UvA

Multi-Agent Systems

Game Theory

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Extensive-Form Games

- So far we have been looking at games in normal form, in which players act *at the same time*
    - And *in ignorance* of each other's actions
- We now look at **extensive-form games**, in which players take turns playing their actions

Perfect-Information Extensive-Form Games: Intuition

Player 1 takes an action

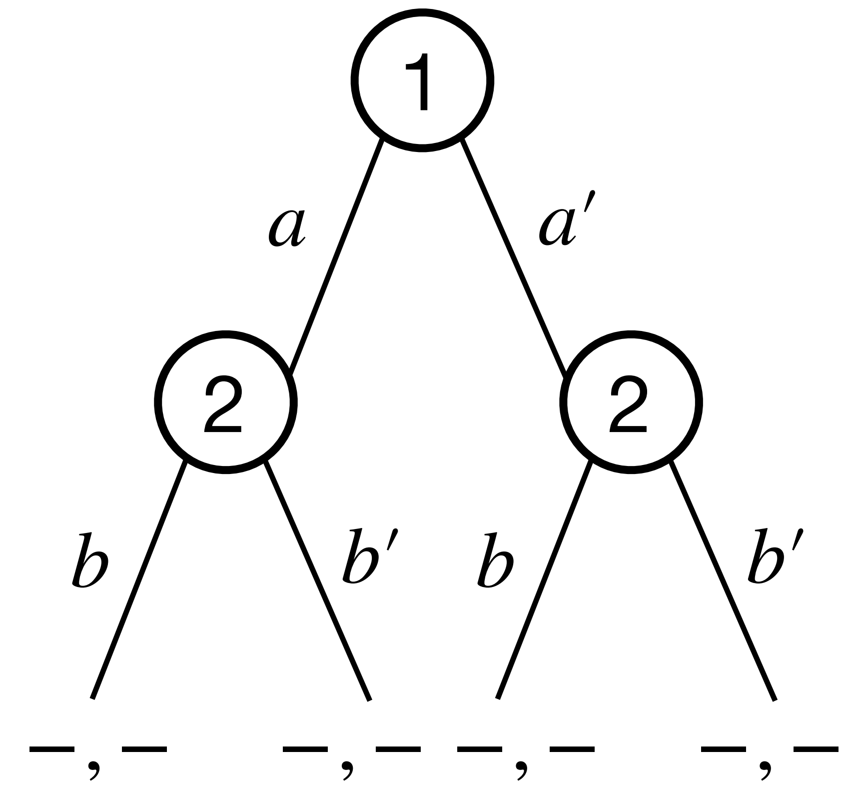
… out of their action set

Player 2 follows up

… knowing the action P1 has taken

Every player receives a payoff

… specific to the branch taken



The whole game tree is known to all players

Perfect-Information Extensive-Form Games

- Extensive-form games with perfect information are modeled as **trees**, where non-terminal nodes (called **choice nodes**) correspond to **players**

- At every one of its choice nodes, an agent has some **actions** available

- Each edge is labeled with the action taken by the parent agent at that node

- Terminal nodes are labeled with the **utilities** of the players for the combination of actions that led to that particular outcome

- A **strategy** for an agent is a combination of actions, one for each node corresponding to that agent

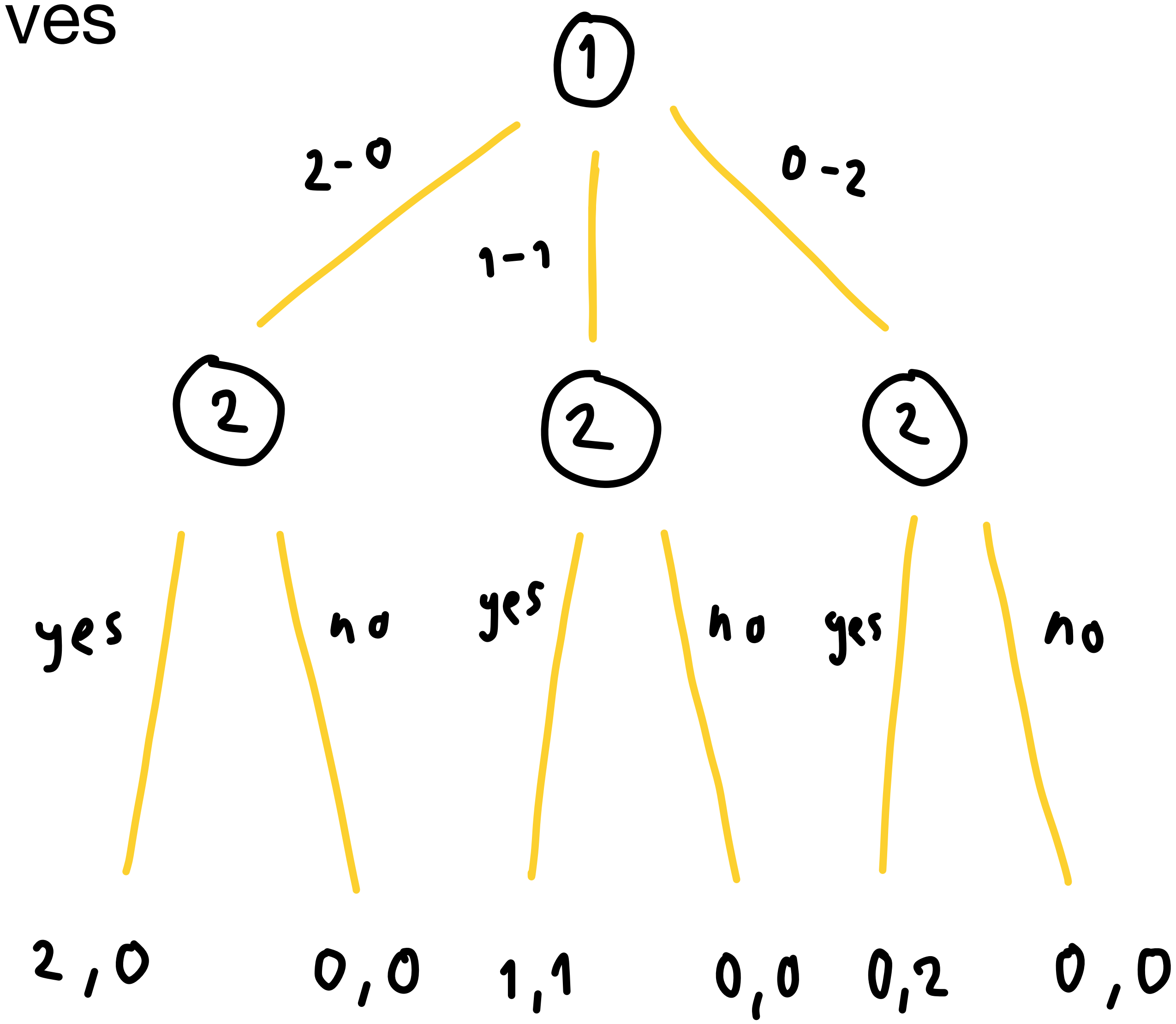
The Ultimatum Game

- Player 1 has two Euros, which they have to divide between themselves and Player 2

- Player 1 makes an offer, which Player 2 can accept or reject

- If Player 2 accepts, money is divided according to Player 1's offer

- If Player 2 rejects, no one gets anything



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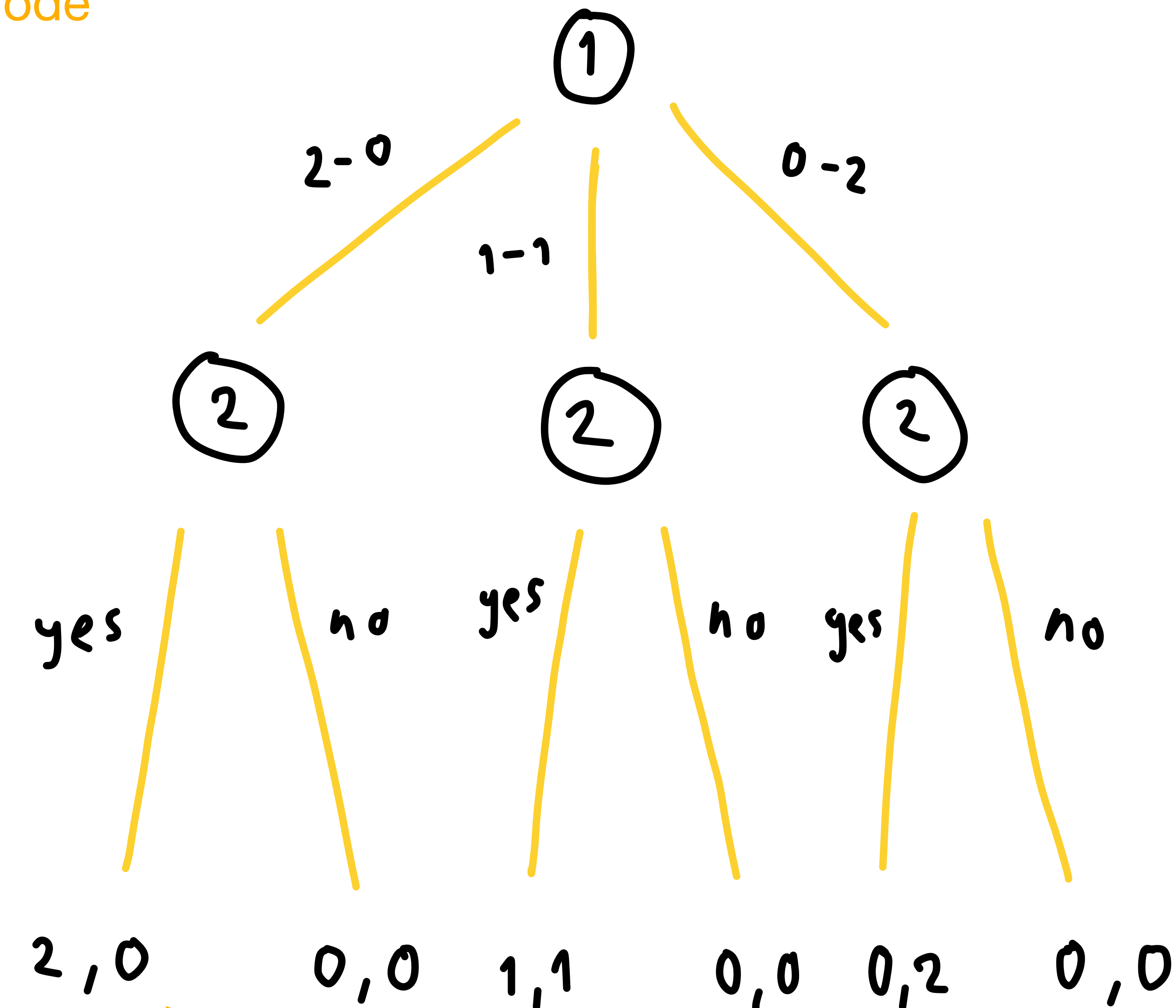
The Ultimatum Game

It is usually clear from the context what actions are available to each agent at its corresponding node

Players: 1 and 2
Actions: 2-0, 1-1, 0-2, yes, no
Utilities: (2,0), (0,0), …

Strategies of P1: 2-0, 1-1, 0-2

Strategies of P2: (yes, yes, yes), (yes, yes, no), (yes, no, yes), (yes, no, no), (no, yes, yes), (no, yes, no), (no, no, yes), (no, no, no)



Utility of P1 with the combination of the actions (2-0, yes)

Utility of P2 with the same combination of actions

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Notes on Strategies

- Note that there is a subtlety in the definition of strategies
- The strategies of each player need to be defined at every choice node of that player
- Even if there is no way to reach that node, given the other choice nodes

Notes on Strategies: Example

Strategies of P2?

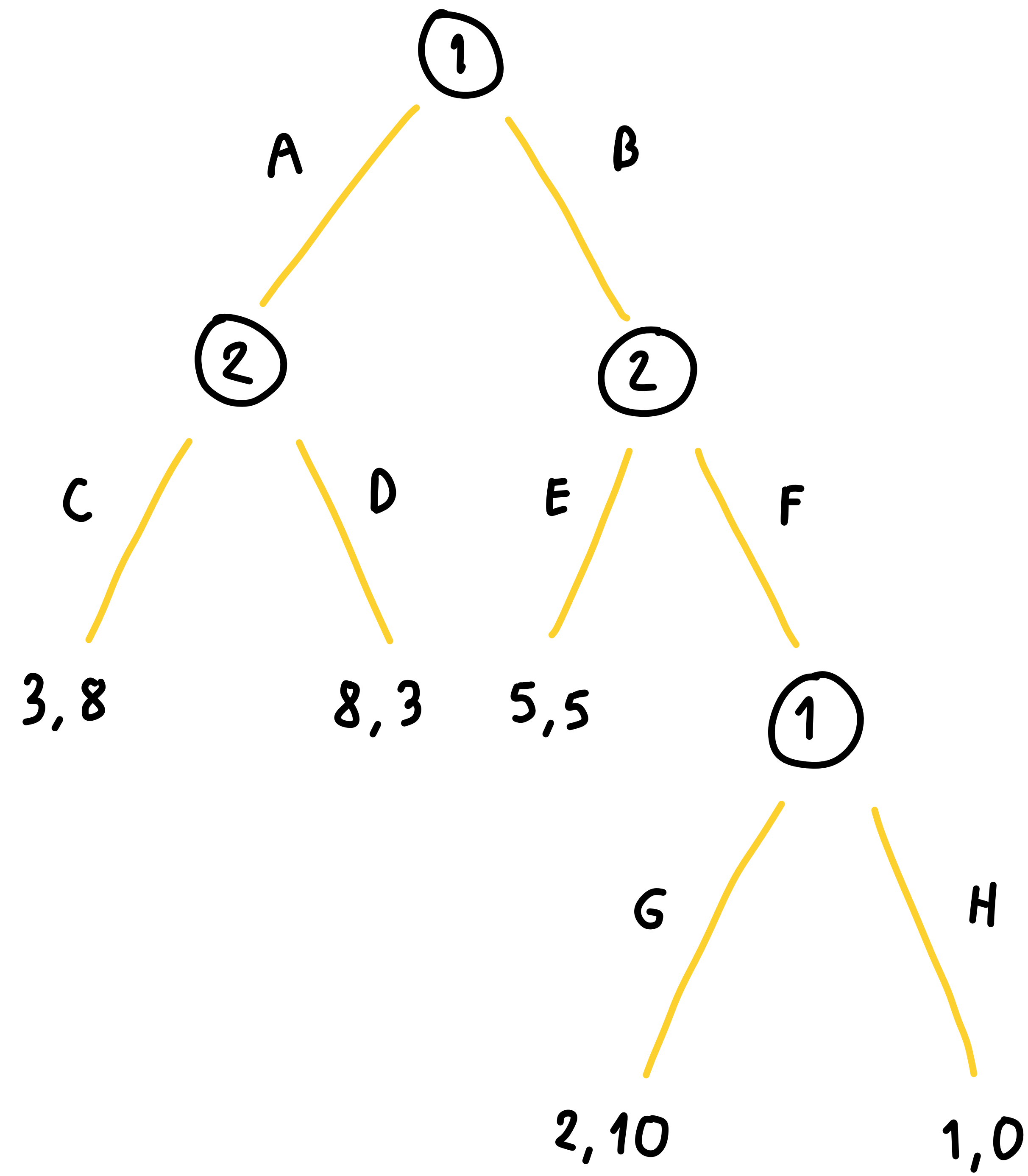
(C, E), (C, F), (D, E), (D, F)

Strategies of P1?

(A, G), (A, H), (B, G), (B, H)

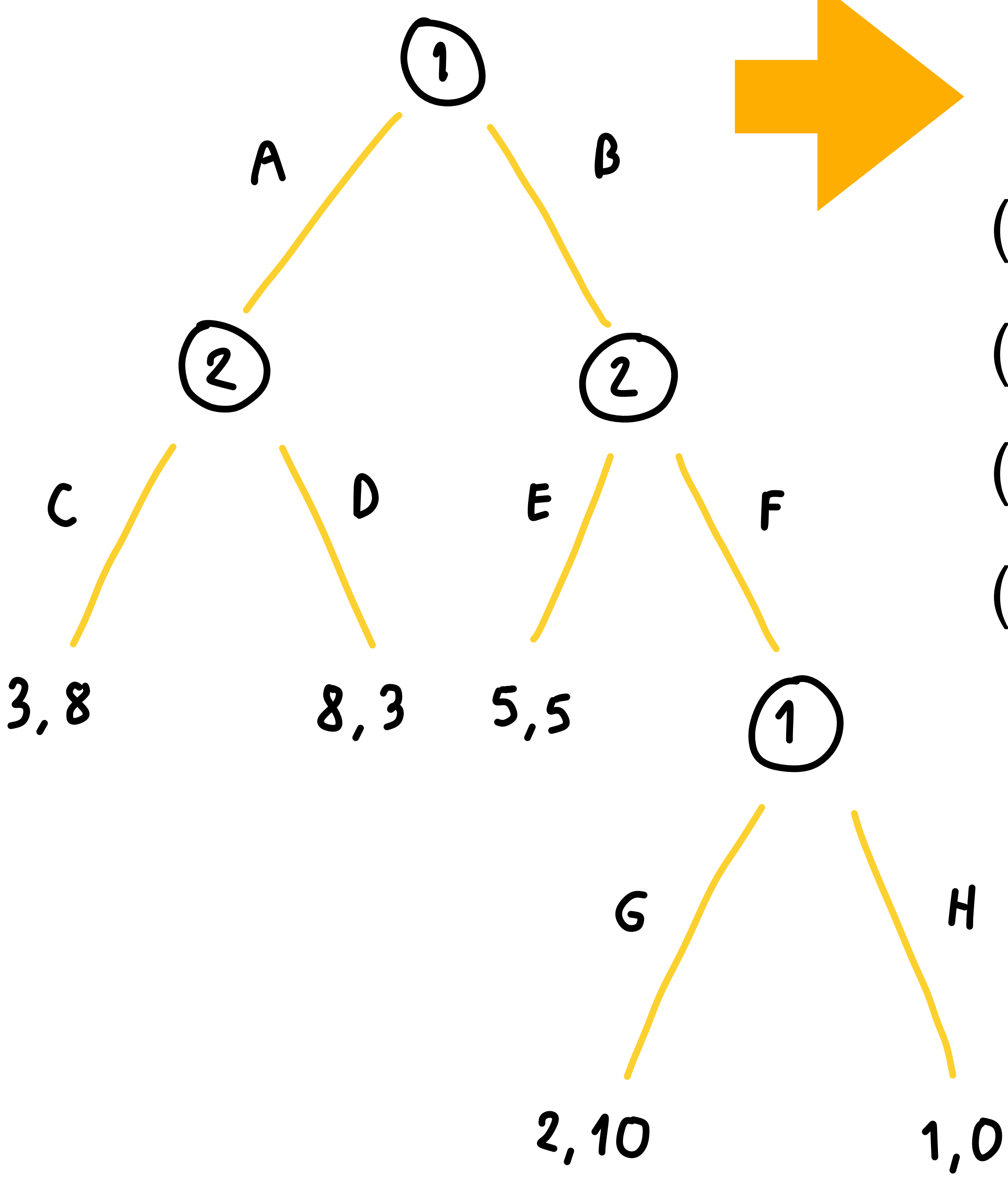
Once P1 takes action A, there is no question of G or H

But (A, G) and (A, H) need to be included anyway!



How do we reason our way through a perfect-information extensive-form game?

Comparison to Normal-Form



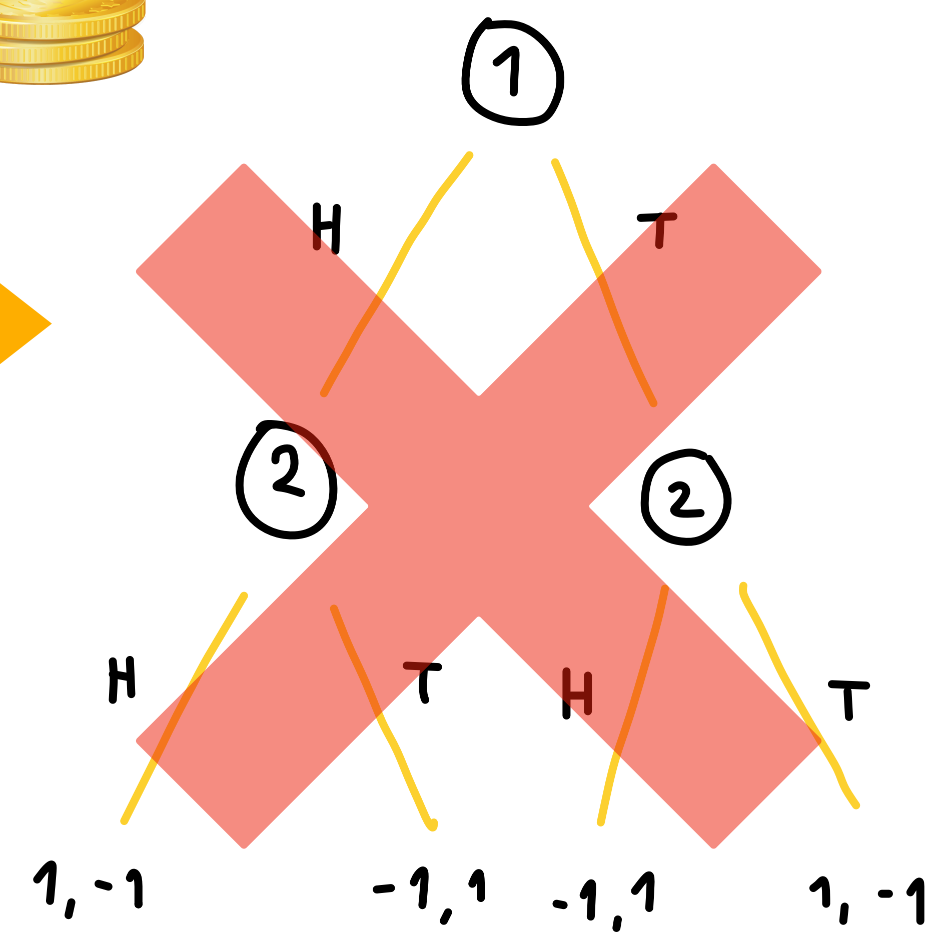
	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	(3,8)	(3,8)	(8,3)	(8,3)
(A, H)	(3,8)	(3,8)	(8,3)	(8,3)
(B, G)	(5,5)	(2,10)	(5,5)	(2,10)
(B, H)	(5,5)	(1,0)	(5,5)	(1,0)

Comparison to Normal-Form

- Every perfect-information extensive-form game can be turned into a normal-form game, called the **induced normal-form game**, by reading out the payoffs for each combination of strategies from the game tree
    - At the cost of introducing redundancies in the normal-form game
    - In fact, the conversion can lead to exponential blow-up
- The other way around does not always work

Comparison to Normal-Form

	Heads	Tails
Heads	(1,-1)	(-1,1)
Tails	(-1,1)	(1,-1)



Perfect-information extensive-form games are not good at modeling simultaneity!

Takeaway:

Perfect-information extensive form $\rightarrow$ Normal-form

Normal-form $\not\rightarrow$ Perfect-information extensive form

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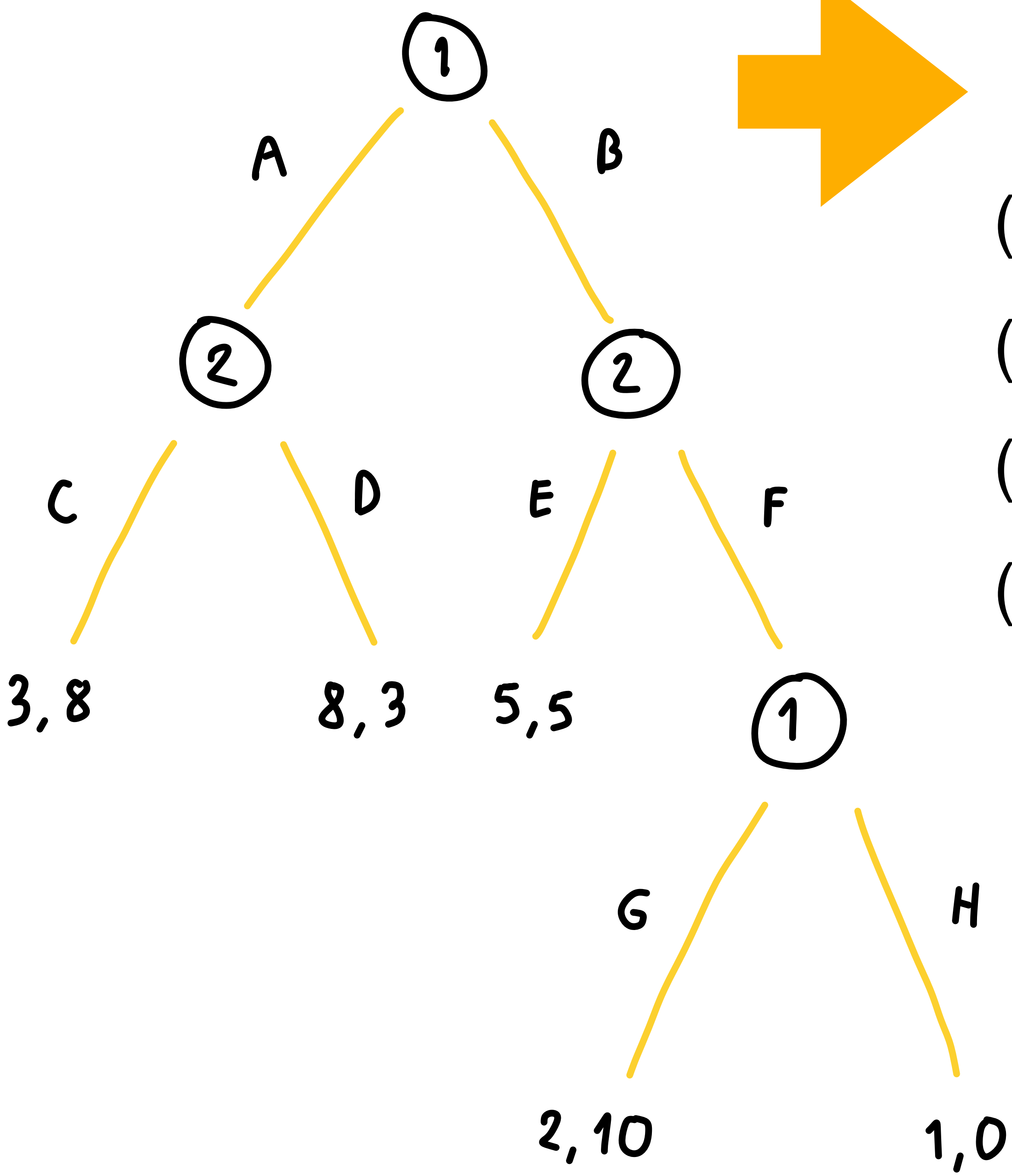
Strategies

- Because of the induced normal-form, concepts like best response, mixed strategies and Nash equilibria carry over naturally to perfect-information extensive-form games

Pure Nash Equilibria



	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	(3,8)	(3,8)	(8,3)	(8,3)
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Pure Nash equilibria?
((A, G), (C, F)),
((A, H), (C, F)),
((B, H), (C, E))

Pure Nash Equilibria

Pure Nash equilibria?
((A, G), (C, F)),
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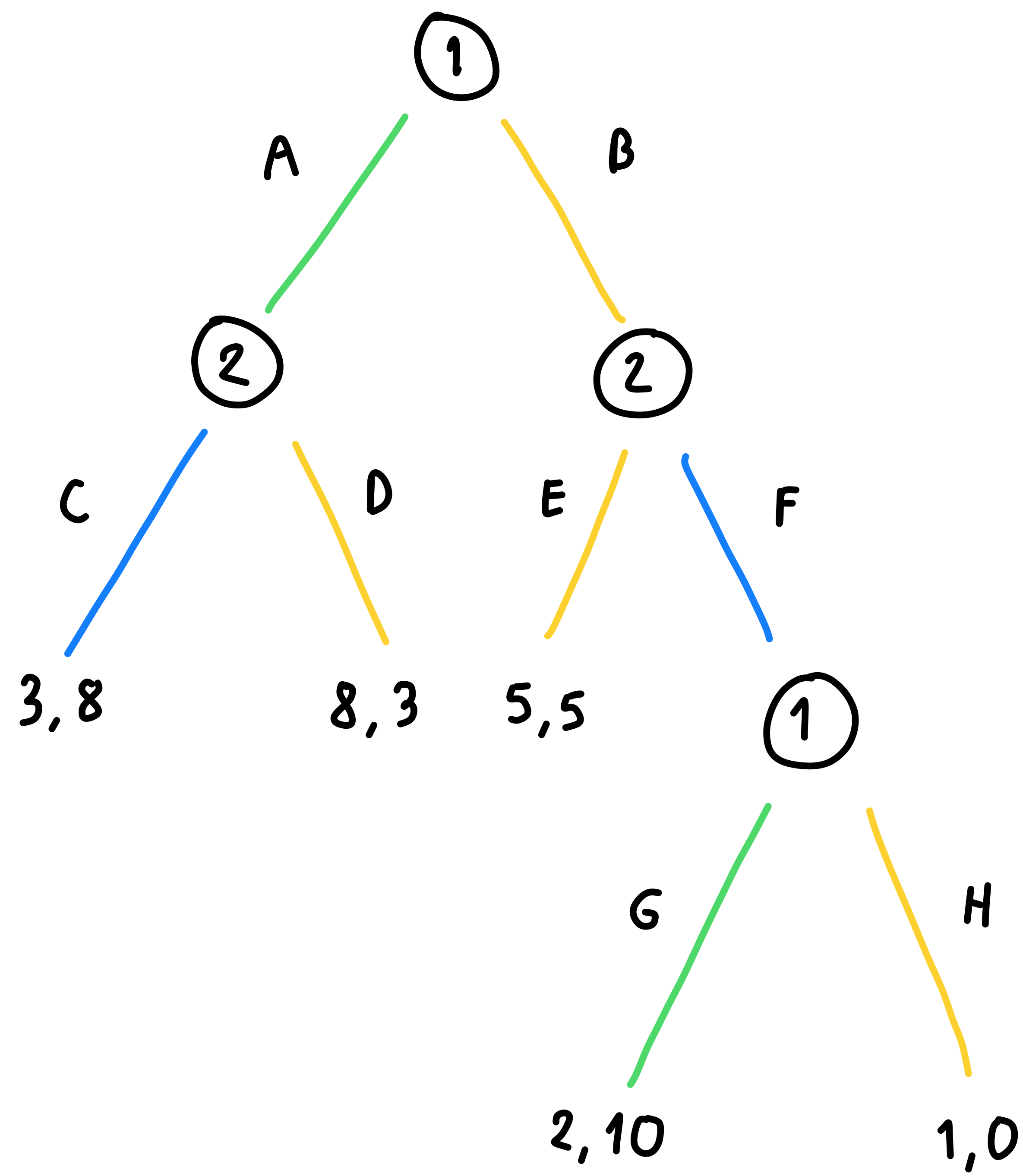
No profitable deviations



Pure Nash Equilibria

Pure Nash equilibria?
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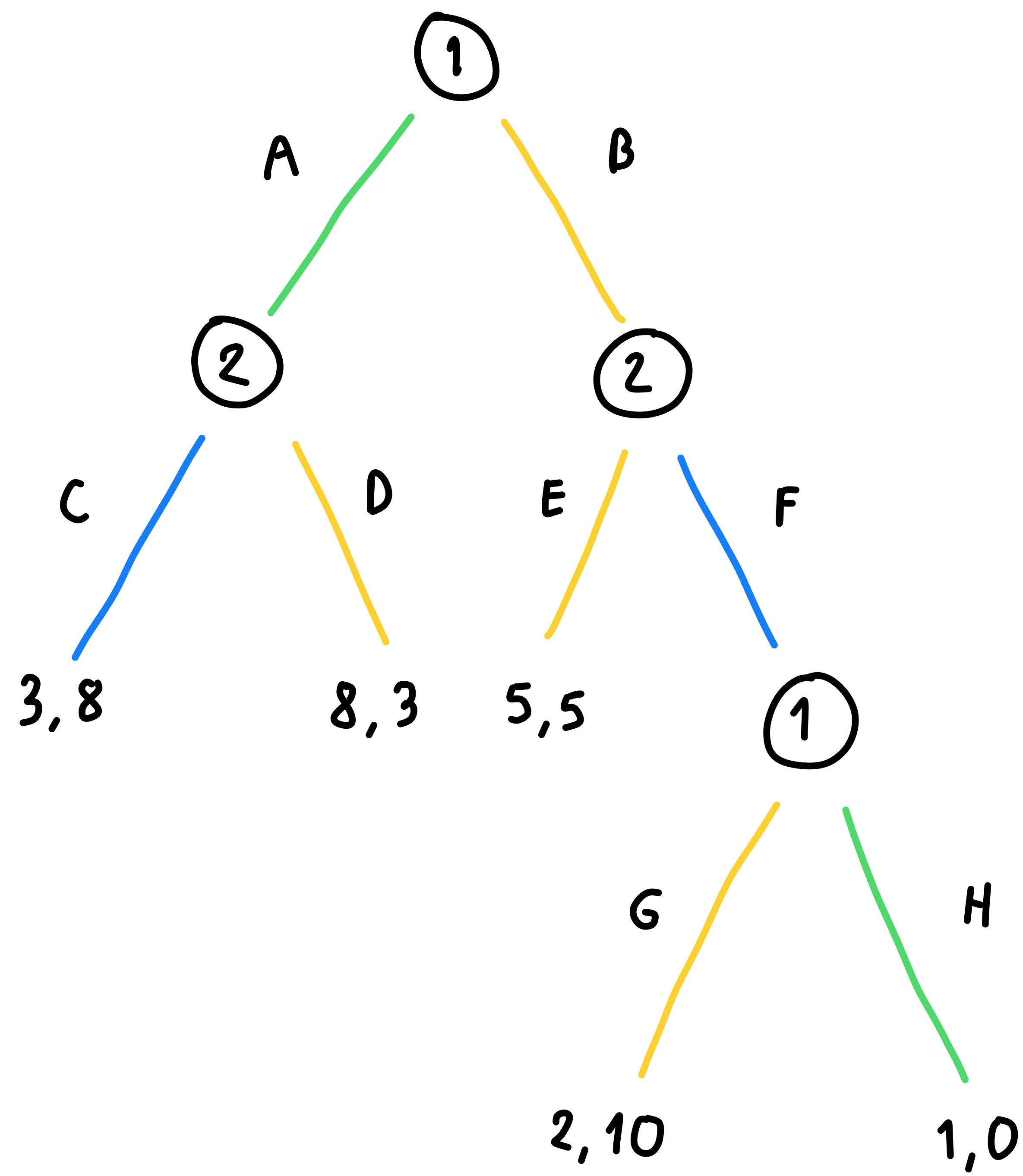


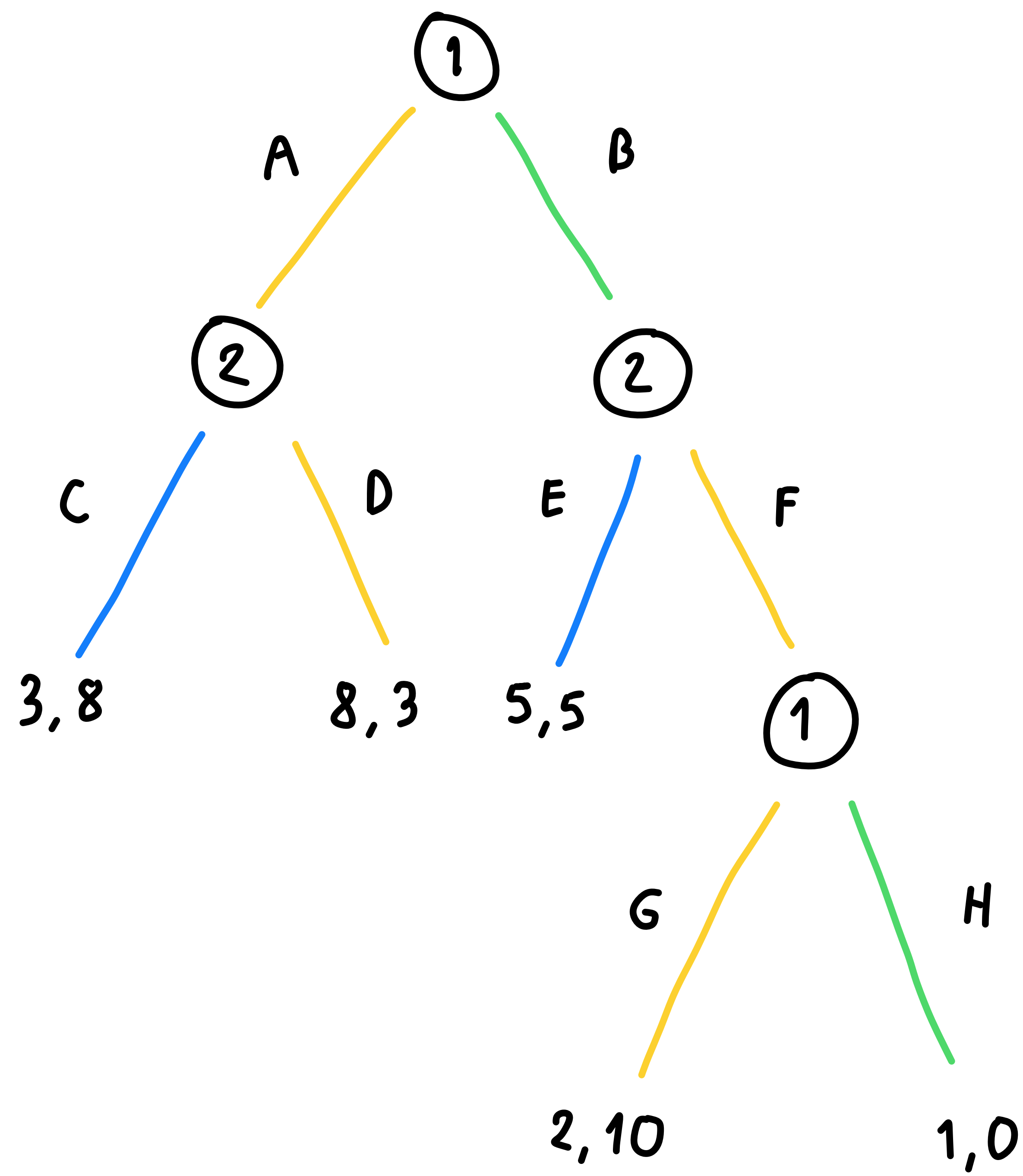
Pure Nash Equilibria

Pure Nash equilibria?
((A, G), (C, F)),
((A, H), (C, F)),
((B, H), (C, E))

No profitable deviations

But also, why is P1 playing H when that leads to a smaller payoff?

P1 is making a *threat*: by committing to play H, they ensure that P2 does not want to take the game there



Back to The Ultimatum Game

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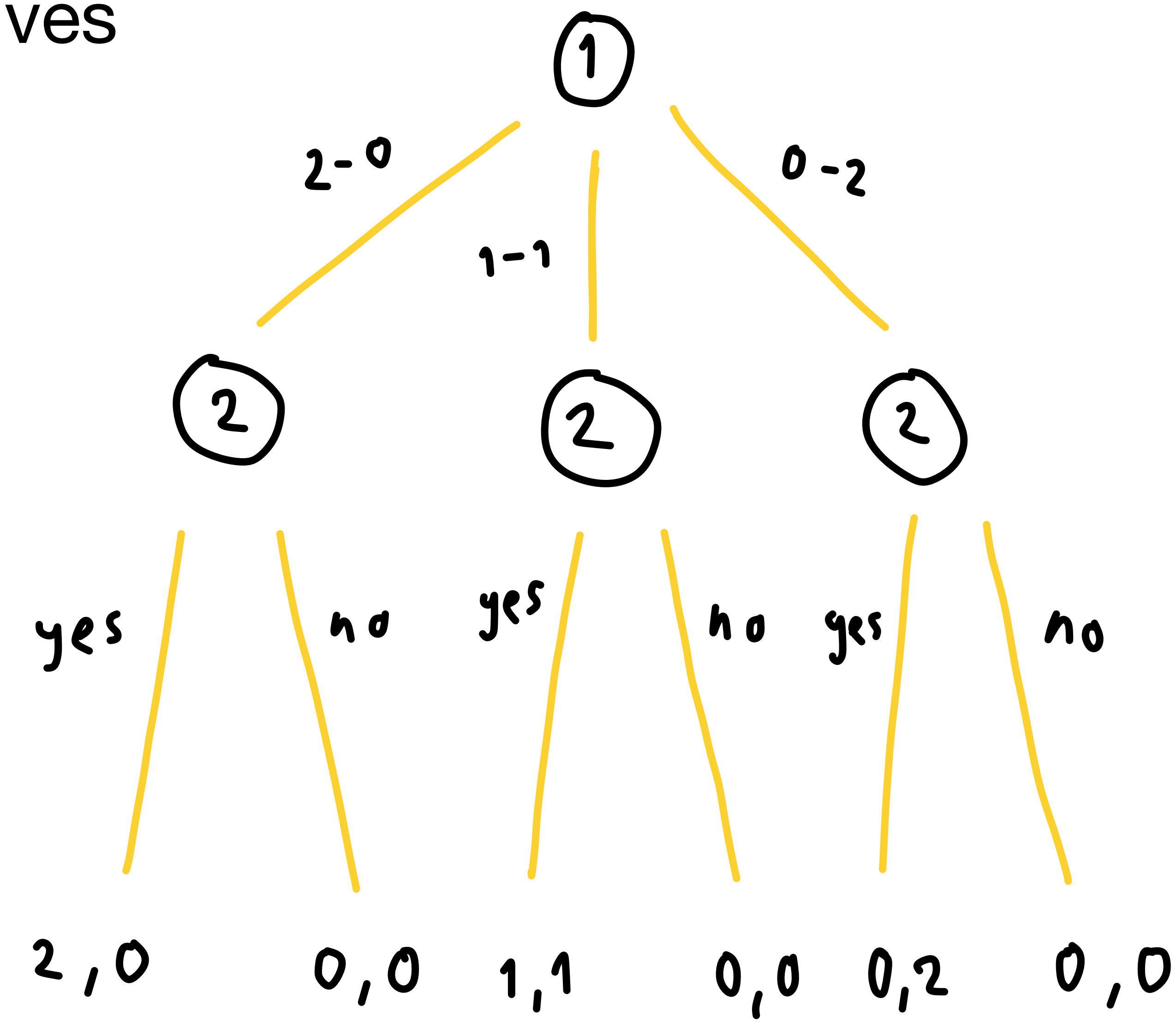

Pure Nash equilibria?

Image: clipartkey.com

Back to The Ultimatum Game

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
2-0	(2,0)	(2,0)	(2,0)	(2,0)	(0,0)	(0,0)	(0,0)	(0,0)
1-1	(1,1)	(1,1)	(0,0)	(0,0)	(1,1)	(1,1)	(0,0)	(0,0)
0-2	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)

Pure Nash equilibria?

What makes this a pure Nash equilibrium depends equally on what the players do at other nodes…

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More on Pure Nash Equilibria in Perfect-Information Extensive-Form Games

- Nash equilibria sometimes involve players being willing to sacrifice some of their utility to persuade the other players not to go down a particular road
    - But perhaps that is too much to ask…
    - And in these cases Nash equilibria might not be what we are looking for

Subgame-Perfect Equilibrium

- The refinement of Nash equilibrium that is often used in analyzing extensive-form games is called subgame perfect equilibrium

- It involves players playing a Nash equilibrium at *every* point in the game, regardless of what happened before

Definition (Subgame)

If G is a perfect-information extensive-form game, the **subgame** of G rooted at node h is the restriction of G to the descendants of h.

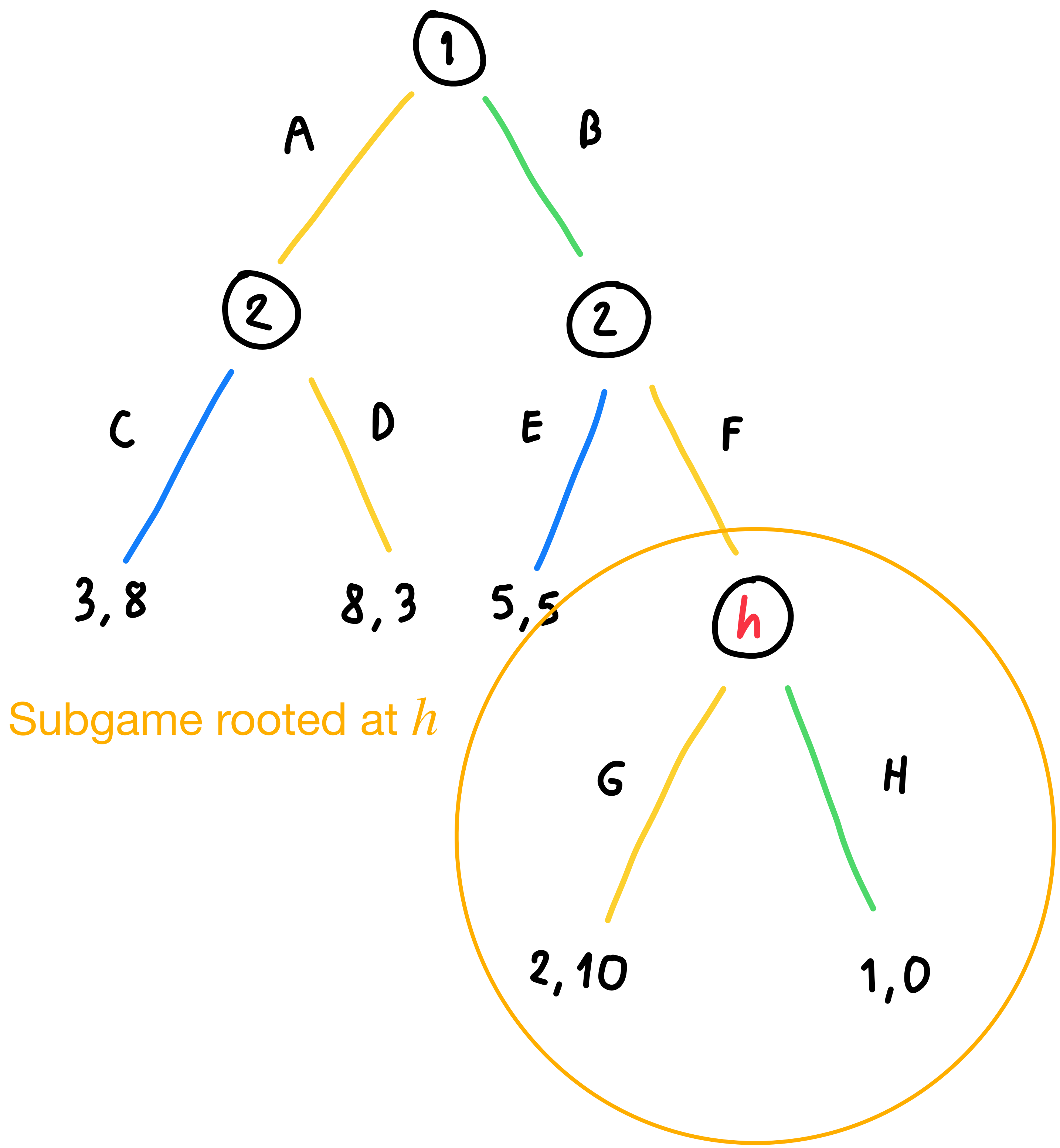
Definition (Subgame-Perfect Equilibrium)

A strategy profile s is a subgame-perfect equilibrium of perfect-information extensive-form game G if for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'.

Subgame-Perfect Equilibrium

Pure Nash equilibria?
((A, G), (C, F)),
((A, H), (C, F)),
((B, H), (C, E))

- P1's (B, H) strategy is **not** a Nash equilibrium for the subgame rooted at h
- So not every Nash equilibrium is a subgame-perfect equilibrium
- But, since every game G is a subgame of itself, every subgame-perfect equilibrium is a Nash equilibrium



Finding Subgame-Perfect Equilibria

- One way to find subgame perfect equilibria is to reason backwards
    - We start from the end stages of a game, to find the optimal action at every intermediate step
- The general procedure goes under the name of **backward induction**

Backward Induction

- Backward induction can be thought of as a procedure that labels every node h with an action a and a vector of utilities

- Action a is one of the optimal actions that can be played at h, and the utilities are those obtained by playing a

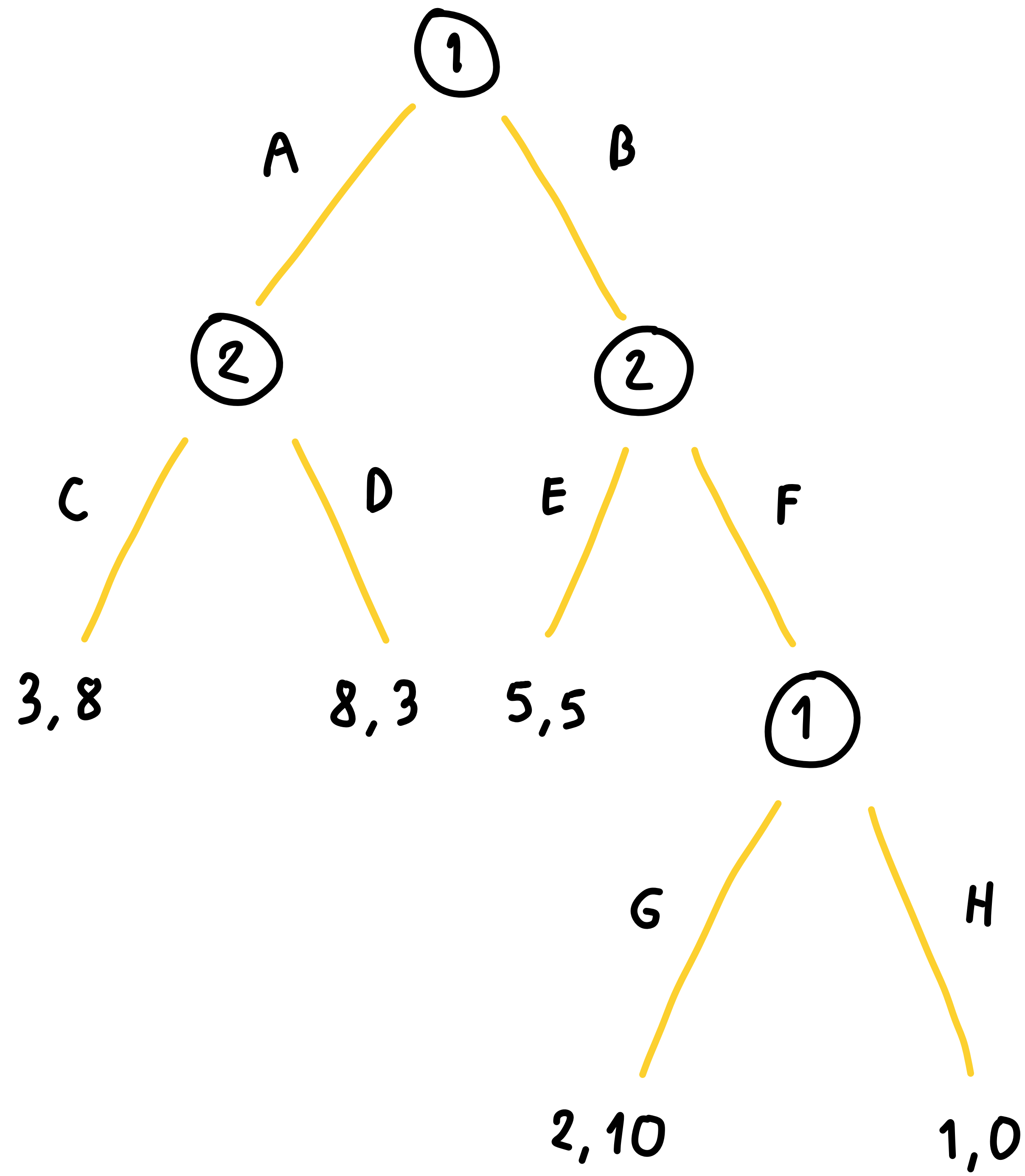
- It works by working upwards from the terminal nodes

```
Input: node h
Output: vector of utilities


if h is a terminal node:
   return utilities at h
else:
   let i be the player at h
   let a be one of i's best
   actions at h
   let h' be the node
   arrived at by playing a
   return utilities at h'
```

Backward Induction: Example

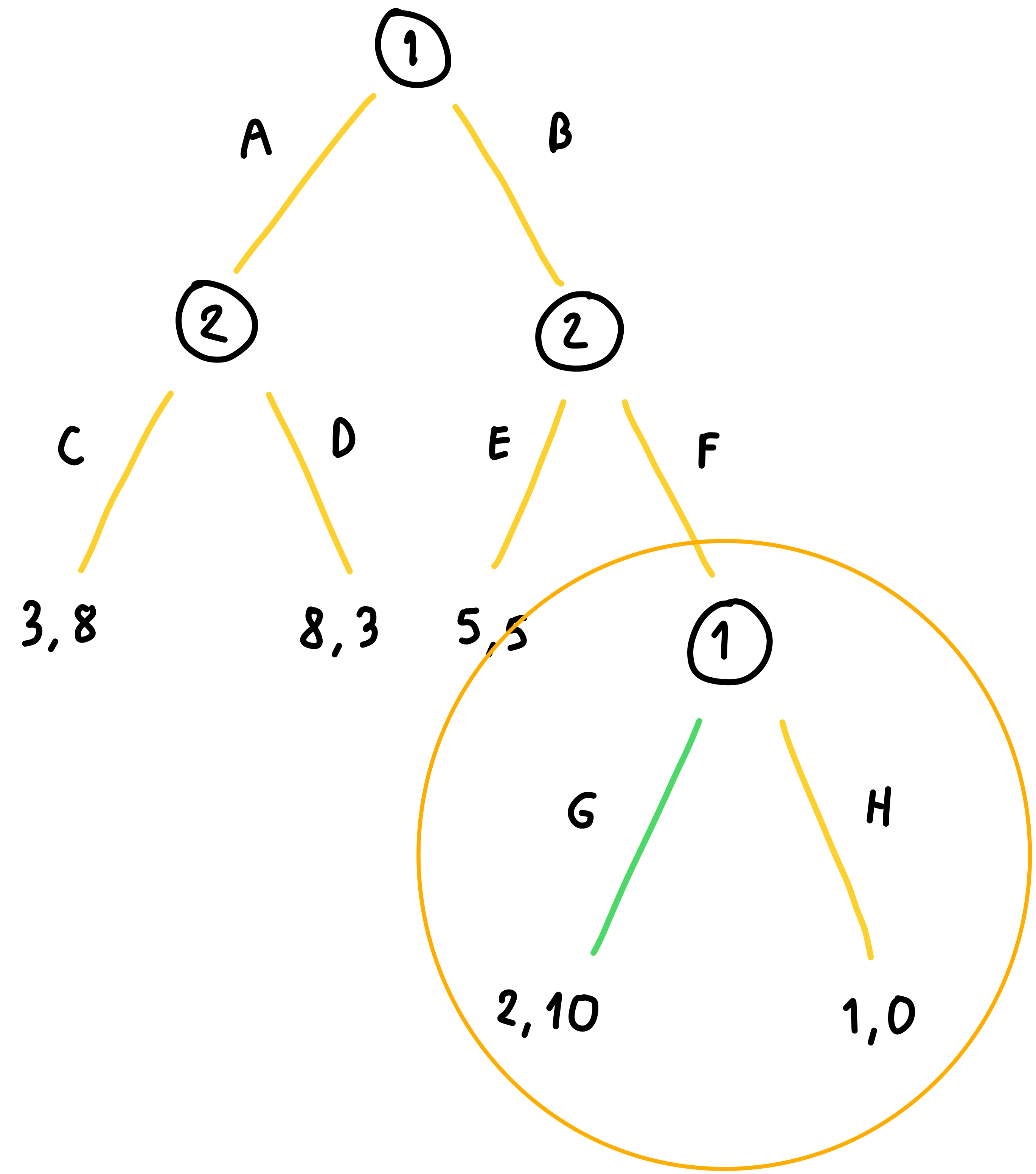
Backward induction on this game:



Backward Induction: Example

Backward induction on this game:

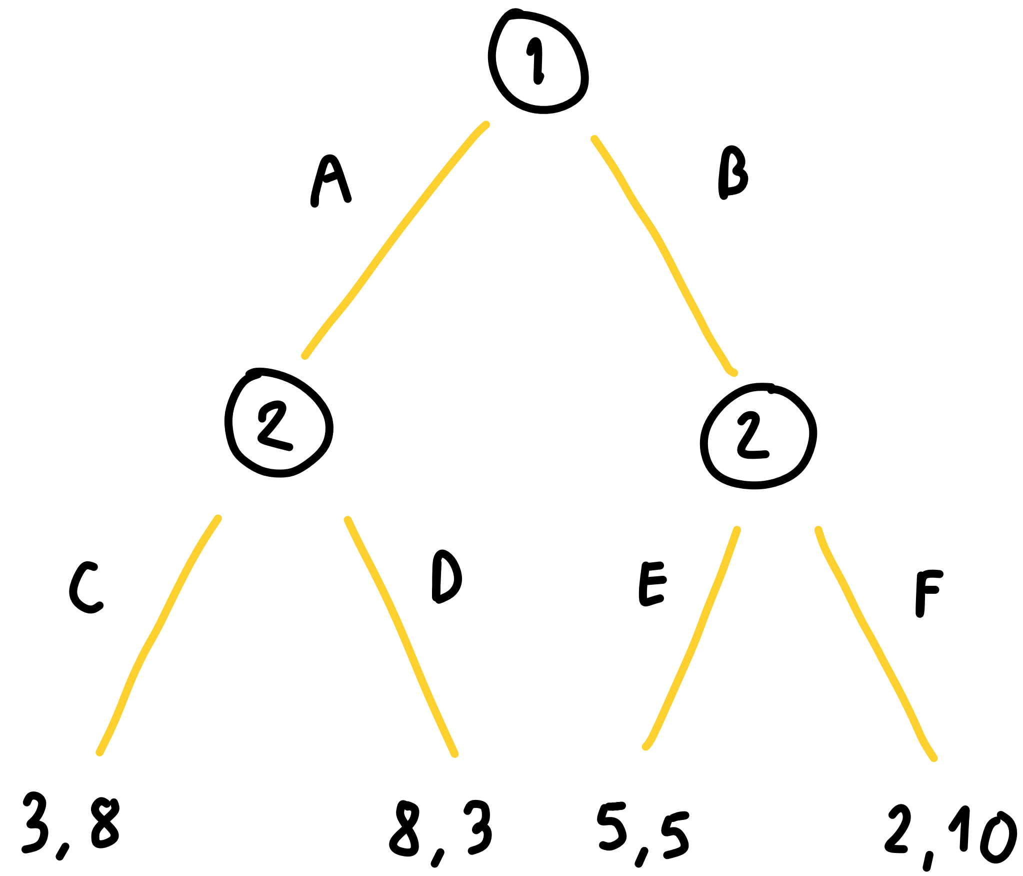
- With a choice only between G and H, P1 chooses G, leading to a payoff of (2,10)



Backward Induction: Example

Backward induction on this game:

- With a choice only between G and H, P1 chooses G, leading to a payoff of (2,10)
- P2 takes into account when making their decision one step earlier, that choosing F leads to a payoff of (2,10)



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- With a choice only between G and H, P1 chooses G, leading to a payoff of (2,10)
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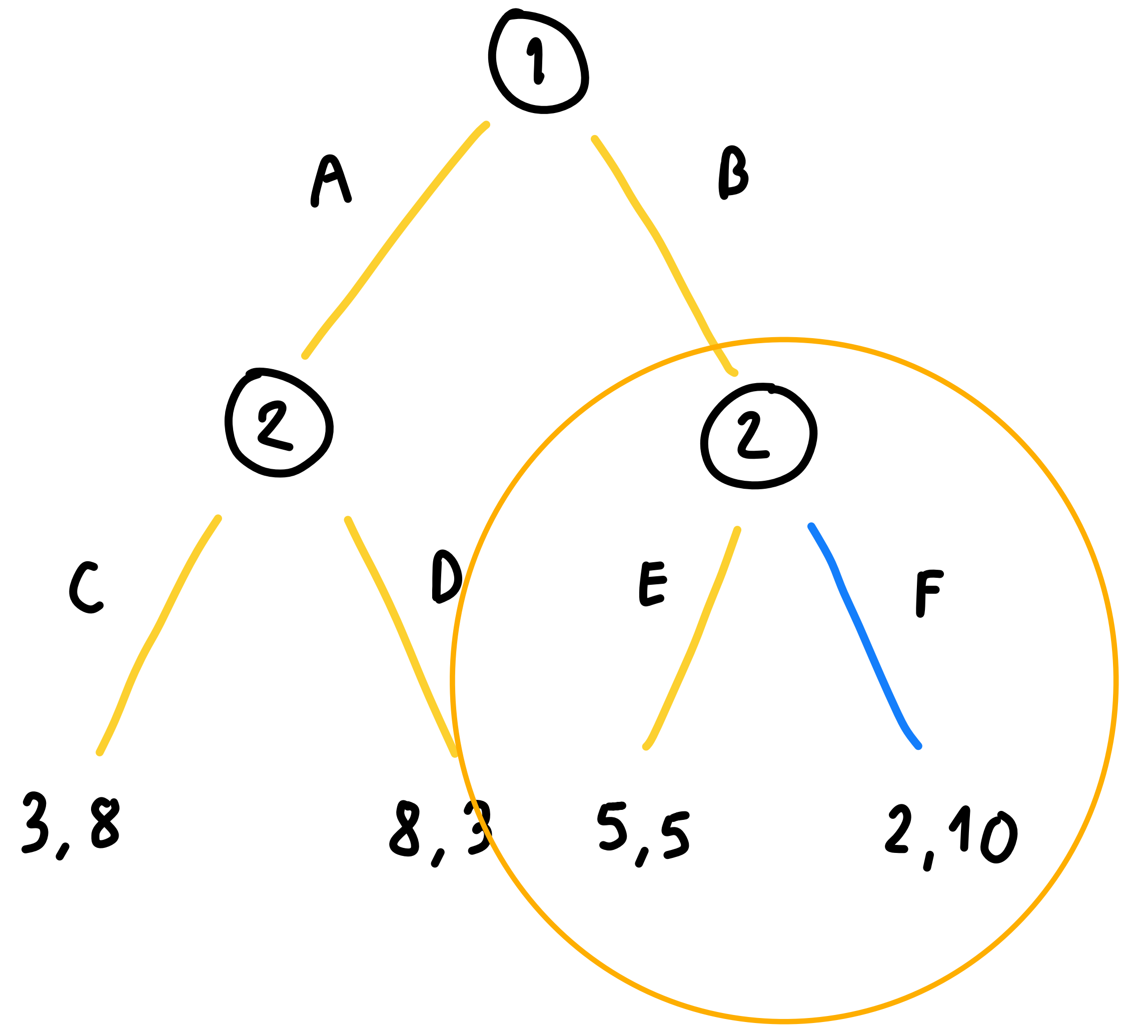
- We infer that P2 chooses F here



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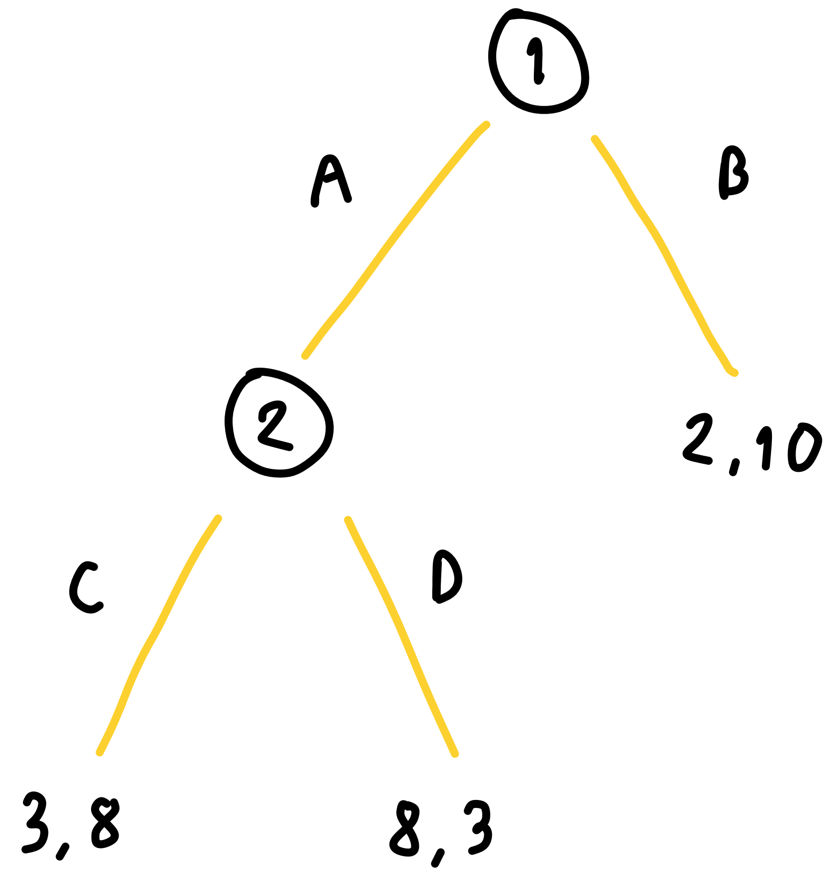
- P2 takes into account when making their decision one step earlier, that choosing F leads to a payoff of (2,10)
- We infer that P2 chooses F here
- Which means that P1 sees a payoff of 2 choosing B



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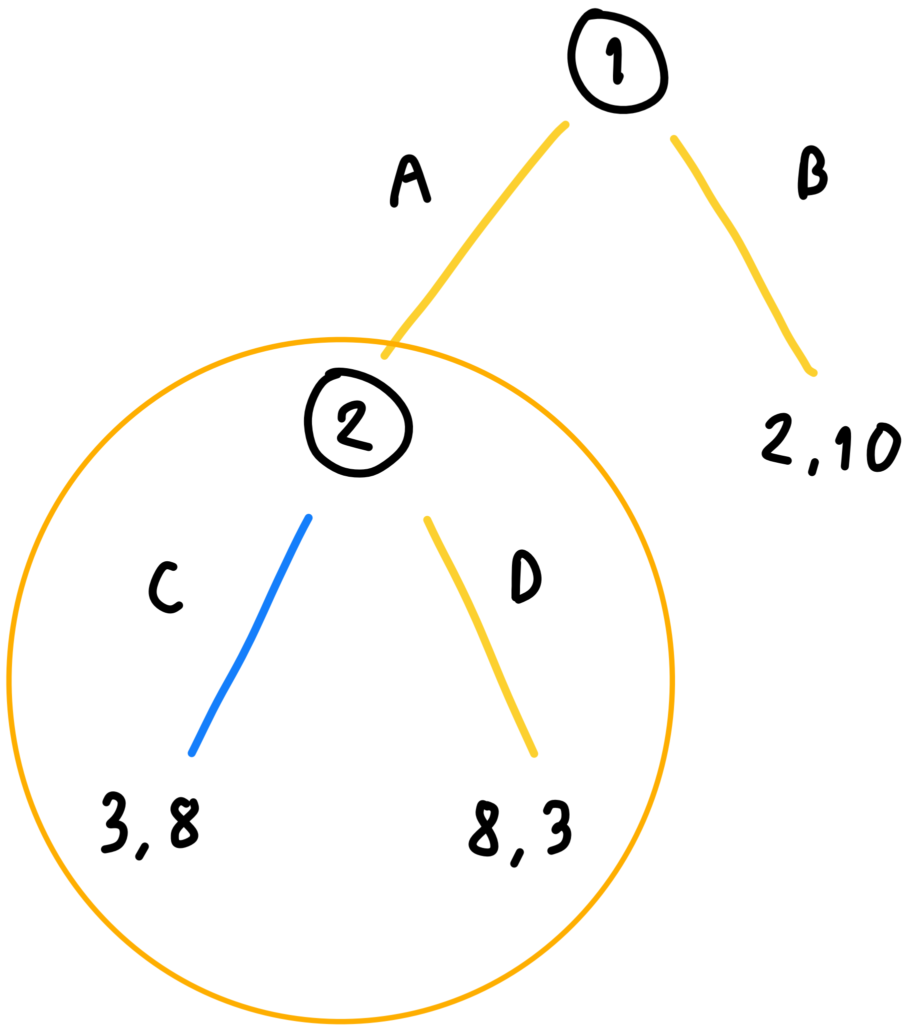
- Which means that P1 sees a payoff of 2 choosing B
- On the other branch, P2 chooses C



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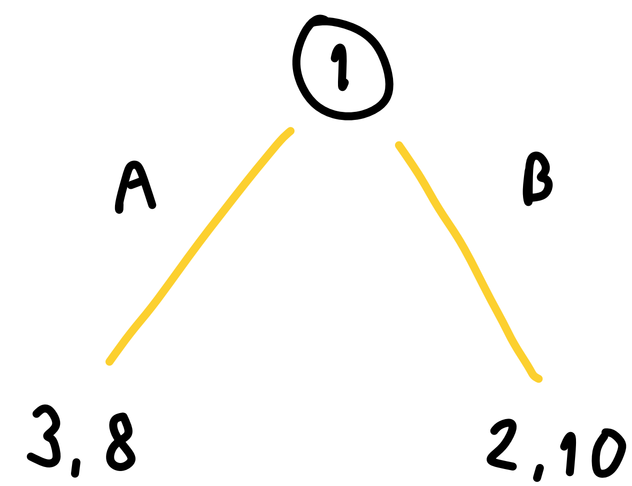

Backward Induction: Example

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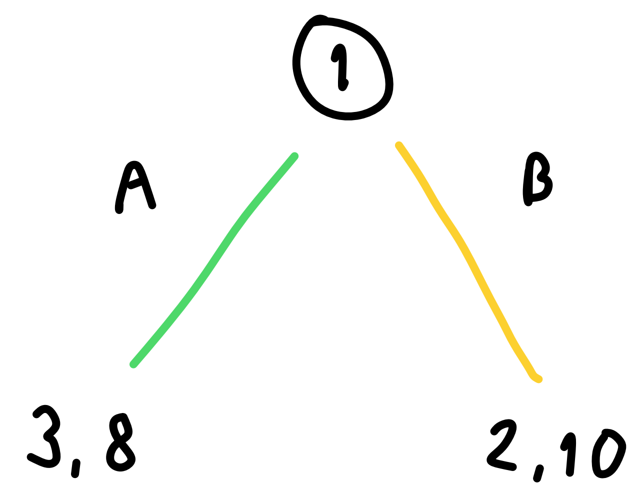
- We infer that P2 chooses F here
- Which means that P1 sees a payoff of 2 choosing B
- On the other branch, P2 chooses C
- Which means P2 chooses A

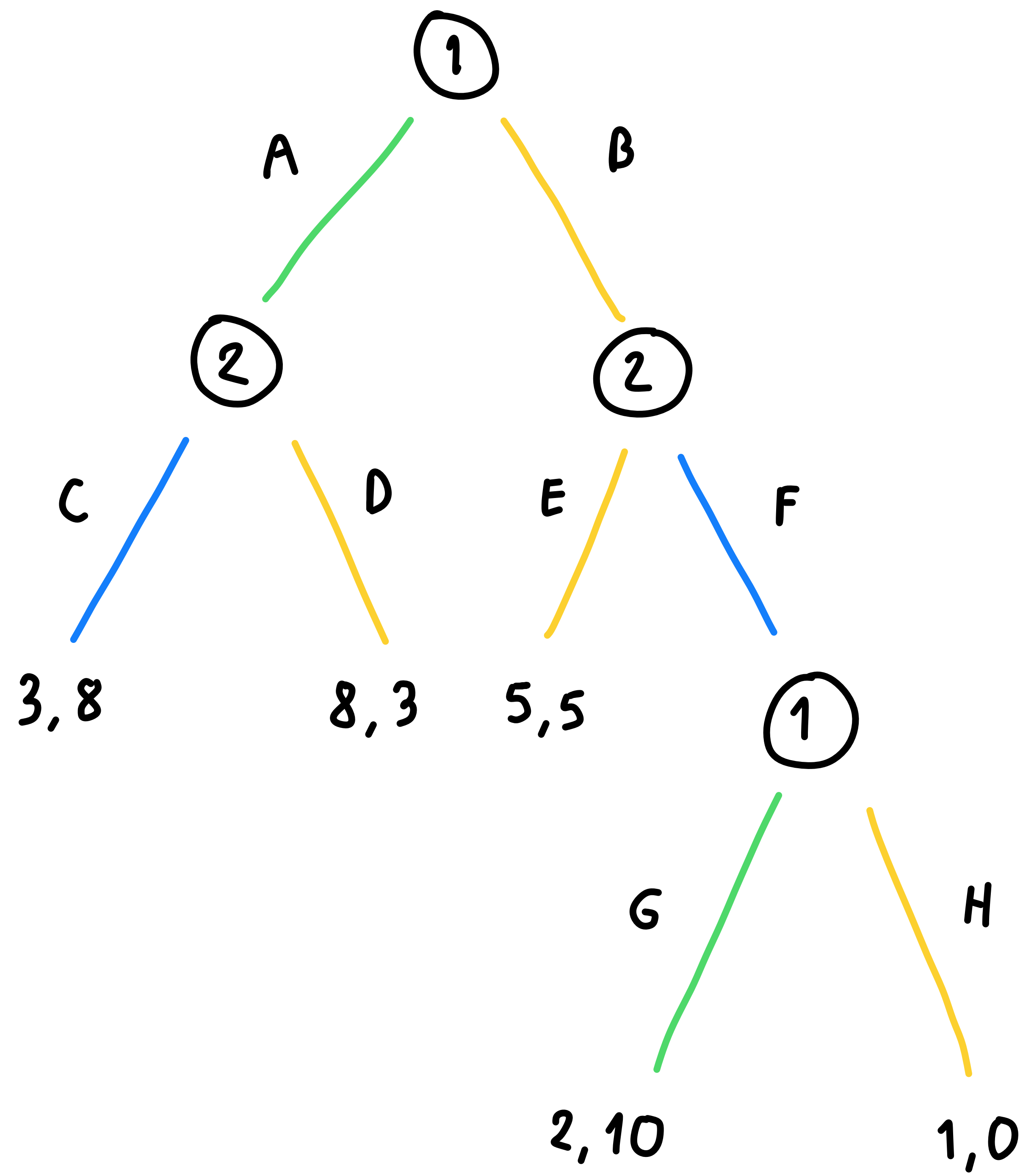


Backward Induction: Example

Backward induction on this game:

- With a choice only between G and H, P1 chooses G, leading to a payoff of (2,10)
- P2 takes into account when making their decision one step earlier, that choosing F leads to a payoff of (2,10)
- We infer that P2 chooses F here
- Which means that P1 sees a payoff of 2 choosing B
- On the other branch, P2 chooses C
- Which means P2 chooses A
- After which we can just read off the subgame-perfect equilibrium

((A, G), (C, F))



Backward induction is well-defined and terminates, if the game tree is finite.

So what have we shown?

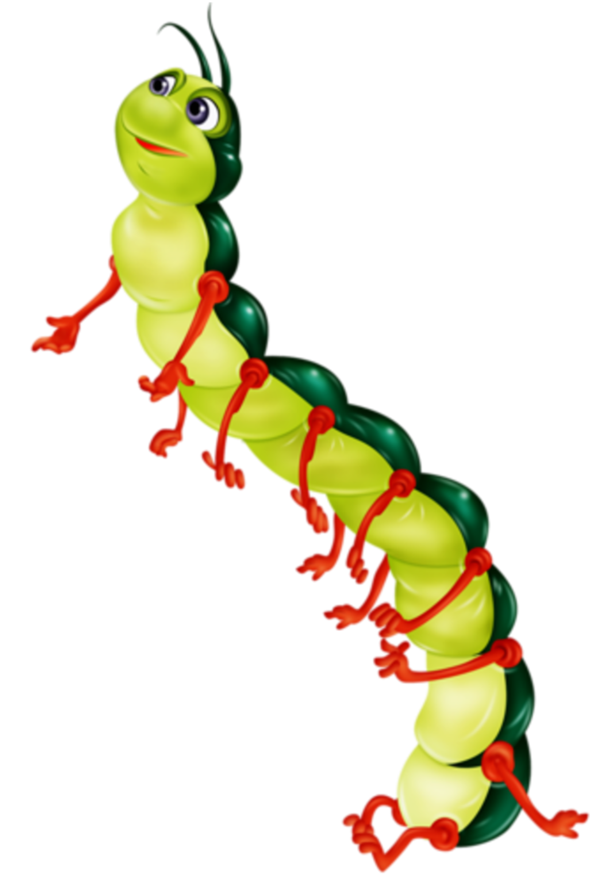
Existence of Equilibria

Theorem (Zermelo, 1913)
Every finite extensive-form game has at least one pure Nash equilibrium.

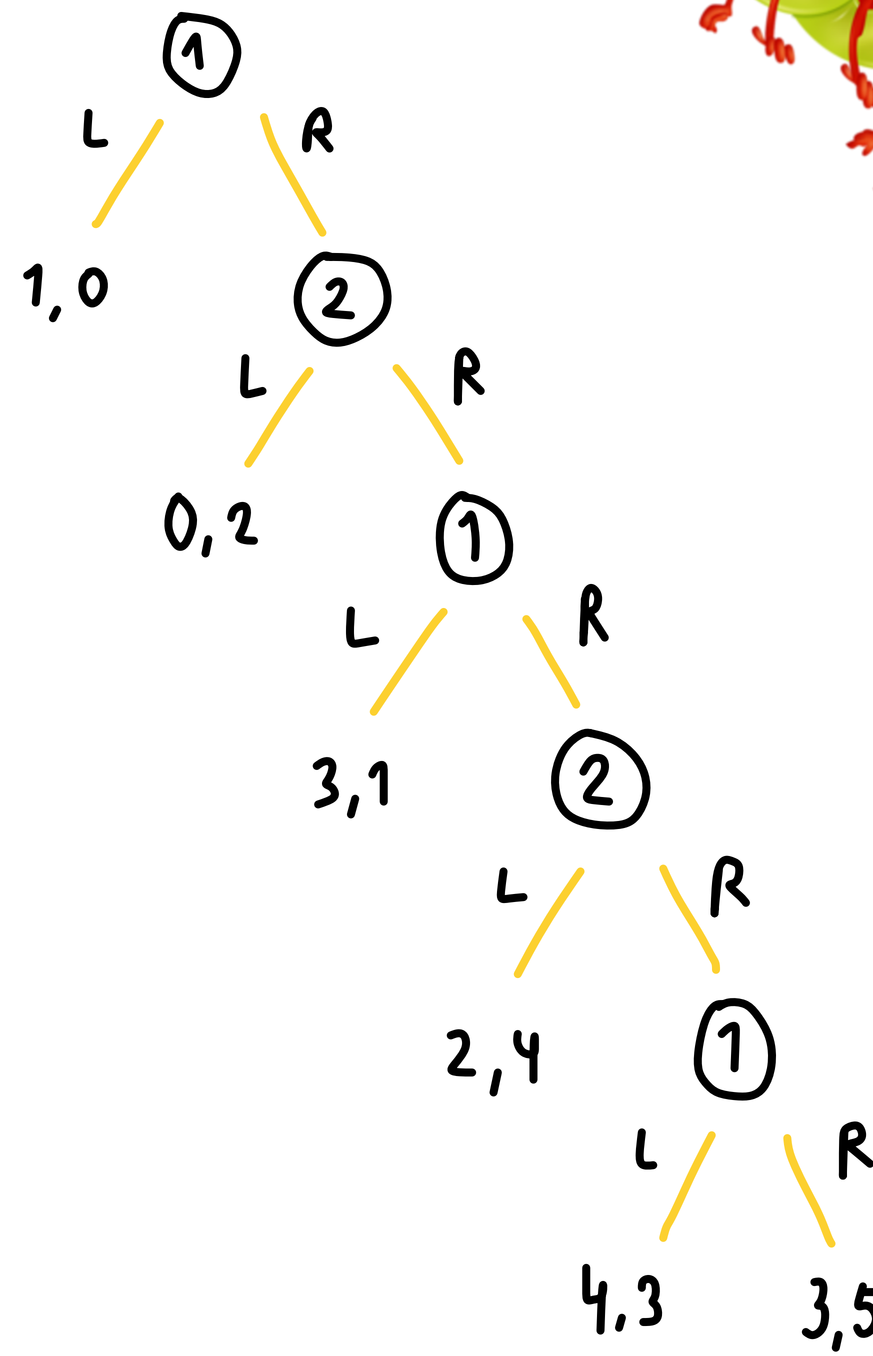
Theorem (Selten, 1965)
Every finite extensive-form game has at least one subgame-perfect equilibrium.

So is subgame-perfect equilibrium what players should, and actually do, play in extensive-form games?

The Centipede Game



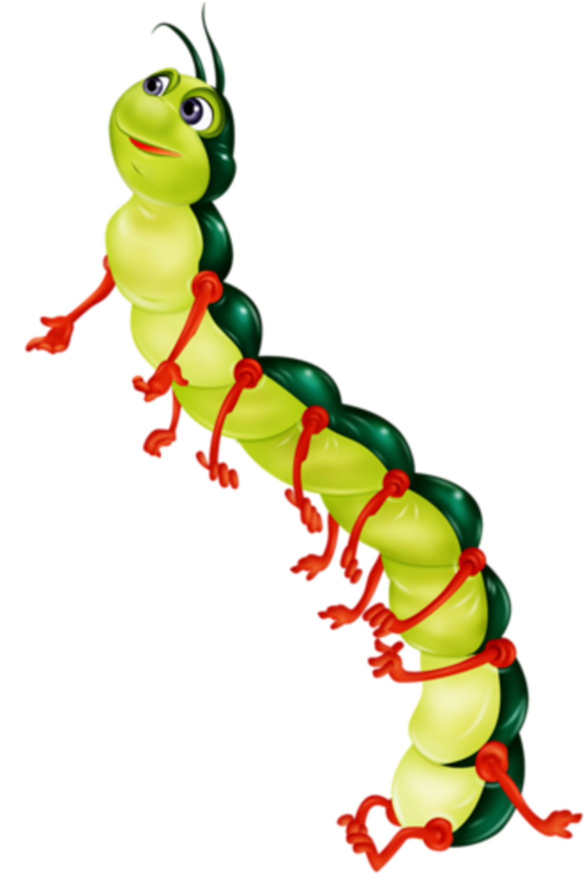
- Player 1 and 2 alternate making decisions between left (L) and right (R)
- Going left ends the game, and players help themselves to a pot
- Going right continues the game (except at the last node), and increases the pot



Subgame-perfect equilibria?

Image: clipartkey.com

The Centipede Game



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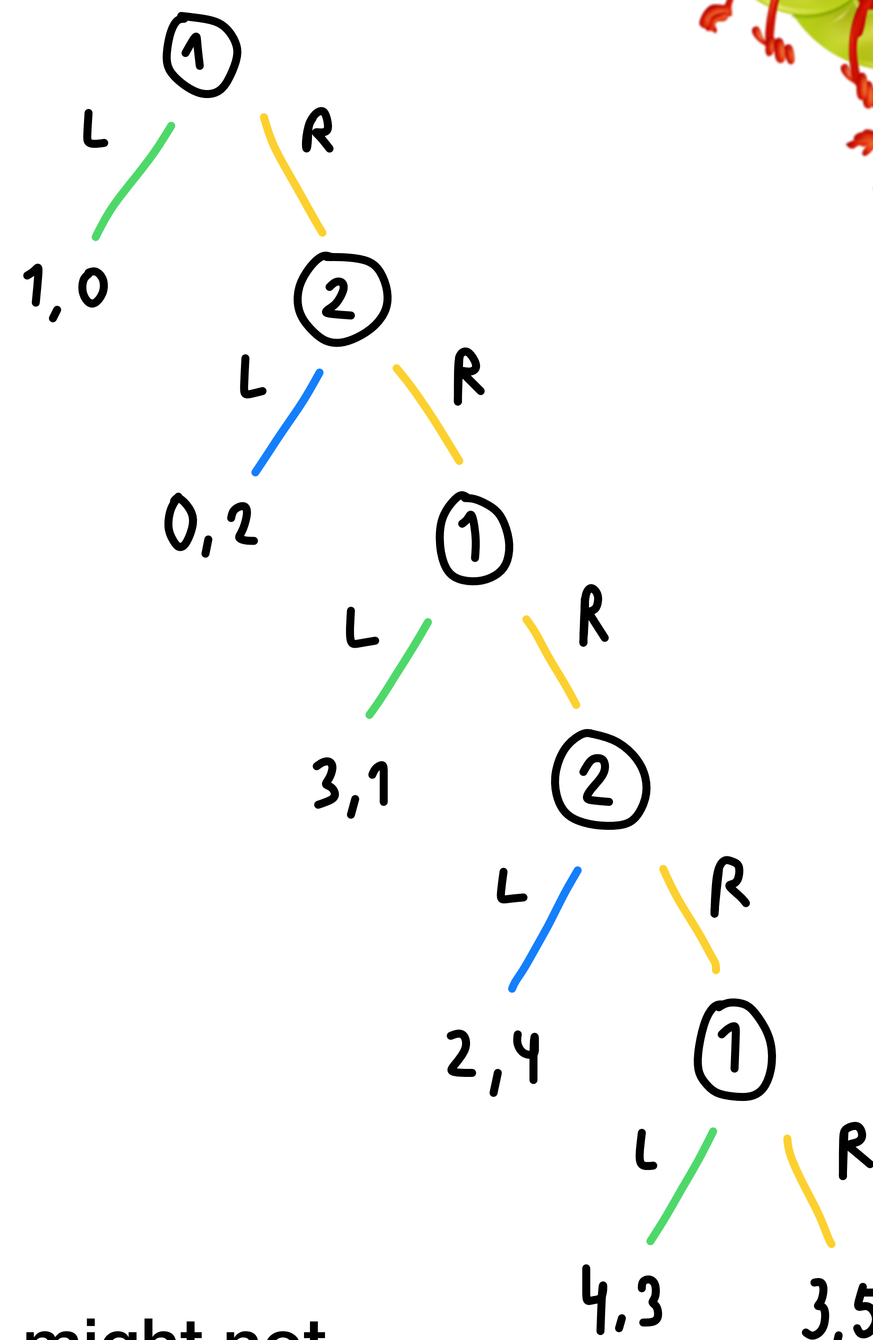

Subgame-perfect equilibria?

The payoffs are (1,0)…

Takeaway: subgame-perfect equilibria might not always picture how people actually reason

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