Factorial Cluster Algebra

Chengpeng Wang

Tsinghua University, China
School of Software
Department of Mathematical Science

June 7, 2016

Outline

Definition-Cluster Algebra

- Let m,n and p be integers with

$$m \geq p \geq n \geq 1 \text{ and } m > 1.$$

Definition-Cluster Algebra

- Let m,n and p be integers with

$$m \geq p \geq n \geq 1 \textit{ and } m > 1.$$

- Assume that K is a field of characteristic 0 or $K = \mathbb{Z}$. Let $\mathcal{F} = K(X_1, ..., X_m)$ be the field of rational functions in m variables.

Definition-Cluster Algebra

- Let m,n and p be integers with

$$m \geq p \geq n \geq 1 \textit{ and } m > 1.$$

- Assume that K is a field of characteristic 0 or $K = \mathbb{Z}$. Let $\mathcal{F} = K(X_1, ..., X_m)$ be the field of rational functions in m variables.
- A seed of $\mathcal{F}$ is a pair $(\mathbf{x}, B)$ such that the following hold:
  (1) $B \in M_{m,n}(\mathbb{Z})$,
  (2) B is connected,
  (3) The principal part of B is skew-symmetrizable,
  (4) $\mathbf{x} = (x_1, ..., x_m)$ is an m-tuple of elements of $\mathcal{F}$ such that $x_1, ...x_m$ are algebraically independent over K.

Definition-Cluster Algebra

- Define the mutation of $(\mathbf{x}, B)$ at k as

$$\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$$

Definition-Cluster Algebra

- Define the mutation of $(\mathbf{x}, B)$ at k as

$$\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$$

- $B' = (b'_{ij})$ is defined as

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise,} \end{cases}$$

and $\mathbf{x}'$=$(x'_1, ..., x'_m)$ is defined as

$$x'_s = \begin{cases} x_k^{-1} \prod_{b_{ik}>0} x_i^{b_{ik}} + x_k^{-1} \prod_{b_{ik}<0} x_i^{-b_{ik}} & \text{if } s = k, \\ x_s & \text{otherwise.} \end{cases}$$

Definition-Cluster Algebra

- Define the mutation of $(\mathbf{x}, B)$ at k as

$$\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$$

- $B' = (b'_{ij})$ is defined as

$$b'_{ij} = \begin{cases} -b_{ij} & \textit{if } i = k \textit{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \textit{otherwise,} \end{cases}$$

and $\mathbf{x}'$=$(x'_1, ..., x'_m)$ is defined as

$$x'_s = \begin{cases} x_k^{-1} \prod_{b_{ik}>0} x_i^{b_{ik}} + x_k^{-1} \prod_{b_{ik}<0} x_i^{-b_{ik}} & \textit{if } s = k, \\ x_s & \textit{otherwise.} \end{cases}$$

- The equality

$$x_k \, x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \tag{1}$$

is called an exchange relation.

Definition-Cluster Algebra

- **Proposition 1.2** Denote $\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$, where $(\mathbf{x}, B)$ is a seed. Then $(\mathbf{x}', B')$ is a seed.

Definition-Cluster Algebra

- **Proposition 1.2** Denote $\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$, where $(\mathbf{x}, B)$ is a seed. Then $(\mathbf{x}', B')$ is a seed.
- **Proposition 1.3** $\mu_k \mu_k(\mathbf{x}, B) = (\mathbf{x}, B)$

Definition-Cluster Algebra

- **Proposition 1.2** Denote $\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$, where $(\mathbf{x}, B)$ is a seed. Then $(\mathbf{x}', B')$ is a seed.
- **Proposition 1.3** $\mu_k \mu_k(\mathbf{x}, B) = (\mathbf{x}, B)$
- Mutation equivalent

$$(\mathbf{x}, B) \sim (\mathbf{y}, C) \Longleftrightarrow \mu_{i_t} \cdots \mu_{i_2} \mu_{i_1}(\mathbf{x}, B) = (\mathbf{y}, C)$$

Definition-Cluster Algebra

- **Proposition 1.2** Denote $\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$, where $(\mathbf{x}, B)$ is a seed. Then $(\mathbf{x}', B')$ is a seed.
- **Proposition 1.3** $\mu_k \mu_k(\mathbf{x}, B) = (\mathbf{x}, B)$
- Mutation equivalent

$$(\mathbf{x}, B) \sim (\mathbf{y}, C) \Longleftrightarrow \mu_{i_t} \cdots \mu_{i_2} \mu_{i_1}(\mathbf{x}, B) = (\mathbf{y}, C)$$

- For a seed $(\mathbf{x}, B)$ of $\mathcal{F}$ let

$$\mathcal{X}_{(\mathbf{x},B)} = \bigcup_{(\mathbf{y},C) \sim (\mathbf{x},B)} \{y_1, ..., y_n\}$$

Definition-Cluster Algebra

- **cluster algebra** $\mathcal{A}(\mathbf{x}, B)$ associated to $(\mathbf{x}, B)$ is the L-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}_{(\mathbf{x},B)}$, where

$$L = K[x_{n+1}^{\pm 1}, ..., x_p^{\pm 1}, x_{p+1}, ...x_m]$$

Definition-Cluster Algebra

- **cluster algebra** $\mathcal{A}(\mathbf{x}, B)$ associated to $(\mathbf{x}, B)$ is the L-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}_{(\mathbf{x},B)}$, where

$$L = K[x_{n+1}^{\pm 1}, ..., x_p^{\pm 1}, x_{p+1}, ...x_m]$$

- **cluster variables**: elements in $\mathcal{X}_{(\mathbf{x},B)}$

Definition-Cluster Algebra

- **cluster algebra** $\mathcal{A}(\mathbf{x}, B)$ associated to $(\mathbf{x}, B)$ is the L-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}_{(\mathbf{x},B)}$, where

$$L = K[x_{n+1}^{\pm 1}, ..., x_p^{\pm 1}, x_{p+1}, ...x_m]$$

- **cluster variables**: elements in $\mathcal{X}_{(\mathbf{x},B)}$
- **cluster coefficients**: x_i for $n+1 \leq i \leq m$

Definition-Acyclic Cluster Algebra

- Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$ with $B = (b_{ij})$. Let $\sum(B)$ be the quiver with vertices 1,...,n, and arrows $i \to j$ for all $1 \leq i, j \leq n$ with $b_{ij} > 0$. So $\sum(B)$ encodes the sign-pattern of the principal part of B.

Definition-Acyclic Cluster Algebra

- Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$ with $B = (b_{ij})$. Let $\sum(B)$ be the quiver with vertices 1,...,n, and arrows $i \to j$ for all $1 \leq i, j \leq n$ with $b_{ij} > 0$. So $\sum(B)$ encodes the sign-pattern of the principal part of B.
- The seed $(\mathbf{x}, B)$ and B are called acyclic if $\sum(B)$ does not contain any oriented cycle. The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is acyclic if there exists an acylic seed $(\mathbf{y}, C)$ with $(\mathbf{y}, C) \sim (\mathbf{x}, B)$

Laurent Phenomenon

- For a seed $(\mathbf{x}, B)$ of $\mathcal{F}$ let

$$\mathcal{L}_{\mathbf{x}} = K[x_1^{\pm 1}, ..., x_n^{\pm 1}, x_{n+1}^{\pm 1}, ..., x_p^{\pm 1}, x_{p+1}, ...x_m]$$

be the location of $K[x_1, ..., x_m]$ at $x_1 x_2 \cdots x_p$, and let

$$\mathcal{L}_{\mathbf{x},\mathcal{Z}} = K[x_1^{\pm 1}, ..., x_n^{\pm 1}, x_{n+1}, ..., x_m]$$

be the localization of $\mathcal{Z}[x_1, ..., x_m]$ at $x_1 x_2 \cdots x_n$. Let's consider $\mathcal{L}_{\mathbf{x}}$ and $\mathcal{L}_{\mathbf{x},\mathcal{Z}}$ as subrings the field of $\mathcal{F}$.

Laurent Phenomenon

- For a seed $(\mathbf{x}, B)$ of $\mathcal{F}$ let

$$\mathcal{L}_{\mathbf{x}} = K[x_1^{\pm 1}, ..., x_n^{\pm 1}, x_{n+1}^{\pm 1}, ..., x_p^{\pm 1}, x_{p+1}, ...x_m]$$

  be the location of $K[x_1, ..., x_m]$ at $x_1 x_2 \cdots x_p$, and let

$$\mathcal{L}_{\mathbf{x},\mathcal{Z}} = K[x_1^{\pm 1}, ..., x_n^{\pm 1}, x_{n+1}, ..., x_m]$$

  be the localization of $\mathcal{Z}[x_1, ..., x_m]$ at $x_1 x_2 \cdots x_n$. Let's consider $\mathcal{L}_{\mathbf{x}}$ and $\mathcal{L}_{\mathbf{x},\mathcal{Z}}$ as subrings the field of $\mathcal{F}$.
- Let y be a cluster variable of $\mathcal{A}(\mathbf{x}, B)$. We have

$$y \in \bigcap_{(\mathbf{y},C)\sim(\mathbf{x},B)} \mathcal{L}_{\mathbf{y},\mathcal{Z}}$$

$$\mathcal{A}(\mathbf{x}, B) \subseteq \bigcap_{(\mathbf{y},C)\sim(\mathbf{x},B)} \mathcal{L}_{\mathbf{y}}$$

  .

Invertible elements in cluster algebras

- **Lemma 2.1** For a seed $(\mathbf{x}, B)$ of $\mathcal{F}$ we have

$$\mathcal{L}_{\mathbf{x}}^{\times} = \{\lambda x_1^{a_1} \cdots x_p^{a_p} | \lambda \in K^{\times}, a_i \in \mathbb{Z}\}.$$

Invertible elements in cluster algebras

- **Lemma 2.1** For a seed $(\mathbf{x}, B)$ of $\mathcal{F}$ we have

$$\mathcal{L}_{\mathbf{x}}^{\times} = \{\lambda x_1^{a_1} \cdots x_p^{a_p} | \lambda \in K^{\times}, a_i \in \mathbb{Z}\}.$$

- **Theorem 2.2** For any seed $(\mathbf{x}, B)$ of $\mathcal{F}$ we have

$$\mathcal{A}(\mathbf{x}, B)^{\times} = \{\lambda x_{n+1}^{a_{n+1}} \cdots x_p^{a_p} | \lambda \in K^{\times}, a_i \in \mathbb{Z}\}.$$

Invertible elements in cluster algebras

- **Lemma 2.1** For a seed $(\mathbf{x}, B)$ of $\mathcal{F}$ we have

$$\mathcal{L}_{\mathbf{x}}^{\times} = \{\lambda x_1^{a_1} \cdots x_p^{a_p} | \lambda \in K^{\times}, a_i \in \mathbb{Z}\}.$$

- **Theorem 2.2** For any seed $(\mathbf{x}, B)$ of $\mathcal{F}$ we have

$$\mathcal{A}(\mathbf{x}, B)^{\times} = \{\lambda x_{n+1}^{a_{n+1}} \cdots x_p^{a_p} | \lambda \in K^{\times}, a_i \in \mathbb{Z}\}.$$

- **Corollary 2.3** For any seed $(\mathbf{x}, B)$ of $\mathcal{F}$ the following propositions hold:
(1)Let y and z be non-zero elements in $\mathcal{A}(\mathbf{x}, B)$. Then y and z are associated if and only if there exist $a_{n+1}, \cdots, a_p \in \mathbb{Z}$ *and* $\lambda \in K^{\times}$ with

$$y = \lambda x_{n+1}^{a_{n+1}} \cdots x_p^{a_p} z.$$

(2)Let y and z be cluster variables of $\mathcal{A}(\mathbf{x}, B)$. Then y and z are associated if and only if y=z.

Irreducibility of Cluster Variables

- **Theorem 3.1** Let ($\mathbf{x}$,B) be a seed of $\mathcal{F}$. Then any cluster variable in $\mathcal{A}(\mathbf{x}, B)$ is irreducible.

Outline

not UFD: Sufficient Condition I

- Denote the polynomial

$$f_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}$$

as exchange polynomial.

not UFD: Sufficient Condition I

- Denote the polynomial

$$f_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}$$

  as exchange polynomial.
- For a matrix $A \in M_{m,n}(\mathbb{Z})$ and $1 \leq i \leq n$, denote $c_i(A)$ to be the ith column of A.

not UFD: Sufficient Condition I

- Denote the polynomial

$$f_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}$$

as exchange polynomial.

- For a matrix $A \in M_{m,n}(\mathbb{Z})$ and $1 \leq i \leq n$, denote $c_i(A)$ to be the ith column of A.
- **Proposition 4.1** Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$. Assume that $c_k(B) = c_s(B)$ or $c_k(B) = -c_s(B)$ for some $k \neq s$ with $b_{ks} = 0$. Then $\mathcal{A}(\mathbf{x}, B)$ is not factorial.

not UFD: Sufficient Condition I

- Denote the polynomial

$$f_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}$$

  as exchange polynomial.
- For a matrix $A \in M_{m,n}(\mathbb{Z})$ and $1 \leq i \leq n$, denote $c_i(A)$ to be the ith column of A.
- **Proposition 4.1** Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$. Assume that $c_k(B) = c_s(B)$ or $c_k(B) = -c_s(B)$ for some $k \neq s$ with $b_{ks} = 0$. Then $\mathcal{A}(\mathbf{x}, B)$ is not factorial.
- **Proposition 4.2** If there are two distinct indices $i, j \in \{1, \cdots, n\}$ such that $f_i = f_j$, then $\mathcal{A}(\mathbf{x}, B)$ is not a unique factorization domain.

not UFD: Sufficient Condition I

- Denote the polynomial

$$f_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}$$

  as exchange polynomial.
- For a matrix $A \in M_{m,n}(\mathbb{Z})$ and $1 \leq i \leq n$, denote $c_i(A)$ to be the ith column of A.
- **Proposition 4.1** Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$. Assume that $c_k(B) = c_s(B)$ or $c_k(B) = -c_s(B)$ for some $k \neq s$ with $b_{ks} = 0$. Then $\mathcal{A}(\mathbf{x}, B)$ is not factorial.
- **Proposition 4.2** If there are two distinct indices $i, j \in \{1, \cdots, n\}$ such that $f_i = f_j$, then $\mathcal{A}(\mathbf{x}, B)$ is not a unique factorization domain.
- The propositions above are equivalent.

not UFD: Sufficient Condition II

- **Proposition 4.3** Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$. Assume that there exists some $1 \leq k \leq n$ such that the polynomial $X^d + Y^d$ is not irreducible in K[X,Y], where $d = gcd(b_{1k}, ..., b_{mk})$ is the greatest common divisor of $b_{1k}, ..., b_{mk}$. Then $\mathcal{A}(\mathbf{x}, B)$ is not factorial.

not UFD: Sufficient Condition II

- **Proposition 4.3** Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$. Assume that there exists some $1 \leq k \leq n$ such that the polynomial $X^d + Y^d$ is not irreducible in K[X,Y], where $d = gcd(b_{1k}, ..., b_{mk})$ is the greatest common divisor of $b_{1k}, ..., b_{mk}$. Then $\mathcal{A}(\mathbf{x}, B)$ is not factorial.
- **Proposition 4.4** If there is some $i \in \{1, \cdots, n\}$ such that f_i is reducible in $K[x_i : 1 \leq i \leq m]$, then $\mathcal{A}(\mathbf{x}, B)$ is not a unique factorization domain.

not UFD: Sufficient Condition II

- **Proposition 4.3** Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$. Assume that there exists some $1 \leq k \leq n$ such that the polynomial $X^d + Y^d$ is not irreducible in K[X,Y], where $d = gcd(b_{1k}, ..., b_{mk})$ is the greatest common divisor of $b_{1k}, ..., b_{mk}$. Then $\mathcal{A}(\mathbf{x}, B)$ is not factorial.
- **Proposition 4.4** If there is some $i \in \{1, \cdots, n\}$ such that f_i is reducible in $K[x_i : 1 \leq i \leq m]$, then $\mathcal{A}(\mathbf{x}, B)$ is not a unique factorization domain.
- The propositions above are equivalent.

Outline

UFD: Sufficient Condition

- **Theorem 5.1** Let $\mathbf{y}$ and $\mathbf{z}$ be disjoint clusters of $\mathcal{A}(\mathbf{x}, B)$ and let U be a factorial subalgebra of $\mathcal{A}(\mathbf{x}, B)$ such that

$$\{y_1, \cdots, y_n, z_1, \cdots, z_n, x_{n+1}^{\pm 1}, ..., x_p^{\pm 1}, x_{p+1}, ..., x_m\} \subset U.$$

Then we have

$$U = \mathcal{A}(\mathbf{x}, B) = U(\mathbf{y}, \mathbf{z}).$$

Outline

Some Definitions

- For all $i \in \{1, 2, ..., n\}$ define an ideal

$$I_i = (x_i, f_i) \subseteq K[x_1, x_2, ..., x_m].$$

Denote abbreviation $R = K[x_i : 1 \leq i \leq m]$ for the polynomial ring.

Some Definitions

- For all $i \in \{1, 2, ..., n\}$ define an ideal

$$I_i = (x_i, f_i) \subseteq K[x_1, x_2, ..., x_m].$$

  Denote abbreviation $R = K[x_i : 1 \leq i \leq m]$ for the polynomial ring.
- **Remark 6.9** We have

$$\mathcal{A}(\mathbf{x}, B) = \bigcup_{\mathbf{a} \in \mathbb{N}^n} \{\frac{\lambda P}{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}} : P \in I_1^{a_1} I_2^{a_2} \cdots I_n^{a_n}. \lambda \in \mathcal{A}(\mathbf{x}, B)^{\times}\}.$$

Some Definitions

- For all $i \in \{1, 2, ..., n\}$ define an ideal

$$I_i = (x_i, f_i) \subseteq K[x_1, x_2, ..., x_m].$$

  Denote abbreviation $R = K[x_i : 1 \leq i \leq m]$ for the polynomial ring.
- **Remark 6.9** We have

$$\mathcal{A}(\mathbf{x}, B) = \bigcup_{\mathbf{a} \in \mathbb{N}^n} \{\frac{\lambda P}{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}} : P \in I_1^{a_1} I_2^{a_2} \cdots I_n^{a_n}.\lambda \in \mathcal{A}(\mathbf{x}, B)^{\times}\}.$$

- Remark 6.9 can be obtained by the Laurent Phenomenon.

A Conjectured Decomposition

- **Conjecture 6.10** For all $\mathbf{a} \in \mathbb{N}^n$ we have $\mathbf{I}^{\mathbf{a}} = I_1^{a_1} \cap I_2^{a_2} \cap \ldots \cap I_n^{a_n}$.

A Conjectured Decomposition

- **Conjecture 6.10** For all $\mathbf{a} \in \mathbb{N}^n$ we have $\mathbf{I}^{\mathbf{a}} = I_1^{a_1} \cap I_2^{a_2} \cap \ldots \cap I_n^{a_n}$.
- **Remark 6.12** For all $\mathbf{a} \in \mathbb{N}^n$, define a set

$$S(\mathbf{a}) = \{P \in I_1^{a_1} \cap I_2^{a_2} \cap \ldots \cap I_n^{a_n} : P \textit{ is not divided by } x_i \textit{ if } 1 \leq i \leq n, a_i \neq 0\}$$

If the conjecture above holds for all $\mathbf{a} \in \mathbb{N}^n$, then it yields a decomposition

$$\mathcal{A}(\mathbf{x}, B) = \bigcup_{\mathbf{a} \in \mathbb{N}^n} \frac{\mathcal{A}(\mathbf{x}, B)^{\times} S(\mathbf{a})}{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}.$$

Main Theorem

- For each none-zero polynomial $P \in R$ and each $1 \leq i \leq n$ there is a largest natural number $a_i \in \mathbb{N}$ s.t. $P \in I_i^{a_i}$. Define $m_i(P)$ to be the unique natural number such that $P \in I_i^{m_i(P)} \backslash I_i^{m_i(P)+1}$. In particular, we define a momomial

$$M(P) = \prod_{i=1}^{n} x_i^{m_i(P)} \in R.$$

Main Theorem

- For each none-zero polynomial $P \in R$ and each $1 \leq i \leq n$ there is a largest natural number $a_i \in \mathbb{N}$ s.t. $P \in I_i^{a_i}$. Define $m_i(P)$ to be the unique natural number such that $P \in I_i^{m_i(P)} \backslash I_i^{m_i(P)+1}$. In particular, we define a momomial

$$M(P) = \prod_{i=1}^{n} x_i^{m_i(P)} \in R.$$

- **Theorem 6.13** If $\mathbf{I}^{\mathbf{a}} = I_1^{a_1} \cap I_2^{a_2} \cap \ldots \cap I_n^{a_n}$ holds for all $\mathbf{a} \in \mathbb{N}^n$, then $\mathcal{A}(\mathbf{x}, B)$ is a unique factorization domain. Moreover, the set irreducible elements in $\mathcal{A}(\mathbf{x}, B)$ is

$$(\{\lambda x_i : 1 \leq i \leq n, \lambda \in \mathcal{A}(\mathbf{x}, B)^{\times}\}$$

$$\cup \{\frac{\lambda P}{M(P)} : P \in R \ irreducible, \lambda \in \mathcal{A}(\mathbf{x}, B)^{\times}\} \setminus \mathcal{A}(\mathbf{x}, B)^{\times}.$$

Main Theorem

- **Lemma 7.1** Assume $\mathbf{a} \in \mathbb{N}^n$ and $\mathbf{I^b} = I_1^{b_1} \cap I_2^{b_2} \cap \cdots \cap I_n^{b_n}$ holds for all $\mathbf{b} \in \mathbb{N}^n$ s.t. $\sum_{i=1}^n b_i < \sum_{i=1}^n a_i$. Assume that the index $i \in \{1, 2, \cdots, n\}$ is either a sink or a source s.t. $a_i \neq 0$. Suppose that there exists an index $1 \leq j \leq n$ with $a_j \neq 0$ s.t. i and j are adjacent. Then $\mathbf{I^a} = I_1^{a_1} \cap I_2^{a_2} \cap \cdots \cap I_n^{a_n}$

Main Theorem

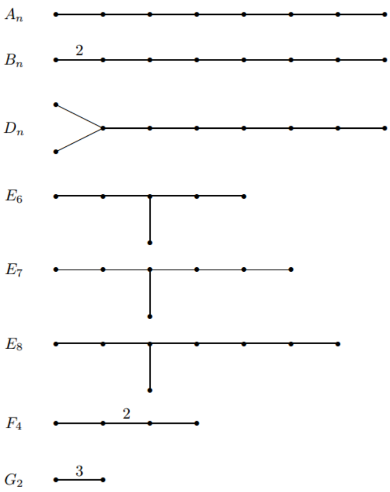
- **Lemma 7.1** Assume $\mathbf{a} \in \mathbb{N}^n$ and $\mathbf{I}^{\mathbf{b}} = I_1^{b_1} \cap I_2^{b_2} \cap \cdots \cap I_n^{b_n}$ holds for all $\mathbf{b} \in \mathbb{N}^n$ s.t. $\sum_{i=1}^n b_i < \sum_{i=1}^n a_i$. Assume that the index $i \in \{1, 2, \cdots, n\}$ is either a sink or a source s.t. $a_i \neq 0$. Suppose that there exists an index $1 \leq j \leq n$ with $a_j \neq 0$ s.t. i and j are adjacent. Then $\mathbf{I}^{\mathbf{a}} = I_1^{a_1} \cap I_2^{a_2} \cap \cdots \cap I_n^{a_n}$
- $N(i) = \{j : 1 \leq j \leq n, b_{ij} \neq 0\}$ for all $i \in \{1, 2, \cdots, m\}$.

Main Theorem

- **Lemma 7.1** Assume $\mathbf{a} \in \mathbb{N}^n$ and $\mathbf{I^b} = I_1^{b_1} \cap I_2^{b_2} \cap \cdots \cap I_n^{b_n}$ holds for all $\mathbf{b} \in \mathbb{N}^n$ s.t. $\sum_{i=1}^n b_i < \sum_{i=1}^n a_i$. Assume that the index $i \in \{1, 2, \cdots, n\}$ is either a sink or a source s.t. $a_i \neq 0$. Suppose that there exists an index $1 \leq j \leq n$ with $a_j \neq 0$ s.t. i and j are adjacent. Then $\mathbf{I^a} = I_1^{a_1} \cap I_2^{a_2} \cap \cdots \cap I_n^{a_n}$
- $N(i) = \{j : 1 \leq j \leq n, b_{ij} \neq 0\}$ for all $i \in \{1, 2, \cdots, m\}$.
- **Lemma 7.2** Put $\mathbf{a} \in \mathbb{N}^n$. Suppose that i is a mutable index s.t. $a_i \neq 0$ and $\mathbf{I^b} = I_1^{b_1} \cap I_2^{b_2} \cap \cdots \cap I_n^{b_n}$ holds for all $\mathbf{b} \in \mathbb{N}^n$ such that $\sum_{i=1}^n b_i < \sum_{i=1}^n a_i$. Suppose that one of the following conditions holds:
  (1) For all indices $j \in N(i)$, we have $a_j = 0$.
  (2) The initial exchange polynomial f_i has the form $f_i = x_k + M_i$ for some index k and some monomials $M_i \in R$. Suppose that for all neighbors $j \in N(k) \setminus \{i\}$, we have $a_j = 0$.
  Then $\mathbf{I^a} = I_1^{a_1} \cap I_2^{a_2} \cap \cdots \cap I_n^{a_n}$

Dynkin Type

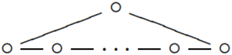
- Dynkin type is finite type of cluster algebras, which have the finite number of cluster variables. And each finite-type cluster algebra is equivalent to a Dynkin type.



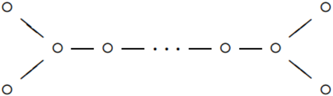
Euclidean Type

- Euclidean type is a infinite type of cluster algebras, which have the infinite number of cluster variables.

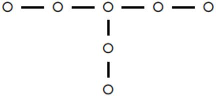
$\widetilde{A}_n$:



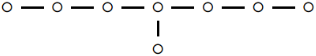
$\widetilde{D}_n$:



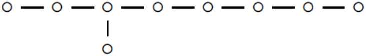
$\widetilde{E}_6$:



$\widetilde{E}_7$:



$\widetilde{E}_8$:



Factorization of Dynkin Type

- For n $\neq 3$, A_n is a unique factorial type, while for n = 3, A_n is not a unique factorial type.
For all $n \geq 4$, D_n is not a unique factorial type.
For E_6, E_7, E_8, they are all unique factorial types.

Factorization of $\widetilde{A_n}$

- **Theorem 10.1** The cluster algebra generated by the type $\widetilde{A_n}$ is a unique domain.

Factorization of $\widetilde{A_n}$

- **Theorem 10.1** The cluster algebra generated by the type $\widetilde{A_n}$ is a unique domain.
- **Poof** Suppose there are k($1 \leq k \leq n/2$) clockwise edges and n+1-k anticlockwise edges. In this orientation, the diagram has the most sources and sinks, which means clockwise and anticlockwise edges are presented in turn and other n+1-2k anticlockwise edges are presented in sequences. Suppose that the points are labeled from 1 to n+1 in a anticlockwise direction. Suppose index 1 to k+1 is a source or a sink, and index k+2 to n+1 is neither a source nor a sink.

Factorization of $\widetilde{A_n}$

- **Theorem 10.1** The cluster algebra generated by the type $\widetilde{A_n}$ is a unique domain.
- **Poof** Suppose there are k($1 \leq k \leq n/2$) clockwise edges and n+1-k anticlockwise edges. In this orientation, the diagram has the most sources and sinks, which means clockwise and anticlockwise edges are presented in turn and other n+1-2k anticlockwise edges are presented in sequences. Suppose that the points are labeled from 1 to n+1 in a anticlockwise direction. Suppose index 1 to k+1 is a source or a sink, and index k+2 to n+1 is neither a source nor a sink.
- Let P be a polynomial in the intersection $I_1^{a_1} \cap I_2^{a_2} \cap \cdots I_n^{a_n}$. We have to show that $P \in \mathbf{I^a}$ which we will prove by mathematical induction on $\sum_{i=1}^{n} a_i$. The base case is trival. Assume that the statement holds for all sequences with a smaller sum.

Factorization of $\widetilde{A_n}$

- If $k > 1$, then index $1, 2, 3$ is either a source or a sink.

Factorization of $\widetilde{A_n}$

- If $k > 1$, then index 1, 2, 3 is either a source or a sink.
- (1) If $a_1 > 0$ and $a_2 > 0$ or $a_{n+1} > 0$, then the claim follows from Lemma 7.1, because index 1 is a source or a sink and index 2 or index n is adjacent to index 0.

Factorization of $\widetilde{A_n}$

- If $k > 1$, then index $1, 2, 3$ is either a source or a sink.
- (1) If $a_1 > 0$ and $a_2 > 0$ or $a_{n+1} > 0$, then the claim follows from Lemma 7.1, because index 1 is a source or a sink and index 2 or index n is adjacent to index 0.
- (2) If $a_1 > 0$ and $a_2 = 0$ and $a_{n+1} = 0$, then the claim follows from Lemma 7.2(a), because $a_1 > 0$ and for all $j \in N(1)$, $a_j = 0$.

Factorization of $\widetilde{A_n}$

- If $k > 1$, then index $1, 2, 3$ is either a source or a sink.
- (1) If $a_1 > 0$ and $a_2 > 0$ or $a_{n+1} > 0$, then the claim follows from Lemma 7.1, because index 1 is a source or a sink and index 2 or index n is adjacent to index 0.
- (2) If $a_1 > 0$ and $a_2 = 0$ and $a_{n+1} = 0$, then the claim follows from Lemma 7.2(a), because $a_1 > 0$ and for all $j \in N(1)$, $a_j = 0$.
- (3) If $a_1 = 0$ and $a_2 = 0$, then the sequences of I_k is of the same form as in type A_{n+1} and can yield to the conclusion.

Factorization of $\widetilde{A_n}$

- If $k > 1$, then index $1, 2, 3$ is either a source or a sink.
- (1) If $a_1 > 0$ and $a_2 > 0$ or $a_{n+1} > 0$, then the claim follows from Lemma 7.1, because index 1 is a source or a sink and index 2 or index n is adjacent to index 0.
- (2) If $a_1 > 0$ and $a_2 = 0$ and $a_{n+1} = 0$, then the claim follows from Lemma 7.2(a), because $a_1 > 0$ and for all $j \in N(1), a_j = 0$.
- (3) If $a_1 = 0$ and $a_2 = 0$, then the sequences of I_k is of the same form as in type A_{n+1} and can yield to the conclusion.
- (4) If $a_1 = 0$ and $a_2 > 0, a_3 = 0$, then the claim follows from Lemma 7.2(a), because $a_2 > 0$ and for all $j \in N(2), a_j = 0$.

Factorization of $\widetilde{A}_n$

- If $k > 1$, then index $1, 2, 3$ is either a source or a sink.
- (1) If $a_1 > 0$ and $a_2 > 0$ or $a_{n+1} > 0$, then the claim follows from Lemma 7.1, because index 1 is a source or a sink and index 2 or index n is adjacent to index 0.
- (2) If $a_1 > 0$ and $a_2 = 0$ and $a_{n+1} = 0$, then the claim follows from Lemma 7.2(a), because $a_1 > 0$ and for all $j \in N(1)$, $a_j = 0$.
- (3) If $a_1 = 0$ and $a_2 = 0$, then the sequences of I_k is of the same form as in type A_{n+1} and can yield to the conclusion.
- (4) If $a_1 = 0$ and $a_2 > 0, a_3 = 0$, then the claim follows from Lemma 7.2(a), because $a_2 > 0$ and for all $j \in N(2), a_j = 0$.
- (5) If $a_1 = 0$, $a_2 > 0$ and $a_3 > 0$, then the claim follows form Lemma 7.1, because index 3 is a sink or a source which is adjacent to index 2 and $a_2 > 0, a_3 > 0$.

Factorization of $\widetilde{A_n}$

- If $k = 1$, then the same argument for the situation (1)-(4). In situation (5), it is obvious that index 3 is not a sink nor a source, then $f_3 = x_2 + x_4$, and the claim follows from Lemma 7.2(b), because k=2, i=3, N(k)=$\{1, 3\}$ and $a_1 = 0$.

Factorization of $\widetilde{A_n}$

- If $k = 1$, then the same argument for the situation (1)-(4). In situation (5), it is obvious that index 3 is not a sink nor a source, then $f_3 = x_2 + x_4$, and the claim follows from Lemma 7.2(b), because k=2, i=3, N(k)=$\{1, 3\}$ and $a_1 = 0$.
- In all situation, we can safely get the conclusion that $P \in \mathbf{I^a}$. According to the main theorem, we can get the unique factorization of type $\widetilde{A_n}$.

Factorization of $\widetilde{D_n}$

- **Theorem 10.2** The cluster algebra generated by the type $\widetilde{D_n}$ is not a unique domain.

$$\begin{tikzcd}
n+1 \arrow[dr] & & & & & n \\
& 2 \arrow[r] & 3 & \cdots & n-2 \arrow[ur] \arrow[dr] & \\
1 \arrow[ur] & & & & & n-1
\end{tikzcd}$$

Factorization of $\widetilde{D_n}$

- **Theorem 10.2** The cluster algebra generated by the type $\widetilde{D_n}$ is not a unique domain.

$$\begin{tikzcd}
n+1 \arrow[dr] & & & & & n \\
& 2 \arrow[r] & 3 & \cdots & n-2 \arrow[ur] \arrow[dr] & \\
1 \arrow[ur] & & & & & n-1
\end{tikzcd}$$

- The diagram of $\widetilde{D_n}$ is presented above. No matter how to orient the diagram, the exchange polynomial of index n+1 and index 1 are both $1 + x_2$. According to Proposition 4.2, we can claim that the cluster algebra generated by the type $\widetilde{D_n}$ is not a unique domain.

Factorization of $\widetilde{E_6}$

- **Theorem 10.3** The cluster algebra generated by the type $\widetilde{E_6}$ is a unique domain.

$$
\begin{tikzcd}
 & & 6 \arrow[d] & & \\
 & & 5 \arrow[d] & & \\
7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l]
\end{tikzcd}
$$

Factorization of $\widetilde{E_6}$

- **Theorem 10.3** The cluster algebra generated by the type $\widetilde{E_6}$ is a unique domain.

$$
\begin{tikzcd}
 & & 6 \arrow[d] & & \\
 & & 5 \arrow[d] & & \\
7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l]
\end{tikzcd}
$$

- **Proof** Let P be a polynomial in the intersection $I_1^{a_1} \cap I_2^{a_2} \cap \cdots I_n^{a_n}$. We have to show that $P \in \mathbf{I^a}$ which we will prove by mathematical induction on $\sum_{i=1}^{n} a_i$. The base case is trival. Assume that the statement holds for all sequences with a smaller sum.

Factorization of $\widetilde{E}_6$

- $$\begin{tikzcd} & & 6 \arrow[d] & & \\ & & 5 \arrow[d] & & \\ 7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$$

Factorization of $\widetilde{E}_6$

- $$\begin{tikzcd} & & 6 \arrow[d] & & \\ & & 5 \arrow[d] & & \\ 7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$$

- (1) If $a_2, a_1 > 0$, then the claim follows from the Lemma 7.1.

Factorization of $\widetilde{E}_6$

- $$\begin{tikzcd} & & 6 \arrow[d] & & \\ & & 5 \arrow[d] & & \\ 7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$$
- (1) If $a_2, a_1 > 0$, then the claim follows from the Lemma 7.1.
- (2) If $a_2 = 0, a_1 > 0, a_7 = 0$, then the claim follows from Lemma 7.2(a), bacause i=1 and N(i)=$\{0, 2\}$.

Factorization of $\widetilde{E}_6$

- $$\begin{tikzcd} & & 6 \arrow[d] & & \\ & & 5 \arrow[d] & & \\ 7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$$
- (1) If $a_2, a_1 > 0$, then the claim follows from the Lemma 7.1.
- (2) If $a_2 = 0, a_1 > 0, a_7 = 0$, then the claim follows from Lemma 7.2(a), bacause i=1 and N(i)=$\{0, 2\}$.
- (3) If $a_2 = 0, a_1 > 0, a_7 > 0$, then the claim follows from Lemma 7.1, because index 7 is a source and $a_7, a_1 > 0$.

Factorization of $\widetilde{E}_6$

- $$\begin{tikzcd}
 & & 6 \arrow[d] & & \\
 & & 5 \arrow[d] & & \\
7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l]
\end{tikzcd}$$
- (1) If $a_2, a_1 > 0$, then the claim follows from the Lemma 7.1.
- (2) If $a_2 = 0, a_1 > 0, a_7 = 0$, then the claim follows from Lemma 7.2(a), bacause i=1 and N(i)=$\{0, 2\}$.
- (3) If $a_2 = 0, a_1 > 0, a_7 > 0$, then the claim follows from Lemma 7.1, because index 7 is a source and $a_7, a_1 > 0$.
- (4) If $a_2 = 0, a_1 = 0, a_7 = 0$, then it is the same form as in type A_5, which can get the conclusion.

Factorization of $\widetilde{E}_6$

- $$\begin{tikzcd} & & 6 \arrow[d] & & \\ & & 5 \arrow[d] & & \\ 7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$$

- (5) If $a_2 = 0, a_1 = 0, a_7 > 0$, then the claim follows from Lemma 7.2(a), because i=0 and N(i)=$\{1\}$.

Factorization of $\widetilde{E}_6$

- $$\begin{tikzcd} & & 6 \arrow[d] & & \\ & & 5 \arrow[d] & & \\ 7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$$
- (5) If $a_2 = 0, a_1 = 0, a_7 > 0$, then the claim follows from Lemma 7.2(a), because i=0 and N(i)=$\{1\}$.
- (6) If $a_2 > 0, a_1 = 0, a_7 = 0$, then it is the same form as in type E_6, which can get the conclusion.

Factorization of $\widetilde{E}_6$

- $$\begin{tikzcd} & & 6 \arrow[d] & & \\ & & 5 \arrow[d] & & \\ 7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$$
- (5) If $a_2 = 0, a_1 = 0, a_7 > 0$, then the claim follows from Lemma 7.2(a), because i=0 and N(i)=$\{1\}$.
- (6) If $a_2 > 0, a_1 = 0, a_7 = 0$, then it is the same form as in type E_6, which can get the conclusion.
- (7) If $a_2 > 0, a_1 = 0, a_7 > 0$, then the claim follows from Lemma 7.1(a), because i=0 and N(i)=$\{1\}$.

Factorization of $\widetilde{E}_6$

- $$\begin{tikzcd} & & 6 \arrow[d] & & \\ & & 5 \arrow[d] & & \\ 7 \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$$
- (5) If $a_2 = 0, a_1 = 0, a_7 > 0$, then the claim follows from Lemma 7.2(a), because i=0 and N(i)=$\{1\}$.
- (6) If $a_2 > 0, a_1 = 0, a_7 = 0$, then it is the same form as in type E_6, which can get the conclusion.
- (7) If $a_2 > 0, a_1 = 0, a_7 > 0$, then the claim follows from Lemma 7.1(a), because i=0 and N(i)=$\{1\}$.
- In all situation, we can safely get the conclusion that $P \in \mathbf{I^a}$. According to the main theorem, we can get the unique factorization of type $\widetilde{E}_6$.

Factorization of $\widetilde{E_7}$ and $\widetilde{E_8}$

- **Theorem 10.4** The cluster algebra generated by the type $\widetilde{E_7}$ and $\widetilde{E_8}$ is a unique domain.

$$\begin{tikzcd}
 & & & 8 \arrow[d] & & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 & 5 \arrow[l] & 6 \arrow[l] & 7 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & 9 \arrow[d] & & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l]
\end{tikzcd}$$

The same argument for index 8 and 4 in $\widetilde{E_7}$ and index 9 and 5 in $\widetilde{E_8}$ can get the conclusion.

Q&A

- Thank You For Listening