Exam Preparation Exercises

December, 2025

agents: {1, 2}

actions: $A_1 = A_2 =$ {comedy, doc}

Exercise 1. In the *Battle of the movies* game we are considering a situation where a couple is deciding what to watch on a Friday evening. After 45 minutes of scrolling through a popular streaming platform, they narrow the choice down to two movies: a comedy and a documentary. 1c , 2d

However, one partner prefers the comedy to the documentary, and the other one the documentary to the comedy. Watching one's favourite movie gives a utility of 3, while watching what the other person prefers gives a utility of 2. If they ended up each on their own laptop, watching their favourite movie separately, they would only get a utility of 1. The absolute worst (and a bit silly too) would be for each person to watch their *least* favourite movie alone, giving a utility of 0.

		2: c	d
1	C	(3,2)	(1,1)
	d	(0,0)	(2,3)

1. Write the above game in normal-form, with agents, actions and payoff matrix.
2. For each pair of strategy profiles, find whether one Pareto dominates the other.
3. What are the Pareto optimal strategy profiles? Justify your answer.
4. Can you find pure(-strategy) Nash equilibria for the game? Justify your answer. proof: no one would deviate alone.

Solution 1. Here's a suggestion:

1. We can express the scenario as a game $G = (N, A, u)$ in normal form, as follows:
   - $N = \{1, 2\}$;
   - $A_1 = A_2 = \{\texttt{comedy}, \texttt{documentary}\}$;
   - For ease of notation, write:

$$\begin{aligned}(\texttt{comedy}, \texttt{comedy}) &= (C, C),\\ (\texttt{comedy}, \texttt{documentary}) &= (C, D),\\ (\texttt{documentary}, \texttt{comedy}) &= (D, C),\\ (\texttt{documentary}, \texttt{documentary}) &= (D, D),\end{aligned}$$

in which case:

$$u_1(C,C) = 2, u_1(C,D) = 0, u_1(D,C) = 1, u_1(D,D) = 3,$$
$$u_2(C,C) = 3, u_2(C,D) = 0, u_2(D,C) = 1, u_2(D,D) = 2.$$

The payoff matrix is as follows:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} (2,3) & (0,0) \\ (1,1) & (3,2) \end{bmatrix} \end{array}$$

2. Profiles (C,C) and (D,D) Pareto dominate (D,C), which in turn Pareto dominates (C,D). In fact, for profile (C,D) both agents have utility 0, and they gain each 1 by moving to profile (D,C). Then, if they move to profile (C,C) player 1 gains 1 and player 2 gains 2, and viceversa for profile (D,D).

3. The Pareto optimal profiles are (C,C) and (D,D), as they are not dominated by any other profile.

4. The Nash equilibria are (C,C) and (D,D). In fact, if player 1 deviates in (C,C) by playing action `documentary`, they would get utility 1 instead of 2, and if player 2 deviates they get utility 0 instead of 3. A similar reasoning holds for profile (D,D).

Exercise 2. Consider the set of alternatives (or candidates) $A = \{a,b,c,d,e\}$ and 100 voters in N who express their preferences over the alternatives as per the following profile:

16 voters	$e \succ c \succ b \succ a \succ d$
20 voters	$b \succ d \succ c \succ e \succ a$
19 voters	$d \succ c \succ e \succ b \succ a$
31 voters	$a \succ d \succ c \succ e \succ b$
14 voters	$c \succ e \succ d \succ b \succ a$

(1) Determine the (set of) winners according to the *plurality* rule. a

(2) Determine the (set of) winners according to the *Borda* rule. c

(3) Determine if there is a *Condorcet winner* and/or a *Condorcet loser* in the profile. A Condorcet loser is a candidate that *loses* against every other candidate in a head-to-head contest based on the given profile.

Solution 2. Here is a solution.

(1) The winner according to the *plurality* rule is candidate a, since they are ranked in the first position by the most voters (31).

(2) The winner under the *Borda* rule is c. In fact, we have the following scores for the candidates:

a	$4 \cdot 31 + 3 \cdot 0 + 2 \cdot 0 + 1 \cdot 16 = 140$
b	$4 \cdot 20 + 3 \cdot 0 + 2 \cdot 16 + 1 \cdot 33 = 145$
c	$4 \cdot 14 + 3 \cdot 35 + 2 \cdot 51 + 1 \cdot 0 = \mathbf{263}$
d	$4 \cdot 19 + 3 \cdot 51 + 2 \cdot 14 + 1 \cdot 0 = 257$
e	$4 \cdot 16 + 3 \cdot 14 + 2 \cdot 19 + 1 \cdot 51 = 195$

(3) Candidate d is a *Condorcet winner*, beating every other candidate under pairwise majority:

Voters preferring d to a	$20 + 19 + 14 = 53$
Voters preferring d to b	$19 + 31 + 14 = 64$
Voters preferring d to c	$20 + 19 + 31 = 70$
Voters preferring d to e	$20 + 19 + 31 = 70$

Candidate a is a *Condorcet loser*, being beaten by every other candidate under pairwise majority:[1]

Voters preferring b to a	$16 + 20 + 19 + 14 = 69$
Voters preferring c to a	$16 + 20 + 19 + 14 = 69$
Voters preferring d to a	$20 + 19 + 31 = 14 = 84$
Voters preferring e to a	$16 + 20 + 19 + 14 = 69$

Exercise 3. Consider an auction for multiple goods, where there are 6 potential buyers. The offers for one item are as follows: Ann bids 117, Bob bids 100, Cathy bids 118, Daan bids 107, Eleanor bids 95 and Farid bids 98. Always justify your answers.

(1) Consider a company that is selling 4 of its shares. Given the above bids from the buyers, use the $k+1^{st}$-price auction to find which agents get the shares and how much they have to pay.

(2) Consider a website that is selling 4 of its ad slots $\{s_1, s_2, s_3, s_4\}$, where s_1 is the best-placed and s_4 is the worst-placed slot in the webpage (and the other two slots are ordered accordingly). Given the above bids from the buyers, use the generalized *first*-price auction to find which agents get which slot and how much they have to pay.

(3) Given the above bids from the buyers, use the generalized *second*-price auction to find which agents get which slot and how much they have to pay.

[1] Note that *(i)* the Borda rule does not elect the Condorcet winner, and *(ii)* the plurality rule elects the Condorcet loser.

(4) Consider now a two-sided auction setting, such as a stock market, which comprises the same buyers listed above and also 6 sellers. Patrick wants to sell at 117, Quentin at 108, Raquel at 118, Sophie at 95, Tobias at 98 and Valencia at 99. Given the offers from the buyers and the sellers, use the average-price double auction to find which agents get a stock and how much do they have to pay.

(5) Assuming that these were the agents' truthful offers, that the bids must be positive integers, and assuming that everybody else keeps their current offer, is there a buyer who could have gotten a better outcome by submitting an untruthful offer? Is there a seller for which this is the case?

Solution 3. (1) Since in this case $k = 4$, i.e., there are four shares for sale, the $k + 1^{st}$-price auction will assign one share to each of the four buyers with the highest bids, and they will all pay the price of the $k + 1 = 5^{th}$ highest bidder (e.g., the first losing bid).

Namely, Cathy, Ann, Daan and Bob will receive 1 share each (while Farid and Eleanor will get none) and each of them will pay 98, i.e., the bid offered by Farid who is fifth in the ordering.

(2) For both the generalized first-price auction (GFP) and the generalized second-price auction (GSP) we assign the slots from best to worst to the 4 highest bidders. Namely, we have that Cathy will get the best slot s_1, Ann will get s_2, Daan will get s_3 and Bob will get s_4 (while Farid and Eleanor will receive no ad slot).

For the payments, in the GFP auction every agent i who was assigned an ad slot by the auction will pay their bid b_i: hence, Cathy will pay 118, Ann will pay 117, Daan will pay 107 and Bob will pay 100.

(3) In the GSP auction every agent i who was assigned an ad slot j by the auction will pay the price b_{j+1}: thus, Cathy will pay 117, Ann will pay 107, Daan will pay 100 and Bob will pay 98. In both cases, Farid and Eleanor will pay nothing. The table below summarizes the situation for both the GFP and GSP auctions:

Buyer i	b_i	slot in G	GFP	GSP
Cathy	118	s_1	118	117
Ann	117	s_2	117	107
Daan	107	s_3	107	100
Bob	100	s_4	100	98
Farid	98	-	-	-
Eleanor	95	-	-	-

(4) First we order the sellers in increasing order of their offer next to the buyers:

Buyer i	Buying price	Seller k	Selling price
Cathy	118	Sophie	95
Ann	117	Tobias	98
Daan	107	Valencia	99
Bob	100	Quentin	107
Farid	98	Patrick	117
Eleanor	95	Raquel	118

According to the average-price double auction, we have to respect the constraint that for each buyer-seller pair, the buying price has to be at least as high as the selling price. This constrain is satisfied for the first $k = 3$ pairs: Cathy and Sophie ($118 \geq 95$), Ann and Tobias ($117 \geq 98$), as well as Daan and Valencia ($107 \geq 99$). Hence, Cathy, Sophie and Ann will sell, and Sophie, Tobias and Valencia will buy.

Buyer i	Buying price	Seller k	Selling price
Cathy	118	Sophie	95
Ann	117	Tobias	98
Daan	107	Valencia	99
Bob	100	Quentin	108
Farid	98	Patrick	117
Eleanor	95	Raquel	118

RWU

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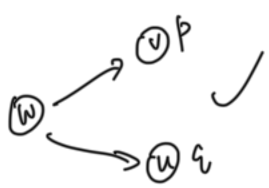
we show there's a case where $\Diamond p \wedge \Diamond q$ but $\neg\Diamond(p \wedge q)$

The payment for the average-price double auction is determined by the last level $k = 3$ at which such a pairing between selling price and buying price is possible, and it consists of the average of the prices at that level. Namely, each buyer in the auction will pay $\frac{107+99}{2} = 103$ to the seller.

(5) From the buyers point of view, Daan could have made a bid of 101: in this way, the average with Valencia's price would have been $\frac{101+99}{2} = 100$, which is better for Daan (and the other buyers) than the current 103 since it is cheaper. Analogously, Valencia could have submitted a selling price of 107, moving the average to 107 which is better for her (and the other sellers) than the current 103 since the stock gets sold at a higher price.



Exercise 4. Is the formula $(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$ valid in the basic modal logic? If it is valid, prove it. If it is not, show a counter-example.

Solution 4. No, the formula is not valid. See a counter-example in model $\mathcal{M}$ in Figure 1. In this model $\mathcal{M}, w \Vdash \Diamond p$ since Rwv and $\mathcal{M}, v \Vdash p$, and $\mathcal{M}, w \Vdash \Diamond q$ since Rwu and $\mathcal{M}, u \Vdash q$. Therefore, $\mathcal{M}, w \Vdash \Diamond p \wedge \Diamond q$. However $\mathcal{M}, w \nVdash \Diamond(p \wedge q)$ since there is no $s \in W$ such that Rws and $\mathcal{M}, s \Vdash p \wedge q$. Thus, it follows that $\mathcal{M}, w \nVdash (\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$.

Exercise 5. Prove the following equivalence in Computation-Tree Logic:

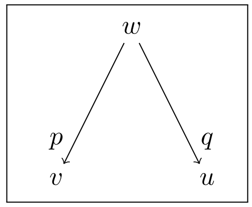


Figure 1: A model $\mathcal{M}$ where $\mathcal{M}, w \nVdash (\Diamond p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$.

(1) $\neg EF\phi \equiv AG\neg\phi$.

Solution 5. Fix an arbitrary model $\mathcal{M} = (S, \rightarrow L)$ and an arbitrary state $s \in S$.

($\Rightarrow$)

Assume that $\mathcal{M}, s \Vdash \neg EF\phi$. Then there is no path $s_1 \rightarrow s_2 \rightarrow \ldots$ such that $s_1 = s$ where there is a state s_i such that $\mathcal{M}, s_i \Vdash \phi$. Thus, for all paths starting in s and for all states s_i: $\mathcal{M}, s_i \nVdash \phi$. Hence, $\mathcal{M}, s \Vdash AG\neg\phi$.

($\Leftarrow$)

Assume that $\mathcal{M}, s \Vdash AG\neg\phi$. Then, for all paths $s_1 \rightarrow s_2 \rightarrow \ldots$ such that $s_1 = s$, for all states s_i: $\mathcal{M}, s_i \Vdash \neg\phi$. Hence, there cannot be a path $s_1 \rightarrow s_2 \rightarrow \ldots$ starting in s such that $\mathcal{M}, s_i \Vdash \phi$ for any s_i along the path. Thus, $\mathcal{M}, s \Vdash \neg EF\phi$.

Since we fixed an arbitrary, $\mathcal{M}$ and s, we conclude that the formulas are equivalent.