Tutorial 5

Auctions, Modal Logic

27 November, 2025

Exercise 1. Consider an auction for multiple goods, where there are 6 potential buyers. The offers for one item are as follows: Ann bids 117, Bob bids 100, Cathy bids 118, Daan bids 107, Eleanor bids 95 and Farid bids 98. Always justify your answers.

(1) Consider a company that is selling 4 of its shares. Given the above bids from the buyers, use the $k+1^{st}$-price auction to find which agents get the shares and how much they have to pay.

(2) Consider a website that is selling 4 of its ad slots $\{s_1, s_2, s_3, s_4\}$, where s_1 is the best-placed and s_4 is the worst-placed slot in the webpage (and the other two slots are ordered accordingly). Given the above bids from the buyers, use the generalized *first*-price auction to find which agents get which slot and how much they have to pay.

(3) Given the above bids from the buyers, use the generalized *second*-price auction to find which agents get which slot and how much they have to pay.

(4) Consider now a two-sided auction setting, such as a stock market, which comprises the same buyers listed above and also 6 sellers. Patrick wants to sell at 117, Quentin at 108, Raquel at 118, Sophie at 95, Tobias at 98 and Valencia at 99. Given the offers from the buyers and the sellers, use the average-price double auction to find which agents get a stock and how much do they have to pay.

(5) Assuming that these were the agents' truthful offers, that the bids must be positive integers, and assuming that everybody else keeps their current offer, is there a buyer who could have gotten a better outcome by submitting an untruthful offer? Is there a seller for which this is the case?

Exercise 2. Recall that the VCG auction is a combinatorial auction in which bidders submit bids for subsets of items, and returns an allocation that maximizes the sum of declared bids (what we called *social welfare*). The payment rule is as follows: bidder i pays an amount equal to the maximum social welfare if i were absent, minus the social welfare of other bidders when i is present.

Consider an auction with two items, $G = \{a, b\}$, and three bidders, $N = \{1, 2, 3\}$, with the following valuations:

$X \subseteq G$	$v_1(X)$	$v_2(X)$	$v_3(X)$
$\emptyset$	0	0	0
$\{a\}$	0	1	0
$\{b\}$	0	0	2
$\{a, b\}$	2	0	0

(1) Suppose bidders bid their true valuations. What is the allocation that maximizes social welfare (with respect to the declared valuations)?

(2) What are the payments p_i, for $i \in \{1, 2, 3\}$, under the VCG auction?

Exercise 3. Below there is a list of formulas of the basic modal language. For each one of them, build a pointed relational model in which the formula is true, and a pointed relational model in which the formula is false; in both cases, there is no restriction on the relation of the model. In case you consider one of the cases is impossible, explain why.

$M, w \models \Box(p \to q)$
if $M, w \not\models \Box p$, RHS always holds
~~if~~ $M, w \models \Box p$
Take v that Rwv:
$M, v \models p$
$M, v \models p \to q$
$\Rightarrow M, v \models q$
$\Rightarrow M, w \models \Box q$

(W, R, V)

(1) $\Diamond\Diamond p \to p$

(2) $p \to \Box\Diamond p$

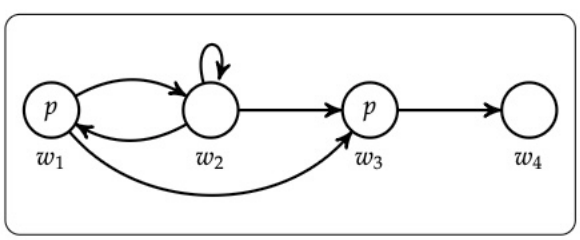
(3) $\Box(p \to q) \to (\Box p \to \Box q)$

impossible to be false
Rwv: v is a successor of w

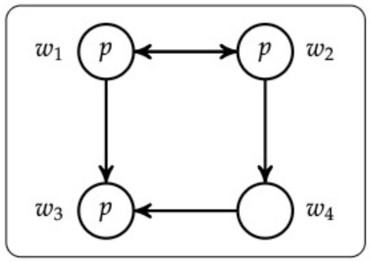
(4) $\Box(p \vee q) \leftrightarrow (\Box p \vee \Box q)$

Exercise 4. Below you find two relational models. For each one of them, find a formula in the basic modal language characterising each one of the model's worlds. In other words, for each listed model M, and for each one world w in M, find a formula χ_w that is true in (M, w) and false in *every other world in* M.

(1)



(2)



Exercise 5. Prove that a modal frame F is reflexive if and only if $F \Vdash p \to \Diamond p$. Recall that a frame is reflexive if and only if for all $w \in W : Rww$.

first try prove $\cancel{Rww}$ reflexive $\Rightarrow F \Vdash p \to \Diamond p$

$\forall w \in W$, if $M, w \models p$,

$\Rightarrow M, w \models \Diamond p$

$\Rightarrow M, w \models p \to \Diamond p$.

if $M, w \not\models p$,

always holds $M, w \models p \to \Diamond p$

then the only if side.

Counter example.

suppose $\exists w \in W$, Rww and $M, w \not\models p \to \Diamond p$

then must $M, w \models p$,

but then $M, w \models \Diamond p$,

contradict.