Multi-Agent Systems

Game Theory

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More on Extensive-Form Games

- In perfect-information extensive-form games players know what actions were played in previous rounds
  - And therefore, what nodes they are in
- But in many other situations, player have to act with partial knowledge
  - This is what we will be modeling now: **imperfect-information extensive-form games**

Imperfect-Information Extensive-Form Games: Intuition

Player 1 takes an action

… out of their action set

Player 2 follows up

… *not* knowing the action P1 has taken

Every player receives a payoff

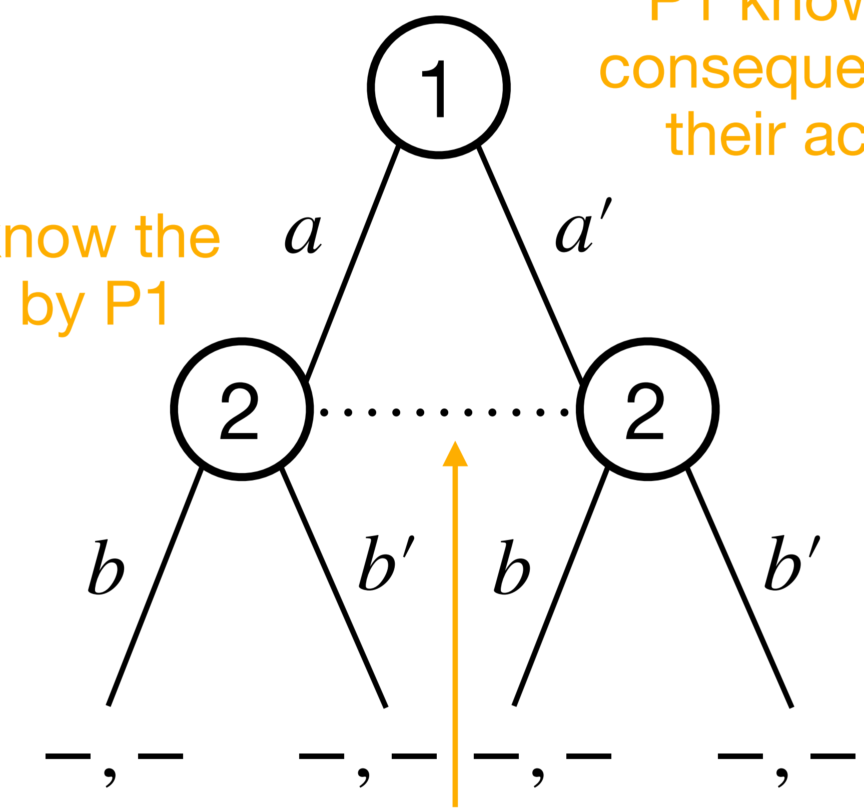
… specific to the branch taken

Players know

… actions available to all players, and the payoffs corresponding to each sequence of actions, i.e., the structure of the game

P1 knows the consequences of their actions

P2 does not know the action taken by P1



Dashed line represents uncertainty over the action taken

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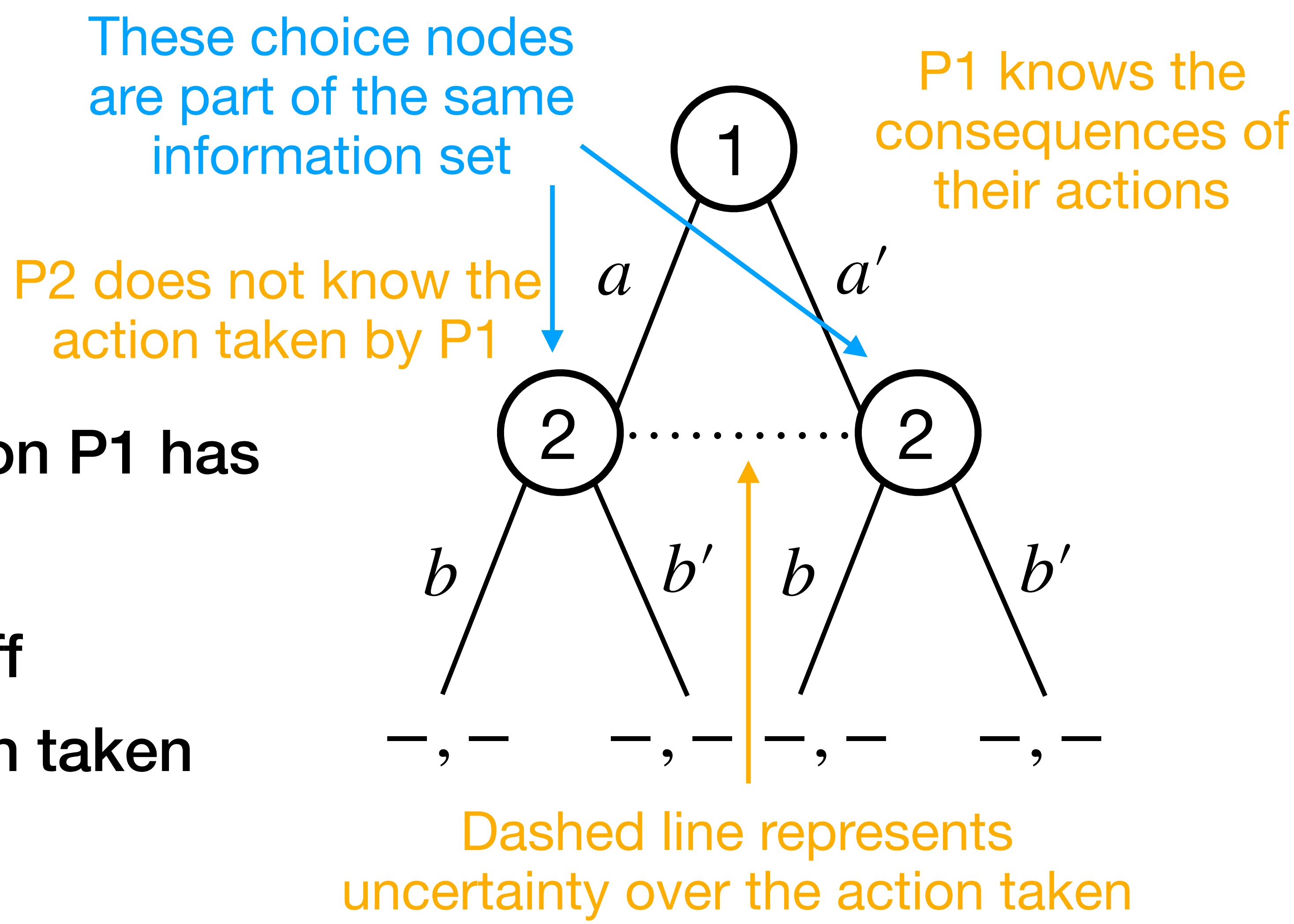


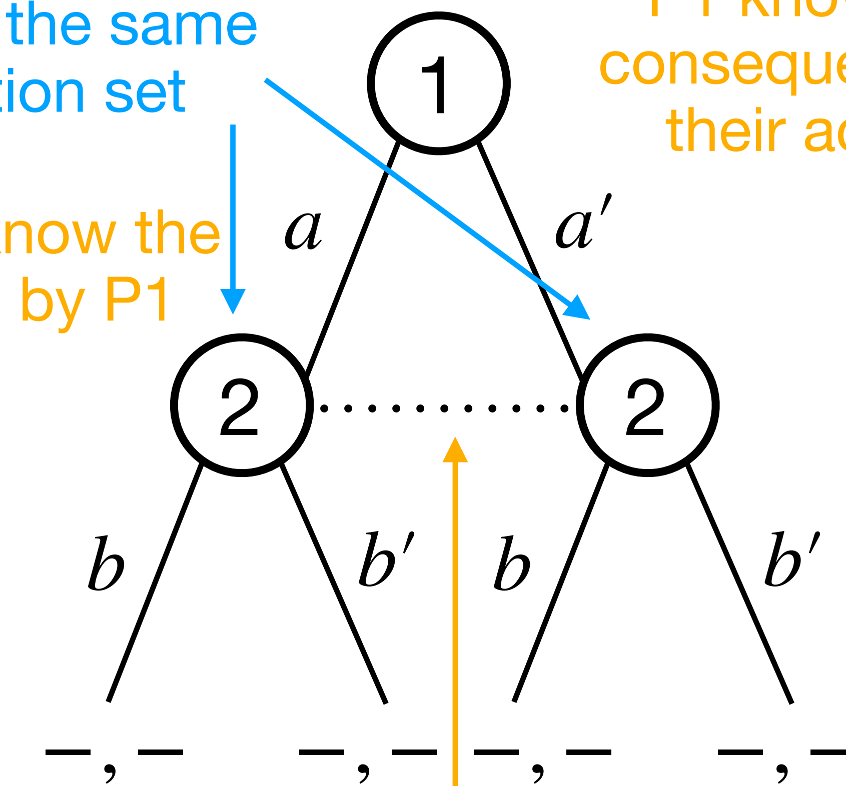
Players know

… actions available to all players, and the payoffs corresponding to each sequence of actions, i.e., the structure of the game

But players do *not* know

… what node from a particular information set they are in

Imperfect-Information Extensive-Form Games: Intuition



Intuitively

… an agent cannot distinguish between the nodes in an information set

Which means (importantly!)

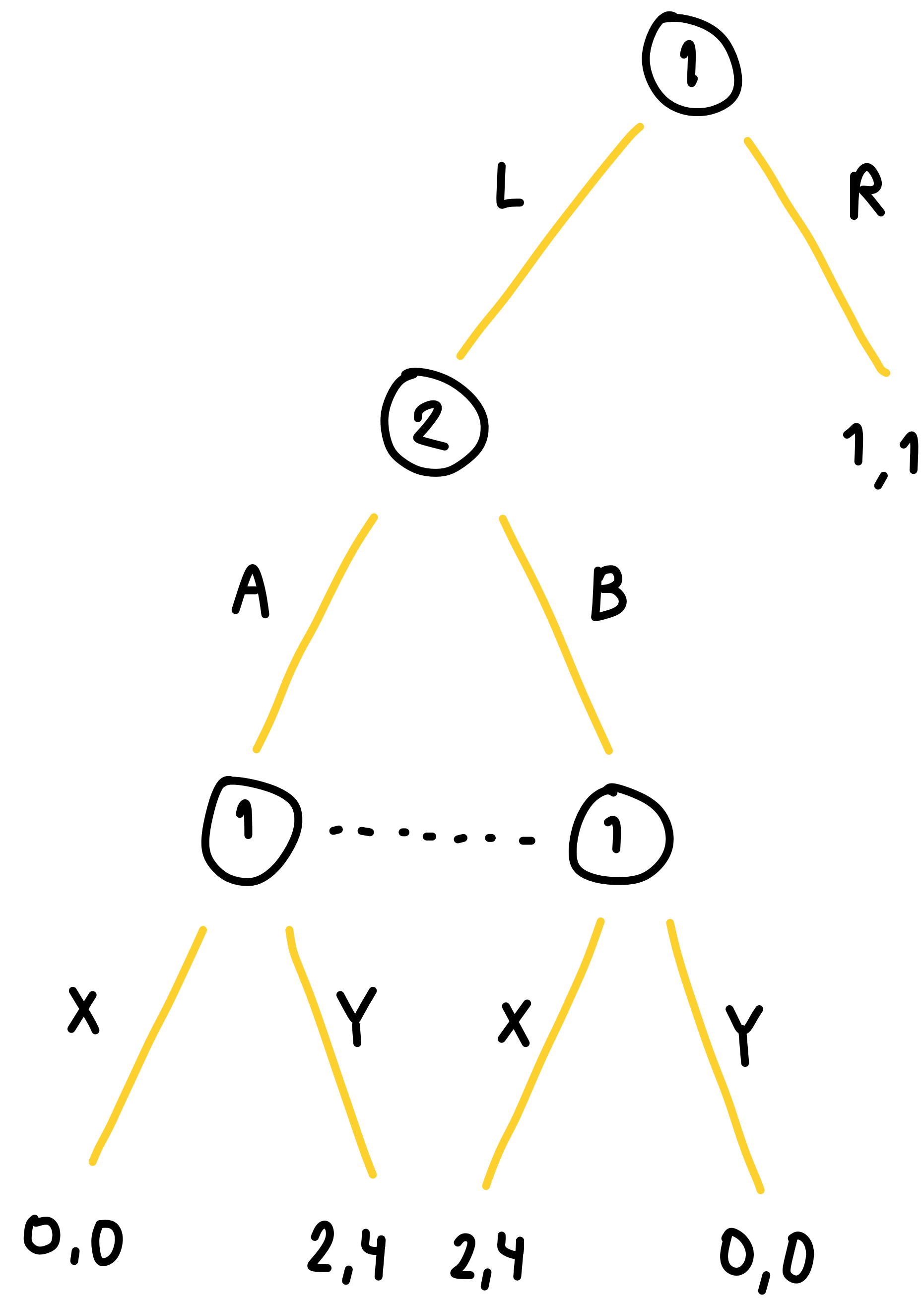
… that the actions available at an information set need to be the same for all nodes in that set

Imperfect-Information Extensive-Form Games

- Like their perfect-information counterparts, extensive-form games with imperfect information are modeled as **trees**
- The main difference is that every agent's choice nodes are partitioned into **information sets**
    - Such that the actions available at every information set are the *same* for all actions in that set
- A **strategy** for an agent is a combination of actions, one for each information set corresponding to that agent
    - Rather than for each choice node

Imperfect-Information Extensive-Form: Example

To illustrate, we need to have individual labels for each node

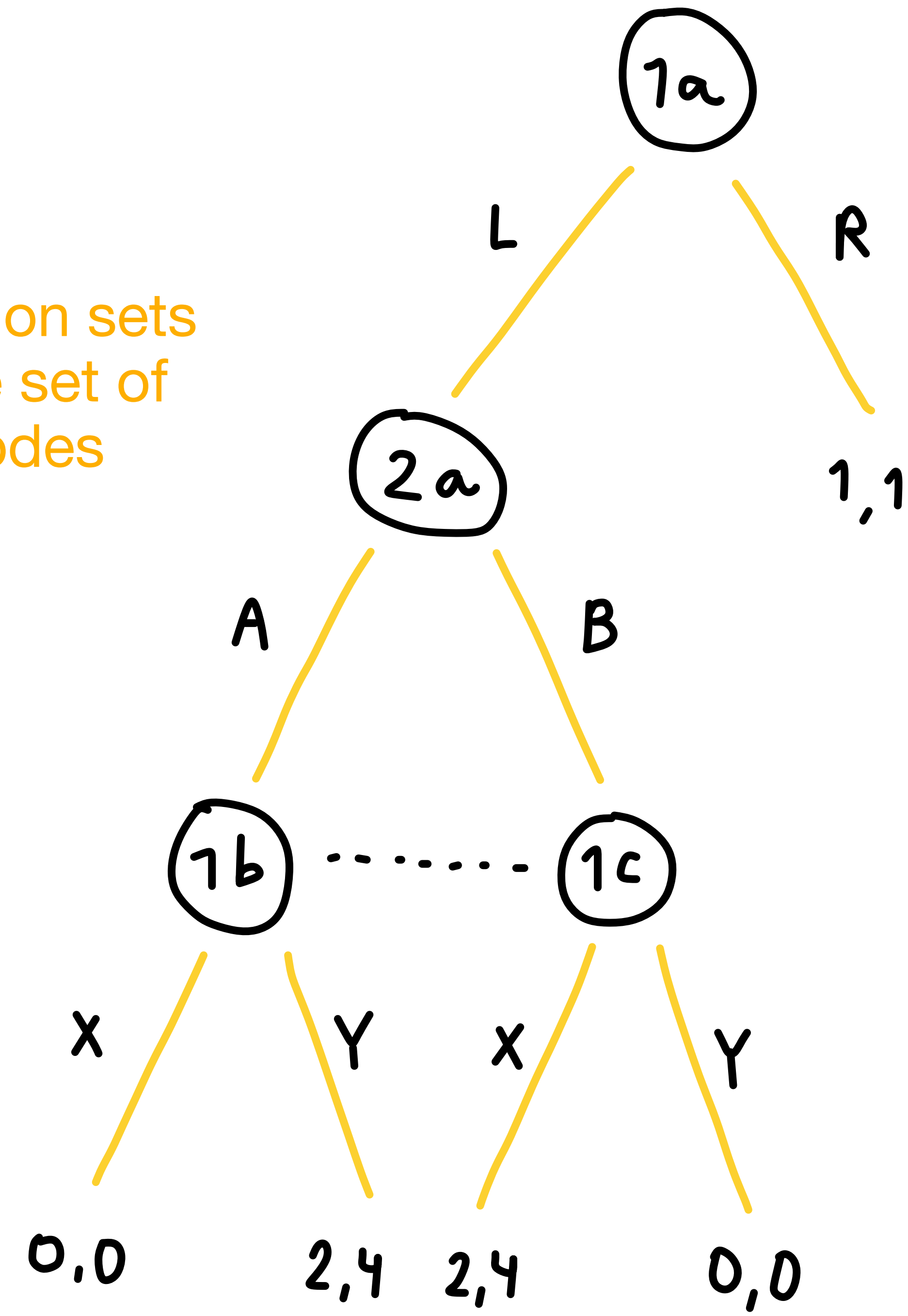


Imperfect-Information Extensive-Form: Example

To illustrate, we need to have individual labels for each node

Players: 1 and 2
Actions: L, R, A, B, X, Y
Utilities: (0,0), (2,4), (1,1)
Choice nodes of P1: 1a, 1b, 1c
Choice nodes of P2: 2a

The information sets partition the set of choice nodes



Information sets of P1: {1a}, {1b, 1c}
Information sets of P2: {2a}

Actions at {1a}: L, R
Actions at {1b, 1c}: X, Y
Actions at {2a}: A, B

Actions are defined at information sets rather than choice nodes

Strategies of P1: (L, X), (L, Y), (R, X), (R, Y)
Strategies of P2: A, B

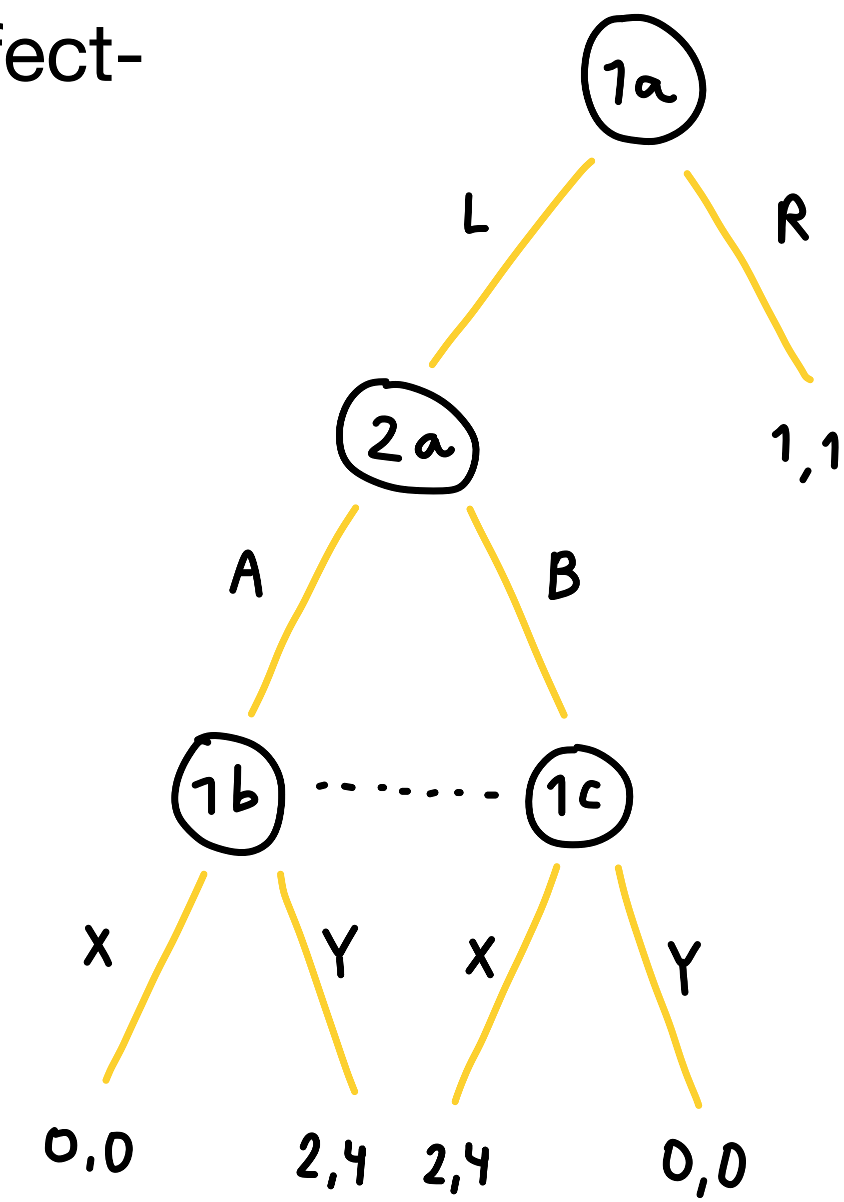
Not (L, X, X), (L, X, Y) … which would be the case with perfect information

Relation Between Perfect-Information and Imperfect-Information Games

- Perfect-information games can be thought of as a special case of imperfect-information games
    - Where the information sets are singletons (they only have one element)

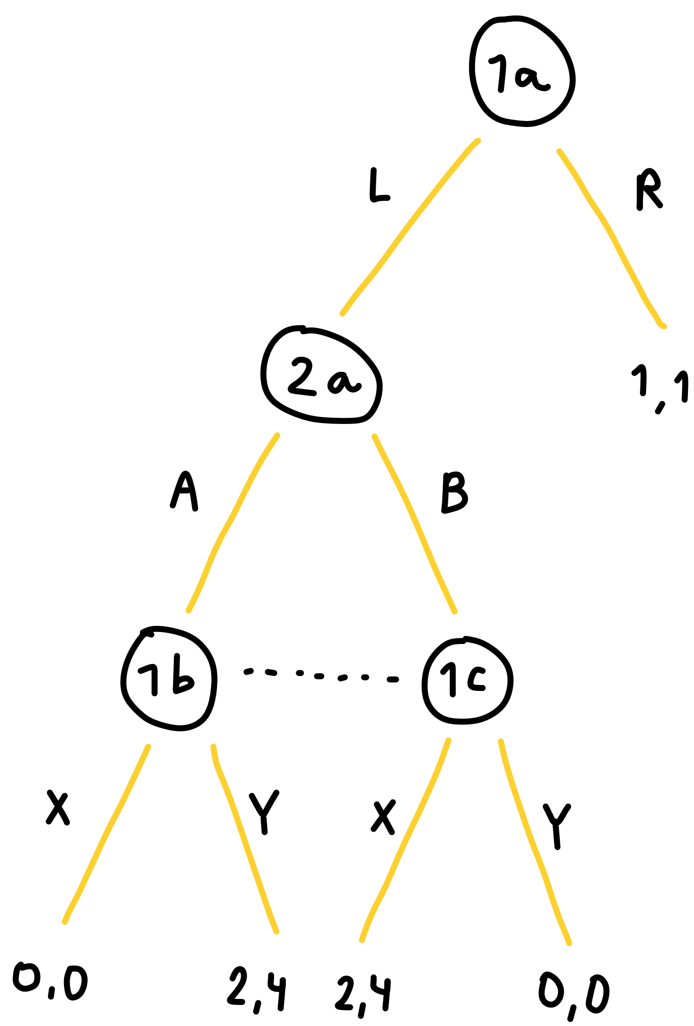
Information sets of P1: {1a}, {1b, 1c}
Information sets of P2: {2a}

Two elements



From Imperfect-Information Extensive-Form to Normal-Form

- We can also translate imperfect-information extensive-form games into normal-form games
    - As was the case when we had perfect information



Strategies of P1: (L, X), (L, Y), (R, X), (R, Y)
Strategies of P2: A, B

	A	B
(L, X)	(0,0)	(2,4)
(L, Y)	(2,4)	(0,0)
(R, X)	(1,1)	(1,1)
(R, Y)	(1,1)	(1,1)

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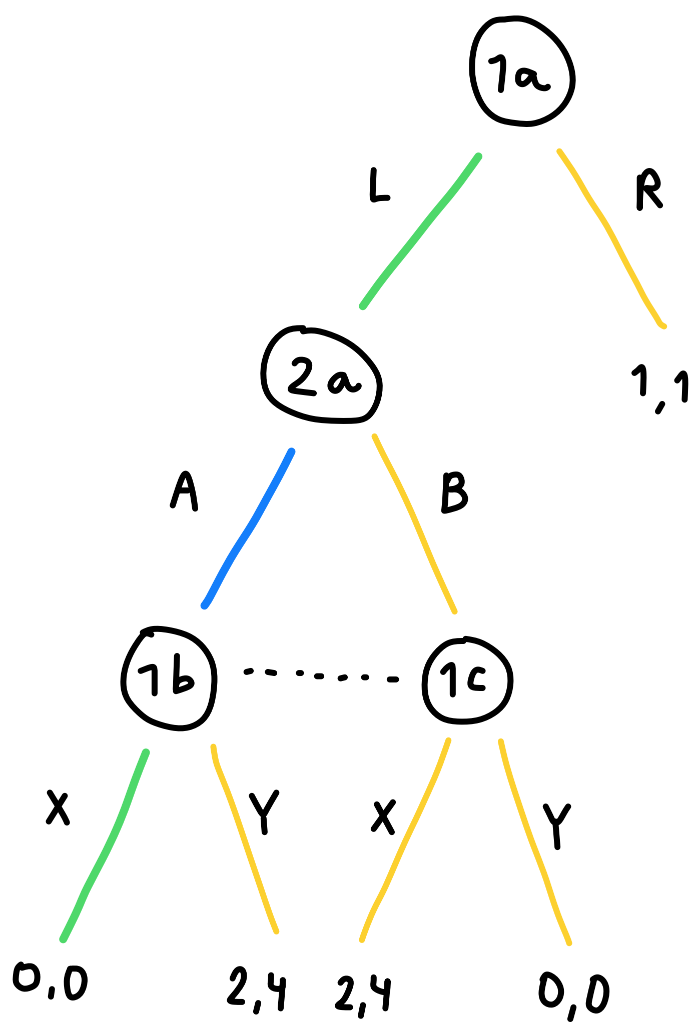

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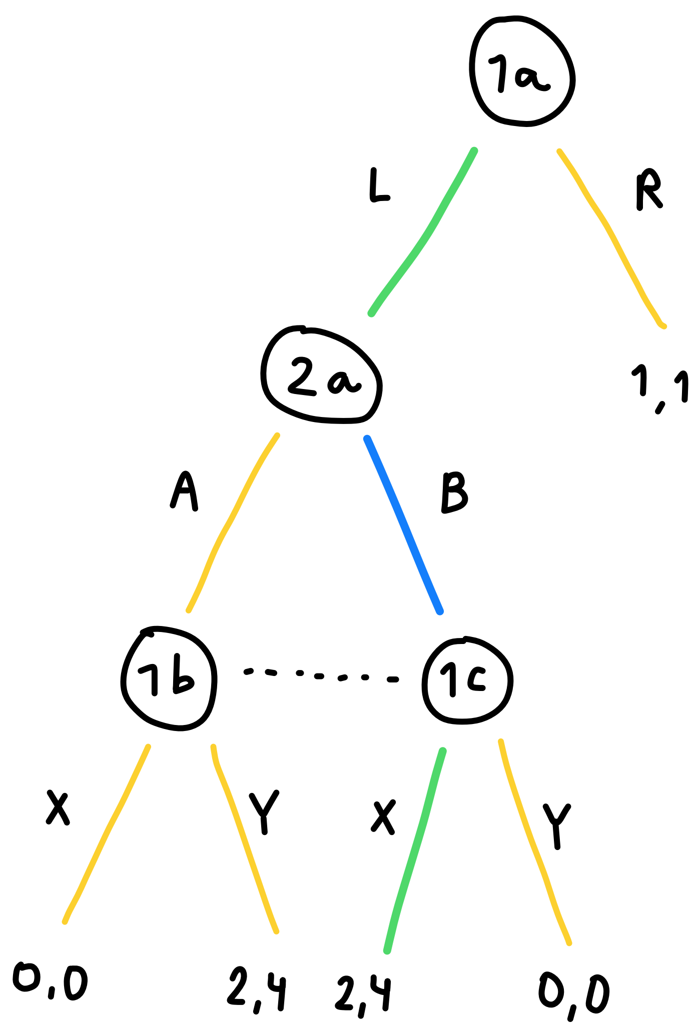

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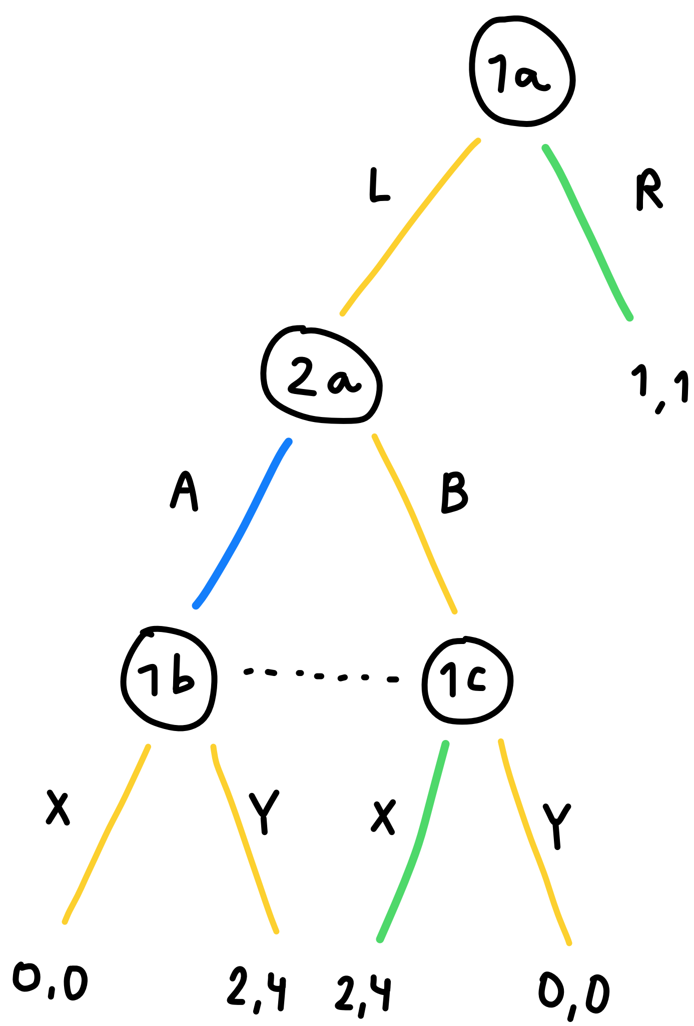

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(L, Y)	(2,4)	(0,0)
(R, X)	(1,1)	(1,1)
(R, Y)	(1,1)	(1,1)

From Normal-Form to Imperfect-Information Extensive-Form

- Recall that we cannot always translate from normal-form to perfect-information extensive-form
    - However, we *can* go from normal-form to imperfect-information extensive-from
    - Using information sets, we can model that players do not know what other players have chosen

The Prisoner’s Dilemma

	Stick to the story	Rat
Stick to the story	(-100,-100)	(-1000,0)
Rat	(0,-1000)	(-500,-500)

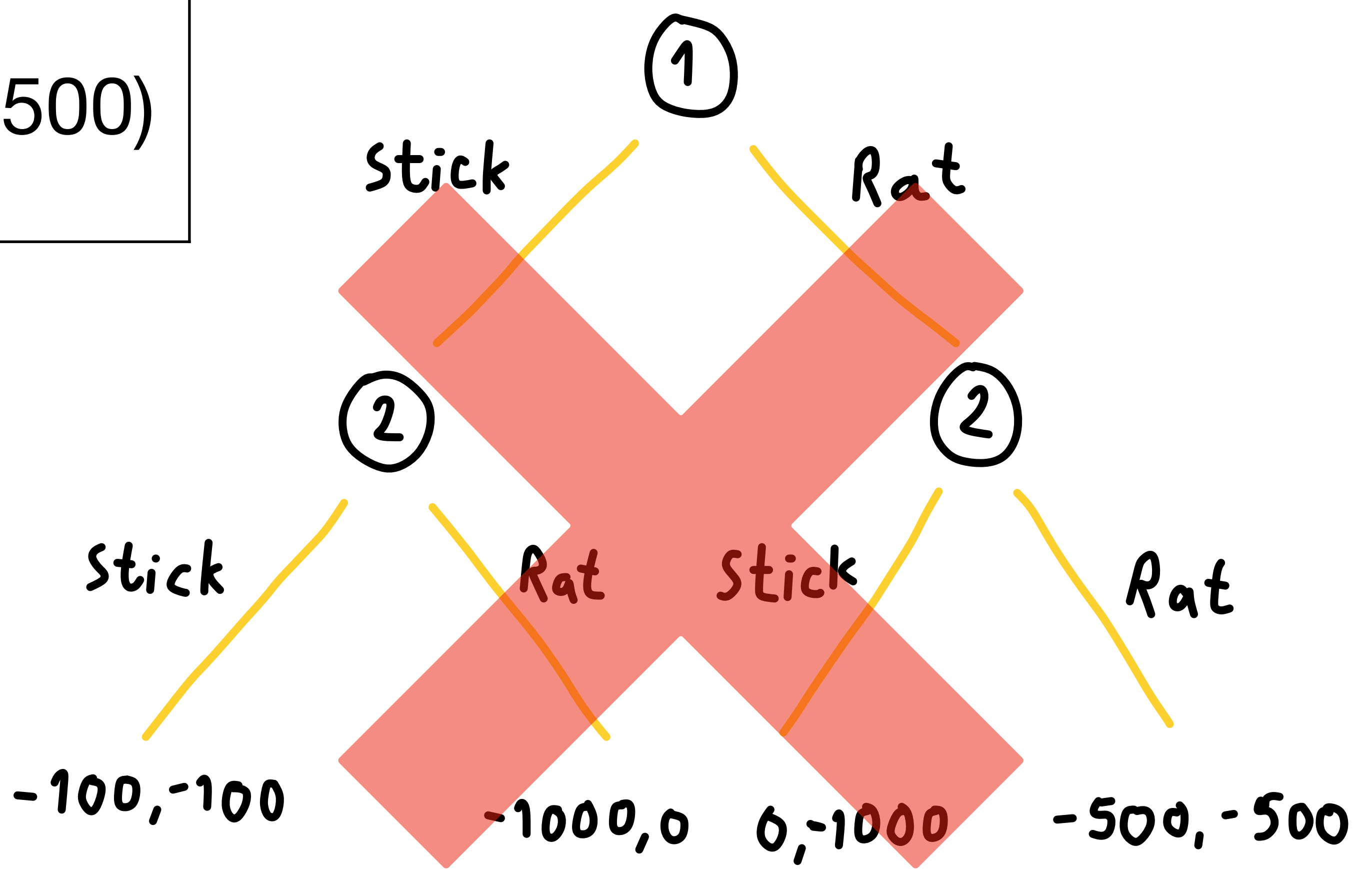

Image: clipartkey.com

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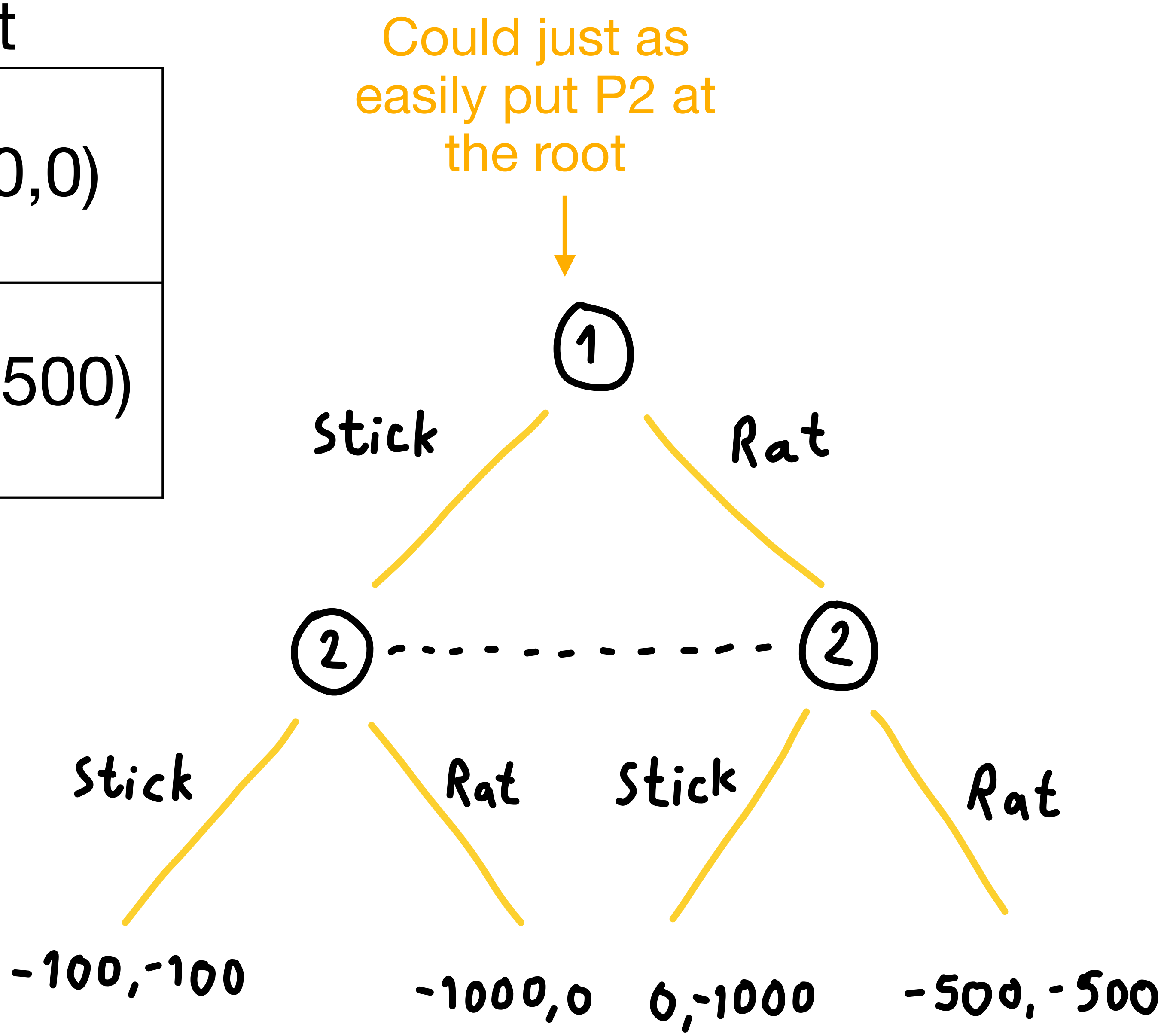

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Best Response and Nash Equilibria in Imperfect-Information Extensive-Form Games

- We have seen that we can translate from (imperfect-information) extensive-form games to normal-form and back
    - And so, concepts like best response and Nash equilibrium are inherited from normal-form games
    - Mixed strategies are also what we would expect: randomizing (playing a mixed strategy) over the pure strategies in the induced normal form

Behavioral Strategies

- But there is another relevant notion of randomized strategy for these kinds of games: **behavioral strategies**

- In a behavioral strategy there is randomization every time an information set is encountered

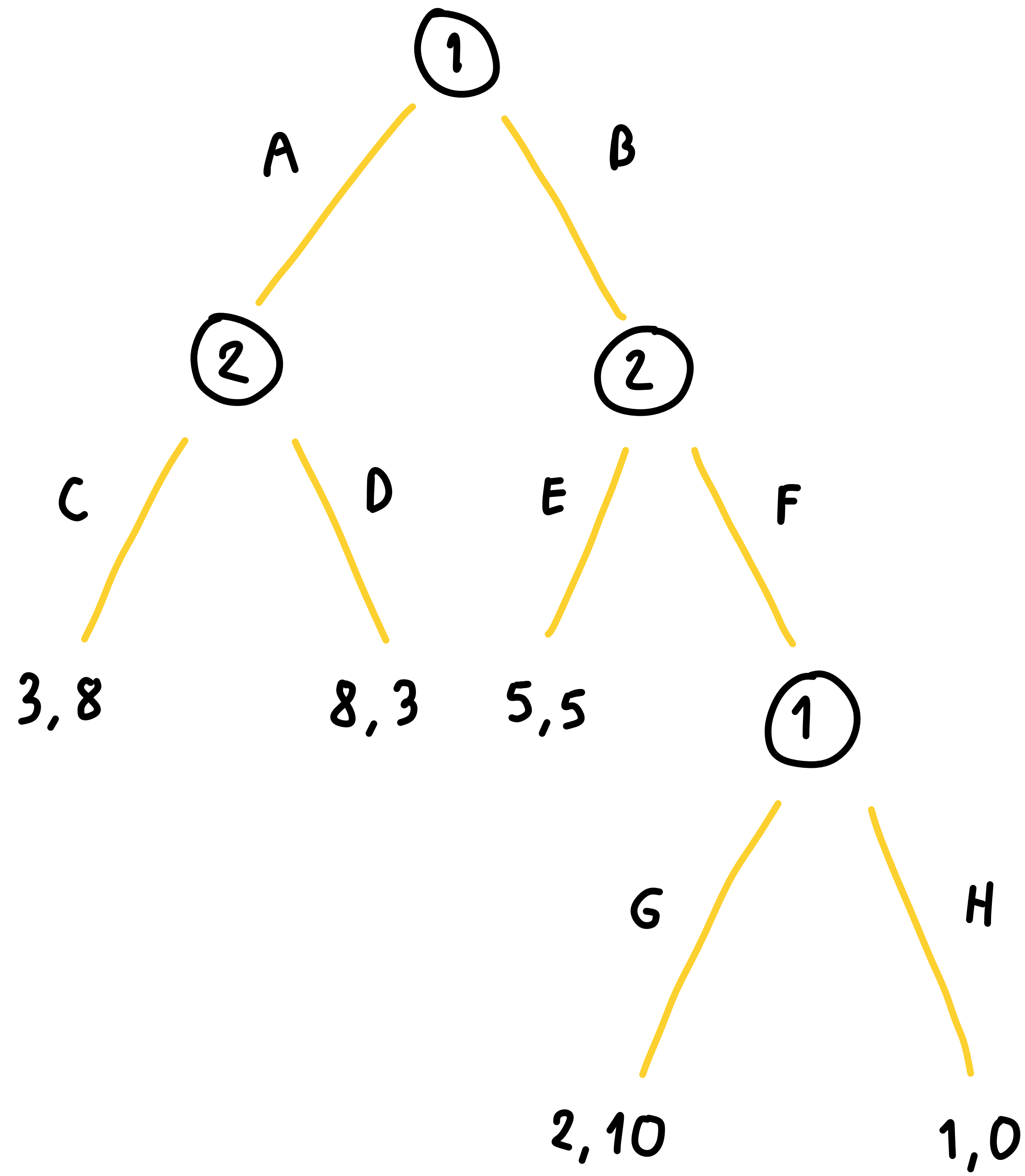
Behavioral Strategies: Example

Pure strategies of P1:

(A, G), (A, H), (B, G), (B, H)

Pure strategies of P2:

(C, E), (C, F), (D, E), (D, F)



Behavioral Strategies: Example

Pure strategies of P1:

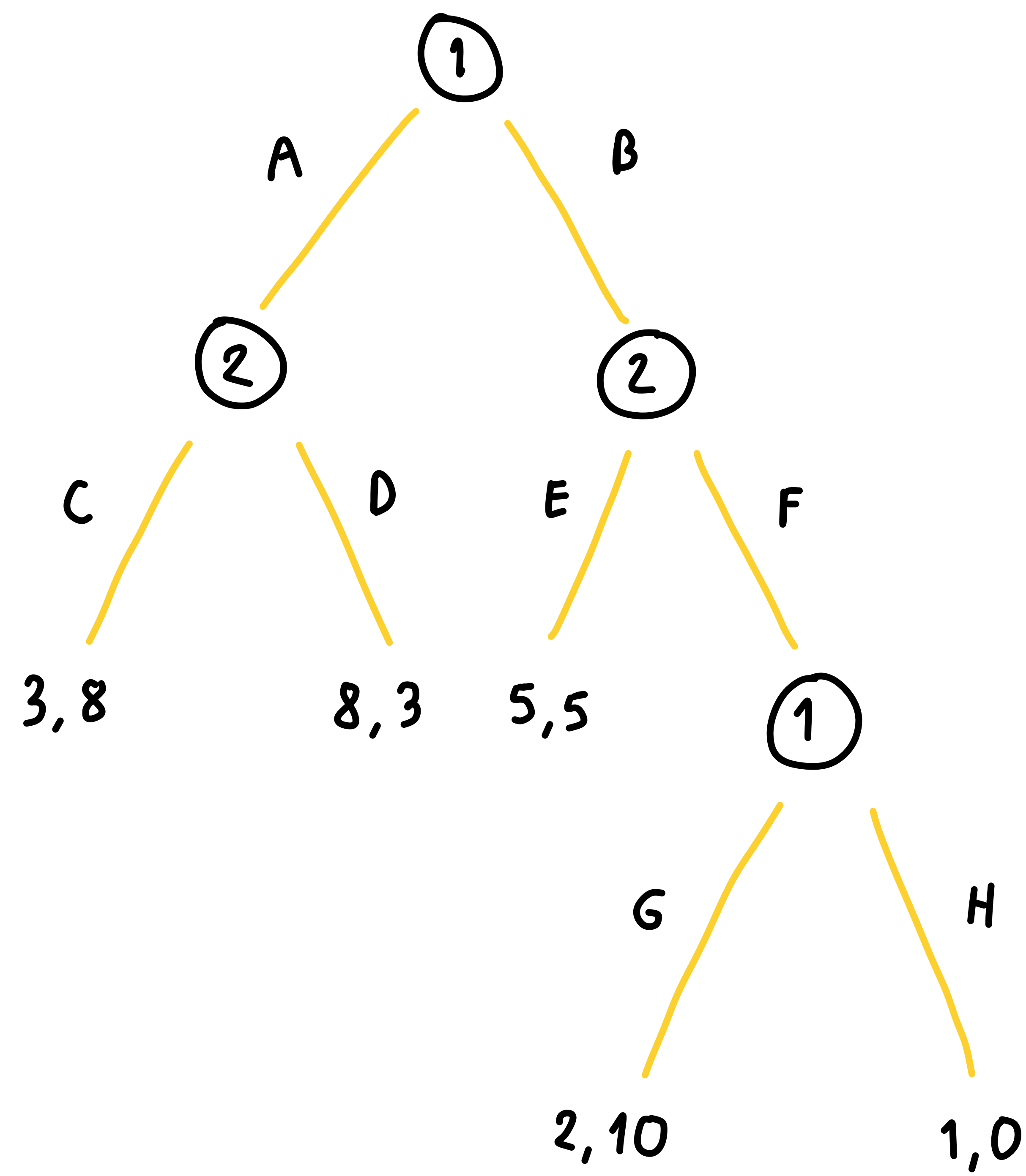
(A, G), (A, H), (B, G), (B, H)

Pure strategies of P2:

(C, E), (C, F), (D, E), (D, F)

Behavioral strategy of P1:

A with probability 0.5
G with probability 0.3



Behavioral Strategies: Example

Pure strategies of P1:

(A, G), (A, H), (B, G), (B, H)

Pure strategies of P2:

(C, E), (C, F), (D, E), (D, F)

Behavioral strategy of P1:

A with probability 0.5
G with probability 0.3

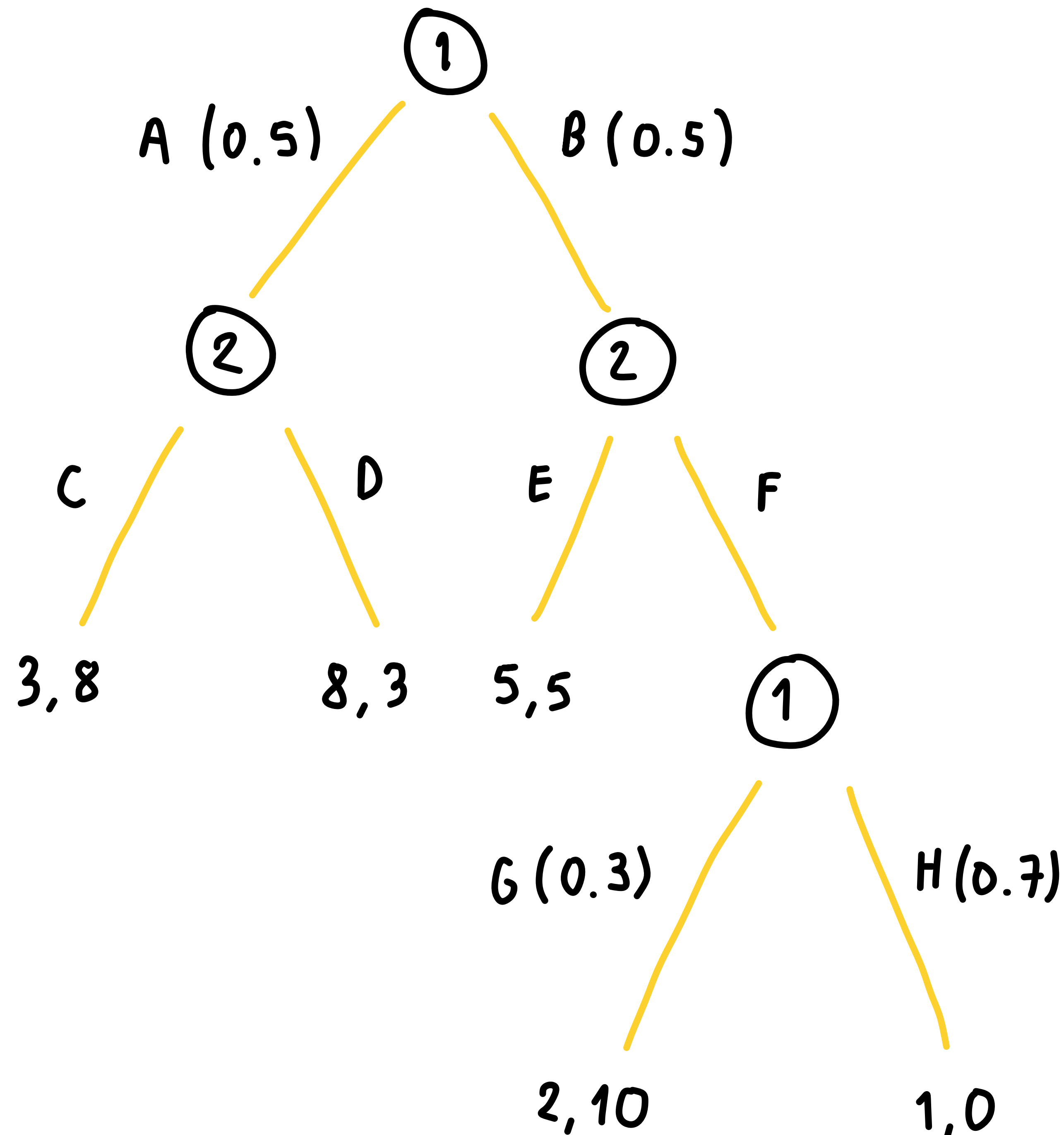
Induces

Mixed strategy of P1:

(A, G) with probability 0.5×0.3

(A, H) with probability 0.5×0.7

(…)



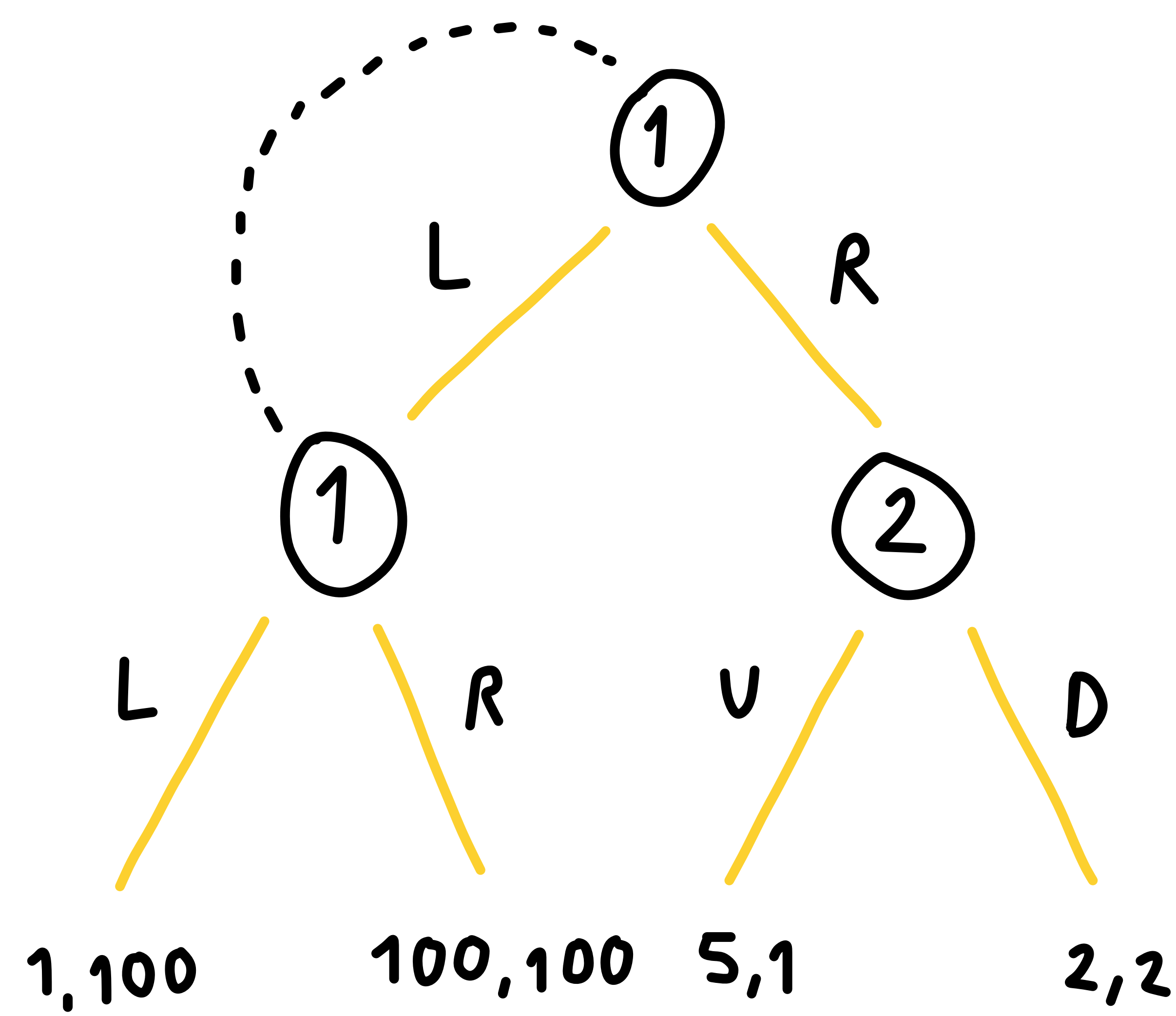
But this equivalence between behavioral strategies and mixed strategies is not ensured in all types of games…

Perfect Recall

- In a behavioral strategy, you decide for each information set, with which probability to play the actions
- Not all mixed strategies coincide with a behavioral strategy, and neither vice versa
- But! It turns out that there is a class of games for which behavioral and mixed strategies coincide: games of **perfect recall**
- In a game of perfect recall all players remember information acquired about moves made so far
  - In other words, they remember the actions they have taken themselves, as well as the information they had when they played that action
- Every perfect-information game is a game of perfect recall
  - But not every imperfect-information game

A Game with Imperfect Recall

Player 1 does not know what node they are in



Perfect Recall

Theorem (Kuhn, 1953)
In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioral strategy, and any behavioral strategy can be replaced by an equivalent mixed strategy.

Two strategies are equivalent if they induce same probabilities on outcomes, for any fixed strategy profile of other agents.

In games of imperfect recall, equilibria for behavioral strategies and mixed strategies can be different.

A Game with Imperfect Recall

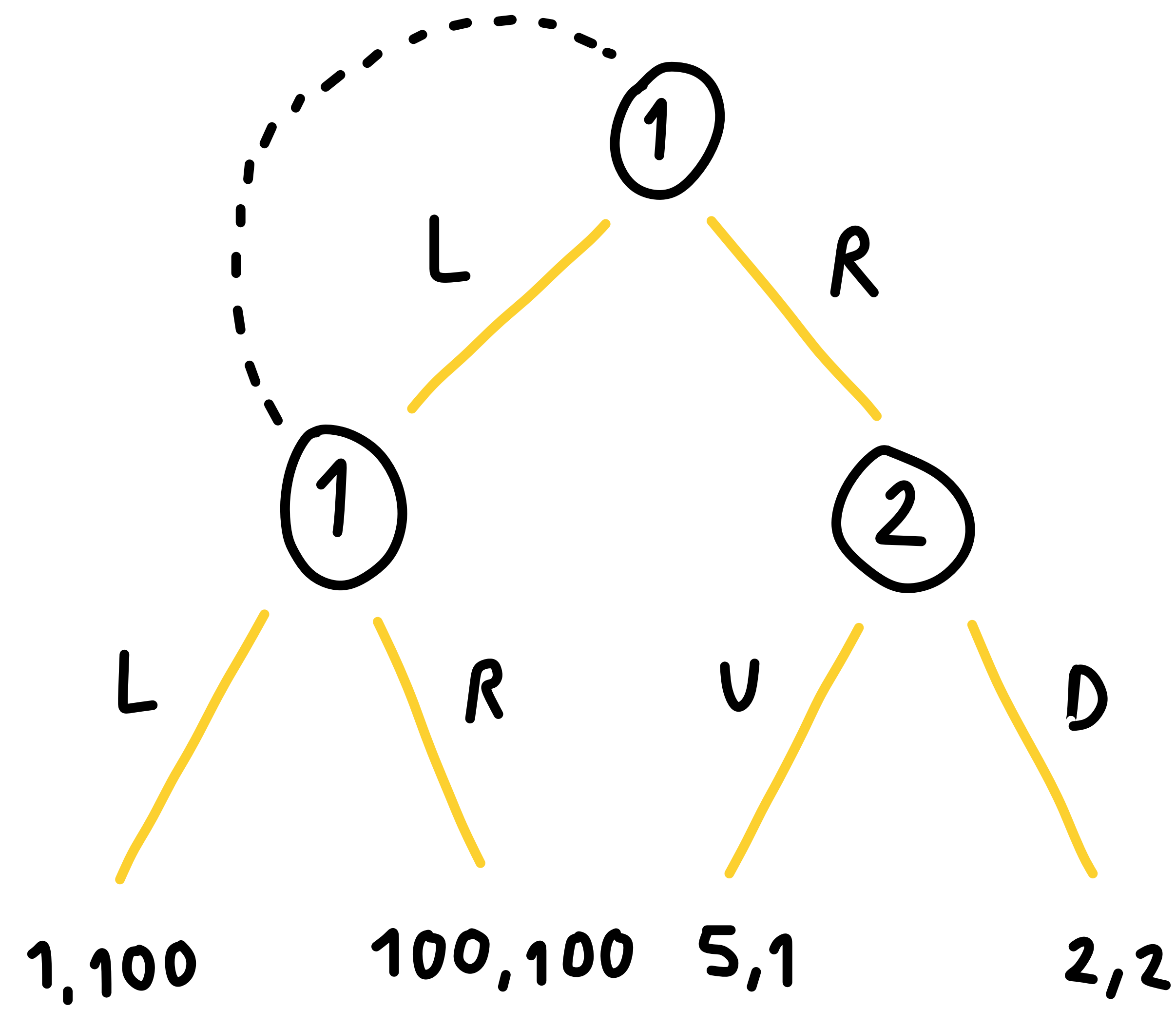
Strategies for Player 1?
{L, R}

Strategies for Player 2?
{U, D}

Pure Nash equilibrium?
(R, D)

Mixed Nash equilibrium?

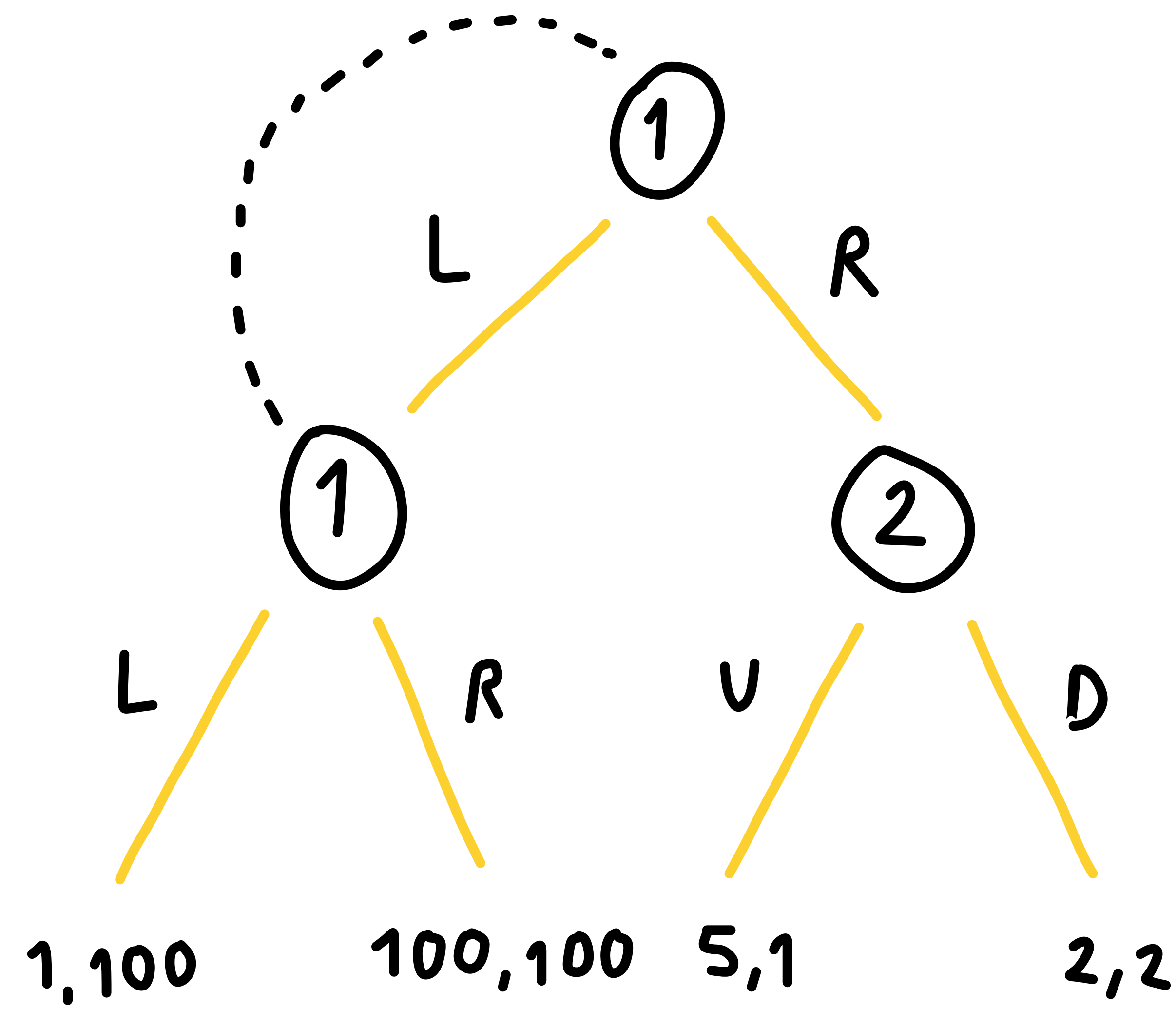
The same! R is strictly dominant, so P1 should always play R, and then P2 should play D



A Game with Imperfect Recall

Mixed Nash:
(R, D) = ((0,1), (0,1))

What about the behavioral equilibrium?



A Game with Imperfect Recall

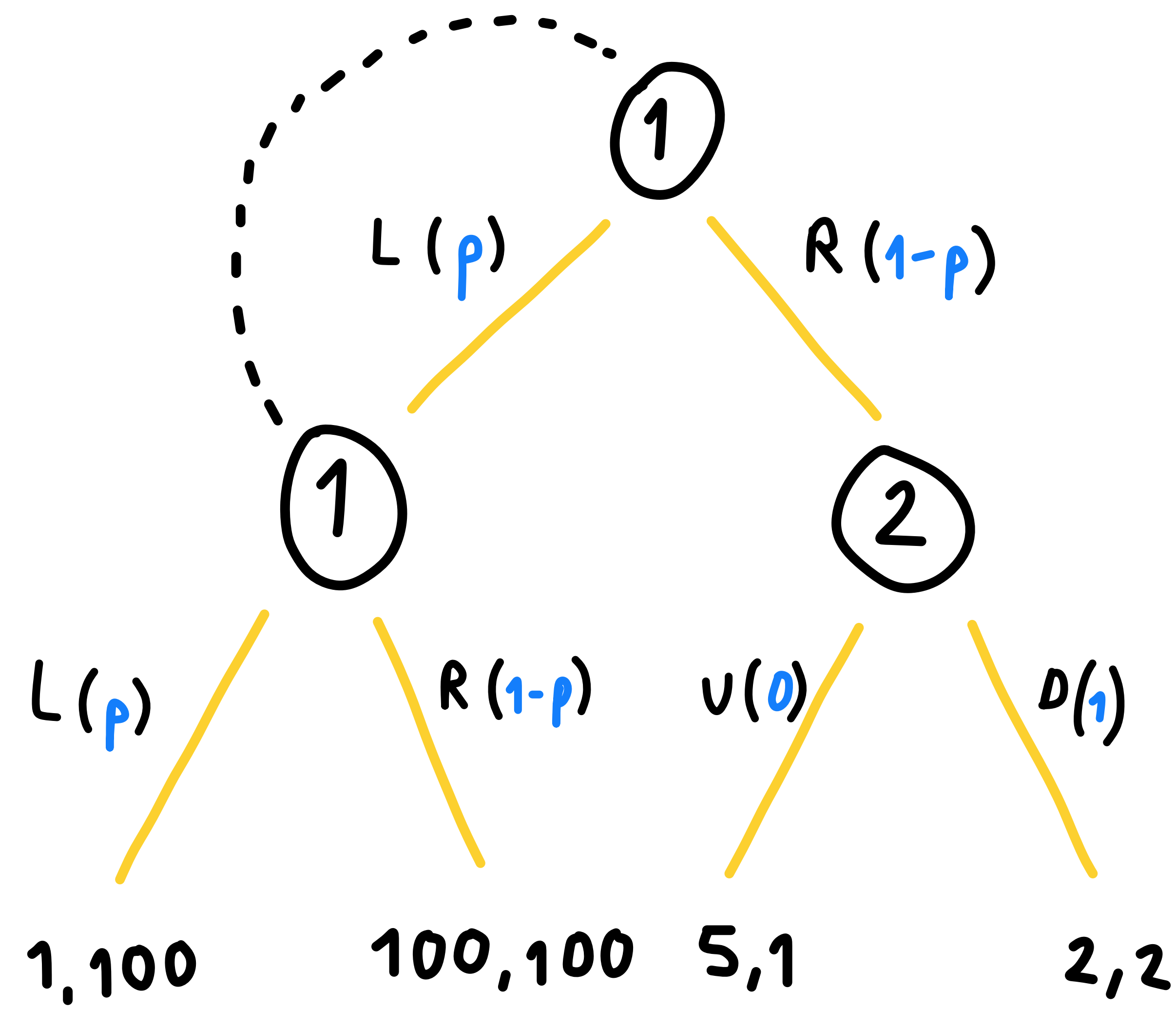
Mixed Nash:
(R, D) = ((0,1), (0,1))

What about the behavioral equilibrium?

P1 randomizes afresh each time they are in the information set

I.e., they play the strategy $(p, 1-p)$

D weakly dominates U for P2



What is P1's best response to P2 playing D?

The expected utility is:

$$p^2 + 100p(1-p) + 2(1-p) = -99p^2 + 98p + 2$$

Maximum is obtained at $p = \dfrac{98}{198}$

Behavioral equilibrium:

$\left(\left(\dfrac{98}{198}, \dfrac{100}{198}\right), (0,1)\right)$

Not the same as mixed Nash!

Game Theory: Summary

- Decision theory
    - Expected utility
- Normal-form games
    - Dominated strategies (and the elimination of them)
    - Pareto optimality
    - Pure and mixed Nash equilibrium

Game Theory: Summary

- Extensive-form games
  - Perfect-information extensive-form games
    - Subgame-perfect equilibrium and backwards induction
  - Imperfect-information extensive-form games
    - Relation to normal-form games
    - Behavioral strategies and perfect recall

Game Theory: Summary

- What we did not cover:
    - Other solution concepts for normal-form games: maxmin, minmax, correlated equilibria …
        - And for extensive-form games: separating equilibria, sequential equilibria …
    - Other representations for games: Bayesian games, repeated games …