Homework 4

Auctions

Due: November 28, 2025, by 23:59

Exercise 1. Your next-door neighbor is moving out and selling that giraffe lamp you've always wanted. Unfortunately, you have competition: another neighbor in the building has their eyes set on the lamp. The owner of the lamp decides to auction it off: they will sell it in a first-price auction (in which, recall, the bidders submit secret bids, the winner is the bidder with the highest bid and they pay their respective bid).

Additionally, the owner of the lamp *insists* that bids be greater than 0, and that they be offered only as multiples of 20 euros. In the case of identical bids, the owner allocates the lamp by flipping a fair coin (i.e., probablity of $\frac{1}{2}$ of landing either side).

As bidder 1, you value the lamp at 48 euros, whereas your competitor (bidder 2) values it at 64 euros. Moreover, your valuations are common knowledge. Each bidder's utility for a specific profile of bids is calculated, as usual, as the value derived (the value of the item in case of winning, 0 otherwise) minus the amount paid. Each player is interested in maximizing their own utility. And in case of equal bids, we will work with the *expected* utility: how much utility each bidder gets on average, depending on the probability of winning.

In this exercise you will figure out yours and your opponent's optimal bids.

(1) Suppose you bid 40 euros and your competitor bids 20 euros, which we write as the profile of bids $(40, 20)$. What are $u_1(40, 20)$ and $u_2(40, 20)$, i.e., yours and your competitor's utilities with this profile of bids for a first-price auction? 1pt

(2) What would the utilities $u_1(40, 20)$ and $u_2(40, 20)$ be under the second-price auction (i.e., where the winner pays the *second*-highest bid)? 1pt

(3) Back to our first-price auction. Is it profitable for you to bid more than 40 euros? Or for your competitor to bid more than 60 euros? Why, or why not? Recall the constraint on bids being multiples of 20 . 1pt

(4) Compute $u_1(20, 20)$ and $u_2(20, 20)$. Keep in mind, here, that the result is determined by the toss of a coin and that we're thinking in terms of expected utilities. 1pt

(5) Write down a payoff matrix for you and your competitor and on the basis of it find yours and your competitor's optimal bids, by iteratively eliminating strictly dominated strategies. 1pt

Exercise 2. In the lectures it was mentioned that a *Generalized Second-Price Auction (GSP)* is not truthful (i.e., bidders can sometimes gain more by not bidding their true valuation), but no proof of this was actually given. This exercise provides an example.

Consider a GSP auction for ad slots, which is very much like the GSP position auction in the lecture, though with a touch more realism. Concretely, say we have two slots that are characterized by their click-through rates $s_1 = 0.3$ and $s_2 = 0.2$, respectively. Click-through rate refers to the probability that a person seeing the ad actually clicks on it, and it determines an ordering of the slots in the natural manner: slot 1 is better than slot 2, i.e., $s_1 \rhd s_2$, because 1 has a higher click-through rate than 2.

Then there are three bidders, i.e., $N = \{1, 2, 3\}$, with valuations $v_1 = 10$, $v_2 = 8$, $v_3 = 6$. Each bidder $i \in N$ submits a bid b_i, which might be different from i's true valuation. The profile of bids is $\boldsymbol{b} = (b_1, b_2, b_3)$.

Each bidder i pays a price p_i. Under the rules of the GSP auction, the i-th highest bid gets the i-th best slot, and pays p_i equal to the $(i+1)$-st highest bid. That is, the highest bid pays the second-highest bid, the second-highest bid pays the third highest bid, etc. If a bidder i gets nothing, they pay $p_i = 0$.

This a pay-per-click model, so bidders get charged only if users actually click on the slot. Consequently, the utility of bidder i from getting a slot j is $u_i(\boldsymbol{b}) = s_j(v_i - p_i)$. For instance, if bidder 1 gets the first slot at price $p_1 = 9$, their utility is:

$$u_1(\boldsymbol{b}) = s_1(v_1 - p_1) = 0.3(10 - 9) = 0.3.$$

If a bidder gets nothing, their utility is, predictably, 0.

(1) Suppose bidders bid their true valuations, i.e., $\boldsymbol{b} = (v_1, v_2, v_3)$. What is the allocation arrived at using the GSP auction for this bidding profile, and how much does each bidder have to pay? 1pt

(2) For the allocation computed above, what are the utilities of the bidders according to the pay-per-click rule described previously, i.e., what is $u_i(v_1, v_2, v_3)$ for $i \in \{1, 2, 3\}$? Write down the computation steps. 1pt

(3) And now for the kicker: showing that truthful bidding is not a Nash Equilibrium. Assuming bidders 2 and 3 bid truthfully, find some bid b_1 for bidder 1 that gets bidder 1 more utility, i.e., such that $u_1(b_1, v_2, v_3) > u_1(v_1, v_2, v_3)$. Explain your answer. 1pt

Exercise 3. In this exercise we explore the VCG auction: the good, the bad and the huh. It turns out that we can illustrate most of the points on the following example, with two items and three bidders:

Table 1 summarizes the (true) valuations of each bidder with each subset of items.

$X \subseteq G$	$v_1(X)$	$v_2(X)$	$v_3(X)$
$\emptyset$	0	0	0
$\{a\}$	0	2	0
$\{b\}$	0	0	2
$\{a, b\}$	2	0	0

Table 1: True valuations

$X \subseteq G$	$b_1(X)$	$v_2(X)$	$v_3(X)$
$\emptyset$	0	0	0
$\{a\}$	0	2	0
$\{b\}$	0	0	2
$\{a, b\}$	5	0	0

Table 2: Untruthful bid from bidder 1

(1) First, to make sure we're on the same page: give the outcome of the VCG auction if bidders bid their true valuations from the table above, i.e., the allocation of items and the prices each bidder pays according to the rules of the VCG auction. Justify your answers, briefly (e.g., if the same reasoning applies to two bidders you can omit the additional explanation). 1pt

(2) We define the *revenue* of the auction as the sum of payments over all bidders. What is the revenue in this example, when bidders bid their true valuations? 1pt

Hint: the answer points to one of the drawbacks of the VCG auction.

(3) Say bidder 1 gets greedy and submits an untruthful bid for the bundle $\{a, b\}$, as follows:

The utility of an agent with a particular bundle is, as expected, the true value they place on the bundle (from Table 1) minus the price paid. Does bidder 1 become better off with the truthful bid relative to the untruthful bid?

Justify your answers by comparing the utilities of bidder 1 under the truthful and untruthful bids. Describe, briefly, how they are obtained. 1pt

Hint: this points to one of the nice properties of the VCG auction.

(4) Describe how the revenue changes if bidder 3 decides to sit the auction out (i.e., is absent). 1pt

Hint: this is the weird.