Homework 3

Voting

Due: November 21, 2025, by 23:59

Exercise 1. We're on the same street as the one from a previous homework assignment, except that now there are five food-trucks (a, b, c, d and e) and their owners have decided to park them at the positions depicted in Figure 1. We also consider three inhabitants on this street (v_1, v_2 and v_3), who live at the positions depicted in Figure 1.

The inhabitants' preferences over the food-trucks are determined by how far the trucks are from where they themselves live: closer is better. Thus, a hypothetical inhabitant v_i living at position 0 would prefer a (the closest food-truck) to b (the second-closest) to c, etc. And we would write this inhabitants' preference order as $abcde$, which is short for $a \succ_i b \succ_i c \succ_i d \succ_i e$.

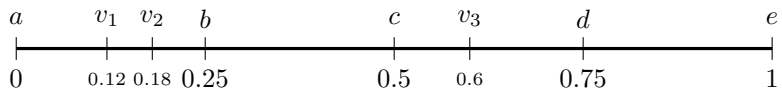

Figure 1: Five food-trucks and inhabitants that prefer the ones that are closer

The three inhabitants are going to throw a party, so they want to order food from one food-truck. Which one? They decide to aggregate their preferences by voting.

(1) Write down the preference orders $\succ_1$, $\succ_2$ and $\succ_3$ of v_1, v_2 and v_3, respectively. 1pt

(2) We now have the profile $\boldsymbol{R} = (\succ_1, \succ_2, \succ_3)$. Recall that a Condorcet winner is a candidate that defeats every other candidate in a head-to-head election. Does a Condorcet winner with respect to profile $\boldsymbol{R}$ exist? If yes, what is it? Explain your answer. 1pt

(3) Recall that the Borda rule is a scoring function that assigns points to each candidate according to their position in the preference orders of the voters (lowest point being 0). For every candidate (i.e., food-truck) compute its Borda score with respect to $\boldsymbol{R}$, i.e., the sum of the Borda scores in every individual preference.

The Borda winners are the candidates with the most overall points (we allow tied winners here). What are the Borda winner with respect to $\boldsymbol{R}$? What is the order on alternatives induced by these scores?[1] 1pt

(4) Suppose voters v_2 and v_3 keep their ballots fixed but voter 1 is strategic, in the sense that they can submit a ranking that is different from their true one (the one computed at (1)). In this case, can voter v_1 succesfully manipulate with respect to the Borda rule?

To be clear, what we are looking for is a situation in which voter v_1 submits a preference $\succ_1'$ that is different from their true one $\succ_1$, and in doing so they change the Borda winners computed at (3). We consider the manipulation successful if the new Borda winners include at least *one* alternative that voter v_1 thinks is better[2] than all the original Borda winners. 1pt

(5) Can voter v_1 manipulate with respect to the Condorcet winner? That is, can v_1 lie about their preference (while everyone else keeps their votes the same), and change the Condorcet winner to an alternative that v_1 prefers over the one obtained at (2)? 1pt

Exercise 2. The *Baldwin* voting rule is a variation of single-transferable vote (STV) where at each step the candidate with the lowest Borda score is eliminated. Analogously, (a modification of the) *Coombs* voting rule is the variant of STV where at each step the candidate ranked last most often is eliminated.

Consider the following profile for 100 voters and 5 alternatives $A = \{a, b, c, d, e\}$:

16 voters	$e \succ c \succ b \succ a \succ d$
20 voters	$b \succ d \succ c \succ e \succ a$
19 voters	$d \succ c \succ e \succ b \succ a$
31 voters	$a \succ d \succ c \succ e \succ b$
14 voters	$c \succ e \succ d \succ b \succ a$

(1) Compute the winner of the *Baldwin* social choice rule. Justify your answer. 1pt

(2) Compute the winner of the *Coombs* social choice rule. Justify your answer. 1pt

Exercise 3. Recall the Anonymity and Neutrality axioms: Anonymity stipulates that the result of a voting rule does not depend on the order of the preferences in the profile—or, put differently, no matter what the order of the preferences, the result is the same. Neutrality is different: it says that if we replace, say, alternative a with b everywhere in the profile, then a should be replaced with b in the result.

Recall, as well, that a resolute social choice function gives a *unique* candidate as the output. We assume that the input consists of linear orders.

[1] That is, the order got by putting the alternatives with the most points on top, followed by the alternatives with the second-highest scores, etc.

[2] Better relative to the true preference of voter v_1.

(1) Consider a setting where with two voters $N = \{1, 2\}$ and two alternatives $A = \{a, b\}$. Provide a formal argument to show that Anonymity and Neutrality are incompatible, i.e., there is no resolute social choice function that satisfies both properties. 1pt

(2) Consider, now, a social choice function F (i.e., a function that returns a *non-empty set* of alternatives, the tied winners), and the Condorcet-cycle profile $\boldsymbol{R} = (abc, bca, cab)$, for three voters and three alternatives.[3] Using only the Anonymity and Neutrality axioms, provide an argument for what $F(\boldsymbol{R})$ should be.

Hint: start from the definition of a social choice function, assume that some alternative is in it and see what follows from applying an appropriately defined permutation. 1pt

(3) Consider the Dictatorship function as a resolute social choice function, i.e., the rule that returns exactly one alternative: the top choice of some voter $i \in N$, the designated dictator. Show that this rule is strategyproof, i.e., that no voter can become (strictly) better off by lying about their preferences. Provide a rigorous argument. 1pt

[3] For clarity: abc on position i is the order $a \succ_i b \succ_i c$ of voter i.