



Multi-Agent Systems

Voting Theory

UvA

Mina Young Pedersen
m.y.pedersen@uva.nl

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What is Voting Theory?

- **Voting** is about aggregating, or combining, information across a group
 - Where the goal is to take a decision that concerns the entire group
 - This often involves finding a compromise between conflicting preferences
- In **voting theory**, we take the role as a *designer*:
 - Instead of asking how an agent should or would act (game theory), we ask what rules should be put in place by an authority deciding for a set of agents

Example

- There are 4 Dutchmen, 3 Frenchmen, and 2 Germans who have to decide which drink will be served for lunch
- Only a single drink will be served to all
- The Dutchmen prefer Milk to Beer to Wine
- The Frenchmen prefer Wine to Beer to Milk
- The Germans prefer Beer to Wine to Milk
- Which drink should be served?
 - Hint: There are many different ways to decide!



Other Related Questions

- How to ...
 - ... combine the website rankings of multiple search engines?
 - ... choose a president given people's preferences?

Back to the Example

- Three possible voting rules:
 - **Plurality:** Each agent announces her most *preferred* drink; the winner is the alternative that is announced most often.
 - **Plurality with runoff:** Each agent submits her *full* drink *preference ordering*. Plurality is applied, and if no one has more than half of the votes, only the two front-runners are kept. Then plurality is applied again.
 - **Borda:** Each agent submits her *full* drink *preference ordering*, with her best choice getting 2 points, her next choice getting 1, and her worst choice getting 0 points. Points are added, and the winner is the drink with the most.



Back to the Example

Milk > Beer > Wine (x 4 Dutchmen)

Wine > Beer > Milk (x 3 Frenchmen)

Beer > Wine > Milk (x 2 German)

Plurality: Each agent announces her most *preferred* drink; the winner is the alternative that is announced most often.

Winner: Milk has the most first-place votes.

(Since the 4 Dutchmen has milk as their most preferred drink)



Back to the Example

Milk > Beer > Wine

(x 4 Dutchmen)

Wine > Beer > Milk

(x 3 Frenchmen)

Beer > Wine > Milk

(x 2 German)

Plurality with runoff: Each agent submits her *full drink preference ordering*. Plurality is applied, and if no one has more than half of the votes, only the two front-runners are kept. Then plurality is applied again.

Winner: **Wine** wins the second round.

(After the first round, Milk and Wine are kept. In the second round, both the Frenchmen and the German prefer Wine over Milk)



Back to the Example

Milk > Beer > Wine (x 4 Dutchmen)

Wine > Beer > Milk (x 3 Frenchmen)

Beer > Wine > Milk (x 2 German)

Borda: Each agent submits her *full drink preference ordering*, with her best choice getting 2 points, her next choice getting 1, and her worst choice getting 0 points. Points are added, and the winner is the drink with the most.

Winner: Beer gets most points.

(Milk: 8, Wine: 8, Beer: 11)

How to decide?



A Formal Model

Definition

- $N = \{1, \dots, n\}$ is a set of agents, or voters
- $A = \{a, b, c, \dots\}, |A| = m$ is a finite set of alternatives, or candidates
- \succ_i is a preference order of voter i (a linear order on alternatives)
- $L = \{ \succ \mid \succ \text{ is a linear order on } A \}$ is the set of all possible preferences
- $(\succ_1, \dots, \succ_n) \in L^n$ is a preference profile
- $F : L^n \rightarrow 2^A \setminus \emptyset$ is a social choice function
- $F : L^n \rightarrow A$ is a resolute social choice function
- $F : L^n \rightarrow L$ is a social welfare function

Note on Voting Rules

- Many voting rules can sometimes result in ties
- If we need to avoid this, we can use some tie-breaking rule, or, sometimes, assume that the number of voters is odd

First Voting Rule: Majority

Definition (Majority)

Every agent reports their top choice. The winners are the alternatives that get at least half of the votes.

Majority

- **Used:** Whenever a group needs to choose between two alternatives a and b
- **Pros:** Simple, intuitive, works perfectly when $|A| = 2$ and n is odd
- **Cons:** Doesn't always deliver an answer for more than two alternatives

55 voters prefer a over b

55	54
a	b
b	a

Majority winner? a

34	33	33
a	b	c
b	c	a
c	a	b

Majority winner? ??

Next Voting Rule: Plurality

Definition (Plurality*)

Every agent reports their top choice. The winners are the alternatives that get the most votes.

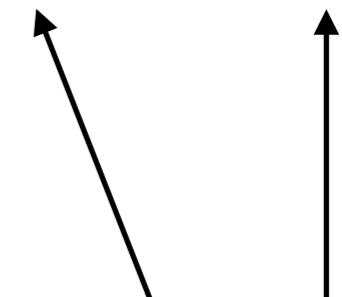
*Also known as *first-past-the-post (FPTP)*

Plurality

- **Used:** To elect representatives (e.g., members of parliament, mayors) in the UK, the US, Canada, India
 - All in all, for local and national elections in 43 of the 193 countries of the United Nations
- **Pros:** Simple, works for any number of alternatives
- **Cons:** May produce very bad results

Plurality winner? *a*

34	33	33
<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>c</i>	<i>b</i>
<i>c</i>	<i>a</i>	<i>a</i>



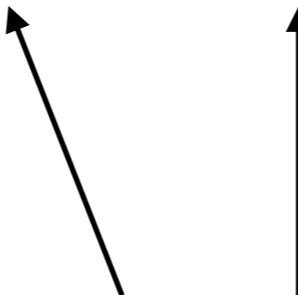
But *a* is hated by two thirds of the electorate!

Plurality

Plurality winner? *a*

b and *c* split the vote
against *a*

34	33	33
<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>c</i>	<i>b</i>
<i>c</i>	<i>a</i>	<i>a</i>



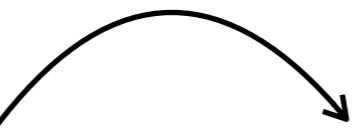
But *a* is hated by two
thirds of the electorate!

This could encourage tactical voting!

Plurality

changing the ballot

34	33	31	2
<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>
<i>b</i>	<i>c</i>	<i>b</i>	<i>c</i>
<i>c</i>	<i>a</i>	<i>a</i>	<i>a</i>



these are voters who still think that *c* is better than *b* and *b* is better than *a*, but they report something else to manipulate the result

Plurality winner? *b*

This could encourage tactical voting!

More on Plurality

- A similar thing happens if the population is divided into districts
- Suppose a , b , and c are political parties and each district gets a seat on the city council

More on Plurality

West

8	7	5
a	b	c
b	c	b
c	a	a

Zuid

8	7	5
a	b	c
b	c	b
c	a	a

Oost

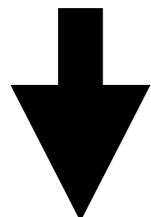
8	7	5
a	b	c
b	c	b
c	a	a

Centrum

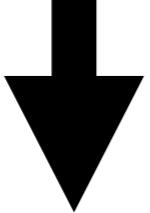
8	7	5
a	b	c
b	c	b
c	a	a

Noord

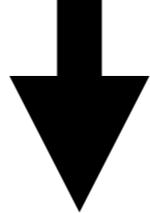
8	7	5
a	b	c
b	c	b
c	a	a



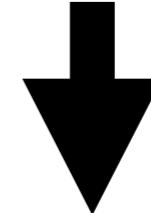
a



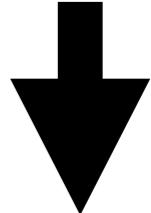
a



a



a



a

With plurality, political party *a* gets all five seats, despite being ranked last by 60% of the voters!

Duverger's Law

8	7	5
a	b	c
b	c	b
c	a	a

- In the long run, c loses support as more and more voters migrate towards b
- In general, smaller parties get squeezed out of power
 - Because no one bothers to vote for them
 - Because they don't win anyway
- A single-ballot plurality-rule election structured within single-member districts tends to favor a two party system
- This is known as **Duverger's Law**

Plurality with Runoff

Definition (Plurality with Runoff)

Every agent reports their top choice. If there is a candidate that gets a majority of the votes, they are declared the winner. If not, hold an **extra round** of voting between the two candidates that get the most votes. The majority winner at this round is declared the winner.

Plurality with Runoff

40	35	25
<i>a</i>	<i>b</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>b</i>
<i>b</i>	<i>a</i>	<i>a</i>

Plurality w. runoff winner?

No majority winner. We go to round 2

40	35	25
<i>a</i>	<i>b</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>b</i>
<i>b</i>	<i>a</i>	<i>a</i>

Plurality w. runoff winner?

b

Plurality with Runoff

- **Used:** To elect presidents in France, Romania, etc.
- **Pros:** Reduces the need for tactical voting.
- **Cons:** Though does not eliminate it: sometimes voters have to lie to make sure a preferred alternative makes it to the second round
 - And may still produce very bad results

40	35	25
a	b	c
c	c	b
b	a	a

40	35	25
a	b	c
c	c	b
b	a	a



65% prefer c to b

Plurality w. runoff
winner? b

More on Plurality

- Turns out plurality, a super-popular voting rule, is not very good
- It allows for alternatives to get elected even when there is some other alternative that a majority thinks is better
 - And having a runoff does not fix the problem

The Condorcet Rule

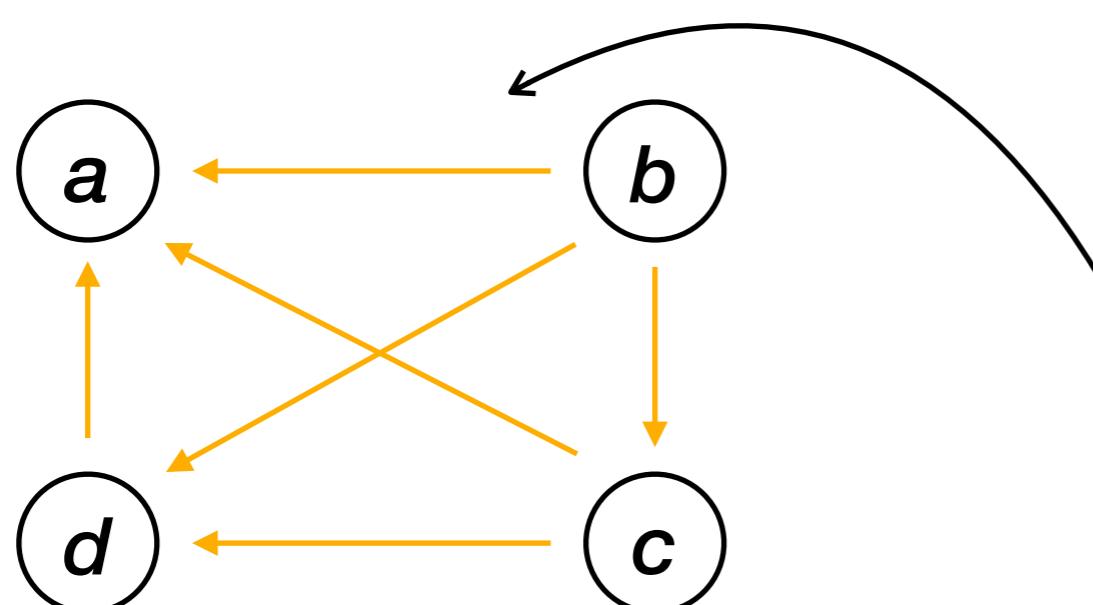
Definition (The Condorcet Rule)

Winners are those alternatives that beat every other alternative in a **head-to-head contest**.

More concretely, we write $n(x, y)$ for the number of agents who prefer alternative x to alternative y . Then, a **Condorcet winner** is an alternative x^* such that $n(x^*, y) > n(y, x^*)$, for any other alternative y .

The Condorcet Rule

4	3	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a



number of agents
who prefer a to b

$$n(a, b) = 4$$

$$n(b, a) = 9$$

$$n(b, c) = 7$$

$$n(c, b) = 6$$

$$n(b, d) = 7$$

$$n(d, b) = 6 \quad (\dots)$$

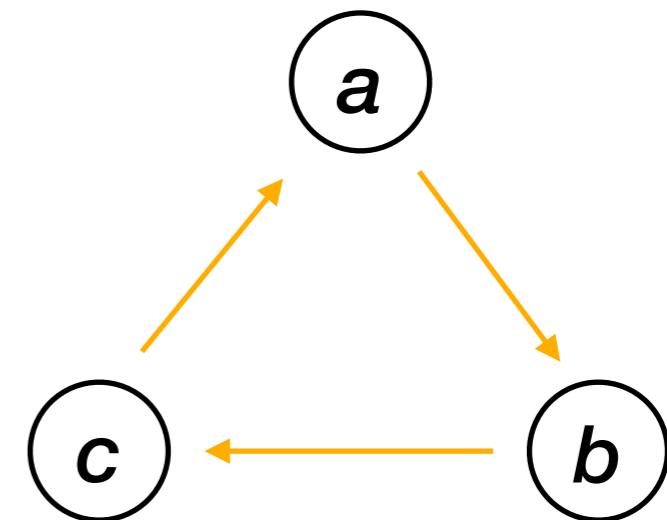
More people prefer b
to a than the other
way around

Condorcet winner? b

The Condorcet Rule

- **Used:** In this form, nowhere (will see in a moment why)
- **Pros:** Makes sense
- **Cons:** Does not always exist!

1	1	1
a	b	c
b	c	a
c	a	b



Condorcet winner?
??

Condorcet Consistent

Definition (Condorcet Consistent Rules)

A voting rule is **Condorcet consistent** if it selects the Condorcet winner, when it exists.

More on Condorcet Consistency

- The Condorcet method needs to be tweaked to make sure it returns an answer when a Condorcet winner does not exist
- As such, Condorcet consistent rules are used in practice
- The Wikimedia Foundation used a Condorcet consistent rule called the *Schulze method* to elect its Board of Trustees until 2013
- The Pirate Party of Sweden uses the Schulze method for its primaries
- The Debian project uses the Schulze method for internal referendums and to elect its leader

What Now?

- Looking at head-to-head contests does not always result in a meaningful ranking
 - As there can be majority cycles
- One suggestion is that we are not using all the information available in the profile...

The Borda Rule

Definition (The Borda Rule)

Every voter i gives a score of $m - pos_i(x)$, called the **Borda score**, to alternative x , where $pos_i(x) \in \{1, \dots, m\}$ is the position of x in i 's preference order.

The **Borda winners** are the alternatives that maximize the sum of the Borda scores.

The Borda Rule

	4	3	3	3
3	a	b	c	d
2	b	c	d	c
1	c	d	b	b
0	d	a	a	a

Borda scores:

$$a : 3 \times 4 = 12$$

$$b : 3 \times 3 + 4 \times 2 + 6 \times 1 = 23$$

$$c : 3 \times 3 + 6 \times 2 + 4 \times 1 = 25$$

$$d : 3 \times 3 + 3 \times 2 + 3 \times 1 = 18$$

Borda winner?

c

The Borda Rule

- **Used:** In the National Assembly of Slovenia, Icelandic parliamentary elections
- **Pros:** Borda winners always exist
- **Cons:** Sensitive to the introduction/removal of irrelevant alternatives

Borda winner?

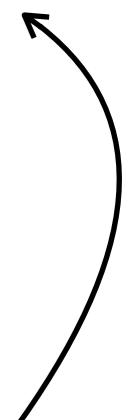
b

35	33	32
a	b	c
c	a	b
b	c	a

Borda winner?
a

35	33	32
a	b	c
c	a	b
b	c	a

Candidate c, who has no chance of winning, acts as a spoiler for b



The Borda Rule

- **Used:** In the National Assembly of Slovenia, Icelandic parliamentary elections
- **Pros:** Borda winners always exist
- **Cons:** Sensitive to the introduction/removal of irrelevant alternatives

And it can be manipulated by strategic agents

1	1	1
b	b	a
a	a	b
c	c	c
d	d	d

Borda winner? b
Changing the ballot

1	1	1	1
b	b	a	a
a	a	b	c
c	c	c	d
d	d	d	b

Borda winner? a
By pushing b down in their ranking, the voter makes a win, to their advantage

Scoring Rules

- The Borda rule is one instance of a broader class of rules: **scoring rules**

Definition (Scoring Rule)

A **scoring rule** uses a scoring vector $s = (s_1, \dots, s_n)$, with $s_1 \geq \dots \geq s_n$ and $s_1 > s_n$ to assign score s_j to the candidate in position j of voter i 's ranking.

For every alternative, we add up the score across all voters.

The **winners** are the alternatives with the highest overall score.

Scoring Rules

- For the Borda rule, the scores are:
 $s = (m - 1, m - 2, \dots, 0)$
- For plurality: $s = (1, 0, \dots, 0)$
- In Nauru, they use the Dowdall scoring vector:
 $s = (1, \frac{1}{2}, \dots, \frac{1}{m})$
- In Eurovision, they use the scores:
 $s = (12, 10, 8, 7, 6, 5, 4, 3, 2, 1)$

Single Transferable Vote

Definition (Single Transferable Vote, or STV*)

The rule proceeds in rounds. If an alternative is selected by majority, it wins. If not, the alternative that shows up on top least often is eliminated (if several, use a tie-breaking rule). Repeat the procedure.

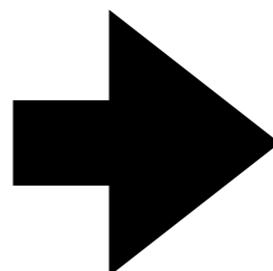
The **STV winner** is the last alternative left standing.

*Also known as *instant run-off voting*, *Alternative Vote (AV)*, or, *ranked choice voting*

Single Transferable Vote

Round 0			
4	2	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

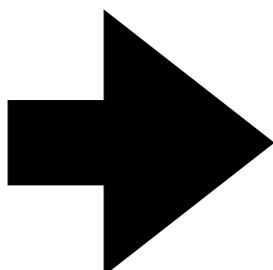
Eliminate b



Round 1			
4	2	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

Round 2

Eliminate d



4	2	3	3
a	b	c	d
b	c	d	c
c	d	b	b
d	a	a	a

STV winner?

c

Single Transferable Vote

- **Used:** To elect members of the Australian House of Representatives, the President of India, the President of Ireland, governors in Alaska
 - To choose the Academy Award for Best Picture
- **Pros:** Less vulnerable to tactical voting
- **Cons:** Not Condorcet consistent
 - Somewhat complicated to get across to people

Approval Voting

Definition (Approval Voting)

Ballots are subsets of alternatives: the ones voters approve of.

Approval winners are the alternatives that have the most approvals.

Approval Voting

- **Used:** In municipal elections in Fargo, North Dakota and St Louis, Missouri
- **Pros:** Is not vulnerable to tactical voting
 - Strikes a good balance between expressiveness and difficulty, prevents minor party candidates from being spoilers

	a	b	c	d	e
voter 1	✓		✓	✓	
voter 2				✓	
voter 3	✓	✓			
voter 4				✓	✓
voter 5		✓			

Approval winner?
d

**There are many other voting
rules out there!**

**Coming up next: What voting
rule should we use?**