UvA

Multi-Agent Systems

Game Theory

Mina Young Pedersen
m.y.pedersen@uva.nl

Fall 2025

Decision Theory

- **Decision theory** is about deciding what is best when one agent acts against nature (not other agents)
- Example: Two lotteries, you can enter one.
  - Lottery 1: 50% chance to win 150€
  - Lottery 2: 90% chance to win 100€
  - Which one would you choose?

- You want to **maximize expected utility**

utility of acting a when the state is o

probability that o occurs

$$\mathbb{E}[u(a)] = \sum_{o} (u(a,o) \cdot \mathbb{P}[o])$$

expected utility

possible state of nature

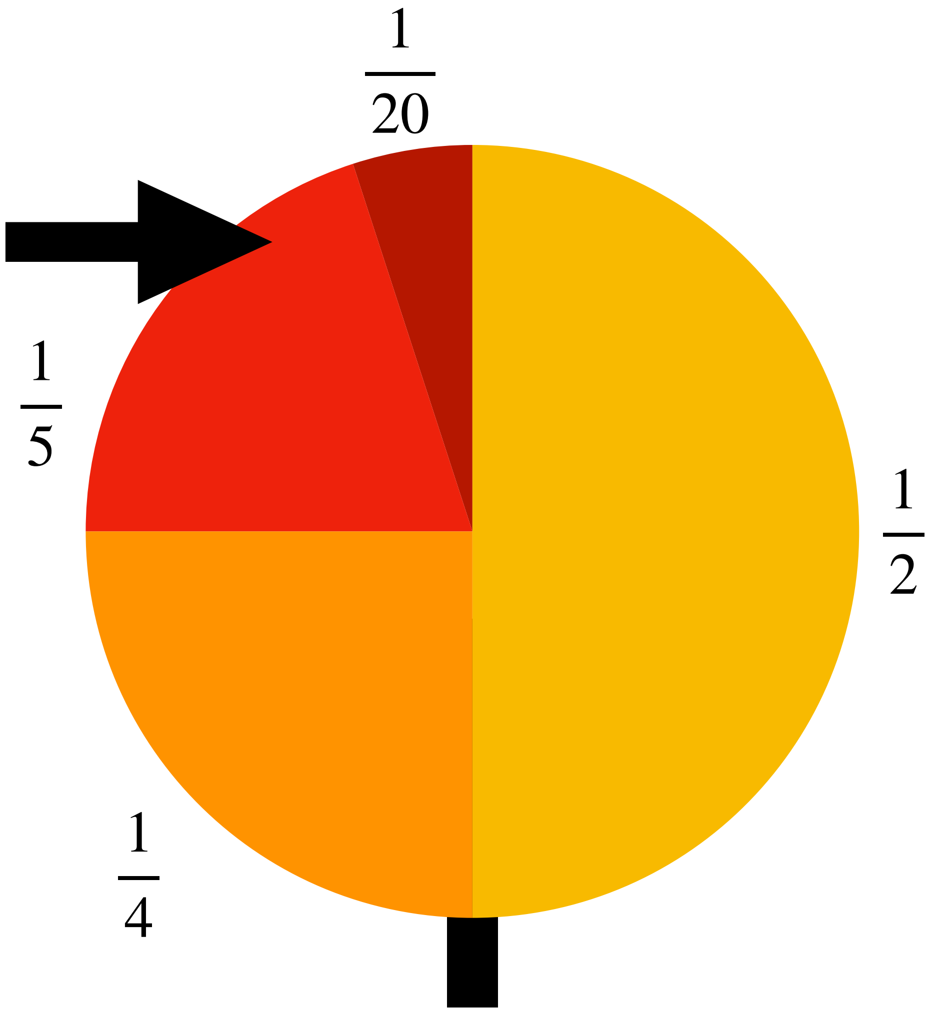
$$\mathbb{E}[u(L1)] = 150 \cdot \frac{1}{2} = 75$$

$$\mathbb{E}[u(L2)] = 100 \cdot \frac{9}{10} = 90$$

Image: clipartkey.com

Decision Theory: Another Example

$$\mathbb{E}[u(a)] = \sum_{o} (u(a,o) \cdot \mathbb{P}[o])$$



$\mathbb{E}[u(\text{spinning the wheel})] =$

$u(\text{spinning,win 0}) \cdot \mathbb{P}[\text{win 0}] + u(\text{spinning,win 5}) \cdot \mathbb{P}[\text{win 5}] + (\ldots) =$

$$0 \cdot \frac{1}{2} + 5 \cdot \frac{1}{4} + 10 \cdot \frac{1}{5} + 100 \cdot \frac{1}{20} = \frac{33}{4} = 8{,}25$$

Coming up: Playing with others…

What is Game Theory?

- **Game theory** is about interactions among independent, self-interested agents

- In a way: thinking about conflict, natural phenomena or daily-life scenarios as games

- **Cooperative** and **non-cooperative** game theory

    - In this course: non-cooperative

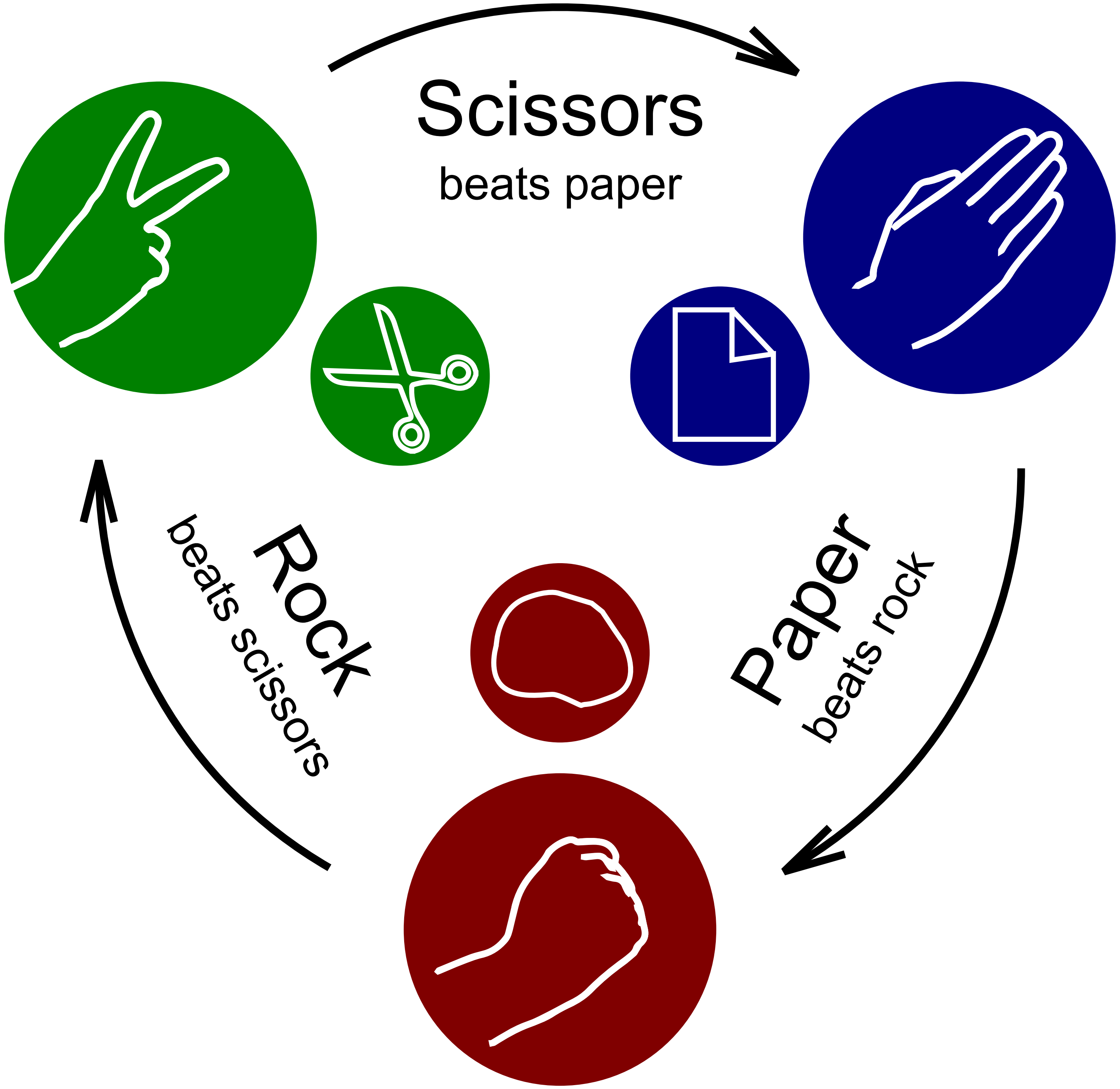
    - Players are:

        1. **Intelligent** (reason perfectly and quickly)

        2. **Rational** (always want to maximize their payoff)

        3. **Selfish** (only care for their own payoff)

Example: Rock Paper Scissors



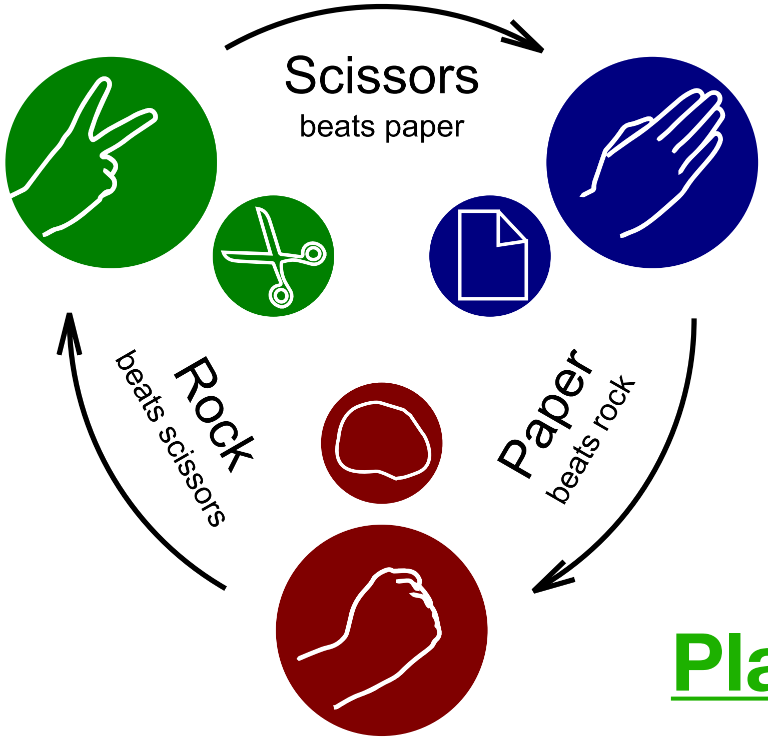
Scissors
beats paper

Paper
beats rock

Rock
beats scissors

Image: Enzoklop, Wikipedia

Example: Rock Paper Scissors



You get 1 point for winning, and -1 for losing.

Player 2 (columns), **Player 1** (rows)

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

This way of presenting a game is called a **normal form game**.

Generally, we assume Player 1 is the row player, and Player 2 is the column player.

Image: Enzoklop, Wikipedia

Another Example: Stag Hunt

- Two hunters
- Have to decide what to hunt: one stag or two hares
- A stag takes two to catch: hunting a stag solo is certain failure
- One hunter alone can catch both hares, but if both hunters go for the hares they end up with one each
- Even combined, the hares are worth less than the stag

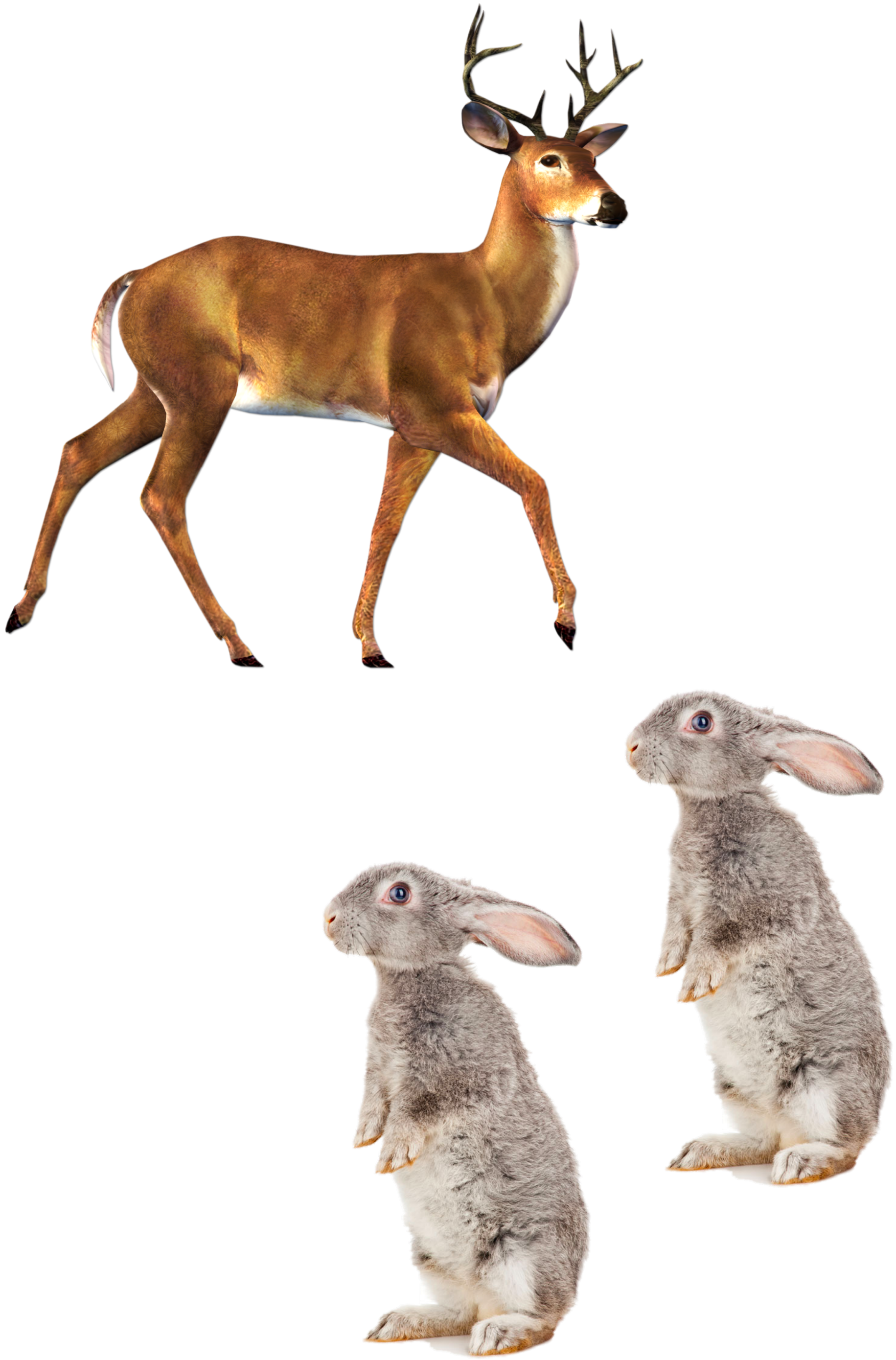


Image: clipartkey.com

Another Example: Stag Hunt

	Hunter 2	
Hunter 1	Stag	Hare
Stag	(10,10)	(0,6)
Hare	(6,0)	(3,3)

Image: clipartkey.com

Games in Normal Form

- A game in normal form consists of **players** who can take **actions**, which lead to **payoffs**

Definition (Normal-Form Game)

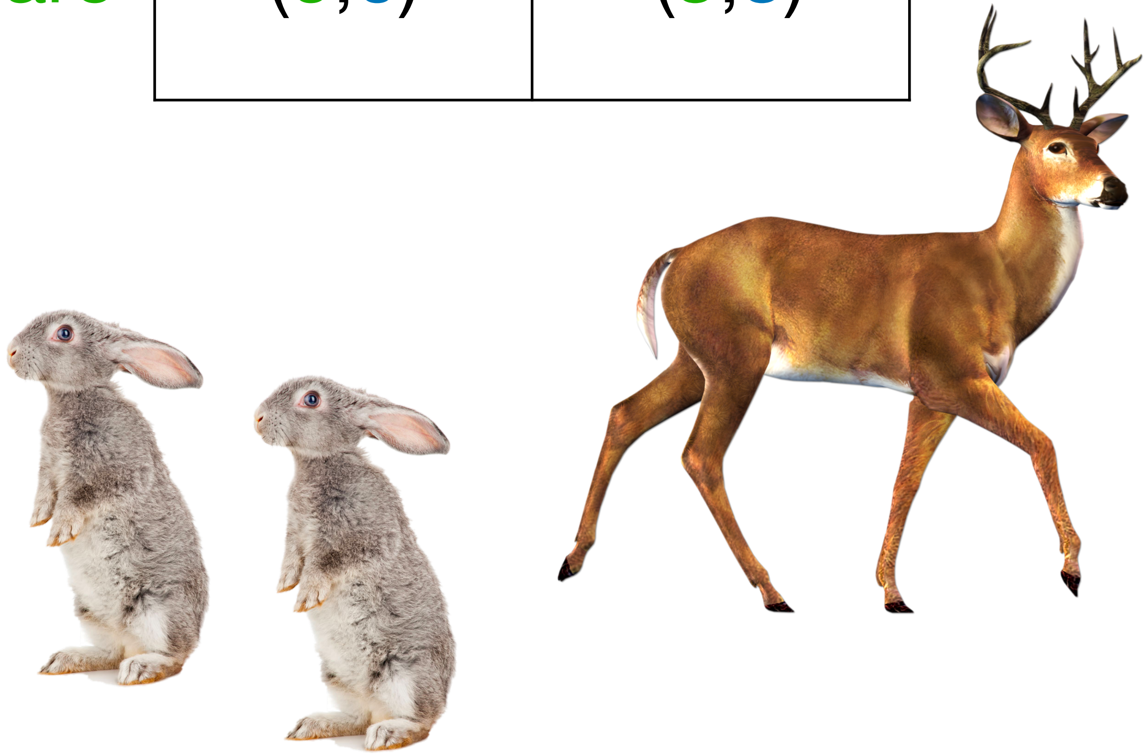
A normal-form game is a tuple (N, A, u), where:

- $N = \{1, \ldots, n\}$ **is a finite set of agents, or players**
- A_i **is a finite set of actions available to player** i
- $a = (a_1, \ldots, a_n)$ **is an action profile**
- $A = A_1 \times \ldots \times A_n$ **is the set of all action profiles**
- $u_i : A \rightarrow \mathbb{R}$ **is a utility (or payoff) function for player** i
- $u = (u_1, \ldots, u_n)$ **is a utility profile**

Back to the Stag Hunt

Hunter 2

Hunter 1	Stag	Hare
Stag	(10,10)	(0,6)
Hare	(6,0)	(3,3)



- $N = \{1,2\}$
- $A_1 = \{Stag, Hare\}$
- $A_2 = \{Stag, Hare\}$
- $A = \{(Stag, Stag), (Stag, Hare), (Hare, Stag), (Hare, Hare)\}$
- $u_1(Stag, Stag) = 10$
- $u_2(Stag, Stag) = 10$
- $u_1(Stag, Hare) = 0$
- $u_2(Stag, Hare) = 6$
- (…)

Image: clipartkey.com

What Should Players Do: Strategies

- A **pure strategy** for an agent is to select a single action and play it

- $N = \{1, \ldots, n\}$ is a finite set of agents, or players
- A_i is a finite set of actions available to player i
- $a = (a_1, \ldots, a_n)$ is an action profile
- $A = A_1 \times \ldots \times A_n$ is the set of all action profiles
- $u_i : A \rightarrow \mathbb{R}$ is a utility (or payoff) function for player i
- $u = (u_1, \ldots, u_n)$ is a utility profile

Definition (Pure Strategy)

- $S_i = A_i$ are the pure strategies of player i
- $s = (s_1, \ldots, s_n)$ is a strategy profile
- $S = S_1 \times \ldots \times S_n$ is the set of all strategy profiles
- $u_i(s) = u_i(a)$ is the utility of i wrt. strategy profile s
- $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ is s without s_i
- $s = (s_i, s_{-i})$ is another way to write s

What Should Players Do: Solution Concepts

- A **solution concept** describes what strategies we might expect the players will adopt
    - And therefore, the result of the game

Dominance Among Strategies

- A player has a **dominated strategy** if the player would always be as well (or even better) off playing something else

Strict Dominance

Definition (Strict Dominance Among Strategies)

- Strategy s_i **strictly dominates** s_i' if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$, for any profile s_{-i} of other agents' strategies
- Strategy s_i is **strictly dominant** if it strictly dominates every other strategy s_i'

Example

	L	C	R
T	(3,0)	(2,1)	(0,0)
M	(1,1)	(1,1)	(5,0)
B	(0,1)	(4,2)	(0,1)

- For Player 1, does T strictly dominate M?
    - If P2 plays L: $3 > 1$ ✓
    - If P2 plays C: $2 > 1$ ✓
    - If P2 plays R: $0 < 5$ ✗
    - No

- For Player 2, does C strictly dominate L?
    - If P1 plays T: $1 > 0$ ✓
    - If P1 plays M: $1 = 1$ ✗
    - If P1 plays B: $2 > 1$ ✓
    - No

Example

	L	C	R
T	(3,0)	(2,1)	(0,0)
M	(1,1)	(1,1)	(5,0)
B	(0,1)	(4,2)	(0,1)

- **For Player 2, does C strictly dominate R?**
  - **If P1 plays T:** $1 > 0$ ✓
  - **If P1 plays M:** $1 > 0$ ✓
  - **If P1 plays B:** $2 > 1$ ✓
  - Yes

Player 2 should always play C over R…

…this suggests a way of filtering out strategies

Eliminating Strictly Dominated Strategies

Definition (Iterated Elimination of Strictly Dominated Strategies, or IESDS)

- For each player, go through its strategies. If any are strictly dominated, eliminate them. Repeat until there are no further strategies to eliminate.

Back to the Example

- What can we eliminate here?
    - C strictly dominates R:
        - Eliminate R

	L	C	R
T	(3,0)	(2,1)	(0,0)
M	(1,1)	(1,1)	(5,0)
B	(0,1)	(4,2)	(0,1)

Back to the Example

- What can we eliminate here?
  - C strictly dominates R:
    - Eliminate R
  - T strictly dominates M:
    - Eliminate M

	L	C
T	(3,0)	(2,1)
M	(1,1)	(1,1)
B	(0,1)	(4,2)

Back to the Example

- What can we eliminate here?
  - C strictly dominates R:
    - Eliminate R
  - T strictly dominates M:
    - Eliminate M
  - C strictly dominates L:
    - Eliminate L

	L	C
T	(3,0)	(2,1)
B	(0,1)	(4,2)

Back to the Example

- What can we eliminate here?
    - C strictly dominates R:
        - Eliminate R
    - T strictly dominates M:
        - Eliminate M
    - C strictly dominates L:
        - Eliminate L
    - B strictly dominates T:
        - Eliminate T

	C
T	(2,1)
B	(4,2)

Back to the Example

- What can we eliminate here?
    - C strictly dominates R:
        - Eliminate R
    - T strictly dominates M:
        - Eliminate M
    - C strictly dominates L:
        - Eliminate L
    - B strictly dominates T:
        - Eliminate T
    - Nothing left:
        - Stop

	C
B	(4,2)

Back to the Example

- What can we eliminate here?
    - C strictly dominates R:
        - Eliminate R
    - T strictly dominates M:
        - Eliminate M
    - C strictly dominates L:
        - Eliminate L
    - B strictly dominates T:
        - Eliminate T
    - Nothing left:
        - Stop

	L	C	R
T	(3,0)	(2,1)	(0,0)
M	(1,1)	(1,1)	(5,0)
B	(0,1)	(4,2)	(0,1)

- IESDS solution: (B, C)

More on IESDS

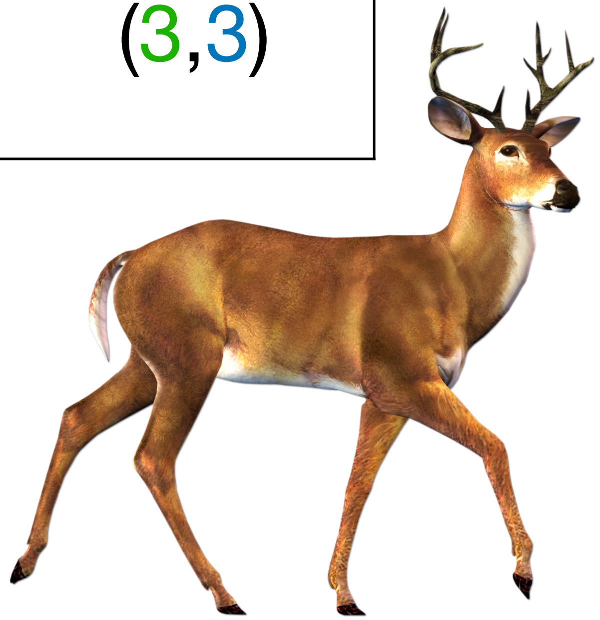
- The order in which strictly dominated strategies are eliminated does not matter
- We say that a game is **solvable** when IESDS results in a unique strategy profile

Return to the Stag Hunt

Hunter 2

Hunter 1	Stag	Hare
Stag	(10,10)	(0,6)
Hare	(6,0)	(3,3)

- Are there any strictly dominated strategies to eliminate here?
    - No



Not all games are solvable…

Image: clipartkey.com

Weak Dominance

Definition (Weak Dominance Among Strategies)

- Strategy s_i **weakly dominates** s_i' if:
    - $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$, for any profile s_{-i} of other agents' strategies, and
    - there is some strategy profile $\tilde{s}_{-i}$ such that $u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$
- Strategy s_i is **weakly dominant** if it weakly dominates every other strategy s_i'

Example

	L	C	R
T	(3,0)	(2,1)	(0,0)
M	(1,1)	(1,1)	(5,0)
B	(0,1)	(4,2)	(0,1)

- For Player 2, does C weakly dominate L?
  - If P1 plays T: $1 > 0$ ✓
  - If P1 plays M: $1 = 1$ ✓
  - If P1 plays B: $2 > 1$ ✓
  - Yes

More on Weakly Dominated Strategies

- We can eliminate weakly dominated strategies, just like we eliminate strictly dominated strategies

- This time, though, the order in which we eliminate matters

This will be further explored in the homework…

Pareto Optimality

- In a **Pareto optimal outcome** no one can be made better off without making someone else worse off

Definition (Pareto Optimality)

- Strategy profile s **Pareto dominates** strategy profile s' if:
  - $u_i(s) \geq u_i(s')$ for every player $i \in N$, and
  - there exists an agent j such that $u_j(s) > u_j(s')$
- Strategy profile s is **Pareto optimal** if there is no profile s' that Pareto dominates s

The Coordination Game

- A country with no traffic rules
- Two cars on the road, driving towards each other
- They have to decide what side of the road to take
- If they choose the same side, all is well
- If they choose different sides, they crash into each other

Image: clipartkey.com

The Coordination Game

What are the Pareto optimal outcomes?

	Left	Right
Left	(1,1)	(0,0)
Right	(0,0)	(1,1)

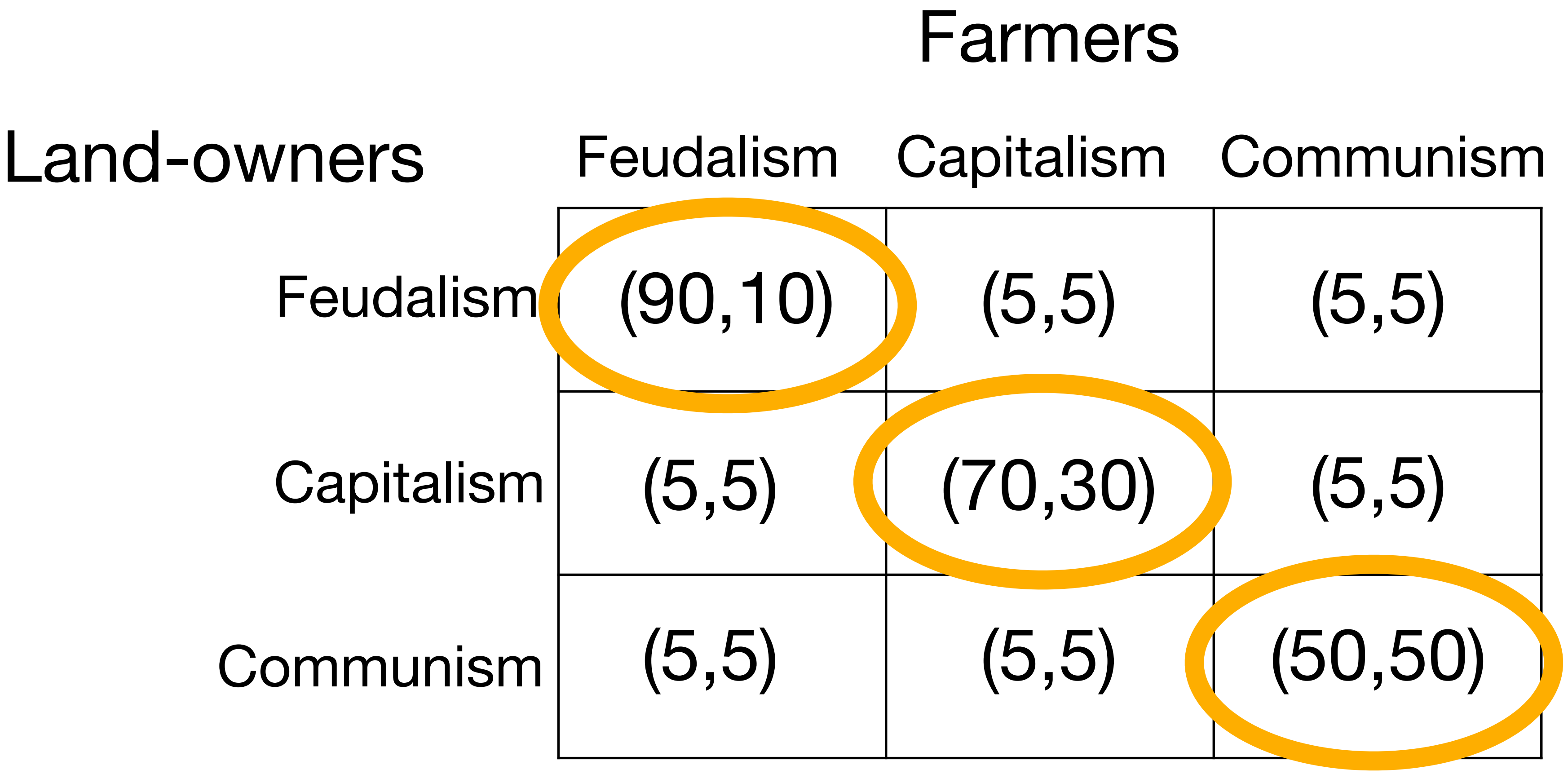
Pareto optimal

Pareto dominated by (Left, Left) and (Right, Right)

Image: clipartkey.com

More on Pareto Optimality

- Pareto optimal outcomes always exist
- They are not always unique (there can be more than one)
- Pareto optimality does not necessarily imply that outcomes are fair



Farmers

Land-owners	Feudalism	Capitalism	Communism
Feudalism	(90,10)	(5,5)	(5,5)
Capitalism	(5,5)	(70,30)	(5,5)
Communism	(5,5)	(5,5)	(50,50)

Pareto optimal