UvA

Multi-Agent Systems

Voting Theory

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Which Voting Rules?

- We need some general principles to distinguish between voting rules
  - Let’s take the point of view of someone who wants to design a voting rule from scratch, and think about what properties, or **axioms**, we would want the voting rule to satisfy

Recall: A Formal Model

Definition

- $N = \{1,\ldots,n\}$ is a set of agents, or voters
- $A = \{a, b, c, \ldots\}, |A| = m$ is a finite set of alternatives, or candidates
- $\succ_i$ is a preference order of voter i (a linear order on alternatives)
- $L = \{ \succ \mid \succ$ is a linear order on $A\}$ is the set of all possible preferences
- $R = (\succ_1, \ldots, \succ_n) \in L^n$ is a preference profile
- $F : L^n \to 2^A \backslash \varnothing$ is a social choice function
- $F : L^n \to A$ is a resolute social choice function
- $F : L^n \to L$ is a social welfare function

Anonymity

- The first axiom we look at is **anonymity**
- It says that the order in which we arrange the voters does not matter for the final result

Definition (Anonymity Axiom)

A voting rule F satisfies **anonymity** if, for any permutation σ of the set N of voters, it holds that:

$$F(\succ_1, \ldots, \succ_n) = F(\succ_{\sigma(1)}, \ldots, \succ_{\sigma(n)})$$

Anonymity

- **Anonymity** requires invariance under permutations of the voters in the profile

If

Ann	Bob	Cat
a	*a*	*b*
b	*c*	*a*
c	*b*	*c*

$\xrightarrow{F}$ ***a***

Permutation here is
$\sigma(\text{Ann}) = \text{Bob}$,
$\sigma(\text{Bob}) = \text{Cat}$,
$\sigma(\text{Cat}) = \text{Ann}$

Then

Bob	Cat	Ann
a	*b*	*a*
c	*a*	*b*
b	*c*	*c*

$\xrightarrow{F}$ ***a***

Neutrality

- The next property is **neutrality**
- It says that the names we give to alternatives do not matter

Definition (Neutrality Axiom)

A voting rule F satisfies **neutrality** if, for any permutation σ of the set A of alternatives, it holds that:

$$\sigma(F(\succ_1, \ldots, \succ_n)) = F(\sigma(\succ_1), \ldots, \sigma(\succ_n))$$

Every alternative is replaced with its image under σ

Neutrality

- **Neutrality** requires that permutations of the alternatives in the profile are reflected by permutations of the alternatives in the result

If

Ann	Bob	Cat
a	*a*	*b*
b	*c*	*a*
c	*b*	*c*

$\xrightarrow{F}$ ***a***

Permutation here is $\sigma(a) = b$, $\sigma(b) = c$, $\sigma(c) = a$

Then

Ann	Bob	Cat
b	*b*	*c*
c	*a*	*b*
a	*c*	*a*

$\xrightarrow{F}$ ***b***

Positive Responsiveness

- Now, we look at something a bit more involved: **positive responsiveness**
- It says, roughly, that increased support for some alternative has the power to break a tie in favor of that alternative

Definition (Positive Responsiveness Axiom)

A social choice function F satisfies positive responsiveness if, for any distinct profiles R and R', and alternative $x*$, we have that R and R' are the same except that in R' some voters move $x*$ up some positions in their preference rankings, then it holds that if $x* \in F(R)$, then $F(R') = \{x*\}$.

Positive Responsiveness

- If in R' some voters raise x^*, while leaving everything else untouched, then x^* goes from being a (possibly tied) winner to the unique winner

If

Ann	Bob	Cat	Dov
a	*a*	*b*	*b*
b	*b*	*a*	*a*

F → ***{a, b}***

Then

Ann	Bob	Cat	Dov
a	*a*	*a*	*b*
b	*b*	*b*	*a*

F → ***{a}***

May’s Theorem

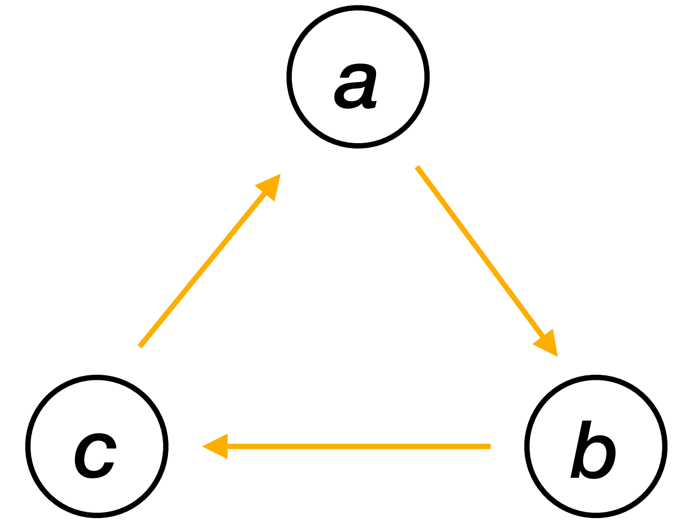
- For two alternatives, it turns out that these properties are satisfied only by the majority voting rule
  - Note that when there are only two alternatives, the majority rule is well-defined

Theorem (May, 1952)
If there are only two alternatives, then the only social choice function that satisfies anonymity, neutrality, and positive responsiveness is the majority rule.

More on May's Theorem

- For two alternatives, we cannot do better than using majority
    - And note that when there are only two alternatives, all the voting rules we have looked at so far are equivalent to the majority rule
    - Now for more than two alternatives…
- We know that majority comparisons can get us into troubles with cycles
    - But maybe there is some other clever way to combine preferences into a coherent social ranking?

1	**1**	**1**
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*



Reasonable Properties of Voting Rules

- For the next results we will focus on **social welfare functions**: voting rules that return a raking of the alternatives
- Let’s write down some more reasonable properties…

Pareto Efficiency

- Pareto efficiency captures the intuition that if everyone thinks some alternative is better than another, then this should be reflected in the result

Definition (Pareto Efficiency Axiom)

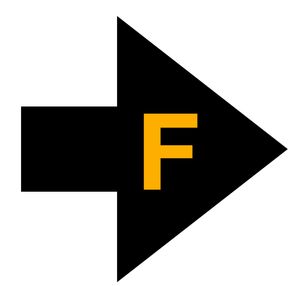
A social welfare function F satisfies **Pareto efficiency** if, for any alternatives x and y, it holds that if $x \succ_i y$, for every voter $i \in N$, then $x \succ_{F(R)} y$.

Society's ranking

Pareto Efficiency

- If there is unanimous agreement that x is better than y, then x is ranked above y in the aggregated ranking

$\succ_1$	$\succ_2$	$\dots$	$\succ_n$
			x
x	x		
			y
y			
	y		



$\succ_{F(R)}$
x
y

Independence of Irrelevant Alternatives

- Society's ranking between two alternatives x and y should depend on how voters in the profile rank x and y… and nothing else

Definition (Independence of Irrelevant Alternatives, or IIA, Axiom)
A social welfare function F satisfies **independence of irrelevant alternatives (IIA)** if, for any alternatives x and y, and profiles R, R' such that for any agent $i \in N$ it holds that $x \succ_i y$ if and only if $x \succ'_i y$, then it holds that $x \succ_{F(R)} y$ if and only if $x \succ_{F(R')} y$.

Independence of Irrelevant Alternatives

- If voters rank x and y in the same way across the two profiles, then the final ranking between x and y is the same for both profiles

If

$\succ_1$	$\succ_2$	$\dots$	$\succ_n$
x	x		y
y			
	y		x

F

$\succ_{F(R)}$
y
x

Then

$\succ'_1$	$\succ'_2$	$\dots$	$\succ'_n$
x	x		y
y			
	y		x

F

$\succ_{F(R')}$
y
x

Non-Dictatorship

- Non-dictatorship is about making sure that there is no one voter who has a final say, regardless of the preferences of the other voters

Definition (Dictator)

An agent $i \in N$ is a **dictator** for a social welfare function F if, for any alternatives x and y, and profile R, it holds that if $x \succ_i y$, then $x \succ_{F(R)} y$.

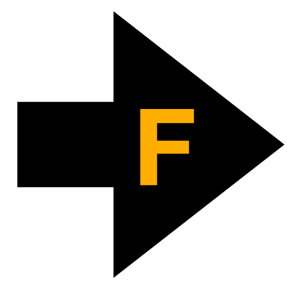
Definition (Non-Dictatorship Axiom)

A social welfare function F satisfies **non-dictatorship** if no agent is a dictator.

Dictatorship

- A dictator decides the final ranking of every pair of alternatives, and thus the full final ranking

Ann	Bob	Cat
a	*a*	*b*
b	*c*	*a*
c	*b*	*c*



$\succ_{F(R)}$
a
b
c

Designated dictator

Arrow's Theorem

- These properties seem reasonable enough. But it turns out that, together, they spell trouble…

Theorem (Arrow, 1951)

If there are at least three alternatives, then any social welfare function that satisfies **Pareto efficiency** and **independence of irrelevant alternatives** is a **dictatorship.**

What now?

Let's look closer at another type of strategy: lying…

Lying

- We've seen that many voting rules are afflicted by a common problem: they create incentives for voters to lie about their preferences
- For now, we return to resolute social choice functions $F : L^n \to A$, where F outputs a single candidate
- Recall…

Plurality

- Under plurality, voters don't want to support a losing candidate

49	48	3
a	*b*	*c*
b	*c*	*b*
c	*a*	*a*

Plurality → *a*

Submitting insincere ballot

49	48	3
a	*b*	*b*
b	*c*	*c*
c	*a*	*a*

Plurality → *b*

Borda

- Under Borda, voters can manipulate by pushing alternatives they don't like down their list

2	**1**
b	*a*
a	*b*
c	*c*
d	*d*

Borda → *b*

Submitting insincere ballot

2	**1**
b	*a*
a	*c*
c	*d*
d	*b*

Borda → *a*

It would be great if truthfulness turned out to be a dominant strategy…

Strategyproofness

Definition (Strategyproofness)

A resolute social choice function F is **strategyproof** if for all voters $i \in N$ it holds that there does not exist a profile R (containing i's truthful preference) and some order $\succ_i'$ (representing some untruthful preference of i) such that:

$$F(\succ_i', R_{-i}) \succ_i F(R)$$

profile without i

profile with i's truthful preference

In other words, it is not possible for any voter i to get a better result by submitting a preference order $\succ_i'$ instead of its true preference order $\succ_i$.

Can we design strategyproof voting rules?

Dictatorship

Definition (Dictatorship)

Choose an agent $i \in N$, called **the dictator.** The winner is the top choice of the dictator.

Dictatorship

- Under dictatorship there is no point in manipulating

Dictator

Ann	Bob	Cat
a	*a*	*c*
b	*c*	*a*
c	*b*	*b*

Dictatorship → *a*

Dictator

Ann	Bob	Cat
a	*a*	*b*
b	*c*	*a*
c	*b*	*c*

Submitting insincere ballot

Dictatorship → *a*

Another Impossibility Result!

Theorem (Gibbard-Satterthwaite, 1973, 1975)

If a resolute social choice function F has at least three possible outcomes, then F is strategyproof if and only if it is a dictatorship.

What Now?

- What can we take from the impossibility theorems?
  - Certain intuitive desirable properties are incompatible
  - Illustrates that there are trade-offs when designing a voting system
  - Democracy is mathematically limited
  - It does not mean that voting is pointless, but we must decide what kind of unfairness we can live with