Multi-Agent Systems

Game Theory

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John Nash

- (1928-2015)
- Made fundamental contributions to game theory (and several other fields)
- Won the Nobel Prize in Economics with fellow game theorists John Harsanyi and Reinhard Selten in 1994
- Won the Abel Prize with Louis Nirenberg in 2015
- Battled with schizophrenia

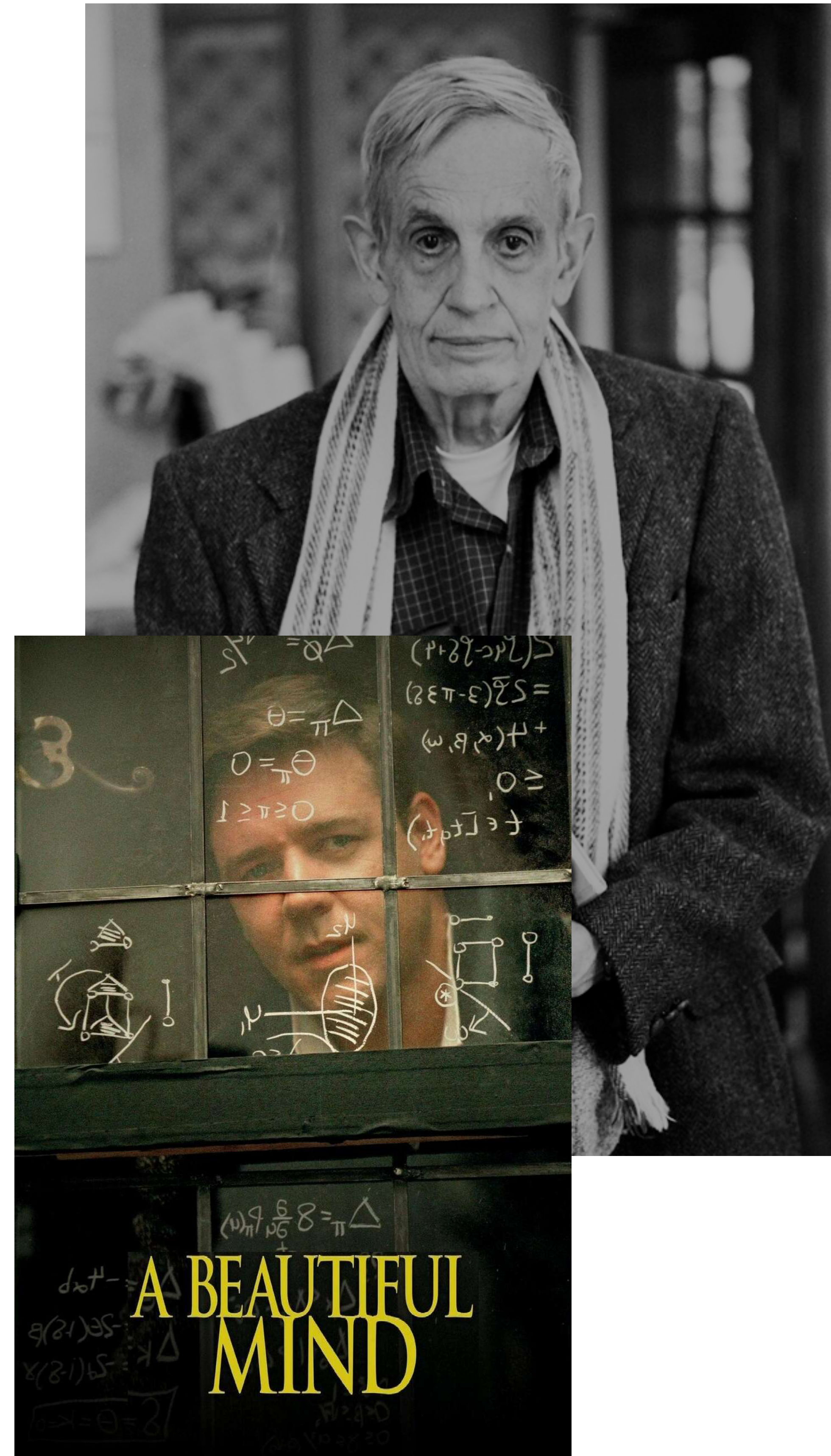

Image: Wikipedia, Jimmy Wales

Nash Equilibria in Pure Strategies

- In a **Nash equilibrium**, no one has an incentive to change their strategy, given the other players' strategies

Definition (Best Response)

Player i's **best response** to strategy profile s_{-i} is a strategy s_i^* such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$, for any strategy s_i that i can play.

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a **pure Nash equilibrium** if, for every player i, s_i^* is a best response to s_{-i}^*.

In other words, there is no player i and strategy s_i' such that:

$$u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$$

The Prisoner’s Dilemma

- You and a friend are at the police station
- You are the main suspects in a string of bicycle thefts
- You are interrogated at the same time, in separate rooms
- If both of you stick to the common story that you did not do it, you get off with a small fine
- But if you tell on your friend you get off free, while they get a hefty fine
- Your friend faces the same situation
- If you rat each other out, you split the large fine

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The Prisoner's Dilemma

Both players want to deviate

Player 1 wants to deviate

	Stick to the story	Rat
Stick to the story	(-100,-100)	(-1000,0)
Rat	(0,-1000)	(-500,-500)

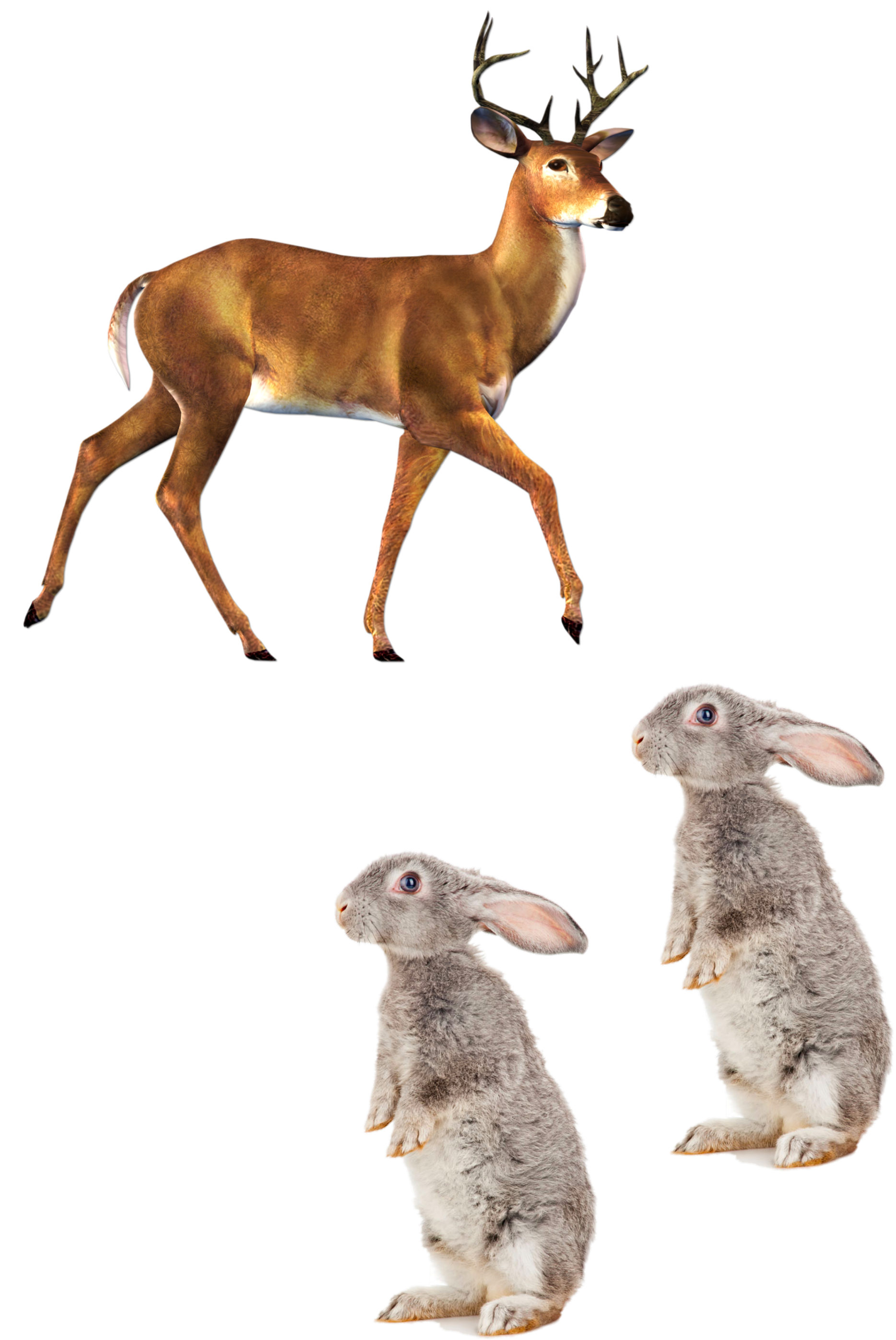
Player 2 wants to deviate

What are the pure Nash equilibria?

Image: clipartkey.com

Stag Hunt

- Two hunters
- Have to decide what to hunt: one stag or two hares
- A stag takes two to catch: hunting a stag solo is certain failure
- One hunter alone can catch both hares, but if both hunters go for the hares they end up with one each
- Even combined, the hares are worth less than the stag



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Stag Hunt

Player 1 wants to deviate

	Stag	Hare
Stag	(10,10)	(0,6)
Hare	(6,0)	(3,3)

Player 2 wants to deviate

What are the pure Nash equilibria?

Image: clipartkey.com

More on Pure Nash Equilibria

- Pure Nash equilibria does not always exist
- A strictly dominating strategy profile, if it exists, is a pure Nash equilibrium
- Nash equilibria is not necessarily Pareto optimal

The Prisoner's Dilemma Again

	Stick to the story	Rat
Stick to the story	(-100,-100)	(-1000,0)
Rat	(0,-1000)	(-500,-500)

What are the Pareto optimal outcomes?

In the Prisoner's Dilemma every outcome *except* the Nash equilibrium is Pareto optimal!

Image: clipartkey.com

More on The Prisoner's Dilemma

- One way to look at it: In the Prisoner's Dilemma, the temptation to be selfish and defect is greater than the payoff from pursuing the common good by cooperating

- Was created by researchers Flood and Dresher in the 50s to show that Nash Equilibrium sometimes makes strange predictions

- It is an abstract interaction: players know the rules, but don't interact outside the game

	Stick to the story	Rat
Stick to the story	(-100,-100)	(-1000,0)
Rat	(0,-1000)	(-500,-500)

Mixed Strategies

- So far, we have been assuming that strategies are pure
    - That players choose and action and stick with it
- But it also makes sense for players to have a strategy where they state with which probability they want to play each action
    - We call these **mixed strategies**

Mixed Strategies

- $N = \{1, \ldots, n\}$ is a finite set of agents, or players
- A_i is a finite set of actions available to player i
- $a = (a_1, \ldots, a_n)$ is an action profile
- $A = A_1 \times \ldots \times A_n$ is the set of all action profiles
- $u_i : A \to \mathbb{R}$ is a utility (or payoff) function for player i
- $u = (u_1, \ldots, u_n)$ is a utility profile

Definition (Mixed Strategy)

- $\mathbb{P} : A_i \to [0,1]$ is a probability distribution over A_i such that $\sum_{a \in A_i} \mathbb{P}(a) = 1$
- $\Pi(A_i)$ is the set of probability distributions over A_i
- $s_i \in \Pi(A_i)$ is a **mixed strategy** of player i
- $S_i = \Pi(A_i)$ is the set of mixed strategies of player i
- $s = (s_1, \ldots, s_n)$ is a strategy profile
- $S = S_1 \times \ldots \times S_n$ is the set of all strategy profiles
- $u_i(s) = \sum_{a \in A} (u_i(a) \cdot \prod_{j=1} s_j(a_j))$ is the utility of i w.r.t. strategy profile s

Stag Hunt Again

	Stag $\frac{1}{5}$	Hare $\frac{4}{5}$
$\frac{1}{10}$ Stag	(10,10)	(0,6)
$\frac{9}{10}$ Hare	(6,0)	(3,3)

As an example: $s_1 = (\frac{1}{10}, \frac{9}{10})$ **and** $s_2 = (\frac{1}{5}, \frac{4}{5})$

Mixed strategy of P1

Mixed strategy of P2

$$s = (s_1, s_2)$$

Mixed strategy profile

$$u_i(s) = \sum_{a \in A} (u_i(a) \cdot \prod_{j=1} s_j(a_j))$$

Probability that P1 plays Stag

$$\begin{aligned} u_1(s) = \; & u_1(Stag, Stag) \cdot s_1(Stag) \cdot s_2(Stag) + \\ & u_1(Stag, Hare) \cdot s_1(Stag) \cdot s_2(Hare) + \\ & u_1(Hare, Stag) \cdot s_1(Hare) \cdot s_2(Stag) + \\ & u_1(Hare, Hare) \cdot s_1(Hare) \cdot s_2(Hare) = \end{aligned}$$

Expected utility of P1

$$10 \cdot \frac{1}{10} \cdot \frac{1}{5} + 0 \cdot \frac{1}{10} \cdot \frac{4}{5} + 6 \cdot \frac{9}{10} \cdot \frac{1}{5} + 3 \cdot \frac{9}{10} \cdot \frac{4}{5} = \quad 3.44$$

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Stag Hunt Again

	Stag $\frac{1}{5}$	Hare $\frac{4}{5}$
$\frac{1}{10}$ Stag	(10,10)	(0,6)
$\frac{9}{10}$ Hare	(6,0)	(3,3)

Another way to think about the utility with mixed strategies:

We we write $u_1(Stag, s_2)$ for the expected utility P1 gets by playing Stag, if P2 plays mixed strategy s_2, which is:

$$u_1(Stag, s_2) = u_1(Stag, Stag) \cdot s_2(Stag) + u_1(Stag, Hare) \cdot s_2(Hare)$$

Similarly for expected utility of P1 playing Hare:

$$u_1(Hare, s_2) = u_1(Hare, Stag) \cdot s_2(Stag) + u_1(Hare, Hare) \cdot s_2(Hare)$$

Then the utility of P1 from the mixed strategy is the expected utility of playing its actions with the given probabilities:

$$u_1(s_1, s_2) = u_1(Stag, s_2) \cdot s_1(Stag) + u_1(Hare, s_2) \cdot s_1(Hare)$$

Image: clipartkey.com

Nash Equilibria in Mixed Strategies

- By defining mixed strategies as we did, best responses and Nash equilibria are defined as for pure strategies

Mixed strategies

Definition (Best Response)

Player i's best response to strategy profile s_{-i} is a strategy s_i^* such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$, for any strategy s_i that i can play.

Expected utility of playing actions with given probabilities

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a pure Nash equilibrium if, for every player i, s_i^* is a best response to s_{-i}^*.

In other words, there is no player i and strategy s_i' such that:

$$u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$$

How do we find mixed Nash equilibria?

Intuition

As an example: $s_1 = (\frac{1}{10}, \frac{9}{10})$ and $s_2 = (\frac{1}{5}, \frac{4}{5})$

$s = (s_1, s_2)$

- We are looking for a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ of mixed strategies
- Say P2 randomizes among its actions (they play a mixed strategy)
- Then at equilibrium P1 needs to be **indifferent** among its actions, given P2's strategy
- Otherwise, if P1 gets more expected utility from one of the actions, then P1 can switch to playing that action all the time
- For a mixed profile to be a NE there must be no profitable deviations
- Same for P2, if P1 randomizes

	q Left	$(1-q)$ Right
p Left	(0,1)	(3,0)
$(1-p)$ Right	(4,0)	(0,2)

For example, if P2 plays Left and Right each 50% of the time $(q = \frac{1}{2})$, then it makes sense for P1 to always play Right…

But, then it makes sense for P2 to always play Right…

(…)

Finding Mixed Nash: Calculation

- We are looking for a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ of mixed strategies
- Intuition: At equilibrium, both players are indifferent

	q Left	$(1-q)$ Right
p Left	(0,1)	(3,0)
$(1-p)$ Right	(4,0)	(0,2)

P1's choice of p: Make P2 indifferent between L and R

$$u_2(s_1, L) = u_2(s_1, R)$$

$$u_2(L, L) \cdot s_1(L) + u_2(R, L) \cdot s_1(R) = u_2(L, R) \cdot s_1(L) + u_2(R, R) \cdot s_1(R)$$

$$1 \cdot p + 0 \cdot (1-p) = 0 \cdot p + 2 \cdot (1-p)$$

$$p = 2 - 2p$$

$$3p = 2$$

$$p = \frac{2}{3}$$

Finding Mixed Nash: Calculation

- We are looking for a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ of mixed strategies
- Intuition: At equilibrium, both players are indifferent

P2's choice of q: Make P1 indifferent between L and R

	q Left	$(1-q)$ Right
p Left	(0,1)	(3,0)
$(1-p)$ Right	(4,0)	(0,2)

$$u_1(L, s_2) = u_1(R, s_2)$$

$$u_1(L, L) \cdot s_2(L) + u_1(L, R) \cdot s_2(R) = u_1(R, L) \cdot s_2(L) + u_1(R, R) \cdot s_2(R)$$

$$0 \cdot q + 3 \cdot (1-q) = 4 \cdot q + 0 \cdot (1-q)$$

$$3 - 3q = 4q$$

$$7q = 3$$

$$q = \frac{3}{7}$$

Finding Mixed Nash: Calculation

- We are looking for a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ of mixed strategies
- Intuition: At equilibrium, both players are indifferent

	q Left	$(1-q)$ Right
p Left	(0,1)	(3,0)
$(1-p)$ Right	(4,0)	(0,2)

$$p = \frac{2}{3} \qquad q = \frac{3}{7}$$

$$s_1^* = (p, (1-p)) = (\frac{2}{3}, \frac{1}{3}) \qquad s_2^* = (q, (1-q)) = (\frac{3}{7}, \frac{4}{7})$$

Mixed Nash equilibrium: $s^* = (s_1^*, s_2^*)$

Matching Pennies

- Two players
- Each has a penny and secretly turns it to heads or tails
- Players reveal their choices at the same time
- If pennies match, player 1 keeps both
- If they don't, player 2 keeps both

	Heads	Tails
Heads	(1,-1)	(-1,1)
Tails	(-1,1)	(1,-1)

Any pure strategy Nash equilibria?

No

Image: clipartkey.com

Matching Pennies

Let's look at mixed strategies:

	q Heads	$(1-q)$ Tails
p Heads	(1,-1)	(-1,1)
$(1-p)$ Tails	(-1,1)	(1,-1)

P2's choice of q: Make P1 indifferent between H and T

$$u_1(H, s_2) = u_1(T, s_2)$$

$$u_1(H, H) \cdot s_2(H) + u_1(H, T) \cdot s_2(T) = u_1(T, H) \cdot s_2(H) + u_1(T, T) \cdot s_2(T)$$

$$1 \cdot q + (-1) \cdot (1 - q) = (-1) \cdot q + 1 \cdot (1 - q)$$

$$q - 1 + q = -q + 1 - q$$

$$4q = 2$$

$$q = \frac{1}{2}$$

Image: clipartkey.com

Matching Pennies

Let's look at mixed strategies:

	q Heads	$(1-q)$ Tails
p Heads	(1,-1)	(-1,1)
$(1-p)$ Tails	(-1,1)	(1,-1)

P1's choice of p: Make P2 indifferent between H and T

$$u_2(s_1, H) = u_2(s_1, T)$$

$$u_2(H,H) \cdot s_1(H) + u_2(T,H) \cdot s_1(T) = u_2(H,T) \cdot s_1(H) + u_2(T,T) \cdot s_1(T)$$

$$-1 \cdot p + 1 \cdot (1-p) = 1 \cdot p + (-1) \cdot (1-p)$$

$$-p + 1 - p = p - 1 + p$$

$$4p = 2$$

$$p = \frac{1}{2}$$

The moral is: both players should play each action with equal probability, if not, the other player can exploit them…

Image: clipartkey.com

We have seen that pure strategy Nash equilibria do not always exist.

What about Nash equilibria with mixed strategies?

Existence of Mixed Nash Equilibria

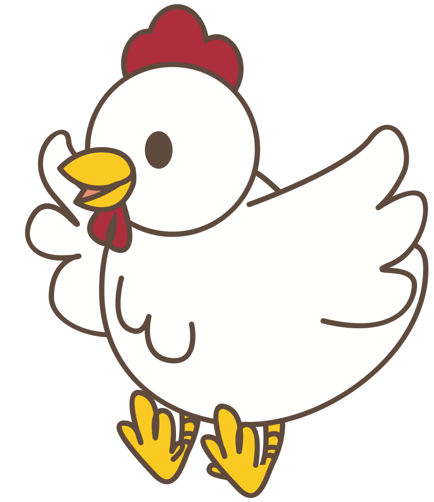
Theorem (Nash, 1951)
Any game with a finite number of players and a finite number of actions has a Nash equilibrium in mixed strategies.

Proof (Idea)

- Define a function f that maps strategy profile s to strategy profile s', such that in s', the actions that are better responses to s get more probability;
- By Brouwer's Fixed Point Theorem, f has at least one fixed point;
- The fixed point of f are Nash equilibria.

Game of Chicken

- Two drivers racing towards each other on a single-lane street, at full speed
- The first to swerve avoids a collision
- Both of them going straight results in a collision



	Swerve	Straight
Swerve	(0,0)	(-1,1)
Straight	(1,-1)	(-100,-100)

Any pure strategy Nash equilibria?

Yes

$$s_1^* = (Swerve, Straight)$$

$$s_2^* = (Straight, Swerve)$$

Any mixed strategy Nash equilibria?

Yes

$$s_3^* = ((0.99, 0.01), (0.99, 0.01))$$

Image: clipartkey.com

Main Takeaways About Nash Equilibria

- For games in normal form we look at, there can be none, or one, or many pure Nash equilibria
- On top of that, there always exists a mixed Nash equilibrium
- Generally, we look for *all* of them

Coming up next: games in extensive form…