f,r))$, which depends on the cost vector (f,r), can be represented as the following ILP:

$$\pi_{\mathrm{UFL}}(S;(f,r)) = \min \sum_{i\in M} f_i z_i + \sum_{i\in M}\sum_{j\in N} r_{ij}x_{ij} \tag{3a}$$

$$\text{s.t.} \quad \sum_{i\in M} x_{ij} = y_j(S), \qquad \forall j \in N, \tag{3b}$$

$$x_{ij} \leq z_i, \quad \forall i \in M,\ \forall j \in N, \tag{3c}$$

$$x_{ij} \in \{0,1\}, \qquad \forall i \in M,\ \forall j \in N, \tag{3d}$$

$$z_i \in \{0,1\}, \qquad \forall i \in M, \tag{3e}$$

where each binary variable x_{ij} indicates whether facility i serves player j, each binary variable z_i indicates whether facility i is opened, Constraint (3b) indicates that players and only players in S need to be served, Constraint (3c) indicates that only an opened facility can serve players, and Objective Function (3a) aims to minimize the total cost of facility opening and services provided.

2.2. Formulation: Constrained Inverse Optimization Problem

Consider an IM game $(N, \pi(\cdot\,;c))$ as defined in Definition 1 with $N = \{1, 2, \ldots, n\}$, where the characteristic function $\pi(\cdot\,;c) : \mathcal{S} \to \mathbb{R}$ with $\mathcal{S} \subseteq 2^N \setminus \{\emptyset\}$ is defined as an ILP in (2) with $c \in \mathbb{R}^q$ being the cost vector, which can be adjusted. From the example shown in Figure 2, we have seen that when the IM game $(N, \pi(\cdot\,;c))$ is unbalanced, to stabilize the grand coalition, one can adjust c to a new cost vector $d \in \mathbb{R}^q$ to obtain a new IM game $(N, \pi(\cdot\,;d))$, which is desired to be balanced, that is, to have a nonempty core. Accordingly, we require the new cost vector $d \in \mathbb{R}^q$ to satisfy the following constraint, which is the so-called *balancedness constraint*:

$$|\mathrm{Core}(N, \pi(\cdot\,;d))| \geq 1. \tag{4}$$

It is easy to verify that the zero cost vector $\mathbf{0}$ satisfies the balancedness constraint and that if a cost vector d satisfies the balancedness constraint, then every cost vector μd for $\mu > 0$ also satisfies the balancedness constraint. Thus, either the zero cost vector **0** is the only cost vector satisfying the balancedness constraint, or there exist infinitely many cost vectors satisfying the balancedness constraint.

Accordingly, the cost adjustment needs to be further regulated to achieve, for the grand coalition, (i) a desirable way of cooperation and (ii) a desirable amount of total cost to be shared. To achieve (i), we are given $x^0 \in \mathbb{Z}^q$ satisfying $Ax^0 \geq By(N) + E$, which indicates a *desirable cooperation scheme*. When adjusting the cost vector from c to d, to ensure that the cooperation of the grand coalition follows x^0 as desired, we require x^0 to be an optimal solution to the ILP defined as in (2) for $\pi(N;d)$ under the new cost vector d, or, equivalently, to satisfy the following constraint, which is the so-called *desirable cooperation constraint*:v

$$\sum_{k=1}^{q} d_k x_k^0 = \pi(N;d). \tag{5}$$

In practice, the outside authority, who determines the cost adjustments, can set x^0 to be the optimal solution, or the best-known solution, to the ILP defined for $\pi(N;c)$ under the original cost vector c. When c represents the actual cost issued, such a setting of x^0 ensures the stability of the grand coalition that follows the socially optimal or best-known cooperation scheme.

To achieve (ii), we are given a range $[l,u]$, which is called *a desirable range of cost sharing*. When adjusting the cost vector from c to d, to ensure that the cost to be shared for the grand coalition falls within the range $[l,u]$ as desired, we require $\pi(N;d)$ to satisfy the following constraint, which is the so-called *desirable cost sharing constraint*:

$$l \leq \pi(N;d) \leq u. \tag{6}$$

In practice, the outside authority can set the range $[l,u]$ to be around $\sum_{k=1}^{q} c_k x_k^0$, or even to be $l = u = \sum_{k=1}^{q} c_k x_k^0$. When c represents the actual cost issued, setting $l = u = \sum_{k=1}^{q} c_k x_k^0$ ensures stabilizing the cooperation of the grand coalition such that the actual total cost issued is fully covered by sharing among all the players. Similarly, setting $u \leq \sum_{k=1}^{q} c_k x_k^0$ indicates that not all the actual cost is shared and certain amount of outside subsidy is needed from the central authority. However, setting $\sum_{k=1}^{q} c_k x_k^0 \leq l$ implies that the central operator can collect a certain amount of revenue from the players. In this case, the collected revenue may be used as the expenditures of coordinating players.

Moreover, in this study, we measure the total deviation effort of the new cost vector d from the original cost vector c by the weighted L_1 norm, in the form of $\sum_{k=1}^{q} \omega_k |d_k - c_k|$, where each $\omega_k \geq 0$ for $k \in \{1,2,\ldots,q\}$ represents the unit penalty for adjustment of c_k because it could be interpreted as being the total operating effort of adjustment for the outside authority in practice. Although we do not explicitly demonstrate it within this paper, it is doable to explore the idea of cost adjustment by using other types of norms, such as L_∞ norm in the form of $\max_{k \in \{1,2,\ldots,q\}} |d_k - c_k|$, to measure the deviation effort of d from c.

Accordingly, given an IM game $(N, \pi(\cdot;c))$, a desirable cooperation scheme x^0 satisfying $Ax^0 \geq By(N) + E$, and a desirable range $[l,u]$ of cost sharing, to adjust the cost vector c to stabilize the grand coalition for desirable cooperation and cost sharing, we need to solve a *constrained inverse optimization problem* (CIOP), with the aim of finding a new cost vector d that minimizes the total deviation effort $\sum_{k=1}^{q} \omega_k |d_k - c_k|$ from c, such that d satisfies the balancedness constraint (4), the desirable cooperation constraint (5), and the desirable cost sharing constraint (6). The CIOP can thus be formulated as follows:

$$\text{CIOP} \quad \min \sum_{k=1}^{q} \omega_k |d_k - c_k| \tag{7a}$$

$$\text{s.t.} \quad |\mathrm{Core}(N, \pi(\cdot\,;d))| \geq 1, \tag{7b}$$

$$\sum_{k=1}^{q} d_k x_k^0 = \pi(N;d), \tag{7c}$$

$$l \leq \pi(N;d) \leq u, \tag{7d}$$

$$d \in \mathbb{R}^q. \tag{7e}$$

The CIOP falls into the category of inverse optimization problems. Similar to the classic inverse optimization problems known in the literature (Ahuja and Orlin 2001, Heuberger 2004), the CIOP aims to minimize the adjustment (or perturbation) of the cost vector for a (forward) optimization problem, which is defined for the grand coalition in the CIOP, to achieve a desirable optimal solution and a desirable optimal objective value, due to (7c) and (7d). However, unlike the classic inverse optimization problems, the CIOP proposed in this paper imposes a new balancedness constraint (7b) on the cost vector obtained after the adjustment, which involves an exponential number of optimization problems defined for all the coalitions, making the solution to the inverse optimization problem much more complicated.

3. Solving the CIOP for Instrument of Cost Adjustment

In this section, we first prove the $\mathcal{NP}$-hardness of the CIOP and then develop solution algorithms for the CIOP by two linear programming reformulations, upon which we derive some conditions on the existence of feasible solutions for the CIOP.

3.1. Complexity Result

We are going to show that solving the CIOP in general is $\mathcal{NP}$-hard. This result is not surprising, given that the characteristic function $\pi(S;c)$ of the IM game used to define the CIOP is an ILP, which in general is $\mathcal{NP}$-hard to solve.

However, establishing this $\mathcal{NP}$-hard result is not a trivial task. On one hand, the corresponding ILP of the characteristic function of an IM game cannot be reduced directly to an instance of the CIOP of the same IM game. In fact, as we will demonstrate in Section 4, the tractability of the CIOP may or may not be the same as that of the corresponding ILP. For some IM games (such as the weighted matching (WM) game in Section 4.1), both the CIOP and the corresponding ILP are polynomial time solvable. For some other IM games (such as the UFL game in Section 4.2), the CIOP is polynomial time solvable, but the corresponding ILP is NP-hard. This is consistent with analysis of $\mathcal{NP}$-hardness of other inverse optimization problems, which cannot also be directly derived from the $\mathcal{NP}$-hardness of their original optimization problems (Ahmed and Guan 2005, Schulz and Uhan 2010).

On the other hand, one may falsely argue that the $\mathcal{NP}$-hard result of the CIOP in general can be shown by a reduction from a core nonemptiness test problem, which examines whether a given IM game has a nonempty core, and is known to be $\mathcal{NP}$-hard (Deng and Papadimitriou 1994). Specifically, by setting $l=-\infty$ and $u=+\infty$, and evaluating whether the optimal objective value of (7a) is zero, one can reduce the core nonemptiness test problem to a relaxation of the CIOP that drops the desirable cooperation constraint (7c). Such a reduction implies only that solving a relaxation of the CIOP that drops Constraint (7c) is $\mathcal{NP}$-hard, which, however, cannot show that the CIOP defined in (7a)–(7e) is $\mathcal{NP}$-hard.

To show that solving the CIOP in general is $\mathcal{NP}$-hard, we need to reduce a well-known $\mathcal{NP}$-complete problem, PARTITION (Garey and Johnson 1979), to the CIOP: Consider a set $N=\{1,2,\ldots,n\}$, a positive integer s_j for each $j\in N$. The question is whether there exists $N'\subseteq N$ such that $\sum_{j\in N'}s_j=\sum_{j\in N\setminus N'}s_j$. Given any instance of PARTITION, we assume that $\sum_{j\in N}s_j$ is even, because otherwise no partition can exist.

We now construct an instance of the CIOP by specifying an IM game in which the characteristic function is formulated by an ILP as in (2), as well as a desirable cooperation scheme x^0 and a desirable range $[l,u]$ of cost sharing. For the desirable range $[l,u]$, we set $l=-\infty$ and $u=+\infty$ so that the desirable cost sharing constraint (7d) is always satisfied.

For the IM game to be constructed, we consider $N=\{1,2,\ldots,n\}$ to be the set of players. To define the characteristic function of the IM game, we suppose that each player $j\in N$ has an item of size s_j, corresponding to each integer given in PARTITION. There are three bins for packing the items, with each bin having a capacity of W, where

$$W:=\sum_{j\in N}s_j/2, \tag{8}$$

which is an integer. For each coalition $S\in\mathcal{S}$ where $\mathcal{S}=2^N\setminus\{\emptyset\}$, the players in S need to solve a variant of the bin packing problem, denoted by VBP for short, which aims to pack their items into some bins, with the total packing cost, from the following three sources, minimized:

- Bin cost f_i incurred if bin i is used, for $i=1,2,3$;
- Item cost t_{ij} incurred if item j is placed in bin i, for $i=1,2,3$ and $j\in S$;
- Fixed cost, equal to τ if there are less than n players, and equal to $\min\{0,\tau\}$ otherwise.

Accordingly, for any cost vector $d = (f, t, \tau)$, we define the value of function $\pi_{\mathrm{VBP}}(S; d)$ for $S \in \mathcal{S}$ as the minimum total packing cost of the VBP defined above for S, which can be formulated as the following ILP:

$$\pi_{\mathrm{VBP}}(S; d) := \min \sum_{i \in \{1,2,3\}} f_i \cdot v_i + \sum_{i \in \{1,2,3\}} \sum_{j \in N} t_{ij} \cdot u_{ij} + \tau \cdot h \tag{9a}$$

$$\text{s.t.} \quad \sum_{j \in N} s_j u_{ij} \le W, \quad \forall i \in \{1,2,3\}, \tag{9b}$$

$$u_{ij} \le v_i, \quad \forall i \in \{1,2,3\}, \ \forall j \in N, \tag{9c}$$

$$u_{1j} + u_{2j} + u_{3j} = y_j(S), \ \forall j \in N, \tag{9d}$$

$$nh \ge n - \sum_{j \in N} y_j(S), \tag{9e}$$

$$u_{ij} \in \{0,1\}, \ \forall i \in \{1,2,3\}, \ \forall j \in N, \tag{9f}$$

$$v_i \in \{0,1\}, \ \forall i \in \{1,2,3\}, \tag{9g}$$

$$h \in \{0,1\}, \tag{9h}$$

where each binary variable v_i indicates whether bin $i \in \{1,2,3\}$ is used, and each binary variable u_{ij} indicates whether the item of player $j \in S$ is packed into bin $i \in \{1,2,3\}$. Moreover, from (9e) and (9h), we know that in any feasible solution to the previous ILP for $\pi_{\mathrm{VBP}}(S; d)$, the binary variable h must equal one if $S \subset N$. In addition, because $n - \sum_{j \in N} y_j(N) = 0$, by (9e) and (9h), we have that any optimal solution to the ILP for $\pi_{\mathrm{VBP}}(N; d)$ satisfies that $\tau \cdot h = \min\{0, \tau\}$.

Now, for the instance of the CIOP to be reduced from PARTITION, we use c to denote the cost vector, in which we set the bin cost $f_i = 1$ for $i \in \{1,2,3\}$, the item cost $t_{ij} = 0$ for $i \in \{1,2,3\}$ and $j \in N$, and the fixed cost $\tau = 3$. As a result, the characteristic function $\pi_{\mathrm{VBP}}(S; c)$ for $S \in \mathcal{S}$ of the IM game defined for the CIOP can be formulated as an ILP as follows:

$$\pi_{\mathrm{VBP}}(S; c) = \min \left\{ \sum_{i \in \{1,2,3\}} 1 \cdot v_i + \sum_{i \in \{1,2,3\}} \sum_{j \in N} 0 \cdot u_{ij} + 3 \cdot h : \ (9b)–(9h) \right\}. \tag{10}$$

Moreover, to construct the desirable cooperation scheme x^0 for the CIOP, we consider the variant of the bin packing problem defined earlier, for the n players in N, and under the cost vector c. We denote this problem by VBP$(N; c)$, for which we can establish Lemma 1.

Lemma 1. *For problem* VBP$(N; c)$, *one can obtain a feasible bin packing of the n items in polynomial time such that the total packing cost is not greater than three.*

With the feasible packing obtained according to Lemma 1, we can construct a feasible solution as the desirable cooperation scheme, denoted by $x^0 = (u^0, v^0, h^0)$, to the ILP defined in (10) for $\pi_{\mathrm{VBP}}(N; c)$, by setting $u^0_{ij} = 1$ if and only if item j is packed to bin i, setting $v^0_i = 1$ if and only if bin i is not empty, and setting $h^0 = 0$. The objective value of this solution (u^0, v^0, h^0) equals the total packing cost, which, by Lemma 1, is less than or equal to three. Thus,

$$\sum_{i \in \{1,2,3\}} 1 \cdot v^0_i + \sum_{i \in \{1,2,3\}} \sum_{j \in N} 0 \cdot u^0_{ij} + 3 \cdot h^0 \le 3. \tag{11}$$

Now, consider the CIOP defined for the IM game $(N, \pi_{\mathrm{VBP}}(\cdot; c))$, the interval $[l, u]$ with $l = -\infty$ and $u = +\infty$, and the solution $x^0 = (u^0, v^0, h^0)$ defined previously, which, as shown earlier, can all be constructed from the instance of PARTITION in polynomial time. Based on (11), we can establish Lemma 2.

Lemma 2. *For the given instance of* PARTITION, *there exists* $N' \subseteq N$ *such that* $\sum_{j \in N'} s_j = \sum_{j \in N \setminus N'} s_j$, *if and only if at least one of the following conditions is satisfied:* (i) $\sum_{i \in \{1,2,3\}} v^0_i = 2$; *and* (ii) *the optimal objective value of the* CIOP *is greater than zero.*

From Lemma 2 and the $\mathcal{NP}$-completeness of PARTITION, we have that solving the CIOP in general is $\mathcal{NP}$-hard, and thus, Theorem 1 is obtained.

Theorem 1. *Solving the CIOP in general is $\mathcal{NP}$-hard.*

Remark 1. In our establishment of Theorem 1, we have shown that for any given instance of PARTITION, it has a feasible set partition if and only if the CIOP, which is reduced from the instance of PARTITION, satisfies at least one of the two conditions (i) and (ii) as specified in Lemma 2. It can be seen that condition (ii) is equivalent to a condition that the cost vector c is a feasible solution to the CIOP. This implies that, in general, it is also $\mathcal{NP}$-hard to determine whether a given cost vector is a feasible solution to the CIOP.

3.2. Solution Algorithms Based on Two Linear Programming Reformulations

Consider a CIOP defined in (7a)–(7e) for an IM game $(N;\pi(\cdot;c))$, a given desirable cooperation scheme x^0 satisfying $Ax^0 \geq By(N)+E$, and a desirable range $[l,u]$ of cost sharing. We develop solution algorithms to solve the CIOP based on two linear programming (LP) reformulations of the CIOP, which are illustrated in Sections 3.2.1 and 3.2.2, respectively.

3.2.1. Reformulation LP1.

Reformulation LP1 corresponds to a linear program derived from the CIOP by linearizing the objective function (7a) and the constraints (7b)–(7d) as follows.

Remember that c and d are the respective cost vectors before and after adjustment. We introduce some new variables, including $\nu \in \mathbb{R}$ to indicate the value of $\pi(N;d)$, $s_k^+ \in \mathbb{R}_+$ and $s_k^- \in \mathbb{R}_+$ to indicate the positive and negative parts of the difference $(d_k - c_k)$ for $k \in \{1,2,\ldots,q\}$, and α_j for $j \in N$ to indicate the cost allocation. Let Q_{xy} denote the overall set of pairs (x,y) such that x is a feasible solution to the ILP defined for some coalition $S \in \mathcal{S}$ with $y = y(S)$, that is,

$$Q_{xy} := \left\{(x,y) : Ax \geq By + E,\ y = y(S) \text{ for some } S \in \mathcal{S},\ x \in \mathbb{Z}^q,\ y \in \{0,1\}^n\right\}. \tag{12}$$

We obtain the following linear programming reformulation (LP1) of the CIOP:

$$\text{LP1} \quad \min \sum_{k=1}^{q} \omega_k s_k^+ + \sum_{k=1}^{q} \omega_k s_k^- \tag{13a}$$

$$\text{s.t.} \quad d_k - (s_k^+ - s_k^-) = c_k, \qquad \forall k \in \{1,2,\ldots,q\}, \tag{13b}$$

$$\sum_{j=1}^{n} \alpha_j - \nu = 0, \tag{13c}$$

$$\sum_{k=1}^{q} \overline{x}_k d_k - \sum_{j=1}^{n} \overline{y}_j \alpha_j \geq 0,\ \forall (\overline{x},\overline{y}) \in Q_{xy}, \tag{13d}$$

$$\sum_{k=1}^{q} x_k^0 d_k - \nu = 0, \tag{13e}$$

$$\nu \geq l, \tag{13f}$$

$$-\nu \geq -u, \tag{13g}$$

$$\nu \in \mathbb{R},\ \alpha \in \mathbb{R}^n,\ s^+ \in \mathbb{R}_+^q,\ s^- \in \mathbb{R}_+^q,\ d \in \mathbb{R}^q. \tag{13h}$$

Here, Constraints (13b) ensure that $s_k^+ + s_k^- = |d_k - c_k|$ for $k \in \{1,2,\ldots,q\}$, so that the objective function in (13a) is equivalent to that in the CIOP. Constraint (13c) associates with the budget balance constraint, implying that the total shared cost among the players is equal to ν. Constraints (13d) associate with the coalition stability constraints, indicating that no subcoalitions has the incentive to deviate. For any feasible solution x to the ILP defined for $\pi(N;d)$, because $Ax \geq By(N)+E$, we can see from Constraints (13d) and the definition of Q_{xy} in (12) that $\sum_{k=1}^{q} x_k d_k \geq \sum_{j=1}^{n} \alpha_j$, which, together with Constraints (13c) and (13e), ensures that $\nu = \pi(N;d)$, that is, the grand coalition cost under the new cost vector d is equal to ν. Thus, Constraints (13c), (13d), and (13e) together ensure that there exists a cost allocation in the core of the IM game $(N;\pi(\cdot;d))$, so that Constraints (7b) of the CIOP are satisfied. Moreover, Constraints (13e)–(13g) ensure that x^0 is an optimal solution to the ILP for $\pi(N;d)$ with $\sum_{k=1}^{q} d_k x_k^0 = \nu = \pi(N;d)$ being in the range

$[l, u]$, so that Constraints (7c) and (7d) of the CIOP are also satisfied. Thus, we obtain that reformulation LP1 is equivalent to the CIOP.

Because the number of constraints in (13d) of LP1 is $|Q_{xy}|$, which can be exponentially large, to solve LP1, it is natural to follow a standard cutting-plane algorithm (Bertsimas and Tsitsiklis 1997), which is described in Algorithm 1. The cutting-plane algorithm starts with a restricted solution set, denoted by $\hat{Q}_{xy} \subseteq Q_{xy}$, and computes an optimal solution, denoted by $(\hat{v}, \hat{\alpha}, \hat{s}^+, \hat{s}^-, \hat{d})$, to a relaxation of LP1 that excludes constraints in (13d) for all $(\overline{x}, \overline{y}) \in Q_{xy} \setminus \hat{Q}_{xy}$. It then checks whether $\hat{\alpha}$ and $\hat{d}$ violate any of the excluded constraints in (13d) for $(\overline{x}, \overline{y}) \in Q_{xy} \setminus \hat{Q}_{xy}$. To this end, it needs to find an optimal solution (x', y') to the following separation problem:

$$\Delta := \min\left\{\sum_{k=1}^{q} \hat{d}_k x_k - \sum_{j=1}^{n} \hat{\alpha}_j y_j : (x, y) \in Q_{xy}\right\}. \tag{14}$$

From the definition of Q_{xy} in (12) and the definition of the ILP in (2) for $\pi(S; \hat{d})$ with $S \in \mathcal{S}$, we know that the separation problem in (14) is equivalent to finding $S \in \mathcal{S}$ such that $\pi(S; \hat{d}) - \hat{\alpha}(S)$ is minimized, which implies that

$$\Delta = \min\{\pi(S; \hat{d}) - \hat{\alpha}(S) : S \in \mathcal{S}\}. \tag{15}$$

If $\Delta < 0$, then $\hat{\alpha}$ and $\hat{d}$ violate the constraint $\hat{d}x' - \hat{\alpha}y' \geq 0$, and thus, we add (x', y') to $\hat{Q}^{xy}$, and then solve again the relaxation of LP1 that excludes constraints in (13d) for $(\overline{x}, \overline{y}) \in Q_{xy} \setminus \hat{Q}_{xy}$. Otherwise, $\Delta \geq 0$, which implies that $(\hat{v}, \hat{\alpha}, \hat{s}^+, \hat{s}^-, \hat{d})$ is an optimal solution to LP1, and thus, we obtain that $\hat{d}$ is an optimal solution to the CIOP.

Algorithm 1 (Solving Reformulation LP1 of the CIOP by Cutting Plane)

Step 1. Let $\hat{Q}_{xy}$ be a subset of Q_{xy}, which is initialized by including some pairs $(\overline{x}, \overline{y})$ in Q_{xy};

Step 2. Solve a relaxation of LP1 that excludes constraints in (13d) for $(\overline{x}, \overline{y}) \in Q_{xy} \setminus \hat{Q}_{xy}$, and obtain an optimal relaxation solution $(\hat{v}, \hat{\alpha}, \hat{s}^+, \hat{s}^-, \hat{d})$;

Step 3. Solve the separation problem defined in (14), or equivalently in (15), and obtain its optimal objective value Δ, and its optimal solution (x', y');

Step 4. If $\Delta < 0$, then add (x', y') to $\hat{Q}_{xy}$, and go to step 2; otherwise, $(\hat{v}, \hat{\alpha}, \hat{s}^+, \hat{s}^-, \hat{d})$ is an optimal solution to LP1, and return $\hat{d}$ as an optimal solution to the CIOP.

The essential part of the previous cutting-plane algorithm shown in Algorithm 1 is how to solve the separation problem efficiently in Step 3 to find a violated constraint $\hat{d}x' - \hat{\alpha}y' \geq 0$ for $(x', y') \in Q_{xy}$. This depends on the specific game being studied. If the separation problem can be solved to optimum efficiently, we can apply Algorithm 1 to obtain an optimal solution to the CIOP. Otherwise, we can still revise Algorithm 1 to obtain a lower bound on the optimal objective value of the CIOP, as shown in Theorem 2.

Theorem 2. (i) *Algorithm* 1 *returns an optimal solution to the CIOP.* (ii) *When* Step 3 *of Algorithm* 1 *is revised to obtain only an upper bound of* Δ *and a heuristic solution to the separation problem in* (14), *Algorithm* 1 *returns a cost vector* $\hat{d}$ *such that the value of* $\sum_{k=1}^{q} \omega_k |\hat{d}_k - c_k|$ *provides a lower bound on the optimal objective value of the CIOP.*

Moreover, by using the well-known ellipsoid method to solve LP1, which is a more complicated cutting-plane algorithm than the standard one in Algorithm 1 (Grötschel et al. 2012), we can establish Theorem 3, showing that the CIOP can be solved in polynomial time if the separation problem can be solved in polynomial time. (See Section 4.1 for the application of Reformulation LP1 in a weighted matching game.)

Theorem 3. *If the separation problem defined in* (14) *can be solved to optimality in polynomial or pseudo-polynomial time, then the CIOP can be solved to optimum in polynomial or pseudo-polynomial time, respectively, by applying the ellipsoid method to Reformulation LP1.*

3.2.2. Reformulation LP2. As shown in Theorem 3, the tractability of Reformulation LP1 of the CIOP depends on whether the separation problem defined in (14) for Constraint (13d) can be solved efficiently. However, in some situations, the separation problem may not be tractable. We therefore develop an alternative solution algorithm based on the second LP reformulation of the CIOP, which is referred to as LP2 and is introduced as follows.

Reformulation LP2 is derived from LP1 by replacing Constraints (13c) and (13d) with some other valid constraints, which are obtained by an analysis of a polyhedron defined by a set of assignable inequalities for the IM game $(N, \pi(\cdot; c))$. The concept of *assignable inequalities* is first introduced by Caprara and Letchford (2010) and is illustrated in Definition 2, where $\text{Conv}\{x \in \mathbb{Z}^q : Ax \geq B\mathbf{1} + E\}$ represents the convex hull of feasible solutions to the ILP for $\pi(N; c)$, and Conv Q_{xy} represents the convex hull of the set Q_{xy} defined in (12).

Definition 2. An inequality $ax \geq \eta$ which is valid for $\text{Conv}\{x \in \mathbb{Z}^q : Ax \geq B\mathbf{1} + E\}$ is said to be *assignable* if there exists an inequality $ax \geq by$, which is valid for Conv Q_{xy} such that $b\mathbf{1} = \eta$.

Let C_x denote the set of points $x \in \mathbb{R}^q$ that satisfy all assignable inequalities as defined in Definition 2. Because assignable inequalities are all linear, C_x is a convex set of $\mathbb{R}^q$. From Definition 2, we know that C_x includes every feasible solution to the ILP for $\pi(N;c)$, implying that $x^0 \in C_x$.

Caprara and Letchford (2010) have shown that for any cost vector $d \in \mathbb{R}^q$, the minimum value of dx over $x \in C_x$ equals the maximum shared cost $\alpha(N)$ over all cost allocations $\alpha \in \mathbb{R}^n$ that satisfy the coalition stability constraints, that is,

$$\min\left\{\sum_{k=1}^{q} d_k x_k : x \in C_x\right\} = \max\{\alpha(N) : \alpha(S) \leq \pi(S;d) \text{ for all } S \in \mathcal{S},\ \alpha \in \mathbb{R}^n\}.$$

Accordingly, Constraints (13c) and (13d) of Reformulation LP1 of the CIOP can be replaced with $\min\{\sum_{k=1}^{q} d_k x_k : x \in C_x\} = \nu$, which, due to (13e) and $x^0 \in C_x$, can be further relaxed to $\min\{\sum_{k=1}^{q} d_k x_k : x \in C_x\} - \nu \geq 0$. Thus, LP1 can be reformulated as follows:

$$\min \sum_{k=1}^{q} \omega_k s_k^+ + \sum_{k=1}^{q} \omega_k s_k^- \tag{16a}$$

$$\text{s.t.} \quad d_k - (s_k^+ - s_k^-) = c_k, \qquad \forall k \in \{1,2,\ldots,q\}, \tag{16b}$$

$$\min\left\{\sum_{k=1}^{q} d_k x_k : x \in C_x\right\} - \nu \geq 0, \tag{16c}$$

$$\sum_{k=1}^{q} x_k^0 d_k - \nu = 0, \tag{16d}$$

$$\nu \geq l, \tag{16e}$$

$$-\nu \geq -u, \tag{16f}$$

$$\nu \in \mathbb{R},\ d \in \mathbb{R}^q,\ s^+ \in \mathbb{R}_+^q,\ s^- \in \mathbb{R}_+^q. \tag{16g}$$

To further linearize (16c), we establish Lemma 3.

Lemma 3. *Points set C_x is a bounded polyhedron of $\mathbb{R}^q$.*

According to Lemma 3, we know that C_x is a bounded polyhedron, and thus, it can be expressed in the following explicit form:

$$C_x = \{x \in \mathbb{R}^q : \tilde{A}x \geq \tilde{B}\}, \tag{17}$$

where $\tilde{A} \in \mathbb{R}^{\tilde{p}\times q}$ and $\tilde{B} \in \mathbb{R}^{\tilde{p}}$ for some integer $\tilde{p} \in \mathbb{Z}_+$. Let $\rho \in \mathbb{R}_+^{\tilde{q}}$ denote the dual variables associated with constraints $\tilde{A}x \geq \tilde{B}$ that define C_x. By the strong duality theorem, Constraint (16c) can be rewritten equivalently as $\max\{\tilde{B}^T\rho : \tilde{A}^T\rho - d = 0,\ \rho \in \mathbb{R}_+^{\tilde{q}}\} - \nu \geq 0$. Thus, we can linearize Constraint (16c) by replacing it with Constraints (18c) and (18d) and then include variables $\rho \in \mathbb{R}_+^{\tilde{q}}$ to obtain the following linear program LP2, which is equivalent to LP1, and thus, is also a reformulation of the CIOP:

$$\text{LP2} \quad \min \sum_{k=1}^{q} \omega_k s_k^+ + \sum_{k=1}^{q} \omega_k s_k^- \tag{18a}$$

$$\text{s.t.} \quad d_k - (s_k^+ - s_k^-) = c_k, \qquad \forall k \in \{1,2,\ldots,q\}, \tag{18b}$$

$$\tilde{B}^T\rho - \nu \geq 0, \tag{18c}$$

$$\tilde{A}^T\rho - d = 0, \tag{18d}$$

$$\sum_{k=1}^{q} x_k^0 d_k - \nu = 0, \tag{18e}$$

$$\nu \geq l, \tag{18f}$$

$$-\nu \geq -u, \tag{18g}$$

$$\nu \in \mathbb{R},\ s^+ \in \mathbb{R}_+^q,\ s^- \in \mathbb{R}_+^q,\ d \in \mathbb{R}^q,\ \rho \in \mathbb{R}_+^{\tilde{q}}. \tag{18h}$$

Algorithm 2 (Solving Reformulation LP2 of the CIOP by Deriving Assignable Inequalities)

Step 1. Derive the explicit expression $\{x \in \mathbb{R}^q : \tilde{A}x \geq \tilde{B}\}$ as in (17) of the polyhedron C_x of points $x \in \mathbb{R}^q$ that satisfy all assignable inequalities defined in Definition 2;

Step 2. Solve LP2 to find its optimal solution, denoted by $(\tilde{v}, \tilde{s}^+, \tilde{s}^-, \tilde{d}, \tilde{\rho})$;

Step 3. Return $\tilde{d}$ as an optimal solution to the CIOP.

Therefore, to solve the CIOP, we can also solve its Reformulation LP2 instead, as illustrated in Algorithm 2. From the optimal solution $(\tilde{v}, \tilde{s}^+, \tilde{s}^-, \tilde{d}, \tilde{\rho})$ to LP2, which is found in Step 2 of Algorithm 2, we can obtain that $\tilde{d}$ is an optimal solution to the CIOP.

The essential part of Algorithm 2 is how to derive the explicit expression $\{x \in \mathbb{R}^q : \tilde{A}x \geq \tilde{B}\}$ of the polyhedron C_x in Step 1. There are a number of IM games for which one can derive the explicit expression of C_x, for example, the UFL game, the two-matching game, and the unrooted traveling salesman game (Caprara and Letchford 2010). If the explicit expression of C_x consists of a polynomial number of assignable inequalities, then LP2 can be solved in polynomial time, and as a result, Algorithm 2 can solve the CIOP to optimum in polynomial time. (See Section 4.2 for the application of Reformulation LP2 in the UFL game.)

In case it is hard or time consuming to identify all the assignable inequalities to derive the explicit expression of C_x, one can still attempt to identify only some of the assignable inequalities to form a polyhedron $\overline{C}_x = \{x \in \mathbb{R}^q : \overline{A}x \geq \overline{B}\}$. With such $\overline{C}_x$, we can then replace $\tilde{A}^T$ and $\tilde{B}^T$ in LP2 with $\overline{A}^T$ and $\overline{B}^T$ to obtain a new linear program LP2′ and revise Algorithm 2 to solve LP2′ instead. This provides a feasible solution to the CIOP and an upper bound on the optimal objective value of the CIOP, as claimed in Theorem 4.

Theorem 4. (i) *If one can identify the explicit expression $\{x \in \mathbb{R}^q : \tilde{A}x \geq \tilde{B}\}$ of the polyhedron C_x in polynomial time, and if $\tilde{A}x \geq \tilde{B}$ consists of a polynomial number of assignable inequalities, then Algorithm 2 returns an optimal solution to the CIOP in polynomial time.* (ii) *If one can identify only some assignable inequalities to form a polyhedron $\overline{C}_x = \{x \in \mathbb{R}^q : \overline{A}x \geq \overline{B}\}$, then by revising Algorithm 2 to solve LP2′ instead of LP2, one can obtain a feasible solution $\overline{d}$ to the CIOP with $\sum_{k=1}^{q} \omega_k |\overline{d}_k - c_k|$ providing an upper bound on the optimal objective value of the CIOP.*

3.3. Feasibility Conditions

Remind that cost imposition methods (e.g., subsidization or penalization) are always feasible, as one can set the subsidy or penalty to a sufficiently large value to accomplish grand cooperation. However, as shown in Example 1, the instrument of cost adjustment may be infeasible due to the simultaneous consideration of the balancedness constraint, desirable cooperation constraint, and desirable cost sharing constraint, that is, Constraints (4)–(6). Therefore, it is of interest to derive some feasibility conditions for the CIOP.

Example 1. Consider an IM game $(N, \pi(\cdot; c))$ with two players in $N = \{1, 2\}$, and with its characteristic function $\pi(\cdot; c)$ defined as the following ILP:

$$\pi(S; c) = \min_x \{cx : \ x \geq y_1(S) + y_2(S) - 1, \ x \in \{0, 1\}\}, \text{ for } S \in \mathcal{S}, \tag{19}$$

where the original cost coefficient $c = 1$. It can be seen that $\pi(N; c) = 1$ with the optimal solution $x^* = 1$. Consider the CIOP defined for this game with the desirable range of cost sharing $[l, u] = [0.5, 1.5]$ and with the desirable cooperation scheme $x^0 = x^* = 1$. This CIOP has no feasible solution due to the following reason: For any cost coefficient $d \in \mathbb{R}$, from (19), we know that if $d \geq 0$, then $\pi(\{1\}; d) = \pi(\{2\}; d) = 0$, and $\pi(\{1, 2\}; d) = d \cdot 1 = d$; otherwise, $d < 0$, and then $\pi(\{1\}; d) = \pi(\{2\}; d) = \pi(\{1, 2\}; d) = d \cdot 1 = d$. This implies that $\pi(\{1\}; d) + \pi(\{2\}; d) < \pi(\{1, 2\}; d)$ for $d \neq 0$. Thus, if the CIOP has a feasible solution d, then because d satisfies the balancedness constraint (7b), we have $d = 0$, which implies that $\pi(N; d) = 0 \notin [0.5, 1.5]$, violating the desirable cost sharing constraint (7d). Therefore, the CIOP has no feasible solution.

Theorem 5 provides a sufficient and necessary condition on the feasibility of the CIOP.

Theorem 5 (Sufficient and Necessary Condition on Feasibility). *The CIOP defined in (7a)–(7e) has a feasible solution if and only if there exists a value $\nu \in [l, u]$ and an assignable inequality $ax \geq \eta$ as defined in Definition 2 such that $\nu \cdot ax^0 = \nu \cdot \eta \geq 0$.*

According to Theorem 5, the feasibility of the CIOP depends on the assignable inequalities. However, it is sometimes difficult to directly apply Theorem 5 by checking all the assignable inequalities. For this reason, we now consider only the inequalities defined in $Ax \geq B\mathbf{1} + E$ of the ILP (2) for $\pi(N; c)$. Based on these inequalities, which are easy to check, we can derive from Theorem 5 some more sufficient conditions on feasibility of the CIOP, as illustrated in Corollary 1, and we will demonstrate its applications to two IM games in Section 4.

Corollary 1 (Sufficient Conditions on Feasibility). *Consider the CIOP defined in (7a)–(7e), and the inequalities defined in* $Ax \geq B\mathbf{1} + E$ *of the ILP (2) for* $\pi(N;c)$*. The CIOP has a feasible solution if* $0 \in [l,u]$ *or at least one of the following conditions is satisfied:*

(i) *There exists a value* $v \in [l,u]$ *and an inequality* $ax \geq b\mathbf{1} + e$ *among the inequalities defined in* $Ax \geq B\mathbf{1} + E$ *such that* $e \geq 0$, $ax^0 = b\mathbf{1} + e$, *and* $v \cdot (b\mathbf{1} + e) \geq 0$;

(ii) *There exist two inequalities* $ax \geq b\mathbf{1} + 0$ *and* $-ax \geq -b\mathbf{1} + 0$ *among the inequalities defined in* $Ax \geq B\mathbf{1} + E$.

4. Applications to Two IM Games

In this section, we demonstrate the applicability and effectiveness of our alternative instrument of cost adjustment for stabilizing the grand coalition. To this end, we apply our CIOP formulation of the instrument and its solution algorithms to two typical IM games, that is, the WM game and the UFL game, upon which we also present some computational results.

As will be explicitly explained in the following two sections, we will apply the solution algorithm based on LP1 to solve the CIOP defined on the WM game because the associating separation problem is polynomially tractable (Theorem 3) and adopt the solution algorithm based on LP2 to analyze the CIOP defined on the UFL game by using the existing results on its explicit expression of polyhedron C_x (Theorem 4).

4.1. Cost Adjustment for WM Game Using LP1

Consider the following game arising from the WM problem, which is referred to as the WM game in short (Kern and Paulusma 2003). In this game, we are given an undirected graph $G = (N, E)$, where N denotes the set of vertices with each vertex representing a player in the game, and E denotes the set of edges with each edge $e \in E$ having a negative weight denoted by $c_e < 0$. Each coalition $S \subseteq N$ aims to find a matching (i.e., a set of pairwise nonadjacent edges) on vertices in S with the total edge weight of the matching minimized. (In the literature, edge weights are assumed to be positive, with the aim of maximizing the total edge weight of the matching, which is equivalent to our setting.)

The WM game, denoted by $(N, \pi_{\mathrm{WM}}(\cdot;c))$, is an IM game with $\mathcal{S} = 2^N \setminus \{\emptyset\}$ because its characteristic function $\pi_{\mathrm{WM}}(S;c)$ for $S \in \mathcal{S}$ can be formulated as the following ILP:

$$\pi_{\mathrm{WM}}(S;c) := \min \sum_{e \in E} c_e x_e \tag{20a}$$

$$\text{s.t.} \quad -\sum_{e \in \varphi(j)} x_e \geq -y_j(S), \qquad \forall j \in N, \tag{20b}$$

$$x_e \in \{0,1\}, \quad \forall e \in E, \tag{20c}$$

where $\varphi(j)$ for each $j \in N$ denotes the set of edges with one endpoint being vertex j and the other endpoint in $N \setminus \{j\}$, each binary variable x_e for $e \in E$ indicates whether edge e is matched, and Constraints (20b) indicate that only a vertex j in the coalition S, that is, $y_j(S) = 1$, can it be matched.

In the literature, Kern and Paulusma (2003) study the use of penalties to stabilize the grand coalition of the WM game. In the following, we apply our alternative instrument of cost adjustment to the WM game to stabilize its grand coalition by adjusting the edge weights c_e for $e \in E$.

Consider the CIOP defined for the WM game. By applying Corollary 1, we can establish Proposition 1, which implies that under a mild condition, the CIOP defined for the WM game always has a feasible cost adjustment for one to stabilize the grand coalition.

Proposition 1. *Consider any WM game* $(N, \pi_{\mathrm{WM}}(\cdot;c))$*, any desirable cooperation scheme* x^0*, which is a feasible solution to the ILP of* $\pi_{\mathrm{WM}}(N;c)$*, and any desirable range* $[l,u]$ *of cost sharing. The CIOP defined on them has a feasible solution if* $l < 0$ *and* x^0 *matches at least one edge (i.e.,* $\sum_{e \in E} x_e^0 \geq 1$*).*

To solve the CIOP defined for the WM game, we can apply the solution algorithm based on Reformulation LP1 in (13a)–(13h). It is known (Deng et al. 1999) that, for the WM game, if a cost allocation α satisfies the coalition stability constraint $\alpha(S) \leq \pi(S)$ for each nonempty coalition $S \subseteq N$ of at most two members (i.e., with $|S| \leq 2$), then α satisfies the coalition stability constraint for all nonempty coalitions $S \subseteq N$. This implies that the WM game is equivalent to the IM game $(N, \pi_{\mathrm{WM}}(\cdot;c))$ with $\mathcal{S} = \mathcal{S}_{\leq 2}$, where $\mathcal{S}_{\leq 2}$ is defined as follows:

$$\mathcal{S}_{\leq 2} := \{S \subseteq N : 1 \leq |S| \leq 2\}.$$

Consider Reformulation LP1 in (13a)–(13h) of the CIOP defined for the WM game with $\mathcal{S} = \mathcal{S}_{\leq 2}$. According to (15), the corresponding separation problem defined in (14) can be reformulated as

$$\Delta_{\text{WM}} = \min\left\{\pi_{\text{WM}}(S;\hat{d}) - \hat{\alpha}(S) : S \in \mathcal{S}_{\leq 2}\right\}, \tag{21}$$

for any new cost vector $\hat{d} \in \mathbb{R}^{|E|}$ and any cost allocation $\hat{\alpha} \in \mathbb{R}^{|N|}$. For each $S \in \mathcal{S}_{\leq 2}$, if $|S| = 1$, then we can denote S by $\{j\}$ for some $j \in N$, and no edge can be matched for S, implying that $\pi_{\text{WM}}(S;\hat{d}) - \hat{\alpha}(S) = -\hat{\alpha}(S) = -\hat{\alpha}_j$; otherwise, $|S| = 2$, we can denote S by $\{j, j'\}$ for some $j \in N$ and $j' \in N$ with $j \neq j'$, and only the edge (j, j') can be matched, implying that $\pi_{\text{WM}}(S;\hat{d}) - \hat{\alpha}(S) = \min\{c_{j,j'}, 0\} - \hat{\alpha}_j - \hat{\alpha}_{j'}$. Accordingly, because $|\mathcal{S}_{\leq 2}| = |N| + |N|(|N| - 1)/2$, we obtain that the separation problem defined in (21) can be solved in polynomial time.

As we have shown previously, the separation problem defined in (21) can be solved in polynomial time. Thus, by Theorem 3, Theorem 6 is directly obtained.

Theorem 6. *Consider any WM game $(N, \pi_{\text{WM}}(\cdot; c))$, any desirable cooperation scheme x^0, which is a feasible solution to the ILP of $\pi_{\text{WM}}(N; c)$, and any desirable range $[l, u]$ of cost sharing. The CIOP defined on them can be solved in polynomial time.*

Besides the WM game, the solution algorithms based on Reformulation LP1 can also be applied in solving the CIOP defined for other games efficiently, such as some machine scheduling games (Schulz and Uhan 2010, Liu et al. 2018), where the corresponding separation problems can be solved efficiently.

4.2. Cost Adjustment for UFL Game Using LP2

In the example shown in Figures 1 and 2, we have seen an instance of the UFL game, which arises from the uncapacitated facility location problem (Goemans and Skutella 2004). Given a set M of m available facilities to open with an opening cost f_i for each $i \in M$ and a set N of n players to be served with a service cost r_{ij} for each $i \in M$ and $j \in N$, the UFL game $(N, \pi_{\text{UFL}}(\cdot; (f, r)))$ is an IM game for players in N with $\mathcal{S} = 2^N \setminus \{\emptyset\}$, where the characteristic function $\pi_{\text{UFL}}(S; (f, r))$ for each coalition $S \in \mathcal{S}$ aims to open facilities to serve players in S with the total cost of opening and service minimized, and can be formulated as an ILP as shown earlier in (3a)–(3e).

To stabilize the grand coalition of the UFL game, some works have been carried out in the literature on the use of subsidies (Caprara and Letchford 2010, Liu et al. 2016). In the following, we apply the alternative instrument of cost adjustment to the UFL game to stabilize its grand coalition by adjusting the facility opening cost f_i for $i \in M$ and the service cost r_{ij} for $i \in M$ and $j \in N$.

Consider the CIOP defined for the UFL game. By applying Corollary 1, we can establish Proposition 2, which implies that the CIOP defined for the UFL game always has a feasible cost adjustment for one to stabilize the grand coalition.

Proposition 2. *Consider any UFL game $(N, \pi_{\text{UFL}}(\cdot; (f, r)))$, any desirable cooperation scheme (x^0, z^0), which is a feasible solution to the ILP of $\pi_{\text{UFL}}(N; (f, r))$, and any desirable range $[l, u]$ of cost sharing. The CIOP defined on them always has a feasible solution.*

To solve the CIOP defined for the UFL game, we can apply the solution algorithm based on Reformulation LP2 in (18a)–(18h) as follows. To derive matrices $\tilde{A}$ and $\tilde{B}$ for Reformulation LP2, we need to explicitly describe the polyhedron C_x by a set of assignable inequalities $\tilde{A}x \geq \tilde{B}$ as in (17). Caprara and Letchford (2010) have shown that for the UFL game,

$$C_x = \left\{(x, z) \in \mathbb{R}^{mn+m} : \sum_{i \in M} x_{ij} = 1, \;\; \forall j \in N, \; z_i \geq x_{ij} \text{ and } x_{ij} \geq 0, \;\; \forall i \in M, \;\; \forall j \in N\right\},$$

from which we can obtain $\tilde{A}$ and $\tilde{B}$.

Accordingly, let $\bar{f}_i$ and $\bar{r}_{ij}$ denote the opening costs and the service costs after adjustments of f_i and r_{ij}, and let s_i^+, s_i^-, $\hat{s}_{ij}^+$, and $\hat{s}_{ij}^-$ denote the positive and the negative parts of the cost adjustments of f_i and r_{ij}, respectively, where $i \in M$ and $j \in N$. Let ρ_j, $\bar{\rho}_{ij}$, and $\hat{\rho}_{ij}$, denote the dual variables associated with constraints, $\sum_{i \in M} x_{ij} = 1$, $z_i \geq x_{ij}$, and $x_{ij} \geq 0$, of C_x, respectively, where $i \in M$ and $j \in N$. For the given desirable cooperation scheme (x^0, z^0) and desirable range $[l, u]$ of cost sharing, Reformulation LP2 in (18a)–(18h) of the CIOP can be represented as follows for the UFL game:

$$\min \sum_{i \in M} \omega_i s_i^+ + \sum_{i \in M} \sum_{j \in N} \omega_{ij} \hat{s}_{ij}^+ + \sum_{i \in M} \omega_i s_i^- + \sum_{i \in M} \sum_{j \in N} \omega_{ij} \hat{s}_{ij}^- \tag{22a}$$

$$\text{s.t.} \quad \overline{f}_i - (s_i^+ - s_i^-) = f_i, \qquad \forall i \in M, \tag{22b}$$

$$\overline{r}_{ij} - (\hat{s}_{ij}^+ - \hat{s}_{ij}^-) = r_{ij}, \qquad \forall i \in M, \ \ \forall j \in N, \tag{22c}$$

$$\sum_{j \in N} \rho_j - \nu \geq 0, \tag{22d}$$

$$\rho_j - \overline{\rho}_{ij} + \hat{\rho}_{ij} - \overline{r}_{ij} = 0, \ \ \forall i \in M, \ \ \forall j \in N, \tag{22e}$$

$$\sum_{j \in N} \overline{\rho}_{ij} - \overline{f}_i = 0, \ \ \forall i \in M, \tag{22f}$$

$$\sum_{i \in M} z_i^0 \overline{f}_i + \sum_{i \in M} \sum_{j \in N} x_{ij}^0 \overline{r}_{ij} - \nu = 0, \tag{22g}$$

$$\nu \geq l, \tag{22h}$$

$$-\nu \geq -u, \tag{22i}$$

$$\nu \in \mathbb{R}, s^+ \in \mathbb{R}_+^m, s^- \in \mathbb{R}_+^m, \hat{s}^+ \in \mathbb{R}_+^{m \times n}, \hat{s}^- \in \mathbb{R}_+^{m \times n}, \overline{f} \in \mathbb{R}^m, \overline{r} \in \mathbb{R}^{m \times n},$$

$$\rho \in \mathbb{R}_+^m, \overline{\rho} \in \mathbb{R}_+^{m \times n}, \hat{\rho} \in \mathbb{R}_+^{m \times n}. \tag{22j}$$

Thus, because Reformulation LP2 in (22a)–(22j) has a polynomial number of variables and constraints, by Theorem 4, we obtain that Reformulation LP2, as well as the CIOP, can be solved in polynomial time. Thus, Theorem 7 is obtained.

Theorem 7. *Consider any UFL game $(N, \pi_{\mathrm{UFL}}(\cdot; (f, r)))$, any desirable cooperation scheme (x^0, z^0), which is a feasible solution to the ILP of $\pi_{\mathrm{UFL}}(N; (f, r))$, and any desirable range $[l, u]$ of cost sharing. The CIOP defined on them can be solved in polynomial time.*

Besides the UFL game, the solution algorithms based on Reformulation LP2 can also be applied to solving the CIOP defined for other games efficiently, such as the packing and covering games (Deng et al. 1999), the two-matching game, and the unrooted traveling salesman game (Caprara and Letchford 2010), where all the assignable inequalities can be explicitly identified.

4.3. Computational Experiments

To further investigate the effectiveness of using cost adjustments to stabilize the grand cooperation, we conduct some numerical experiments by solving the CIOPs of various sized instances randomly generated for the WM game and the UFL game. All the algorithms are implemented in MATLAB Release 2023a, and Gurobi 10.0.3 is used to solve linear programs. Our results on the WM game and the UFL game are reported in Sections 4.3.1 and 4.3.2, respectively. Data and source codes are available at Liu et al. (2023).

4.3.1. Experiments on the WM Game. For the WM game, we consider four groups of instances with different sizes of the graph (N, E), including $|N| = 30$ and $|E| = 435$, $|N| = 780$ and $|E| = 50$, $|N| = 50$ and $|E| = 1225$, and $|N| = 60$ and $|E| = 1770$. For each group, we generate 100 instances randomly, with each edge weight randomly chosen from $\{-1, -2, \ldots, -100\}$.

As mentioned in Section 4.1, an optimal solution, denoted by x^*, and its optimal objective value, denoted by ν^*, to the ILP of $\pi_{\mathrm{WM}}(N; c)$ for the WM game can be obtained in polynomial time. Thus, in our experiment, we set the desirable cooperation scheme x^0 to be x^*, and the desirable range of cost sharing $[l, u]$ to be either $[\nu^*, \nu^*]$ or $[0.95\nu^*, 1.05\nu^*]$. We set the unit adjustment penalty $\omega_e = 1$ for $e \in E$, so that the total deviation effort to be minimized in the CIOP is $\sum_{e \in E} |d_e - c_e|$.

Our computational results, with respect to $[l, u]$ equal to $[\nu^*, \nu^*]$ and $[0.95\nu^*, 1.05\nu^*]$, are presented in Tables 2 and 3, respectively. For each group of instances with the same size, we solve the CIOP to obtain the optimal cost adjustment using the solution algorithm as illustrated in Section 4.1. From the optimal cost adjustments obtained, we derive and present the results shown in column U, column DV%, column DN%, and column T. Here, column U reports the number of instances that are unbalanced games, of the 100 random instances of each group. Column DV% reports the average, maximal, and minimal values of the minimum total deviation effort $\sum_{e \in E} |d_e - c_e|$ as a

Table 2. Computational Results on the WM Game with $x^0 = x^*$ and $[l, u] = [v^*, v^*]$

		DV%			DN%			
$(\lvert N\rvert, \lvert E\rvert)$	U	Average	Maximum	Minimum	Average	Maximum	Minimum	T (s)
(30, 435)	22	0.011	0.046	0.002	0.366	0.690	0.230	10.642
(40, 780)	22	0.005	0.015	0.001	0.239	0.385	0.128	47.418
(50, 1,225)	26	0.004	0.013	0.001	0.170	0.408	0.082	145.078
(60, 1,770)	34	0.001	0.006	0.001	0.088	0.339	0.056	376.976

percentage of the absolute norm $\sum_{e \in E} |c_e|$ of the original cost vector c, over the instances that are unbalanced games in each group. Column DN% reports the average, maximal, and minimal values of the number of adjusted cost elements as a percentage of the total number of cost elements in c (which equals $|E|$), over the instances that are unbalanced games in each group. Column T reports the average computational time (in seconds) of finding new cost vectors for the WM game, and the majority of the computational effort is dedicated to solving the separation problems.

Tables 2 and 3 reveal that with only some minor cost adjustments, the grand cooperation for the WM game can be effectively stabilized without changing the optimal cooperation scheme as given in x^*. From column U of Table 2 we know that 22%–34% of the random instances of the WM game are unbalanced. When the amount of cost sharing is enforced to be unchanged as v^*, as shown in columns DV% and DN% of Table 2, the percentage of the amount of cost adjustments needed does not exceed 0.046%, and the percentage of the number of cost adjustments needed does not exceed 0.69%. When the amount of cost sharing is allowed to have a 5% change within $[0.95v^*, 1.05v^*]$, as shown in columns DV% and DN% of Table 3, the cost adjustments needed are reduced, with the percentage of the amount of cost adjustments needed not exceeding 0.028%, and the percentage of the number of cost adjustments needed not exceeding 0.69%. For some unbalanced instances of the WM game, we observe that only one cost term needs to be adjusted.

4.3.2. Experiments on the UFL Game. For the UFL game, we consider four groups of instances with different sizes of M and N, including $|M| = |N| = 20, |M| = |N| = 40, |M| = |N| = 60$, and $|M| = |N| = 80$. For each group, we generate 100 instances, with each facility opening cost randomly chosen from $\{100, 101, \ldots, 200\}$, and with each service cost randomly chosen from $\{1, 2, \ldots, 100\}$. In addition, we set the unit adjustment penalties as 10,000 for facility opening costs and as 1 for service costs to avoid the adjustment of facility opening costs.

For the UFL game, we use Gurobi 10.0.3 to solve the optimal solution, denoted by (x^*, z^*), and its optimal objective value, denoted by v^*, to the ILP of $\pi_{\text{UFL}}(N; (f, r))$. Accordingly, we set the desirable cooperation scheme to be (x^*, z^*), and set the desirable range of cost sharing $[l, u]$ to be either $[v^*, v^*]$ or $[0.95v^*, 1.05v^*]$.

Our computational results, with respect to $[l, u]$ equal to $[v^*, v^*]$ and equal to $[0.95v^*, 1.05v^*]$, for the four groups of instances of the UFL game are presented in Tables 4 and 5, respectively. For each group of instances with the same size, we solve the CIOP to obtain the optimal cost adjustment by using the algorithm as illustrated in Section 4.2. From the optimal cost adjustments obtained, we derive and present the results shown in column U, column DV%, column DN%, and column F, where the definitions of column U, column DV%, and column DN% are the same as those in Table 2, and where column F indicates the number of instances with cost adjustments of facility opening costs, among the 100 random instances of each group. Column T reports the average computational time (in seconds) of finding new cost vectors for the UFL game, and most of the instances can be solved within five seconds by computing linear programs defined by (22a)–(22j).

From column U of Table 4 we know that 86%–100% of the random instances of the UFL game are unbalanced, which are more than those of the WM game. However, similar to what we observe in the experiments for the WM

Table 3. Computational Results on the WM Game with $x^0 = x^*$ and $[l, u] = [0.95v^*, 1.05v^*]$

		DV%			DN%			
$(\lvert N\rvert, \lvert E\rvert)$	U	Average	Maximum	Minimum	Average	Maximum	Minimum	T (s)
(30, 435)	22	0.009	0.028	0.002	0.293	0.690	0.230	11.096
(40, 780)	22	0.004	0.015	0.001	0.157	0.256	0.128	50.381
(50, 1,225)	26	0.003	0.012	0.001	0.116	0.408	0.082	149.101
(60, 1,770)	34	0.001	0.004	0.001	0.068	0.169	0.056	374.948

Table 4. Computational Results on the UFL Game with $(x^0, z^0) = (x^*, z^*)$ and $[l, u] = [v^*, v^*]$

		DV%			DN%				
$(\|M\|, \|N\|)$	U	Average	Maximum	Minimum	Average	Maximum	Minimum	F	T (s)
(20, 20)	86	0.308	0.896	0.002	1.631	4.000	0.250	0	0.025
(40, 40)	99	0.185	0.410	0.036	1.169	2.500	0.188	0	0.303
(60, 60)	100	0.140	0.279	0.020	0.970	1.806	0.250	0	1.427
(80, 80)	100	0.127	0.238	0.037	0.954	1.672	0.250	0	4.224

game, as shown in Tables 4 and 5, with only some minor cost adjustments, one can also effectively stabilize the grand cooperation for the UFL game, without changing the optimal cooperation scheme as given in (x^*, z^*) or the amount of cost sharing as given in v^*. When the amount of cost sharing is enforced to be unchanged as v^*, columns DV% and DN% of Table 4 reveal that the percentage of the amount of cost adjustments needed does not exceed 0.896%, and the percentage of the number of cost adjustments needed does not exceed 4%. When the amount of cost sharing is allowed to have a 5% change within $[0.95v^*, 1.05v^*]$, columns DV% and DN% of Table 5 reveal that the cost adjustments needed are significantly reduced, with the percentage of the amount of cost adjustments needed not exceeding 0.444%, and the percentage of the number of cost adjustments needed not exceeding 2.25%. For some unbalanced instances of the UFL game, we also observe that only one cost term needs to be adjusted. For all the instances of the UFL game, no facility opening costs need to be adjusted.

5. Conclusion

In summary, our study proposes a new cost adjustment-based instrument for stabilizing the grand coalition of general IM games, which is different from the traditional instrument of subsidization and penalization-based instruments. This alternative instrument enables adjusting only existing cost coefficients that are shared among optimization problems faced by different coalitions of the game. Its aim is to make the game have an empty core by only using a minimum amount of adjustment, with desirable restrictions imposed on the cooperation scheme and the total shared cost of the grand coalition. We formulate the problem on how to make such cost adjustments for stabilizing the grand coalition as a constrained inverse optimization problem and prove it to be $\mathcal{NP}$-hard. To solve the problem, we develop solution algorithms based on two linear programming reformulations, and derive some feasibility conditions that are easy to check. We demonstrate the applicability and effectiveness of our alternative instrument by applying our models and solution algorithms to two typical unbalanced games. We show that their optimum cost adjustment can be obtained in polynomial time, and the computational results show that their grand coalitions can be stabilized effectively by utilizing only minor cost adjustments.

Our work has opened several promising directions for future research. First, in this paper, we demonstrate the applications of our alternative instrument of cost adjustments in two unbalanced games. When applying it to other games, such as the traveling salesman game (Kimms and Kozeletskyi 2016), capacitated facility location game (Goemans and Skutella 2004), and so on, a number of new and interesting inverse optimization problems will arise for future study.

Second, as shown in this paper, our alternative instrument of cost adjustment can be applied alone to stabilizing the grand coalition in many situations. It can also be utilized alongside other existing instruments designed around cost imposition, such as subsidization-based and penalization-based ones. By using cost adjustment, one can reduce either the subsidy or penalty so as to stabilize the grand coalition. It would be of great interest to study how to take advantage of such a tradeoff among these instruments and how to optimize the use of these instruments simultaneously.

Table 5. Computational Results on the UFL Game with $(x^0, z^0) = (x^*, z^*)$ and $[l, u] = [0.95v^*, 1.05v^*]$

		DV%			DN%				
$(\|M\|, \|N\|)$	U	Average	Maximum	Minimum	Average	Maximum	Minimum	F	T (s)
(20, 20)	86	0.153	0.444	0.002	0.814	2.250	0.250	0	0.025
(40, 40)	99	0.083	0.189	0.012	0.531	1.188	0.063	0	0.303
(60, 60)	100	0.062	0.141	0.012	0.456	1.028	0.083	0	1.428
(80, 80)	100	0.055	0.104	0.015	0.449	0.859	0.078	0	4.234

Third, in this paper, we allow for the adjustment of all cost coefficients to stabilize the grand coalition and control their total adjustment by imposing different penalties on them. However, in some situations, such as the cooperative games defined in transportation markets (Chan et al. 2021), only a subset of cost coefficients can be adjusted, and these cost coefficients can be adjusted only within certain prescribed ranges. This raises a new problem for future research, which can be formulated as a more general variant of the constrained inverse optimization problem presented in this paper.

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