Exercise 4 from Homework 5

Modal Logic

December 2025

Exercise 1. Prove that a modal frame F is symmetric if and only if $F \Vdash p \to \Box\Diamond p$. Recall that a frame is symmetric if and only if for all $w, v \in W$: if Rwv, then Rvw. 2pt

Solution 1.

Since this is an *if and only if*-statement, we need to prove both directions.

($\Rightarrow$) Let $F = (W, R)$ be a symmetric frame. We want to prove that $F \Vdash p \to \Box\Diamond p$. Fix an arbitrary valuation V on F, and an arbitrary $w \in W$. For simplicity, we call $M = (F, V)$. Assume that $M, w \Vdash p$. We want to prove that $M, w \Vdash \Box\Diamond p$. Fix an arbitrary $v \in W$ such that Rwv. By symmetry, we know that Rvw. Now, since Rvw and $M, w \Vdash p$, by the semantics, we have that $M, v \Vdash \Diamond p$. Since we fixed arbitrary $v \in W$ such that Rwv, we have that $M, w \Vdash \Box\Diamond p$. Thus, $M, w \Vdash p \to \Box\Diamond p$, and since we fixed an arbitrary V and w, we conclude that $F \Vdash p \to \Box\Diamond p$.

($\Leftarrow$) We prove this direction by contraposition. Let $F = (W, R)$ be a non-symmetric frame. We want to prove that $F \nVdash p \to \Box\Diamond p$. We want to show that this is the case for all non-symmetric frames. We therefore cannot make any more assumptions about F than that it is not symmetric. But from this fact alone, we know that there exists $w, v \in W$ such that Rwv and **not** Rvw. Now, fix a valuation V on F such that $V(p) = \{w\}$. For simplicity, we call $M = (F, V)$. It follows that $M, w \Vdash p$. Since it is not the case that Rvw, and p is only true in the model at w, we have that $M, v \nVdash \Diamond p$. Thus, by the semantics of the $\Box$-operator, since Rwv, it must be the case that $M, w \nVdash \Box\Diamond p$. Hence, we have shown that $M, w \nVdash p \to \Box\Diamond p$. Since we did not make any other assumptions about F except that it is non-symmetric (it is an arbitrary non-symmetric frame), it follows that $F \nVdash p \to \Box\Diamond p$ when F is non-symmetric.