# Generalised semi-tensor product of matrices

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## Body

**1**

**Introduction**

The standard semi-tensor product (STP) of matrices was first proposed about two decades ago by [1]. Since then a great deal of research effort in control community has been devoted to the study of STP and its applications, including power systems [2–4], Boolean (control) networks [5–7], finite games [8–10], just to mention a few. We refer to [11] for a general introduction to STP. Some recent survey papers may help to grasp the basic concepts and current situations, for instance, [12–14].

First, we briefly review STP, which is called the standard STP in this paper. In the following

$m \vee n$

is used for the least common multiplier of *m* and *n*, and

$m \wedge n$

for the greatest common divisor. We also use

$\mathcal{M}_{m \times n}$

for the set of (real)

$m \times n$

matrices.

Definition 1

Let

$A \in \mathcal{M}_{m \times n}$

,

$B \in \mathcal{M}_{p\times q}$
,
$t = n \vee p$
. Then the (left) STP of matrices is defined as (1)
$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right),$$
where
$\otimes$
is the Kronecker product of matrices [11].

It is obvious that the STP is a generalisation of classical matrix product. Fortunately, it keeps all major properties of classical matrix product available. Moreover, since it allows an expansion of dimensions, which makes it easier to manipulate the matrix entries. This capability brings some new properties, such as dimension-free, pseudo-commutativity, equivalence relation, etc. These nice properties make it very powerful in dealing with multi-linear mappings, dynamics over finite objects, dimension-varying systems, etc.

The purpose of this paper is to explore more general STPs, which helps to reveal some fundamental insides and to expand the possible applications of STP. As an immediate application, they are used to construct cross-dimensional linear dynamic systems. The main idea for this generalisation is described as follows:

Observing (1), one sees easily that a set of square matrices
$$I := \{I_n \mid n = 1, 2, \ldots\}$$
is used as the ‘matrix multiplier’ to make the two factor matrices meet the dimension matching condition required by classical matrix product. A natural question is: can we find another set of square matrices to replace ***I***

Assume we have
$$\Gamma_n :\in \mathcal{M}_{n\times n}, \quad n = 1, 2, \ldots.$$
Using them, we can formally define a new STP as: (2)
$$A \ltimes_\Gamma B := \left(A \otimes \Gamma_{t/n}\right)\left(B \otimes \Gamma_{t/p}\right).$$

Finding certain conditions to make (2) meaningful yields a generalised matrix–matrix (MM) STP. That is the basic idea for producing new MM-STPs.

It is obvious that an MM-STP can be used to a matrix and a vector. However, in general, the product of a matrix with a vector is considered as a linear mapping on vector space. MM-STP is not suitable for this because the result of MM-STP of a matrix with a vector may not be a vector. Hence we need to explore a matrix–vector (MV) STP, which is considered as a linear mapping from vector to vector within a vector space.

Then, we are ready to consider the action of arbitrary dimensional matrices on arbitrary dimensional vectors, which leads to a cross-dimensional linear semi-group (S) system, called the (cross dimensional) generalised linear S-system.

Next, we consider matrix equivalence and vector equivalence, caused by the matrix multiplier and vector multiplier, respectively. The generalised linear systems are also extended to corresponding quotient spaces.

Finally, an inner-product is introduced to the cross-dimensional state space, which poses a metric topology on the state space. In addition, we prove that under this topology the cross-dimensional S-system becomes a dynamic system.

The rest of this paper is organised as follows: Section 2 proposes a concept of matrix multiplier. Using a matrix multiplier, corresponding MM-STP is constructed. Certain properties of generalised MM-STPs are investigated.

Section 3 introduces a concept of vector multiplier. Using both matrix multiplier and vector multiplier, MV-STP is introduced. Set of all matrices, denoted by

$\mathcal{M}$

, with a given MM-STP form a semi-group. Denote the set of various dimensional vectors by

$\mathcal{V}$

. Then a MV-STP can be considered as an action of semi-group

$\mathcal{M}$

on

$\mathcal{V}$

, which yields an S-system. The S-system is discussed in Section 4. When the matrix is fixed, the corresponding S-system becomes a constant (dimension varying) linear S-system, which is discussed in Section 5. Using matrix multiplier, a matrix equivalence is proposed in Section 6. Equivalent classes form a quotient matrix space

$\Sigma$

. Similarly, Section 7 constructs a vector equivalence on

$\mathcal{V}$

, using vector multiplier. The corresponding quotient space is proposed and denoted by

$\Omega$

. In Section 8, the action of

$\Sigma$

on

$\Omega$

is investigated, which produces linear system on quotient space

$\Omega$

. Section 9 introduces a vector space structure on

$\mathcal{V}$

. It is then extended to the quotient space

$\Omega$

. Finally, in Section 10, a generalised inner product is also proposed, which turns

$\mathcal{V}$

to be a topological space. Correspondingly, the inner product and the topological structure are also extended to the quotient space

$\Omega$

. Then, the generalised linear S-systems on both

$\mathcal{V}$

and

$\Omega$

become dynamic systems. Section 10 is a brief conclusion.

## 2

## Matrix multiplier-based MM-STP

Observing (2) again, to make this STP meaningful certain requirements for

$\Gamma$

need to be satisfied. These requirements are

(i) $\ltimes_\Gamma$ is a generalisation of classical matrix product.

(ii) $\ltimes_\Gamma$ is distributive and associative.

To meet these requirements, we give the following definition:

Definition 2

A set of non-zero matrices $\Gamma := \{\Gamma_n \in \mathcal{M}_{n\times n} \mid n \geq 1\}$ is called a matrix multiplier, if it satisfies

(i) (3)
$$\Gamma_n \Gamma_n = \Gamma_n;$$

(ii) (4)
$$\Gamma_p \otimes \Gamma_q = \Gamma_{pq}.$$

Next, we consider some basic properties of matrix multiplier. Hereafter, we use $\mathbf{0}_m$ ( $\mathbf{1}_m$ ) for vector in $\mathbb{R}^m$ with all entries zero (one). use $\mathbf{0}_{m\times n}$ ( $\mathbf{1}_{m\times n}$ ) for matrix in $\mathcal{M}_{m\times n}$ with all entries zero (one).

Proposition 1

The spectrum (set of eigenvalues) of $\Gamma_n$ is either $\sigma(\Gamma_n) = \{1\}$ or $\sigma(\Gamma_n) = \{0, 1\}$. Hence (5)
$$\sigma_{\max}(\Gamma_n) = 1, \quad n = 1, 2, \ldots$$

Proof

From (3) we have that $\Gamma 2_n$-$\Gamma_n$=$\Gamma_n$($\Gamma$-$I_n$)=0. It follows that the eigenvalues of

$\Gamma_n$

can only be either 0 or 1. But

$\Gamma_n \neq \mathbf{0}_{n\times n}$

, hence it has at least one eigenvalue, which is 1. □

As an immediate consequence, we have that

Corollary 1

(6)

$$\Gamma_1 = 1.$$

Definition 3

Assume

$\Gamma = \{\Gamma_n, \mid n \geq 1\}$

is a matrix multiplier,

$A \in \mathcal{M}_{m\times n}$

,

$B \in \mathcal{M}_{p\times q}$

.

(i) The multiplier

$\Gamma$

based left MM-STP of *A* and *B* is defined as (7)

$$A \ltimes_\Gamma B := \left(A \otimes \Gamma_{t/n}\right)\left(B \otimes \Gamma_{t/p}\right).$$

where

$t = n \vee p$

.

(ii) The multiplier

$\Gamma$

based right MM-STP of *A* and *B* is defined as (8)

$$A \rtimes_\Gamma B := \left(\Gamma_{t/n} \otimes A\right)\left(\Gamma_{t/p} \otimes B\right).$$

The generalised MM-product has some basic properties.

Proposition 2

Both left and right MM-STP generated by multiplier

$\Gamma$

are generalisations of classical matrix product.

Proof

It follows from (6) immediately. □

Proposition 3

In the following:

$\bowtie \in \{\ltimes, \rtimes\}$

.

Associativity (9)

$$(A \bowtie_\Gamma B) \bowtie_\Gamma C = A \bowtie_\Gamma (B \bowtie_\Gamma C).$$

Distributivity (10)

$$(A + B) \bowtie_\Gamma C := A \bowtie_\Gamma C + B \bowtie_\Gamma C$$
$$A \bowtie_\Gamma (B + C) = A \bowtie_\Gamma B + A \bowtie_\Gamma C.$$

Transpose (11)

$$(A \bowtie_\Gamma B)^{\mathrm{T}} = B^{\mathrm{T}} \bowtie_\Gamma A^{\mathrm{T}}.$$

Inverse

Assume

$\Gamma_n$

,

$n \geq 1$

are invertible. If *A* and *B* are invertible, then

$A \bowtie_\Gamma B$

is invertible. Moreover, (12)

$$(A \bowtie_\Gamma B)^{-1} = B^{-1} \bowtie_\Gamma A^{-1}.$$

Proof

We prove associativity of

$\ltimes_\Gamma$

only. The others are simple.

Let

$A \in \mathcal{M}_{m \times n}$

,

$B \in \mathcal{M}_{p \times q}$

,

$C \in \mathcal{M}_{r \times s}$

, and denote (13)

$$n \vee p = nn_1 = pp_1, \qquad q \vee r = qq_1 = rr_1,$$
$$r \vee qp_1 = rr_2 = qp_1p_2, \quad n \vee pq_1 = nn_2 = pq_1q_2.$$

Using (3) and (4), the left hand side of equation (9) becomes

$$\begin{aligned}(A \ltimes_\Gamma B) \ltimes_\Gamma C &= ((A \otimes \Gamma_{n_1})(B \otimes \Gamma_{p_1})) \ltimes_\Gamma C \\ &= (((A \otimes \Gamma_{n_1})(B \otimes \Gamma_{p_1})) \otimes \Gamma_{p_2})(C \otimes \Gamma_{r_2}) \\ &= (((A \otimes \Gamma_{n_1})(B \otimes \Gamma_{p_1})) \otimes (\Gamma_{p_2}\Gamma_{p_2}))(C \otimes \Gamma_{r_2}) \\ &= (A \otimes \Gamma_{n_1p_2})(B \otimes \Gamma_{p_1p_2})(C \otimes \Gamma_{r_2}).\end{aligned}$$

Similarly, the right hand side of (9) becomes

$$\begin{aligned} A \ltimes_\Gamma (B \ltimes_\Gamma C) &= A \ltimes_\Gamma \left((B \otimes \Gamma_{q_1})(C \otimes \Gamma_{r_1})\right) \\ &= (A \otimes \Gamma_{n_2}) \left(((B \otimes \Gamma_{q_1})(C \otimes \Gamma_{r_1})) \otimes \Gamma_{q_2}\right) \\ &= (A \otimes \Gamma_{n_2}) \left(B \otimes \Gamma_{q_1 q_2})(C \otimes \Gamma_{r_1 q_2})\right). \end{aligned}$$

Hence, to prove (9) it is enough to show the following three equations: (14)

$$n_1 p_2 = n_2 \qquad \text{(a)}$$
$$p_1 p_2 = q_1 q_2 \qquad \text{(b)}$$
$$r_2 = r_1 q_2 \qquad \text{(c)}$$

Using the associativity of least common multiple [15], three equations in (14) are easily verifiable. (A detailed proof can also be found in [16].) □

The following example provides some matrix multipliers.

Example 1

(i) Assume

$$\Gamma = I := \{I_n\}$$

. It is a matrix multiplier. In fact, this multiplier leads to the standard STP, which is well known as

$$\ltimes_\Gamma = \ltimes; \quad \rtimes_\Gamma = \rtimes.$$

From now on, they are called the first MM-STP, briefly, MM-1 STP.

Set (15)

$$J_n := \frac{1}{n}\mathbf{1}_{n \times n}, \quad n = 1, 2, \dots.$$

It is easy to verify that

$$\Gamma = J := \{J_n \mid n = 1, 2, \dots\}$$

satisfies (3) and (4), hence, it is a matrix multiplier, and it can be used to define a new STP.

Set

$$\Delta_n^U \in \mathcal{M}_{n \times n}$$

as (16)

$$(\Delta_n^U)_{i,j} = \begin{cases} 1, & i = 1, \text{and } j = 1, \\ 0, & \text{Otherwise.} \end{cases}$$

It is easy to verify that

$$\Delta^U := \{\Delta_n^U \mid n = 1, 2, \dots\}$$

satisfies (3) and (4), hence, it is a matrix multiplier.

Set

$$\Delta_n^D \in \mathcal{M}_{n \times n}$$

as (17)

$$(\Delta_n^D)_{i,j} = \begin{cases} 1, & i = n, \text{and } j = n, \\ 0, & \text{Otherwise.} \end{cases}$$

It is easy to verify that

$$\Delta^D := \{\Delta_n^D \mid n = 1, 2, \dots\}$$

satisfies (3) and (4), hence, it is a matrix multiplier.

Remark 1

Assume

$$\{\Gamma_n \mid n = 1, 2, \ldots\}$$

is a multiplier. If there exists a sequence of non-singular matrices

$$\{T_n \mid n = 1, 2, \ldots\}$$

, satisfying (18)

$$T_p \otimes T_q = T_{pq}, \quad p > 0,\ q > 0,$$

then it is easy to verify that

$$\{T_n^{-1}\Gamma_n T_n \mid n = 1, 2, \ldots\}$$

is also a matrix multiplier.

Unfortunately, we did not find such a sequence of non-singular matrices so for.

Using

$$J_n$$

defined by (15), we can define MM-2 STP:

Definition 4

Using

$$\Gamma = J = \{J_n \mid n = 1, 2, \ldots\}$$

, the MM-2 STP is defined as

(i) Left MM-2 STP: (19)

$$A \circ_\ell B := \left(A \otimes J_{t/n}\right)\left(B \otimes J_{t/p}\right).$$

(ii) Right MM-2 STP: (20)

$$A \circ_r B := \left(J_{t/n} \otimes A\right)\left(J_{t/p} \otimes B\right).$$

## 3

## Vector multiplier and MV-STP

**Definition 5**

A non-zero vector sequence

$$\gamma := \{\gamma_r \in \mathbb{R}^r \mid r \geq 1\}$$

is called a vector multiplier, if it satisfies the following:

(i) (21)

$$\gamma_1 = 1;$$

(ii) (22)

$$\gamma_p \otimes \gamma_q = \gamma_{pq}.$$

The following example gives some vector multipliers.

Example 2

(i) (23)

$$\gamma = \mathbf{1} := \{\mathbf{1}_n \mid n = 1, 2, \ldots\}.$$

(ii) (24)

$$\gamma = \delta^U := \left\{ \delta_n^1 \mid n = 1, 2, \dots \right\},$$

where $\delta_n^i$ is the *i*th column of identity matrix $I_n$.

(iii) (25)

$$\gamma = \delta^D := \left\{ \delta_n^n \mid n = 1, 2, \dots \right\}.$$

The following proposition is straightforward verifiable. It provides more vector multipliers.

Proposition 4

If $\gamma = \{\gamma_n \mid n = 1, 2, \dots\}$ is a vector multiplier, then, $\gamma' = \{\gamma'_n \mid n = 1, 2, \dots\}$ is also a vector multiplier,where

$$\gamma'_n = n^k \gamma_n, \quad n = 1, 2, \dots,$$

where *k* is an arbitrary non-negative integer.

Next, we consider MV-STP.

Definition 6

Let $\Gamma$ be a matrix multiplier, $\gamma$ a vector multiplier, $A \in \mathcal{M}_{m \times n}$, $x \in \mathbb{R}^r$, $t = n \vee r$. Then the MV-STP of *A* and *x* related with $\Gamma$ and $\gamma$, denoted by $\vec{\times}$, is defined as

(i) Left MV-STP: (26)

$$A\vec{\times}_\ell x := \left(A \otimes \Gamma_{t/n}\right)\left(x \otimes \gamma_{t/r}\right).$$

(ii) Right MV-STP: (27)

$$A\vec{\times}_r x := \left(\Gamma_{t/n} \otimes A\right)\left(\gamma_{t/r} \otimes x\right).$$

Remark 2

(i) It is well known from Linear Algebra that classical matrix product has basically two functions: (a) matrix

$\times$

matrix, which can be considered as a composition of two linear mappings; (b) matrix

$\times$

vector, which can be considered as realising a linear mapping.

(ii) In conventional case, the classical matrix product can realise these two functions. Unfortunately, when we generalise the classical matrix product to STP, because the dimension restriction has been removed, it is not able to define a 'universal' STP to realise these two functions. We, therefore, need two different kinds of STP.

(iii) MM-STP is used for composition of two linear mappings; MV-STP is used for realising a linear mapping.

(iv) So when we read a matrix *B* as a set of vectors, the MV-STP of *A* with *B* is still not the same as the MM-STP of *A* with *B*. Because they represent different objects.

In the following we define two kinds of useful MV-STPs.

Definition 7

Let

$A \in \mathcal{M}_{m\times n}$

,

$x \in \mathbb{R}^r$

,

$t = n \vee r$

. Then two kinds of MV-STPs are defined as follows:

MV-1 STP:

$$\Gamma = \{I_n \mid n = 1, 2, \ldots\}, \quad \gamma = \{\mathbf{1}_n \mid n = 1, 2, \ldots\}.$$

(i) Left MV-1 STP: (28)

$$A\vec{\ltimes} x := \left(A \otimes I_{t/n}\right)\left(x \otimes \mathbf{1}_{t/r}\right).$$

(ii) Right MV-1 STP: (29)

$$A\vec{\rtimes} x := \left(I_{t/n} \otimes A\right)\left(\mathbf{1}_{t/r} \otimes x\right).$$

MV-2 STP:

$$\Gamma = \{J_n \mid n = 1, 2, \ldots\}, \quad \gamma = \{\mathbf{1}_n \mid n = 1, 2, \ldots\}.$$

(i) Left MV-2 STP: (30)

$$A\vec{\circ}_\ell x := \left(A \otimes J_{t/n}\right)\left(x \otimes \mathbf{1}_{t/r}\right).$$

(ii) Right MV-2 STP: (31)

$$A\vec{\circ}_r x := \left(J_{t/n} \otimes A\right)\left(\mathbf{1}_{t/r} \otimes x\right).$$

Next, we consider the algebraic structure of the set of all matrices under an MM-STP.

Definition 8

Let $G$ be a set and

$$* : G \times G \to G$$

be a binary mapping [17].

(i) Semi-group

$$(G, *)$$

is called a semi-group if (32)

$$(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3).$$

(ii) Monoid (semi-group with identity)

$$(G, *)$$

is a semi-group, and there is an identity element

$$e \in G$$

such that (33)

$$e * g = g * e = g, \quad \forall g \in G.$$

Denote the set of all matrices by

$$\mathcal{M} := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{M}_{m \times n}.$$

Proposition 5

Let

$$\bowtie_\Gamma \in \left\{\ltimes_\Gamma, \rtimes_\Gamma\right\}$$

be the MM-STP generated by matrix multiplier

$$\Gamma$$

. Then

(i)

$$(\mathcal{M}, \ \bowtie_\Gamma)$$

is a semi-group.

(ii) If

$$\Gamma = I := \{I_n \mid n = 1, 2, \ldots\}$$

, then

$$(\mathcal{M}, \ I)$$

is a monoid,where the identity is 1.

Note that if

$$\Gamma = J := \{J_n \mid n = 1, 2, \ldots\}$$

, then

$(\mathcal{M},\ J)$
is not a monoid. It is easy to verify that 1 is not its identity.

## 4

## Cross-dimensional linear S-system

**Definition 9**

Let *G* be a monoid with identity
$e \in G$
, *X* a set, with a mapping
$\varphi:\ G \times X \to X$
.
$(G, \varphi, X)$
is called an S-system, if [18, 19]

(i) (34)
$$\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 \circ g_2, x), \quad g_1, g_2 \in G;\ x \in X.$$

(ii) (35)
$$\varphi(e,\ x) = x, \quad \forall x \in X.$$

Note that if *G* is a semi-group (without identity) and there is a mapping
$\varphi:\ G \times X \to X$
satisfying (34) only, then
$(G, \varphi, X)$
may be called a pseudo S-system. For instance, if
$\Gamma = \{J_n \mid n = 1, 2, \ldots\}$
, then
$(\mathcal{M},\ \times_\Gamma)$
is only a semi-group.

In the following we want to show that a pseudo S-system is essentially the same as an S-system. Precisely speaking, we try to convert a pseudo S-system into an S-system by adding an identity.

Proposition 6

Let
$(G, *)$
be a semi-group without identity. Set
$\bar{G} := \{e\} \cup G$
, where (36)
$$\begin{cases} e * e = e \\ e * g = g * e = g, \quad \forall g \in G, \end{cases}$$
then
$(\bar{G}, *)$

is a monoid.

Proof

We have to only prove the associativity. That is, (37)

$$(a*b)*c = a*(b*c), \quad a,\ b,\ c \in \bar{G}.$$

If

$a,\ b,\ c \in G$

, (37) is obviously true. Now assume some of

$\{a,\ b,\ c\}$

equal to *e*, then an easy one by one verification shows that (37) remains true. □

Corollary 2

Assume *G* is a semi-group without identity, and

$(G,\varphi,X)$

is a pseudo S-system. Then

$(\bar{G},\overline{\varphi},X)$

is an S-system, where

(i)

$\bar{G}$

is defined as in Proposition 6;

(ii)

$\overline{\varphi}$

is defined by (38)

$$\overline{\varphi}(g,x) = \begin{cases} \varphi(g,x), & g \in G \\ x, & g = e. \end{cases}$$

Assume

$(G,\varphi,X)$

is an S-system, if we simply express

$\varphi(g,\ x)$

by

$g*x$

, then we have a standard form of evolutive system.

Definition 10

Assume

$(G,\varphi,X)$

is an S-system, and

$$g*x := \varphi(g,x), \quad g \in G,\ x \in X.$$

Then we have

(i) Discrete time S-system: (39)

$$x(t+1) = g(t)*x(t), \quad t \geq 0.$$

(ii) Discrete time constant S-system: (40)

$$x(t+1) = g * x(t), \quad t \geq 0.$$

(iii) Continuous time S-system: (41)

$$\dot{x}(t) = g(t) * x(t), \quad t \geq 0.$$

(iv) Continuous time constant S-system: (42)

$$\dot{x}(t) = g * x(t), \quad t \geq 0.$$

Remark 3

(i) Because of the semi-group property (34), the solution of (39) can be expressed as (43)

$$x(t) = \times_{i=0}^{t-1} g(i) * x_0, \quad t \geq 0, \ x_0 = x(0),$$

where $\times$ is the group product. Particularly, the solution of (40) can be expressed as (44)

$$x(t) = g^t * x_0, \quad t \geq 0, \ x_0 = x(0).$$

The solution of continuous time S-system will be discussed later.

(ii) Assume the system is a pseudo S-system. Then we can add an identity *e* to convert it into an S-system. Later on, we will not distinct pseudo S-system with S-system.

(iii) Consider (44) again. Let $t = 0$, then we have $g^0$. What is $g^0$? it should be the identity element in *G*. If the system is a pseudo S-system, then to make the solution meaningful, we do need an ‘identity’. That is the physical meaning of artificial *e*.

Define a cross-dimensional state space as:

$$\mathcal{V} := \bigcup_{r=1}^{\infty} \mathbb{R}^r.$$

Let $A \in \mathcal{M}$ and $x \in \mathcal{V}$, with a mapping $\varphi : \mathcal{M} \times \mathcal{V} \to \mathcal{V}$ defined by an MV-STP. Then we have a ‘linear system’ as (45)

$$x(t+1) = A(t) \vec{\times} x(t), \quad x(t) \in \mathcal{V},$$

where $\vec{\times}$

is any MV-STP.

Next, we consider when (45) is an S-system.

Definition 11

(i) A matrix multiplier $\Gamma$ and a vector multiplier $\gamma$ are consistent, if (46)

$$\Gamma_n \gamma_n = \gamma_n, \quad \forall n \geq 1.$$

(ii) An MV-STP $\vec{\times}$ is proper, if it is determined by a pair of consistent $\Gamma$ and $\gamma$.

Using this concept, we have the following result:

Proposition 7

Consider system (45). If $\vec{\times}$ is proper, then (45) is an S-system.

Proof

Assume $\vec{\times}$ is determined by $\Gamma$ and $\gamma$. We have to only prove the semi-group property. That is, (47)

$$A \vec{\times} (B \vec{\times} x) = (A \times_{\Gamma} B) \vec{\times} x.$$

Using the fact that $\Gamma$ and $\gamma$ are consistent, a similar argument as in the proof of equation (9) yields (47). □

The following example presents some consistent multipliers.

Example 3

(i) If

$\Gamma = I = \{I_n \mid n = 1, 2, \ldots\}$

, it is consistent with any

$\gamma$

.

(ii) If

$\Gamma = J = \{J_n \mid n = 1, 2, \ldots\}$

, it is consistent with

$\gamma = \mathbf{1} = \{\mathbf{1}_n \mid n = 1, 2, \ldots\}$

.

(iii) If

$\Gamma = \Delta^U = \{\Delta_n^U \mid n = 1, 2, \ldots\}$

, it is consistent with

$\gamma = \delta^U = \{\delta_n^1 \mid n = 1, 2, \ldots\}$

.

(iv) If

$\Gamma = \Delta^D = \{\Delta_n^D \mid n = 1, 2, \ldots\}$

, it is consistent with

$\gamma = \delta^D = \{\delta_n^n \mid n = 1, 2, \ldots\}$

.

## 5

## Generalised constant linear system

**Definition 12**

Assume

$\Gamma$

and

$\gamma$

are consistent, and

$\vec{\times}$

is determined by them, then we define generalised linear system as follows:

(i) Discrete time generalised linear system: (48)

$$\begin{cases} x(t+1) = A(t) \vec{\times} x(t), \quad A(t) \in \mathcal{M},\ x(t) \in \mathcal{V} \\ x(0) = x_0 \in \mathcal{V}. \end{cases}$$

(ii) Continuous time generalised linear system: (49)

$$\begin{cases} \dot{x}(t) = A(t) \vec{\times} x(t), \quad A(t) \in \mathcal{M},\ x(t) \in \mathcal{V} \\ x(0) = x_0 \in \mathcal{V}. \end{cases}$$

(iii) In (48) if

$A(t) = A$

, it is called a discrete time generalised constant system.

(iv) In (49) if $A(t) = A$, it is called a continuous time generalised constant system.

**Example 4**

(i) Assume $\Gamma = I$, $\gamma = 1$, the corresponding MV-product is $\vec{\bowtie}$, where $\vec{\bowtie} \in \{\vec{\ltimes},\ \vec{\rtimes}\}$. As an example, consider $\vec{\ltimes}$, the corresponding discrete time generalised constant system is (50)

$$\begin{cases} x(t+1) = A\vec{\ltimes}x(t), \quad A \in \mathcal{M},\ x(t) \in \mathcal{V}, \\ x(0) = x_0 \in \mathcal{V}. \end{cases}$$

The corresponding continuous time generalised constant system is (51)

$$\begin{cases} \dot{x}(t) = A\vec{\ltimes}x(t), \quad A \in \mathcal{M},\ x(t) \in \mathcal{V}, \\ x(0) = x_0 \in \mathcal{V} \end{cases}$$

(50) and (51) are called the first generalised constant system.

(ii) Assume $\Gamma = J$, $\gamma = 1$, the corresponding MV-product is $\vec{\circ}$, where $\vec{\circ} \in \{\vec{\circ}_\ell,\ \vec{\circ}_r\}$. As an example, consider $\vec{\circ}_\ell$, the corresponding discrete time generalised constant system is (52)

$$\begin{cases} x(t+1) = A\vec{\circ}_\ell x(t), \quad A \in \mathcal{M},\ x(t) \in \mathcal{V}, \\ x(0) = x_0 \in \mathcal{V} \end{cases}$$

The corresponding continuous time generalised constant system is (53)

$$\begin{cases} \dot{x}(t) = A\vec{\circ}_\ell x(t), \quad A \in \mathcal{M},\ x(t) \in \mathcal{V}, \\ x(0) = x_0 \in \mathcal{V} \end{cases}$$

(52) and (53) are called the second generalised constant system.

**Remark 4**

(i) For second generalised constant system since $(\mathcal{M},\ \circ)$ is not a monoid, to get an S-system,we need to add $\{e\}$, which satisfies

$$e \circ A = A \circ e = A, \quad \forall A \in \mathcal{M}$$

$$e \vec{\circ} x = x, \quad \forall x \in \mathcal{V}.$$

Hence, precisely speaking, second generalised constant systems are defined over $\mathcal{M} \cup \{e\}$.

(ii) In fact, this *e* has its physical meaning. Consider system (52), because of the sime-group property, we have

$$x(t) = A^t \vec{\circ} x_0, \quad t \geq 0.$$

Now what is $A^0$? It should be an identity mapping. But $(\mathcal{M}, \circ)$ has no identity element, hence we need $e := A^0$.

## 6

## Matrix equivalence and quotient space

**Definition 13**

Let $\Gamma$ be a matrix multiplier.

(i) Two matrices *A* and *B* are said to be left $\Gamma$ equivalent, denoted by $A \sim_{\ell}^{\Gamma} B$, if there exist $\Gamma_\alpha$ and $\Gamma_\beta$ such that (54)

$$A \otimes \Gamma_\alpha = B \otimes \Gamma_\beta.$$

(ii) Two matrices *A* and *B* are said to be right $\Gamma$ equivalent, denoted by $A \sim_{r}^{\Gamma} B$, if there exist $\Gamma_\alpha$ and $\Gamma_\beta$ such that (55)

$$\Gamma_\alpha \otimes A = \Gamma_\beta \otimes B.$$

(iii) The equivalent class is denoted by (56)

$$\langle A \rangle_\ell^\Gamma := \{B \mid B \sim_\ell^\Gamma A\},$$

and (57)

$$\langle A \rangle_r^\Gamma := \{B \mid B \sim_r^\Gamma A\}.$$

**Remark 5**

When an equivalence is defined, we need to verify whether it is a equivalent relation. That is, (i) $a \sim a$; (ii) $a \sim b$ implies $b \sim a$; (iii) $a \sim b$ and $b \sim c$ imply $a \sim c$. But in Definition 13 and in the sequel, since it is straightforward verifiable, the verification is omitted.

Hereafter, we consider the left equivalence only. So we use $\sim^\Gamma$ for left $\Gamma$ equivalent. It can also be considered as both left and right $\Gamma$ equivalent. However, for latter, certain obvious modifications need to be done.

Proposition 8

Assume $A \sim^\Gamma B$, where $\Gamma \in \{I,\ J,\ \Delta^U,\ \Delta^D\}$, then there exists $\Lambda$ such that (58)

$$A = \Lambda \otimes \Gamma_\alpha; \quad B = \Lambda \otimes \Gamma_\beta.$$

Proof

It has been proved for $\Gamma = I$

[20, 21]. The proof for other cases is similar. □

Remark 6

(i) As a conjecture, we believe that Proposition 8 is true for any $\Gamma$. But since the verification is one-by-one, we can assure (58) is true only for known $\Gamma$.

(ii) Hereafter, only $\Gamma \in \{I,\ J,\ \Delta^U,\ \Delta^D\}$ are considered.

Definition 14

An order within an equivalent class $\langle A \rangle^\Gamma$ is defined as follows: Let $A, B \in \langle A \rangle^\Gamma$, we define $A \prec^\Gamma B$, if there exists $\Gamma_s$ such that $A \otimes \Gamma_s = B$. With this order, $\langle A \rangle^\Gamma$ becomes a partial order set.

Definition 15

A partial order set $(L,\ \ \prec)$ is a lattice, if [22]

(i) for any $a,\ b \in G$, there is a least supper boundary $s = \sup(a,\ b)$. Precisely speaking, *s* is the smallest one, satisfying (59)

$$a \prec s, \quad b \prec s;$$

(ii) for any $a,\ b \in G$, there is a greatest lower boundary $t = \inf(a,\ b)$

. Precisely speaking, $t$ is the largest one, satisfying (60)

$$t \prec a, \quad t \prec b.$$

Proposition 9

$\langle A \rangle^{\Gamma}$ with the order $\prec^{\Gamma}$ is a lattice.

Proof

In (54), we assume $\alpha \wedge \beta = 1$ (i.e. $\alpha$ and $\beta$ are co-prime), and define (61)

$$\Theta := A \otimes \Gamma_{\alpha} = B \otimes \Gamma_{\beta}.$$

Then it is easy to verify that (62)

$$\Theta = \sup(A,\ B).$$

In (58), assume $\alpha \wedge \beta = 1$, then it is easy to verify that (63)

$$\Lambda = \inf(A,\ B).$$

□

Next, we consider matrix quotient space. Denoted by

$$\Sigma^{\Gamma} := \mathcal{M} / \sim^{\Gamma}.$$

Definition 16

The MM-STP with respect to $\Gamma$ is defined as (64)

$$\langle A \rangle^{\Gamma} \ltimes^{\Gamma} \langle B \rangle^{\Gamma} := \langle A \ltimes^{\Gamma} B \rangle^{\Gamma}.$$

Proposition 10

The product $\ltimes^{\Gamma}$ defined by (64) is well defined.

Proof

We have to only prove (64) is independent of the choice of the representatives. That is, let

$$A,\ \tilde{A} \in \langle A \rangle^{\Gamma}$$

and

$$B,\ \tilde{B} \in \langle B \rangle^{\Gamma}$$

. Then (65)

$$A \ltimes^{\Gamma} B \sim^{\Gamma} \tilde{A} \ltimes^{\Gamma} \tilde{B}.$$

According to Proposition 8, there exist

$$U \in \mathcal{M}_{m\times n}$$

and

$$V \in \mathcal{M}_{p\times q}$$

such that (66)

$$A = U \otimes \Gamma_s; \quad \tilde{A} = U \otimes \Gamma_t;$$

and (67)

$$B = V \otimes \Gamma_\alpha; \quad \tilde{B} = V \otimes \Gamma_\beta.$$

Denote

$$n \vee p = r, \quad ns \vee \alpha p = r\xi, \quad nt \vee \beta p = r\eta.$$

Then

$$\begin{aligned} A \ltimes^{\Gamma} B &= \left(U \otimes \Gamma_s \otimes \Gamma_{r\xi/ns}\right)\left(V \otimes \Gamma_\alpha \otimes \Gamma_{r\xi/\alpha p}\right) \\ &= \left[\left(U \otimes \Gamma_{r/n}\right)\left(V \otimes \Gamma_{r/p}\right)\right] \otimes \Gamma_\xi. \end{aligned}$$

Similarly, we have

$$\tilde{A} \ltimes^{\Gamma} \tilde{B} = \left[\left(U \otimes \Gamma_{r/n}\right)\left(V \otimes \Gamma_{r/p}\right)\right] \otimes \Gamma_\eta.$$

Hence we have (65) □

As immediate consequences, we have the following two corollaries.

Corollary 3

Assume

$$\langle A \rangle^{\Gamma},\ \langle B \rangle^{\Gamma},\ \langle C \rangle^{\Gamma} \in \Sigma^{\Gamma}$$

. Then (68)

$$\langle A \rangle^{\Gamma} \ltimes^{\Gamma} \left(\langle B \rangle^{\Gamma} \ltimes^{\Gamma} \langle C \rangle^{\Gamma}\right) = \left(\langle A \rangle^{\Gamma} \ltimes^{\Gamma} \langle B \rangle^{\Gamma}\right) \ltimes^{\Gamma} \langle C \rangle^{\Gamma}.$$

Corollary 4

(i)

$$\left(\Sigma^{\Gamma},\ \ltimes^{\Gamma}\right)$$

is a semi-group.

(ii) If

$$\left(\mathcal{M},\ \ltimes^{\Gamma}\right)$$

is a monoid, then

$$\left(\Sigma^{\Gamma},\ \ltimes^{\Gamma}\right)$$

is also a monoid.

**Vector equivalence and quotient space**

**Definition 17**

Let
$\gamma$
be a vector multiplier.

(i) Two vectors *x* and *y* are said to be left
$\gamma$
equivalent, denoted by
$x \sim_{\ell}^{\gamma} y$
, if there exist
$\gamma_\alpha$
and
$\gamma_\beta$
such that (69)
$x \otimes \gamma_\alpha = y \otimes \gamma_\beta.$

(ii) Two vectors *x* and *y* are said to be right
$\gamma$
equivalent, denoted by
$x \sim_{r}^{\gamma} y$
, if there exist
$\gamma_\alpha$
and
$\gamma_\beta$
such that (70)
$\gamma_\alpha \otimes x = \gamma_\beta \otimes y.$

(iii) The equivalent class is denoted by (71)
$\bar{x}_{\ell}^{\gamma} := \{y \mid y \sim_{\ell}^{\gamma} x\},$
and (72)
$\bar{x}_{r}^{\gamma} := \{y \mid y \sim_{r}^{\gamma} x\}.$

For statement ease, in the following we consider the left equivalence only. Similar to matrix case, we can easily verify the following properties, which are the vector corresponding properties of matrix ones.

Proposition 11

(i) Define an order on
$\mathcal{V}$
as follows: assume
$x,\ y \in \mathcal{V}$
, it is said that

$x \prec^{\gamma} y$ if there exists a $\gamma_n$ such that $x \otimes \gamma_n = y$. Then $\mathcal{V}$ with order $\prec^{\gamma}$ becomes a partial order set.

(ii) Assume $x \sim^{\gamma} y$, that is, (69) holds. We further assume $\alpha \wedge \beta = 1$, then set (73)

$$\xi := x \otimes \gamma_{\alpha} = y \otimes \gamma_{\beta}.$$

It is ready to verify that $\xi = \sup(x,\ y)$.

(iii) Assume $x \sim^{\gamma} y$, then there exists $\eta \in \mathcal{V}$ such that (74)

$$x = \eta \otimes \gamma_{\beta}; \quad y = \eta \otimes \gamma_{\alpha}.$$

(iv) In (74) assume $\alpha \wedge \beta = 1$, then $\eta = \inf(x, y)$.

(v) $(\mathcal{V},\ \prec^{\gamma})$ is a lattice.

## 8

## Generalised linear system on quotient spaces

First, the quotient vector space is naturally defined as $\Omega^{\gamma} := \mathcal{V} / \sim^{\gamma}$.

Note that in this section the equivalences are assumed to be left equivalences. (They could be right equivalences for both. But the case of one left one right is not allowed.)

Then, we consider the action of $\Sigma^{\Gamma}$ on $\Omega^{\gamma}$.

Definition 18

Let $\langle A\rangle^{\Gamma} \in \Sigma^{\Gamma}$ and $\bar{x}^{\gamma} \in \Omega^{\gamma}$. The MV-product $\vec{\times}$ is defined by $\Gamma$ and $\gamma$. The action of $\Sigma^{\Gamma}$ on $\Omega^{\gamma}$ is defined by (75)

$$\langle A\rangle^{\Gamma} \vec{\times} \bar{x}^{\gamma} := \overline{A \vec{\times} x}^{\gamma}.$$

Proposition 12

Assume $\Gamma$ and $\gamma$ are consistent, then(75) is properly defined.

Proof

Assume $A \sim^{\Gamma} B$ and $x \sim^{\gamma} y$. We have to only show that (76)

$$A \vec{\times} x \sim^{\gamma} B \vec{\times} y.$$

Using Propositions 8 and 11, there exist $\Lambda \in \mathcal{M}_{m\times n}$ and $\xi \in \mathcal{V}_r$, such that

$$A = \Lambda \otimes \Gamma_s; \quad B = \Lambda \otimes \Gamma_t$$

$$x = \xi \otimes \gamma_\alpha; \quad y = \xi \otimes \gamma_\beta.$$

Set

$$\ell = n \vee r,$$

and

$$ns \vee r\alpha = p := a\ell;$$

$$nt \vee r\beta = q := b\ell.$$

Then we have

$$\begin{aligned} A \vec{\times} x &= \left(\Lambda \otimes \Gamma_s \otimes \Gamma_{p/ns}\right)\left(\xi \otimes \gamma_\alpha \otimes \gamma_{p/r\alpha}\right) \\ &= \left(\Lambda \otimes \Gamma_{p/n}\right)\left(\xi \otimes \gamma_{p/r}\right) \\ &= \left[\left(\Lambda \otimes \Gamma_{\ell/n}\right)\left(\xi \otimes \gamma_{\ell/r}\right)\right] \otimes \left(\Gamma_a \gamma_a\right) \\ &= \left[\left(\Lambda \otimes \Gamma_{\ell/n}\right)\left(\xi \otimes \gamma_{\ell/r}\right)\right] \otimes \gamma_a. \end{aligned}$$

A similar calculation shows that

$$B \vec{\times} y = \left[\left(\Lambda \otimes \Gamma_{\ell/n}\right)\left(\xi \otimes \gamma_{\ell/r}\right)\right] \otimes \gamma_b.$$

(76) follows immediately. □

Proposition 13

If $\vec{\times}$ is determined by consistent $\Gamma$ and $\gamma$, then $(\Sigma, \vec{\times}, \Omega)$ is a pseudo S-system.

Proof

Let $\langle A\rangle^\Gamma, \ \langle B\rangle^\Gamma \in \Sigma$ and $\bar{x}^\gamma \in \Omega$. Then

$$\begin{aligned}
&\left(\langle A\rangle^{\Gamma} \times^{\Gamma} \langle B\rangle^{\Gamma}\right) \vec{\times} \bar{x}^{\gamma} \\
&\quad = \left\langle A^{\Gamma} \times^{\Gamma} B^{\Gamma}\right\rangle^{\Gamma} \vec{\times} \bar{x}^{\gamma} \\
&\quad = \overline{\left(A \times^{\Gamma} B\right) \vec{\times} x}^{\gamma} \\
&\quad = \overline{A \vec{\times} \left(B \vec{\times} x\right)}^{\gamma} \\
&\quad = \langle A\rangle^{\Gamma} \vec{\times} \overline{\left(B \vec{\times} x\right)}^{\gamma} \\
&\quad = \langle A\rangle^{\Gamma} \vec{\times} \left(\langle B\rangle^{\Gamma} \vec{\times} \bar{x}^{\gamma}\right).
\end{aligned}$$

□

Remark 7

It is easy to verify that

(i) if $e \in \mathcal{M}$ is the identity, then $\langle e\rangle$ is the identity in $\Sigma$;

(ii) if $(\mathcal{M}, \vec{\times}, \mathcal{V})$ is an S-system, then the corresponding $(\Sigma, \vec{\times}, \Omega)$ is also an S-system;

(iii) if $(\mathcal{M}, \vec{\times}, \mathcal{V})$ is only a pseudo S-system because $\left(\mathcal{M}, \times^{\Gamma}\right)$ has no identity, then by adding an artificial identity *e*, $\left(\left\{\mathcal{M}, e\right\}, \vec{\times}, \mathcal{V}\right)$ becomes an S-system. Moreover, its corresponding $\left(\left\{\Sigma, e\right\}, \vec{\times}, \Omega\right)$ is also an S-system.

## 9

## Vector space structure of ${\mathcal V}$V

This section pose a vector space structure on $\mathcal{V}$ and

$\Omega$

.

Definition 19

Let

$x,\ y \in \mathcal{V}$

. Say,

$x \in \mathbb{R}^p$

,

$y \in \mathbb{R}^q$

, and

$p \vee q = t$

.

$\gamma$

is a vector multiplier. Then (77)

$$x +^{\gamma} y := \left(x \otimes \gamma_{t/p}\right) + \left(y \otimes \gamma_{t/q}\right) \in \mathbb{R}^t \subset \mathcal{V}.$$

(78)

$$x -^{\gamma} y := x +^{\gamma} (-y) \in \mathbb{R}^t \subset \mathcal{V}.$$

It is easy to verify that

$(\mathcal{V}, +^{\gamma})$

satisfies all the requirements of a vector space [23] except that the zero is not unique. Precisely, (79)

$$\vec{0} := \{\mathbf{0}_n \mid n = 1, 2, \ldots\}$$

is the set of zeros. Moreover, because of this, the

$-x$

for each *x* is also not unique. Precisely, (80)

$$-x = \{y \mid x +^{\gamma} y \in \vec{0}\}.$$

We call such a 'vector space' pseudo vector space. Hence,

$(\mathcal{V},\ +^{\gamma})$

is a pseudo vector space.

Next, we transfer this vector space structure to

$\Omega$

.

Definition 20

Let

$\bar{x}^{\gamma},\ \bar{y}^{\gamma} \in \Omega$

. Define (81)

$$\bar{x}^{\gamma} +^{\gamma} \bar{y}^{\gamma} := \overline{x +^{\gamma} y}^{\gamma}.$$

(82)

$$\bar{x}^{\gamma} -^{\gamma} \bar{y}^{\gamma} := \bar{x}^{\gamma} +^{\gamma} \left(\overline{-y}^{\gamma}\right).$$

Proposition 14

The addition (subtraction) defined by (81) (82) is properly defined.

Proof

Assume

$x \sim^{\gamma} u$

and

$y \sim^{\gamma} v$

. According to Proposition 11 (see (74)), we can find

$\xi,\ \eta \in \mathcal{V}$

, say,

$\xi \in \mathbb{R}^a$

and

$\eta \in \mathbb{R}^b$

, such that

$$x = \xi \otimes \gamma_p, \quad u = \xi \otimes \gamma_q$$

$$y = \eta \otimes \gamma_\alpha, \quad v = \eta \otimes \gamma_\beta.$$

Assume

$a \vee b = \ell$

$$ap \vee b\alpha = m\ell, \quad aq \vee b\beta = n\ell.$$

Then

$$\begin{aligned} x +^{\gamma} y &= (\xi \otimes \gamma_p) \otimes \gamma_{m\ell/ap} + (\eta \otimes \gamma_\alpha) \otimes \gamma_{m\ell/b\alpha} \\ &= (\xi \otimes \gamma_{m\ell/a}) + (\eta \otimes \gamma_{m\ell/b}) \\ &= \left[\xi \otimes \gamma_{\ell/a} + \eta \otimes \gamma_{\ell/b}\right] \otimes \gamma_m. \end{aligned}$$

Similar calculation yields that

$$u +^{\gamma} v = \left[\xi \otimes \gamma_{\ell/a} + \eta \otimes \gamma_{\ell/b}\right] \otimes \gamma_n.$$

Hence

$$(x +^{\gamma} y) \sim^{\gamma} (u +^{\gamma} v).$$

□

Proposition 15

$(\Omega,\ +^{\gamma})$

is a vector space.

Proof

By definition, one sees easily that

$\vec{0} = \bar{0}^{\gamma}.$

Hence, there is only a unique zero in

$\Omega$

. Second, it is also easy to verify that

$\{y \mid x +^{\gamma} y = 0\} = \overline{-x}^{\gamma},$

which is also unique. Hence,

$(\Omega, +^{\gamma})$

is a vector space. □

## 10

## Cross-dimensional linear dynamic system

**Definition 21**

Let $(G, \varphi, X)$ be an S-system (or a pseudo S-system).

(i) If *X* is a topological space and for each fixed $g \in G$ the mapping $\varphi|_g : X \to X$ is continuous, then $(G, \varphi, X)$ is called a pseudo dynamic system.
(ii) A pseudo dynamic system $(G, \varphi, X)$ is called a dynamic system, if *X* is a Hausdorff space.

Consider a matrix S-system $(\mathcal{M}, \vec{\times}, \mathcal{V})$, where $\vec{\times}$ is determined by consistent $\Gamma$ and $\gamma$. The purpose of this section is to pose a topology on $\mathcal{V}$ such that the S-system can be considered as a dynamic system.
Definition 22
Let $x,\ y \in \mathcal{V}$. Say, $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, and $t = p \vee q$. Then the ‘inner product’ of *x* and *y*, related to

$\gamma$

, is defined by (83)

$$\langle x,\ y\rangle_\gamma := \frac{1}{\|\gamma_t\|^2}\left\langle x\otimes\gamma_{t/p},\ y\otimes\gamma_{t/q}\right\rangle,$$

where

$\langle\cdot,\cdot\rangle$

is the classical inner product of Euclidean Space.

It is easy to verify that

$\left(\mathcal{V},\ \langle,\rangle_\gamma\right)$

is an inner product space [23]. This inner product can be used to define a norm on

$\mathcal{V}$

.

Definition 23

The norm on

$\mathcal{V}$

is defined as (84)

$$\|x\|_\gamma := \sqrt{\langle x\ ,\ x\rangle_\gamma}.$$

Finally, we define a distance on

$\mathcal{V}$

.

Definition 24

Let

$x,\ y\in\mathcal{V}$

. The distance between *x* and *y* is defined as (85)

$$d_\gamma(x,y) := \|x -^\gamma y\|_\gamma.$$

Remark 8

(i) Using the distance defined by (85),

$(\mathcal{V},\ d_\gamma)$

is a pseudo metric space. That is, it satisfies all the requirements of a metric space except that ‘

$d(x,\ y)=0\Leftrightarrow x=y$

’ is replaced by ‘

$x=y\Rightarrow d(x,\ y)=0$

’ [24].

(ii) It is easy to verify that

$d_\gamma(x,\ y)=0$

, if and only if,

$x\sim_\gamma y$

.

(iii) The topology deduced by this distance $d = d_\gamma$, denoted by $\mathcal{T}_{\text{d}}$, makes $(\mathcal{V}, \mathcal{T}_{\text{d}})$ a topological space [25].

(iv) Because of (ii), $(\mathcal{V}, \mathcal{T}_{\text{d}})$ is not a Hausdorff space [25].

(v) It is easy to see that $(\mathcal{V}, \mathcal{T}_{\text{d}})$ is a path-wise connected topological space. Hence, this distance glue all the Euclidean spaces $\mathbb{R}^n$, $n = 1, 2, \dots$ together to form the dimension free state space $\mathcal{V}$.

Our next purpose is to establish generalised (or cross-dimensional) linear dynamic system. To this end we have to define the norm of a matrix.

Definition 25

Assume $\vec{\times}$ is determined by consistent $\Gamma$ and $\gamma$. The norm of *A* with respect to $\Gamma$ and $\gamma$ is defined by (86)

$$||A||_{\vec{\times}} := \sup_{0 \neq x \in \mathcal{V}} \frac{||A \vec{\times} x||_\gamma}{||x||_\gamma}.$$

Next, we estimate the normal $||A||_{\vec{\times}}$.

Lemma 1

Let

$$x \in \mathbb{R}^t \subset \mathcal{V}$$

. Then

(i) (87)

$$\|x\|_\gamma = \frac{1}{\|\gamma_t\|}\|x\|.$$

(ii) (88)

$$\|x \otimes \gamma_s\|_\gamma = \|x\|_\gamma.$$

Proof

Equation (87) is easily verifiable. We prove (88): first, we claim that (89)

$$\|x \otimes y\| = \|x\|\|y\|.$$

Note that

$$\begin{aligned}\|x \otimes y\| &= \sqrt{(x^{\mathrm{T}} \otimes y^{\mathrm{T}})(x \otimes y)} \\ &= \sqrt{x^{\mathrm{T}}x \otimes y^{\mathrm{T}}y} = \sqrt{(x^{\mathrm{T}}x)(y^{\mathrm{T}}y)} \\ &= \|x\|\|y\|,\end{aligned}$$

which proves the claim. Using (89), we have

$$\|\gamma_{pq}\| = \|\gamma_p\|\|\gamma_q\|.$$

Hence

$$\begin{aligned}\|x \otimes \gamma_r\|_\gamma &= \frac{1}{\|\gamma_{tr}\|}\|x \otimes \gamma_r\| \\ &= \frac{1}{\|\gamma_{tr}\|}\|x\|\|\gamma_r\| \\ &= \frac{1}{\|\gamma_t\|}\|x\| = \|x\|_\gamma.\end{aligned}$$

□

We refer to [26] for the following two Lemmas.

Lemma 2

(90)

$$\|A\| = \sqrt{\sigma_{\max}(A^{\mathrm{T}}A)}.$$

Lemma 3

(91)

$$\sigma(A \otimes B) = \{rs \mid r \in \sigma(A),\ s \in \sigma(B)\}.$$

From Lemma 3 the following claim is clear.

Lemma 4

Assume *A* and *B* are positive semi-definite, then (92)

$\sigma_{\max}(A \otimes B) = \sigma_{\max}(A)\sigma_{\max}(B).$

Now we are ready to present the main result about norm estimation of a matrix. We need the following assumption:

*A-1*: $\Gamma_n$ is symmetric for all $n \geq 1$.

Remark 9

(i) If $\Gamma \in \{I,\ J,\ \Delta^U,\ \Delta^D\}$, then it is obvious that *A-1* is satisfied. So far, we did not meet any $\Gamma$, which is not symmetric. Hence, we can see *A-1* is reasonable.

(ii) If *A-1* is assumed. Then $\Gamma_n^{\mathrm{T}}\Gamma_n = \Gamma_n^2$. Hence, $\sigma_{\max}(\Gamma_n^{\mathrm{T}}\Gamma_n) = 1.$

Theorem 1

Assume A-1 and $A \in \mathcal{M}_{m\times n}$, then (93)

$$\|A\|_{\bar{\times}} = \frac{\|\gamma_n\|}{\|\gamma_m\|}\|A\| = \frac{\|\gamma_n\|}{\|\gamma_m\|}\sqrt{\sigma_{\max}(A)}.$$

Proof

Assume $0 \neq x \in \mathbb{R}^r \subset \mathcal{V}$, and $t = n \vee r$, $s = n \wedge r$. Then

$$\begin{aligned}
\frac{\|A\vec{\times}x\|_\gamma}{\|x\|_\gamma} &\le \sup_{0\neq x\in\mathbb{R}^r} \frac{\|(A\otimes\Gamma_{t/n})(x\otimes\gamma_{t/r})\|_\gamma}{\|x\|_\gamma}\\
&= \sup_{0\neq x\in\mathbb{R}^r} \frac{\|\left[(A\otimes\Gamma_{t/n})(x\otimes\gamma_{t/r})\right]\otimes(\gamma_s)\|_\gamma}{\|x\|_\gamma}\\
&= \sup_{0\neq x\in\mathbb{R}^r} \frac{\|\left[(A\otimes\Gamma_{t/n})(x\otimes\gamma_{t/r})\right]\otimes(\Gamma_s\gamma_s)\|_\gamma}{\|x\|_\gamma}\\
&= \sup_{0\neq x\in\mathbb{R}^r} \frac{\|(A\otimes\Gamma_{ts/n})(x\otimes\gamma_{ts/r})\|_\gamma}{\|x\|_\gamma}\\
&= \sup_{0\neq x\in\mathbb{R}^r} \frac{\|(A\otimes\Gamma_{r})(x\otimes\gamma_{n})\|_\gamma}{\|x\|_\gamma}\\
&= \frac{1}{\|\gamma_{mr}\|}\sup_{0\neq x\in\mathbb{R}^r} \frac{\|(A\otimes\Gamma_{ts/n})(x\otimes\gamma_{ts/r})\|}{\|x\|_\gamma}\\
&\le \frac{\|\gamma_{nr}\|}{\|\gamma_{mr}\|}\sup_{0\neq x\in\mathbb{R}^r} \frac{\|(A\otimes\Gamma_{ts/n})\|\|(x\otimes\gamma_{ts/r})_\gamma\|}{\|x\|_\gamma}\\
&\le \frac{\|\gamma_n\|}{\|\gamma_m\|}\|(A\otimes\Gamma_{ts/n})\|\\
&= \frac{\|\gamma_n\|}{\|\gamma_m\|}\sqrt{\sigma_{\max}(A^{\mathrm{T}}A\otimes\Gamma^{\mathrm{T}}_{ts/n}\Gamma^{\mathrm{T}}_{ts/n})}\\
&= \frac{\|\gamma_n\|}{\|\gamma_m\|}\sqrt{\sigma_{\max}(A^{\mathrm{T}}A)\sigma_{\max}(\Gamma^{\mathrm{T}}_{ts/n}\Gamma^{\mathrm{T}}_{ts/n})}\\
&= \frac{\|\gamma_n\|}{\|\gamma_m\|}\sqrt{\sigma_{\max}(A^{\mathrm{T}}A)}\\
&= \frac{\|\gamma_n\|}{\|\gamma_m\|}\|A\|.
\end{aligned}$$

Since $x\in\mathcal{V}$ is arbitrary, we have (94)

$$\|A\|_{\vec{\times}} \le \frac{\|\gamma_n\|}{\|\gamma_m\|}\|A\|.$$

On the other hand, we have (95)

$$\|A\|_\gamma \geq \sup_{0\neq x\in\mathbb{R}^n} \frac{\|A\vec{\times}x\|_\gamma}{\|x\|_\gamma}$$
$$= \sup_{0\neq x\in\mathbb{R}^n} \frac{\frac{1}{\|\gamma_m\|}\|Ax\|}{\frac{1}{\|\gamma_n\|}\|x\|}$$
$$= \frac{\|\gamma_n\|}{\|\gamma_m\|}\|A\| = \frac{\|\gamma_n\|}{\|\gamma_m\|}\sqrt{\sigma_{\max}(A^{\mathrm{T}}A)}.$$

Combining (94) with (95) yields (93). □

Remark 10

It is easy to see that

(i) if $\Gamma \in \{I,\ J,\ \Delta^U,\ \Delta^D\}$, then $\sigma_{\max}(\Gamma_n^{\mathrm{T}}\Gamma_n) = 1$, $n = 1, 2, \ldots$;

(ii) if $\gamma = \mathbf{1}$ then $\|\gamma_n\| = \sqrt{n}$; if $\gamma = \delta^U$ or $\gamma = \delta^D$, then $\|\gamma_n\| = 1$, $n = 1, 2, \ldots$.

Note that if $\Gamma \in \{I,\ J,\ \Delta^U,\ \Delta^D\}$ and $\gamma \in \{\mathbf{1}, \delta^U,\ \delta^D\}$, $\Gamma$ and $\gamma$ are consistent, and $\vec{\times}$ is determined by such $\Gamma$ and $\gamma$

, then according to Theorem 1, for any fixed

$A \in \mathcal{M}$

, the mapping

$\Pi_A : \mathcal{V} \to \mathcal{V}$

defined by

$x \mapsto A \vec{\times} x$

is continuous. Then we have the following result immediately.

Proposition 16

Assume *A-1* and

$\vec{\times}$

is determined by consistent

$\Gamma$

and

$\gamma$

, then

(i) (96)

$$x(t+1) = A(t) \vec{\times} x(t), \quad A(t) \in \mathcal{M},\ x(t) \in \mathcal{V}$$
$$x(0) = x_0$$

is a discrete-time linear pseudo dynamic S-system (may need to add an identity);

(ii) (97)

$$\dot{x}(t) = A(t) \vec{\times} x(t), \quad A(t) \in \mathcal{M},\ x(t) \in \mathcal{V}$$
$$x(0) = x_0$$

is a continuous-time linear pseudo dynamic S-system (may need to add an identity).

Next, we extend the distance and metric topology to quotient space.

Definition 26

Let

$\bar{x}^{\gamma},\ \bar{y}^{\gamma} \in \Omega$

. Define their inner product by (98)

$$\langle \bar{x}^{\gamma},\ \bar{y}^{\gamma} \rangle_{\gamma} := \langle x,\ y \rangle_{\gamma}.$$

Proposition 17

The inner product (98) is properly defined.

Proof

Assume

$x \sim^{\gamma} u$

and

$y \sim^{\gamma} v$

. We have to only prove (99)

$$\langle x,\ y \rangle_{\gamma} = \langle u,\ v \rangle_{\gamma}.$$

According to Proposition 11 there exist

$\xi \in \mathbb{R}^a$

and

$\eta \in \mathbb{R}^b$

such that

$x = \xi \otimes \gamma_\alpha ; u = \xi \otimes \gamma_\beta$

$y = \eta \otimes \gamma_p ; v = \eta \otimes \gamma_q$

Assume

$a \vee b = \ell$

$a\alpha \vee bp = c\ell; \quad a\beta \vee bq = d\ell.$

Then we have

$$\begin{aligned}
\langle x, y \rangle_\gamma &= \frac{1}{\|\gamma_{c\ell}\|^2} \left(\xi_a \otimes \gamma_\alpha \otimes \gamma_{c\ell/a\alpha}\right)^{\mathrm{T}} \left(\eta_b \otimes \gamma_p \otimes \gamma_{c\ell/bp}\right) \\
&= \frac{1}{\|\gamma_{c\ell}\|^2} \left(\xi_a^{\mathrm{T}} \otimes \gamma_{c\ell/a}^{\mathrm{T}}\right) \left(\eta_b \otimes \gamma_{c\ell/b}\right) \\
&= \frac{1}{\|\gamma_{c\ell}\|^2} \left(\xi_a^{\mathrm{T}} \otimes \gamma_{\ell/a}^{\mathrm{T}}\right) \left(\eta_b \otimes \gamma_{\ell/b}\right) \otimes \left(\gamma_c^{\mathrm{T}} \gamma_c\right) \\
&= \frac{1}{\|\gamma_\ell\|^2} \left(\xi_a^{\mathrm{T}} \otimes \gamma_{\ell/a}^{\mathrm{T}}\right) \left(\eta_b \otimes \gamma_{\ell/b}\right).
\end{aligned}$$

Similarly, we also have

$$\langle u, v \rangle_\gamma = \frac{1}{\|\gamma_\ell\|^2} \left(\xi_a^{\mathrm{T}} \otimes \gamma_{\ell/a}^{\mathrm{T}}\right) \left(\eta_b \otimes \gamma_{\ell/b}\right).$$

□

Similarly, we can define the norm and distance on

$\Omega$

as follows:

Definition 27

Let

$\bar{x}^\gamma \in \Omega$

. Then the norm of

$\bar{x}^\sigma$

is defined as follows. (100)

$\|\bar{x}^\gamma\|_\gamma := \|x\|_\gamma, \quad \bar{x}^\gamma \in \Omega;$

Proposition 18

The norm of

$\bar{x}^\gamma$

, defined by (100) is properly defined.

Proof

Since the inner product is independent of the representatives, the norm is also well defined. □

Definition 28

Let

$\bar{x}^{\gamma},\ \bar{y}^{\gamma} \in \Omega$

. A distance on

$\Omega$

is defined as follows. (101)

$$d_{\gamma}(\bar{x}^{\gamma},\ \bar{y}^{\gamma}) := d_{\gamma}(x, y), \quad \bar{x}^{\gamma}, \bar{y}^{\gamma} \in \Omega.$$

Proposition 19

The distance on

$\Omega$

, defined by (101) is properly defined.

Proof

Assume

$x \sim^{\gamma} u$

and

$y \sim^{\gamma} v$

, we have to only prove that (102)

$$d_{\gamma}(x,\ y) = d_{\gamma}(u,\ v).$$

According to Proposition 11 there exist

$\xi \in \mathbb{R}^{a}$

and

$\eta \in \mathbb{R}^{b}$

such that

$$x = \xi \otimes \gamma_{\alpha}; u = \xi \otimes \gamma_{\beta}$$

$$y = \eta \otimes \gamma_{p}; v = \eta \otimes \gamma_{q}$$

Assume

$$a \vee b = \ell$$

$$a\alpha \vee bp = c\ell; \quad a\beta \vee bq = d\ell.$$

Then we have

$$\begin{aligned} d_{\gamma}(x, y) &= \frac{1}{\|\gamma_{c\ell}\|} \left\| (\xi \otimes \gamma_{\alpha}) \otimes \gamma_{c\ell/a\alpha} - (\eta \otimes \gamma_{p}) \otimes \gamma_{c\ell/bp} \right\| \\ &= \frac{1}{\|\gamma_{c\ell}\|} \sqrt{(\Delta^{\mathrm{T}} \otimes \gamma_{c}^{\mathrm{T}})(\Delta \otimes \gamma_{c})} \\ &= \frac{1}{\|\gamma_{\ell}\|} \sqrt{\Delta^{\mathrm{T}}\Delta} = \|\Delta\|_{\gamma} \\ &= \left\| \xi \otimes \gamma_{\ell/a} - \eta \otimes \gamma_{\ell/b} \right\|_{\gamma}, \end{aligned}$$

where

$$\Delta := \xi \otimes \gamma_{\ell/a} - \eta \otimes \gamma_{\ell/b}.$$

Similarly, we can prove that

$$d_\gamma(u,\ v) = \left\|\xi \otimes \gamma_{\ell/a} - \eta \otimes \gamma_{\ell/b}\right\|_\gamma .$$

Equation (102) follows. □

Finally, we define the operator norm of

$\langle A \rangle^\Gamma \in \Sigma$

acting on

$\Sigma$

:

Definition 29

Let

$\langle A \rangle^\Gamma \in \Sigma$

. Then its operator norm is defined as follows: (103)

$$\| \langle A \rangle^\Gamma \|_\Gamma := \|A\|_\Gamma, \quad \langle A \rangle^\Gamma \in \Omega.$$

Proposition 20

The the operator norm of

$\langle A \rangle^\Gamma \in \Sigma$

on

$\Omega$

, defined by (103) is properly defined.

Proof

It is an immediate consequence of Definition 18 and Proposition 12. □

Finally, we can define dynamic systems on quotient space as follows:

Proposition 21

Assume *A-1* and

$\vec{\times}$

is determined by consistent

$\Gamma$

and

$\gamma$

, then

(i) (104)

$$\begin{aligned} \bar{x}^\gamma(t+1) = \langle A \rangle^\Gamma(t) \vec{\times} \bar{x}^\gamma(t), \quad \bar{x}^\gamma(0) = \bar{x}_0^\gamma, \\ \langle A \rangle^\Gamma(t) \in \Sigma, \quad \bar{x}^\gamma(t) \in \Omega \end{aligned}$$

is a discrete-time linear dynamic S-system (may need to add an identity);

(ii) (105)

$$\begin{aligned} \dot{\bar{x}}^\gamma(t+1) = \langle A \rangle^\Gamma(t) \vec{\times} \bar{x}^\gamma(t), \quad \bar{x}^\gamma(0) = \bar{x}_0^\gamma, \\ \langle A \rangle^\Gamma(t) \in \Sigma, \quad \bar{x}^\gamma(t) \in \Omega \end{aligned}$$

is a continuous-time linear dynamic S-system (may need to add an identity).

**11**

**Conclusion**

The matrix and vector multipliers were introduced first. Using them, general STPs, including MM-STPs and MV-STPs, were constructed. It was proved that the set of all matrices with an MM-STP becomes a semi-group. The set of vectors, consisting of all vectors of different dimensions, forms a dimension-varying state space. It was shown that the action of the set of matrices on the set of vectors described by an MV-STP forms S-systems, which are called generalised linear S-system. Then the general inner product is introduced for the set of vectors, which posed a metric topology on the set of vectors. Moreover, the norm of a matrix, which is considered as an operator acting on the set of vectors was estimated. In the light of the topology on state space and the operator norm of matrices, the linear S-systems become linear dynamic systems. Certain properties were revealed. Finally, the matrix equivalence and the vector equivalence were introduced and the equivalence based quotient spaces (of matrices and of vectors, respectively) were obtained. In addition, the structure of linear dynamic systems was also extended to quotient spaces.

As pointed by anonymous reviews that an interesting topic is how to apply the S-system framework to real (control) problems. This is also a major topic for our further study. In fact, to use this model to deal with cross-dimensional systems some additional works are necessary. For instance, (i) how to calculate the trajectory of an S-system; (ii) the relationship between trajectory on

$\mathbb{R}^n$

and the trajectory on quotient space, etc. We refer to [20] for these and some related topics. We also refer to [16] for an application to clutch control of vehicles.

# Bibliography


**REFERENCES**

1 Cheng D.: ‘Semi-tensor product of matrices and its application to Morgan's problem’, Sci. China, F, Inf. Sci., 2001, 44, (3), pp. 195–212

2 Cheng D., Ma J., Lu Q., : ‘Quadratic form of stable sub-manifold for power systems’, Int. J. Robost Nonlinear Control, 2004, 14, pp. 773–788

3 Mei S., Liu F., Xe A.: ‘Semi-tensor product approach to transient process analysis of power systems’ (Tsinghua Universtiy Press, Beijing, 2010) (in Chinese)

4 Xe A., Wu F., Lu Q., : ‘Power system dynamic security region and its approximations’, IEEE Trans. Circuits Syst. I, 2006, 53, (12), pp. 2849–2859

5 Cheng D., Qi H., Li Z.: ‘Analysis and control of Boolean networks – a semi-tensor product approach’ (Springer, London, 2011)

6 Fornasini E., Valcher M.E.: ‘Observability, reconstructibility and state observers of Boolean control networks,’, IEEE Trans. Autom. Control, 2013, 58, (6), pp. 1390–1401

7 Laschov D., Margaliot M.: ‘A maximum principle for single-input Booolean control notworks’, IEEE Trans. Autom. Control, 2011, 56, (4), pp. 913–917

8 Cheng D.: ‘On finite potential games’, Automatica, 2014, 50, (7), pp. 1793–1801

9 Cheng D., He F., Qi H., : 'Modeling, analysis and control of networked evolutionary games', IEEE Trans. Autom. Control, 2015, 60, (9), pp. 2402–2415

10 Guo P., Wang Y., Li H.: 'Algebraic formulation and strategy optimization for a class of evolutionary networks via semi-tensor product method', Automatica, 2013, 49, pp. 3384–3389

11 Cheng D., Qi H., Zhao Y.: 'An introduction to semi-tensor product of matrices and its applications' (World Scientific, Singapore, 2012)

12 Fornasini E., Valcher M.E.: 'Recent developments in Boolean networks control', J. Control Decis., 2016, 3, (1), pp. 1–18

13 Lu J., Li H., Liu Y., : 'Survey on semi-tensor product method with its applications in logical networks and other finite-valued systems', IET Control Theory Applic., 2017, 11, (13), pp. 2040–2047

14 Muhammad A., Rushdi A., Ghaleb F.A.M.: 'A tutorial exposition of semi-tensor products of matrices with a stress on their representation of Boolean function', JKAU Comput. Sci., 2016, 5, pp. 3–30

15 Hua G.: 'Fundation of number theory' (Science Press, Beijing, 1957), (in Chinese)

16 Cheng D., Xu Z., Shen T.: 'Equivalence-based model of dimension-varying linear systems', IEEE TAC, under revision, (preprint: https://arxiv.org/submit/2414749)

17 Hungerford T.W.: 'Algebra' (Springer-Verlag, New York, 1974)

18 Ahsan J.: 'Monoids characterized by their quasi-injective S-systems', Semigroup Fourm, 1987, 36, pp. 285–292

19 Liu Z., Qiao H.: 'S-system theory of semigroup' (Science Press, Beijing, 2008, 2nd edn.), (in Chinese)

20 Cheng D.: 'On equivalence of matrices', Asian J. Math., 2019, 23, (2), pp. 257–348

21 Cheng D.: 'From dimension-free matrix theory to cross dimensional dynamic systems' (Elsevier, London, 2019)

22 Burris S., Sankappanavar H.P.: 'A course in universal algebra' (Springer, New York, 1981)

23 Greub W.: 'Linear algebra' (Springer-Verlag, New York, 1981, 4th edn.)

24 Abraham R., Marsden J.: 'Foundations of mechanics' (Benjamin/Cummings Pub., London, 1978, 2nd edn.)

25 Jamich K.: 'Topology' (Springer-Verlag, New York, 1984)

26 Horn R.A., Johnson C.R.: 'Matrix analysis' (Cambridge University Press, Cambridge, 1985)

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