UvA

Multi-Agent Systems

Auctions

Mina Young Pedersen
m.y.pedersen@uva.nl

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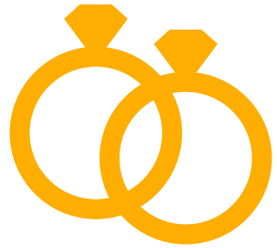
Until now, we have only seen single-good auctions.

Let’s also look at multi-unit auctions.

Multi-Unit Auctions

- In the first **multi-unit auctions** we will look at, we are selling many copies of the same good, to multiple buyers
- To make sense of this we need to generalize single-good auctions, but this is not straightforward
- Let's look first at the simple case of **single-unit demand**
    - In single-unit demand multi-unit auctions there are *k* identical items to be sold, and bidders want only one unit
    - In which case there is an obvious generalization of second-price auctions, the single-item version

k+1st-Price Auction  ?

- Each bidder submits a secret bid for one unit
    - Keep in mind that we are allocating *k* identical items, one per bidder
- Winners are the *k* highest bids
- Price paid, by each, is the (k+1)-highest bid

Position Auctions

- Then, in **position auctions** we are selling multiple goods, again one per bidder, but with the added specification that goods are ordered in terms of quality

- Position auctions are used for ad auctions, to sell differently placed ads on a webpage

    - Important, because better positions have higher clickthrough rates

Generalized First-Price Auction

- There are slots $s_1 \triangleright s_2 \triangleright \ldots$ ordered according to how valuable they are
- Every bidder submits a secret bid for one unit
- The *j*-th highest bid gets the *j*-th highest slot
- Bidder *i* that gets a slot pays their bid

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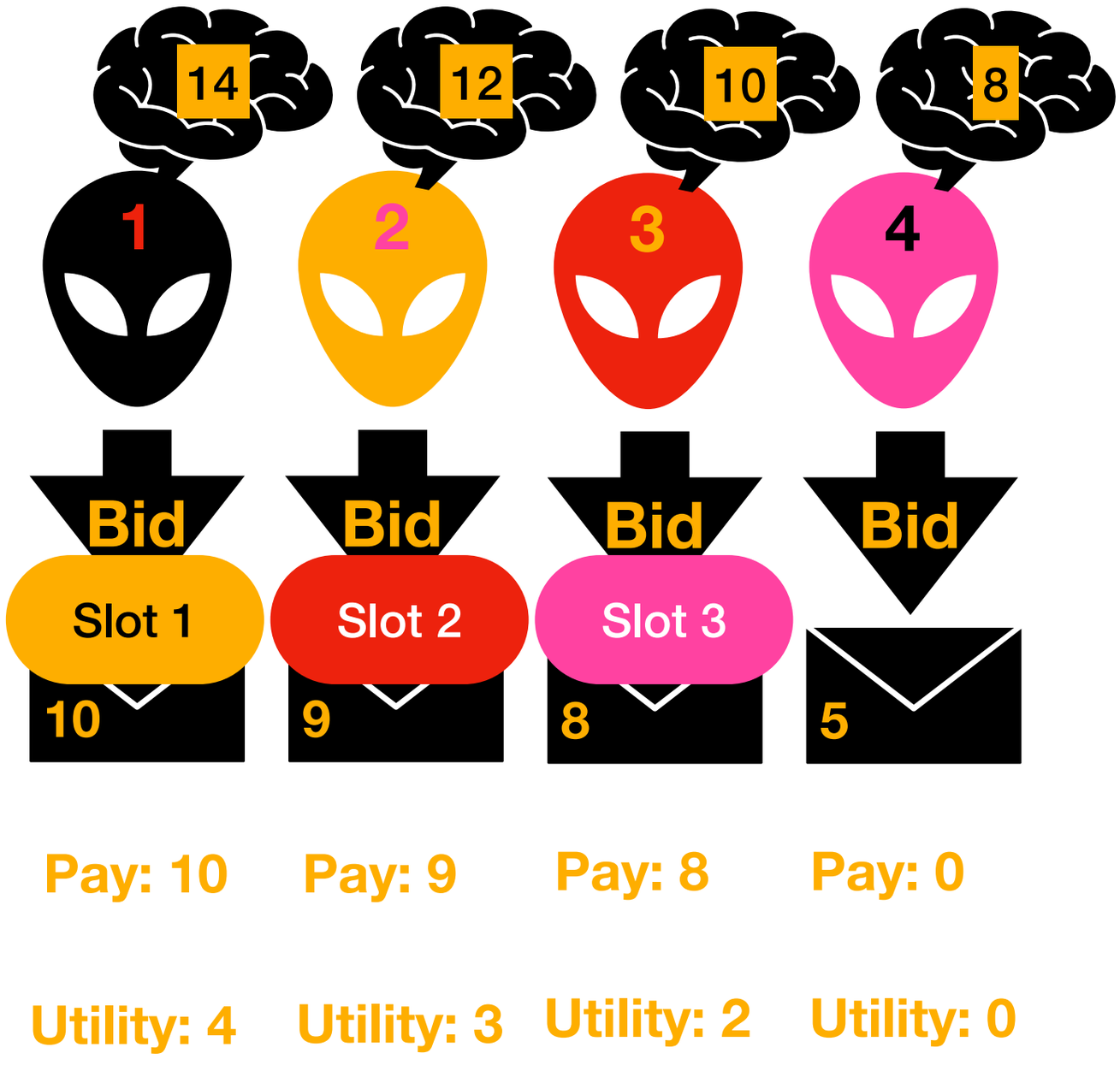

Generalized Second-Price Auction

- There are slots $s_1 \triangleright s_2 \triangleright \ldots$ ordered according to how valuable they are
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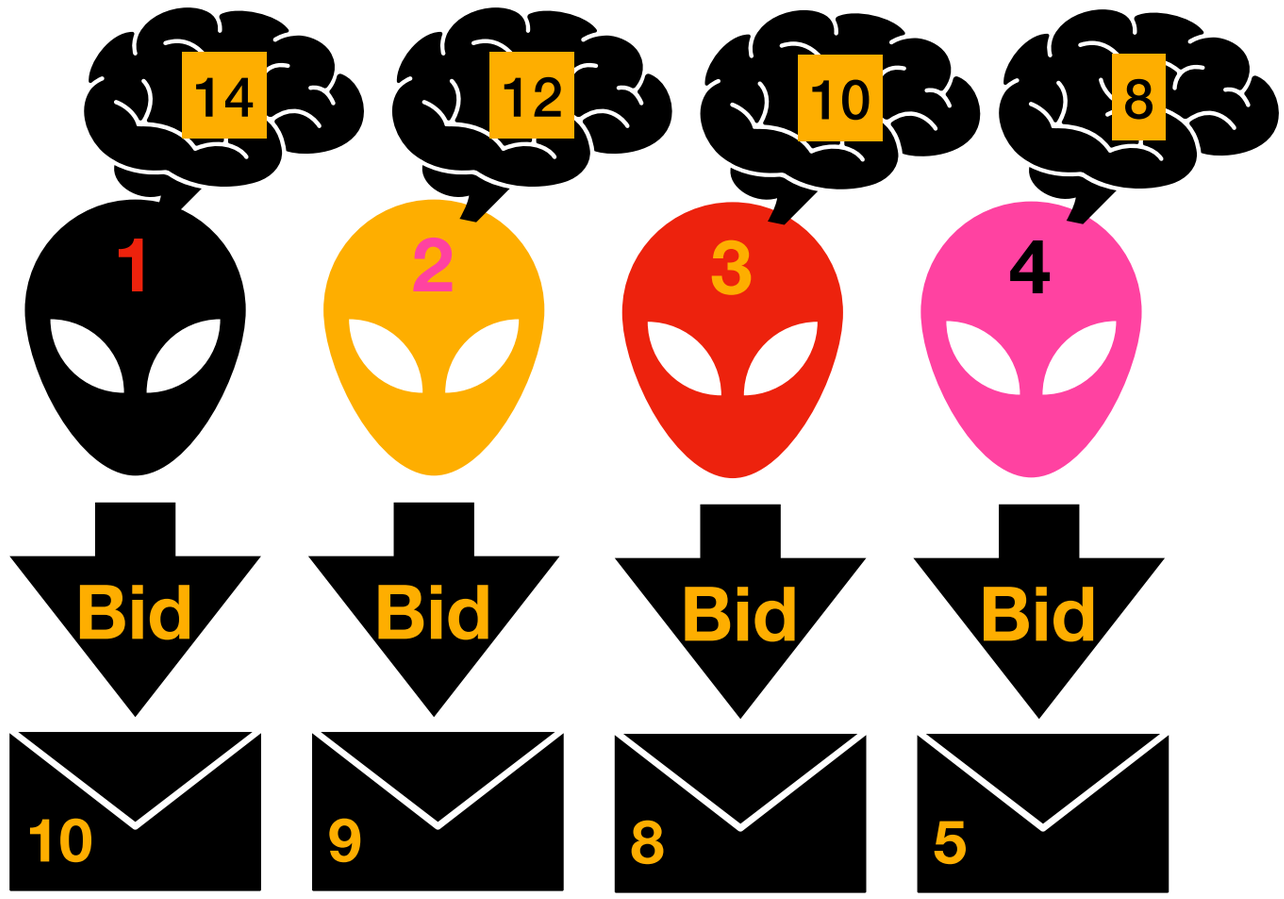

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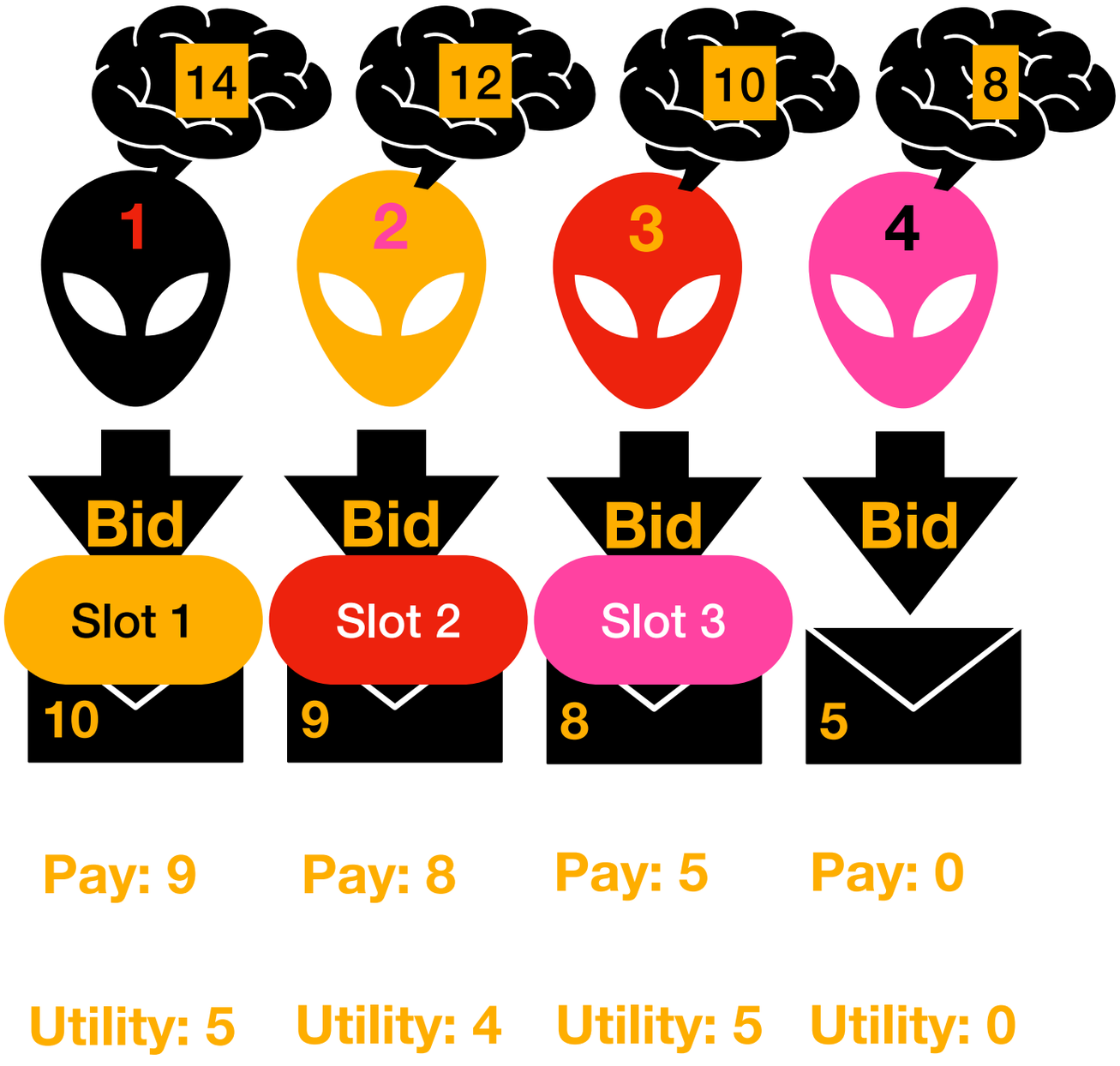


Generalized First- and Second-Price Auctions

- The generalized first-price mechanism was the first one to be used in practice
    - But bidders quickly learned how to manipulate it
    - As it is not truthful (the best strategy is not to bid your true valuation)
- Generalized second-price auctions took over, and are behind the dominant mechanisms used to this day
    - Also not truthful, but more stable than the first-price version

Combinatorial Auctions

- A bit more general, in **combinatorial auctions** there are several goods for sale and bidders bid for **packages** (i.e., subsets) of goods
- Used in a variety of industries: truckload transportation, bus routes, airport arrival and departure slots, allocating radio spectrum for wireless communication services
- One motivation is that, e.g., a mobile phone operator may value licenses in two adjacent cities more than the sum of the individual license values, since customers value roaming between cities
- Such effects are called **complementaries**, and are a special case of a broader phenomenon

Valuation Properties

- Valuations for subsets of goods do not always decompose nicely

Definition (Additive)

Bidder i's valuation $v_i(X)$ with a set $X \subseteq G$ of items is **additive** if:

$$v_i(X) = \sum_{j \in X} v_i(j)$$

Definition (Complementary)

Bidder i's valuation is **complementary** if there exist disjoint $X, Y \subseteq G$ such that:

$$v_i(X \cup Y) \geq v_i(X) + v_i(Y)$$

Definition (Substitutable)

Bidder i's valuation is **substitutable** if there exist disjoint $X, Y \subseteq G$ such that:

$$v_i(X \cup Y) \leq v_i(X) + v_i(Y)$$

Examples

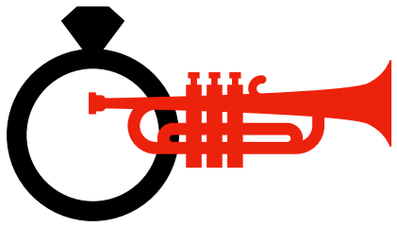
Complementarity: Greater than the sum of its parts

$$v_i(\{\textbf{left shoe, right shoe}\}) \geq v_i(\{\textbf{left shoe}\}) + v_i(\{\textbf{right shoe}\})$$

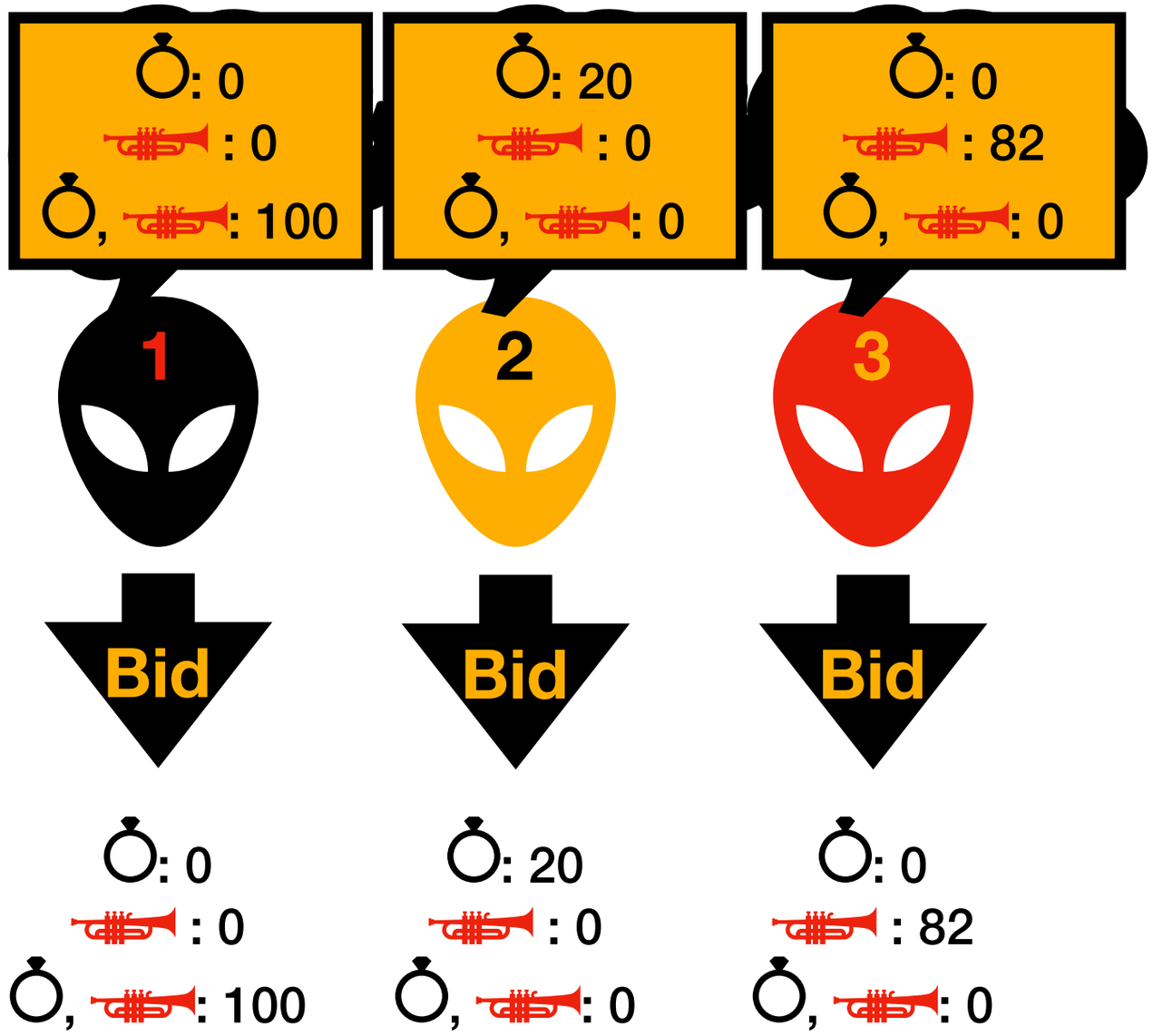
Substitutability: Less than the sum of its parts

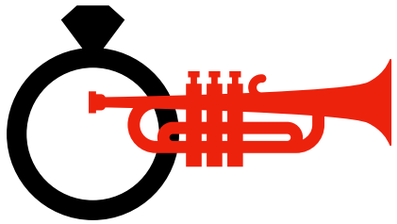
$$v_i(\{\textbf{iPhone, Android phone}\}) \leq v_i(\{\textbf{iPhone}\}) + v_i(\{\textbf{Android phone}\})$$

Image: clipartkey.com, iStock

A Simple Mechanism  ?

- Everyone submits bids for subsets of items
- Choose allocation that maximizes social welfare, i.e., sum of declared valuations
- Charge winners their bids



A Simple Mechanism  ?

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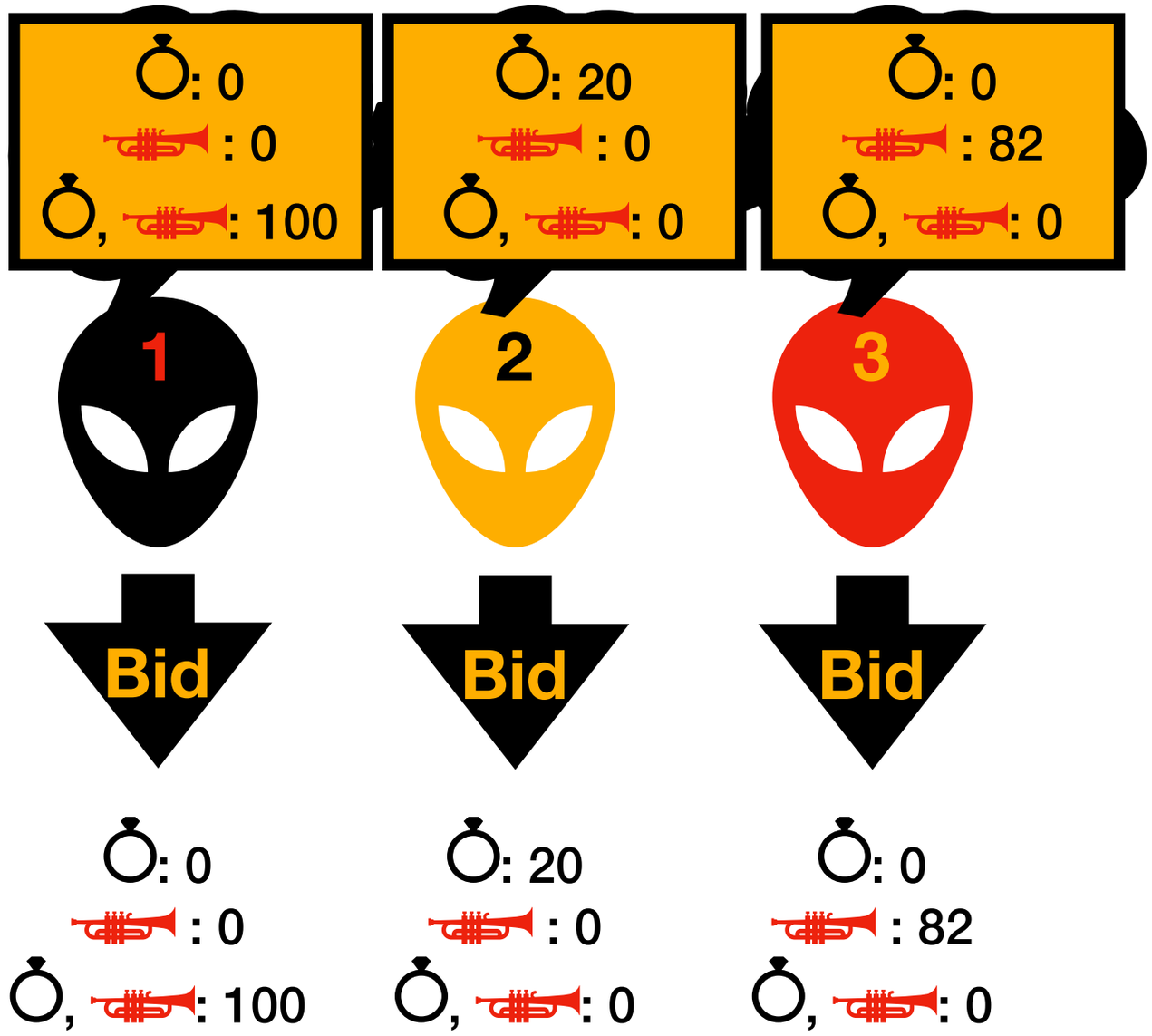
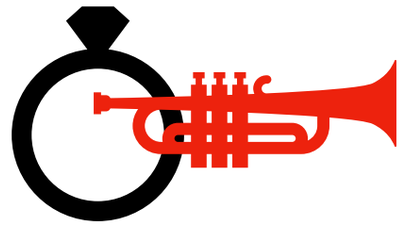


1	2	3	Social Welfare
			100
			102
			82
(…)	(…)	(…)	(…)

A Simple Mechanism  ?

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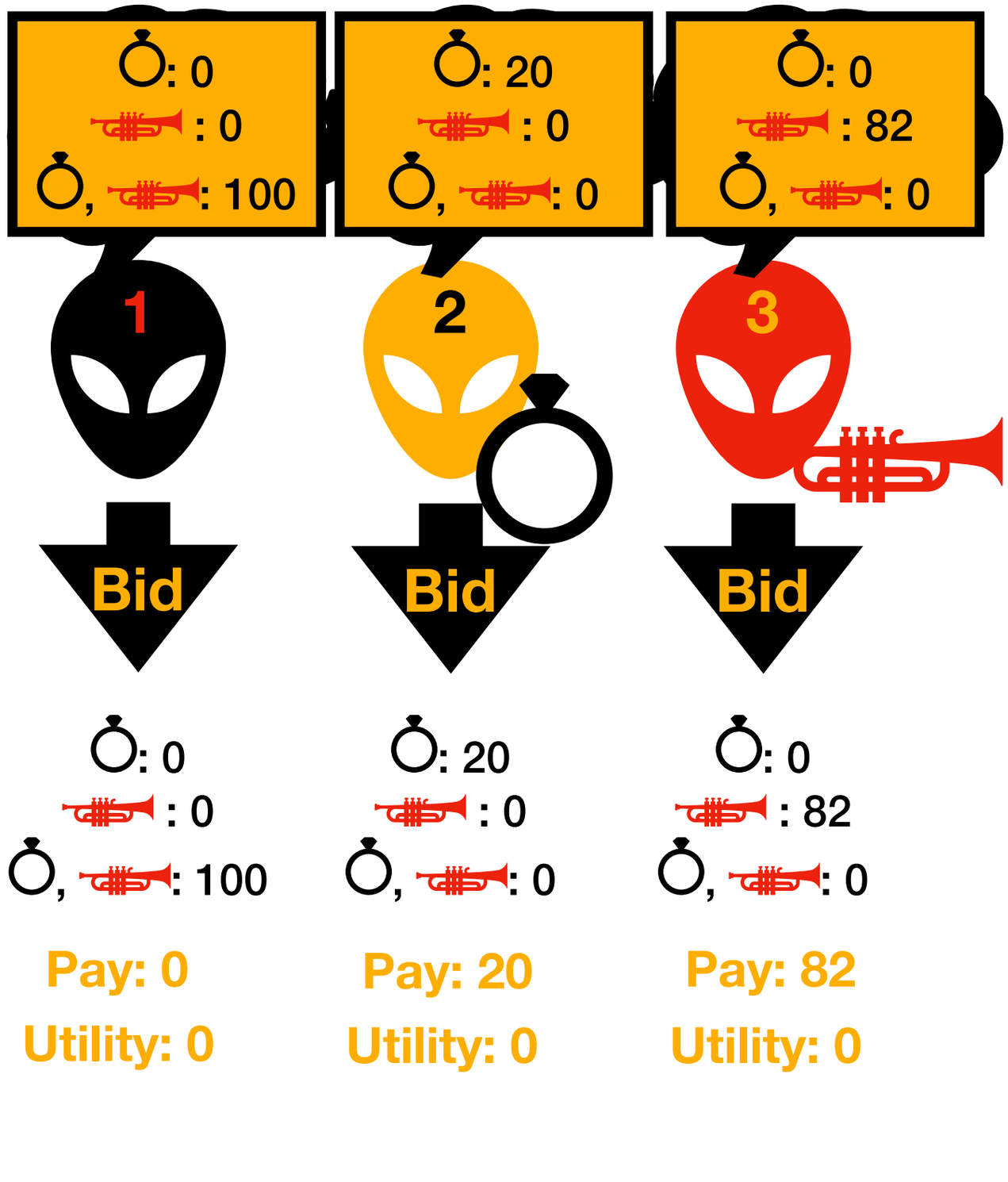
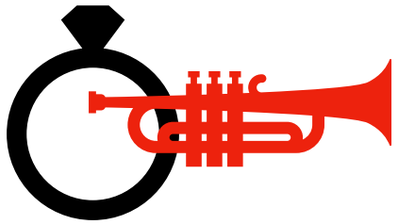


A Simple Mechanism  ?

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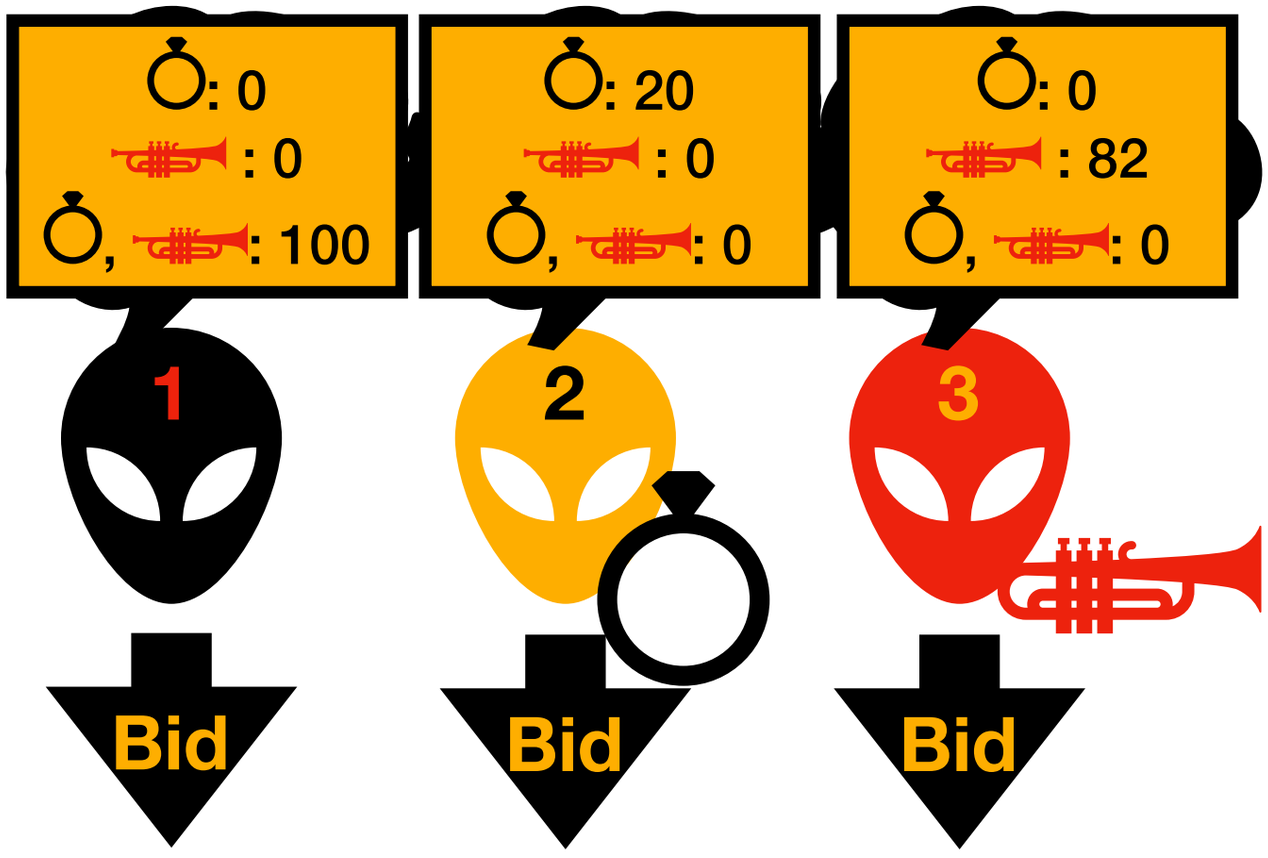

: 0
: 0
, : 100

Pay: 0

Utility: 0

: 20
: 0
, : 0

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: 0
: 82
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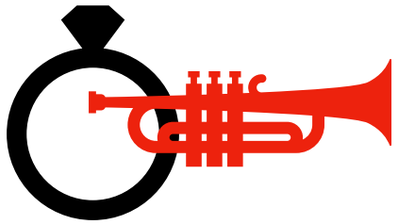
Not strategyproof!

A Simple Mechanism  ?

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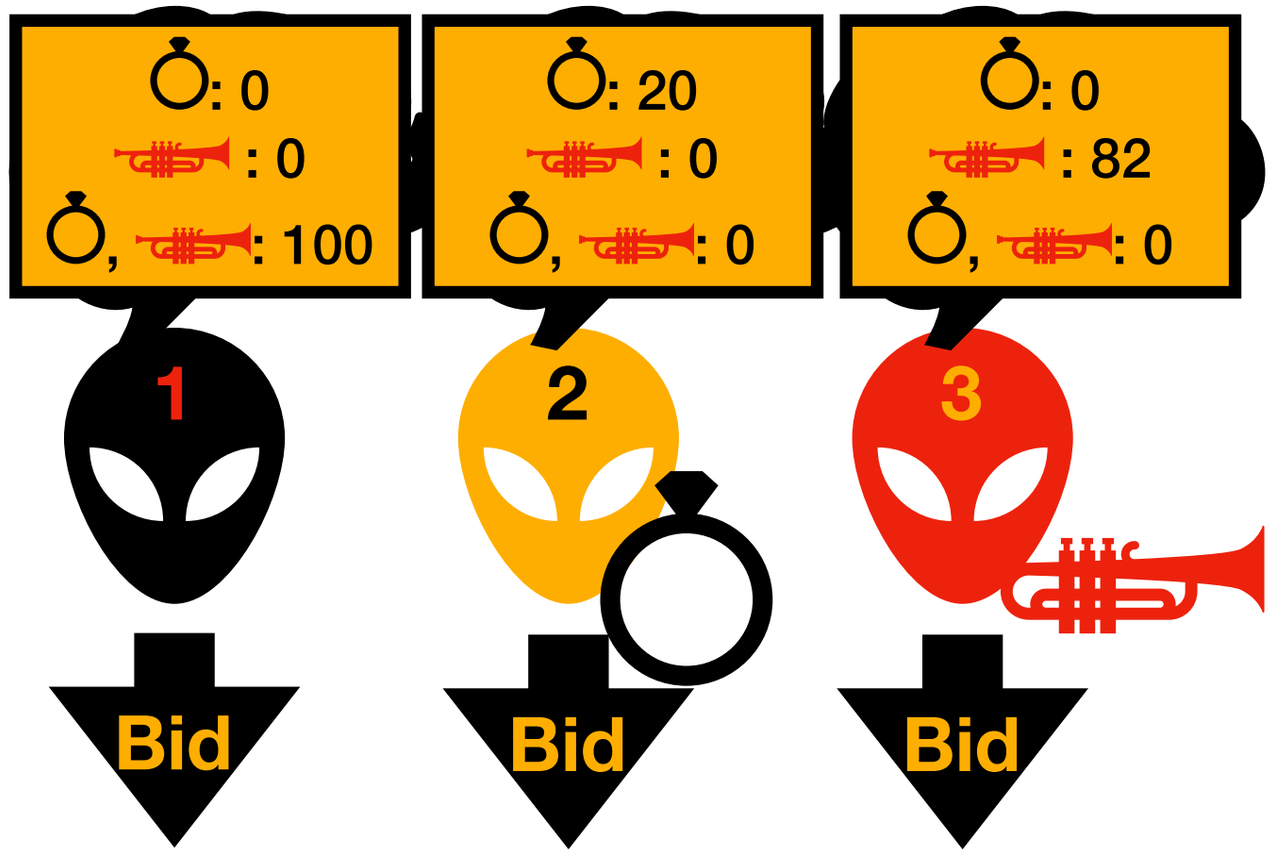


: 0
: 0
, : 100

Pay: 0

Utility: 0

: 20
: 0
, : 0

Pay: 20

Utility: 0

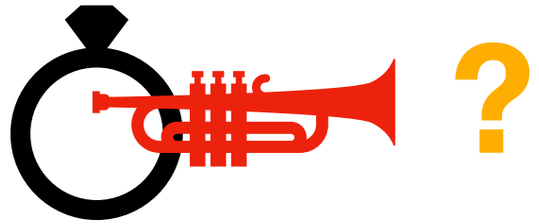
: 0
: 81
, : 0

Pay: 81

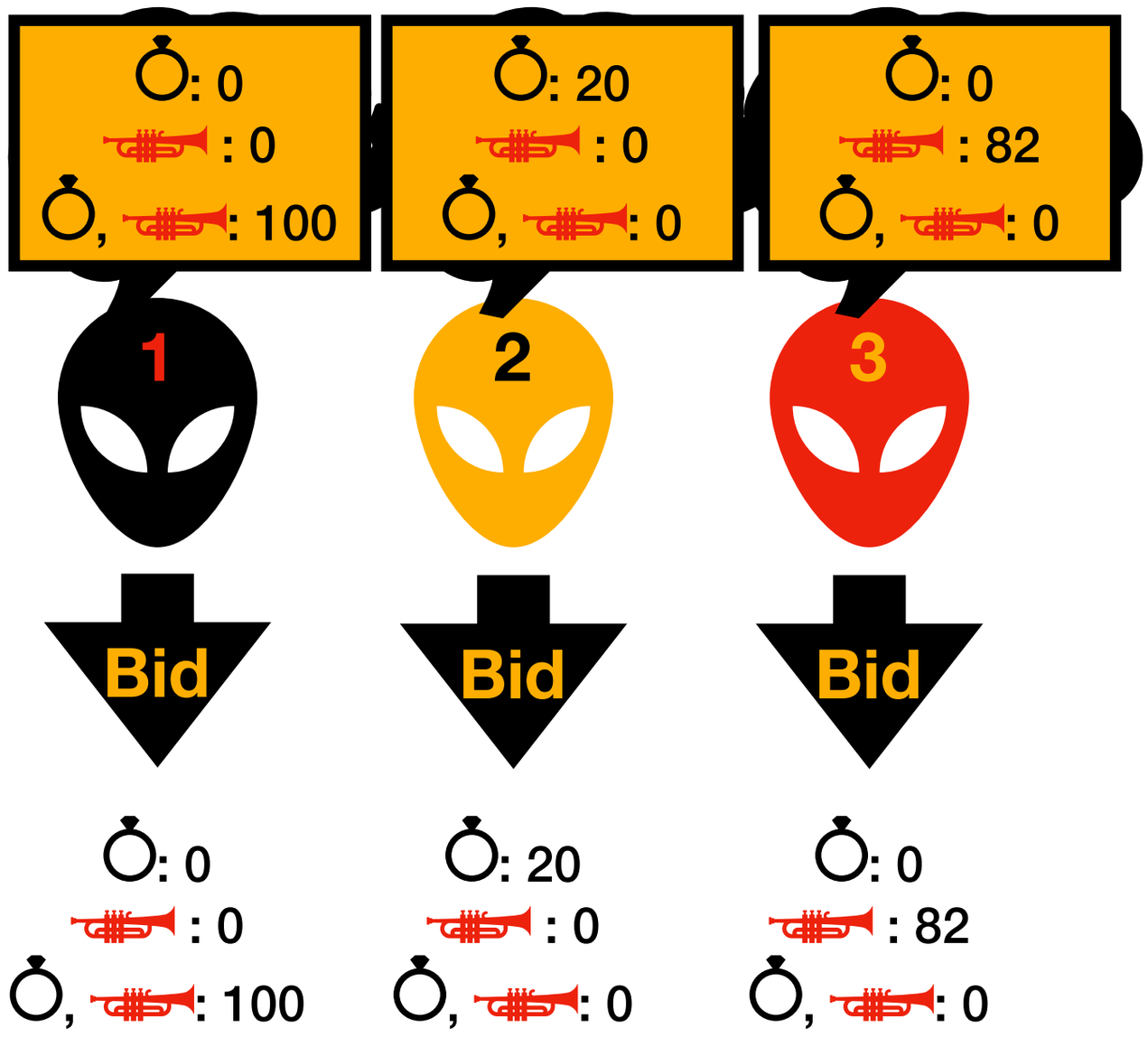
Utility: 1

Not strategyproof!

Vickrey-Clarke-Groves (VCG) Auction



- In a VCG auction bidders are charged their **externalities**, i.e., the harm they cause others
- Everyone submits bids for subsets of items
- Choose allocation that maximizes social welfare, i.e., sum of declared valuations
- Charge bidder i the maximum social welfare if i were absent, minus the social welfare of the other agents when i is present



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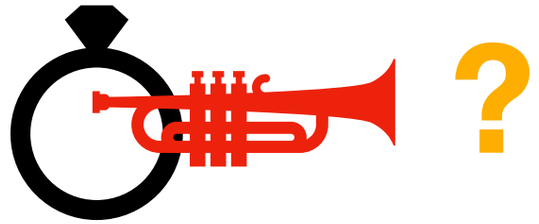

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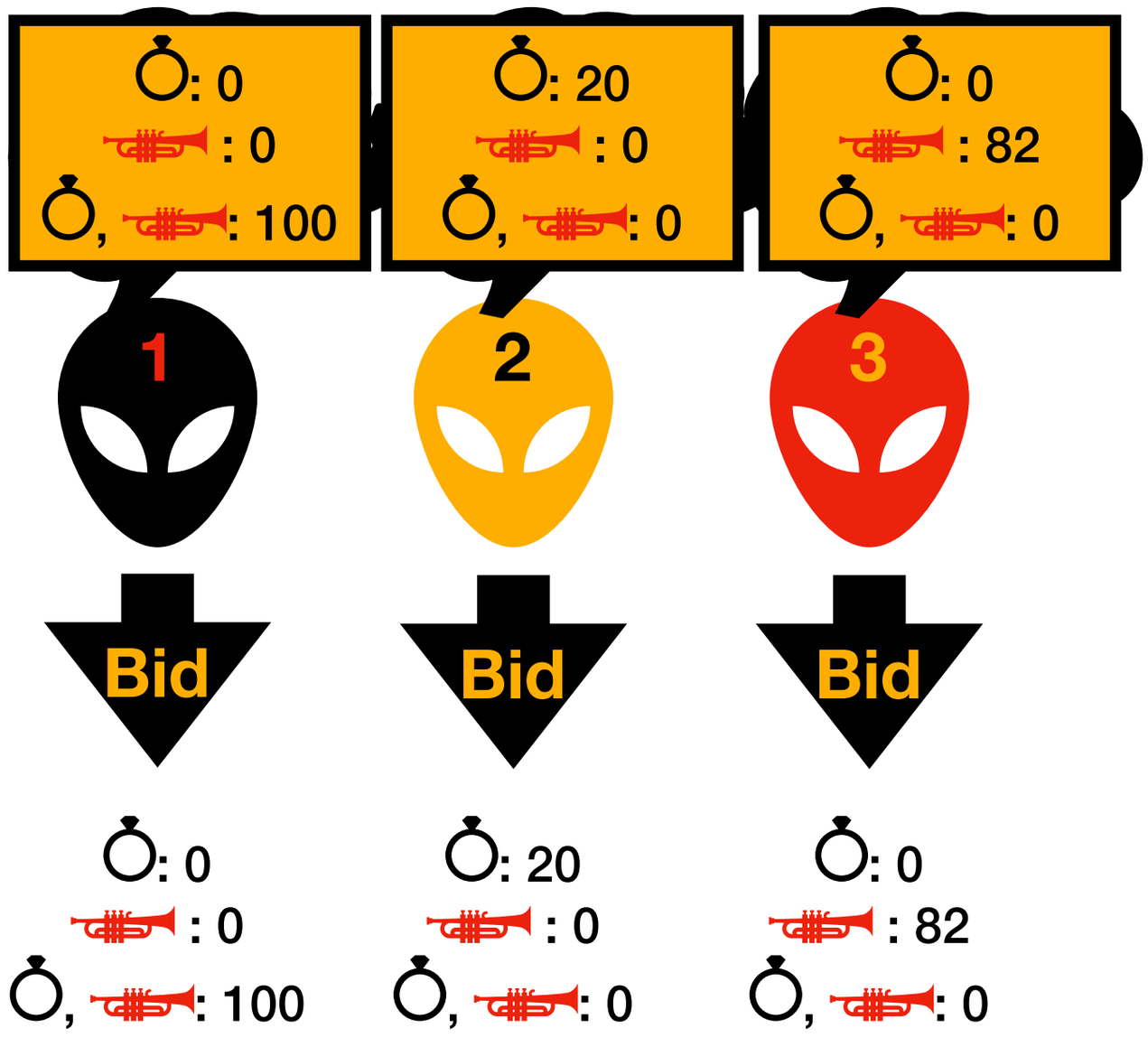

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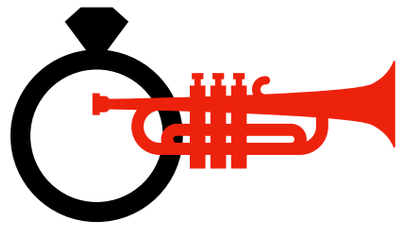

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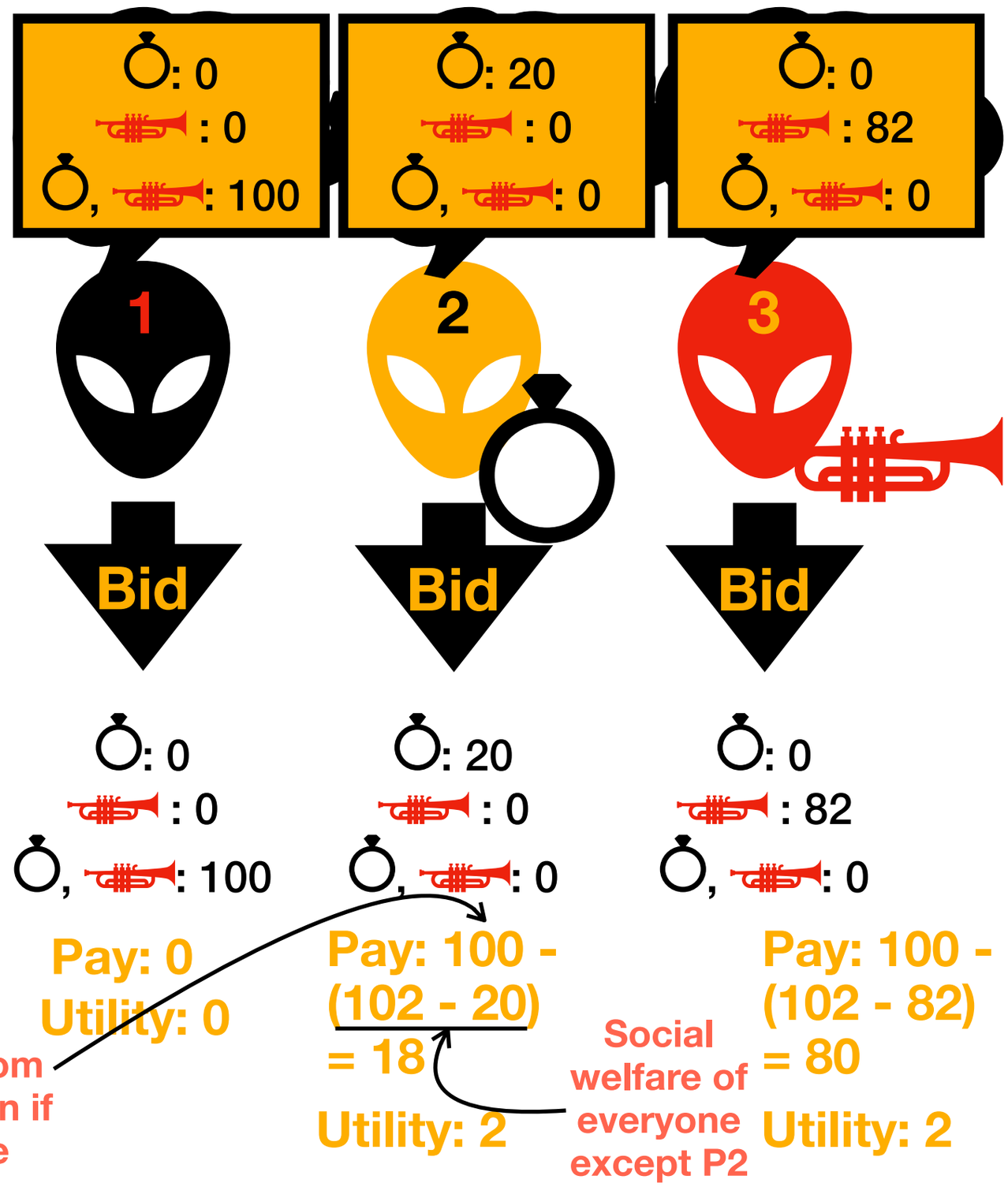

Vickrey-Clarke-Groves (VCG) Auctions

- Unlike the generalized second-price auction, in a VCG auction bidders do not have an incentive to misreport their valuations
- When there is a single item, the VCG auction becomes equivalent to the Vickrey second-price auction
- The VCG auction is an instance of the more general **VCG mechanism**, used to select a socially optimal outcome out of a set of outcomes

Double Auctions

- Another important class of auctions is **double auctions**
- This is where there are multiple buyers *and* sellers, like on the stock market
- Both sides emit bids: buy and sell
    - And the data is recorded in a centralized register

Image: clipartkey.com

Average-Price Double Auction

- Order buyers and sellers by their buying/selling prices: ascending order for sellers, descending order for buyers
- Match as many buyers and sellers as possible (say, k) such that the buying price exceeds the selling price
- Matches correspond to exchanges
- Use a uniform pricing rule: everyone pays the average of the buying and selling price at level k

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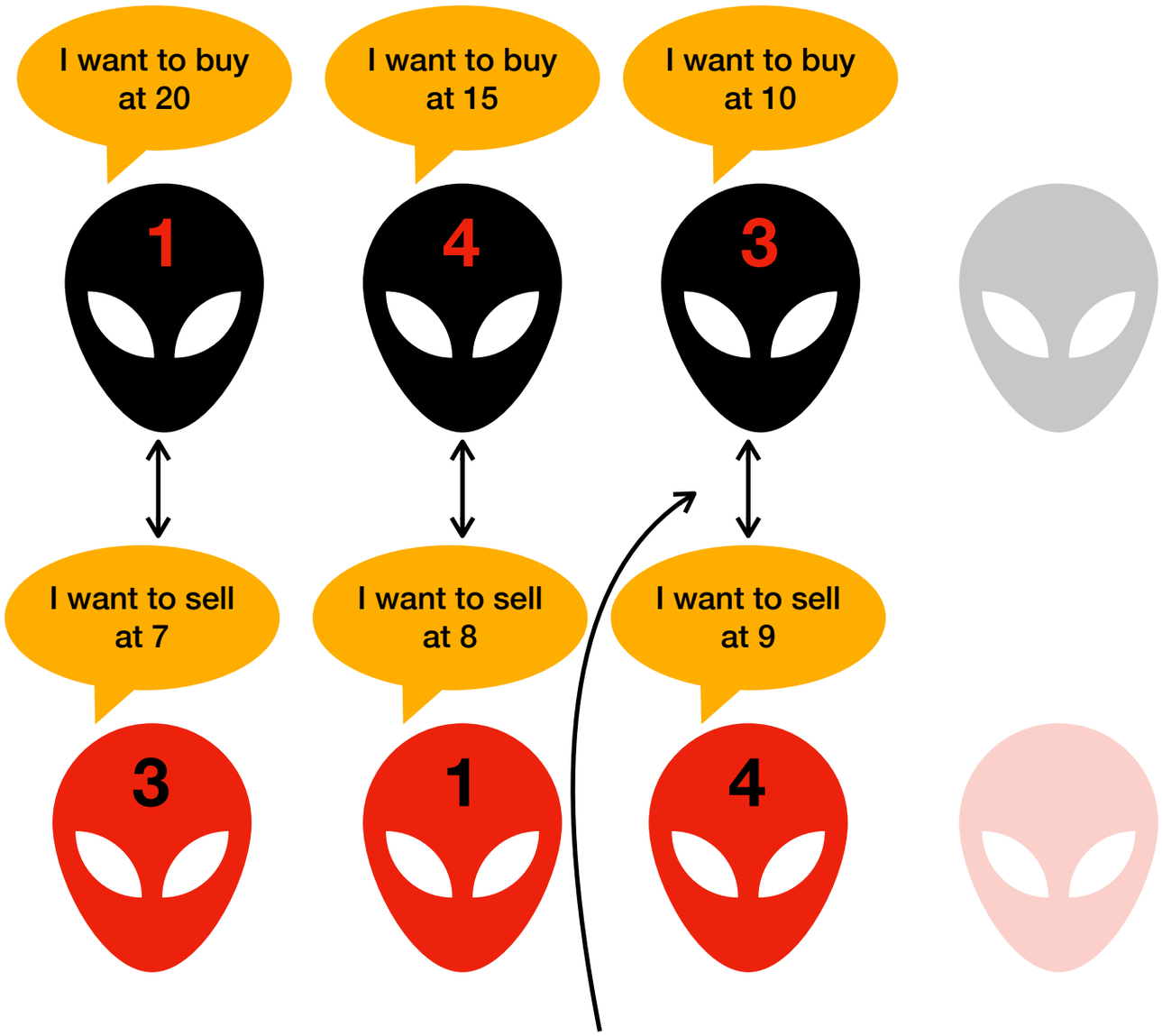

Price: $\frac{1}{2}(9 + 10) = 9.5$

Some Possible Axioms

Definition (Individual Rationality)

No agent loses from joining the auction. In particular, for every buyer the price paid is smaller than their (true) valuation; for every seller, the price is higher than their (true) valuation.

Definition (Weak Balanced Budget)

The total payments collected from buyers is bigger or equal to the total payments made to sellers. In particular, the auctioneer does not lose (buy may gain) money.

Some Possible Axioms

Definition (Strategyproofness, or Truthfulness)
Reporting one's true value (for buyers, their true willingness to pay and for sellers, their actual minimum acceptable price) is the best strategy for all.

Definition (Economic Efficiency)
The total social welfare (i.e., sum of values of all players) is maximum.

Double Auctions

- For one buyer and one seller who do not know each other's valuations, but have probabilistic beliefs about them, there is no mechanism that satisfies individual rationality, weak balanced budget, truthfulness and economic efficiency
- Average-price double auctions satisfy individual rationality, weak balanced budget and economic efficiency, but *not* truthfulness

Last Remarks about Auctions

- We have seen many types of auctions, but the general idea throughout has been that resources get assigned to the highest bidder(s)
- Doing this efficiently ensures that resources go to the people who can derive the most value out of the items
- And the price one is willing to pay serves as a proxy for this value