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§2.1 Weak Form Problem (Formulas, Spaces, Lemma)

1. Green's Formulas

$$(f'\cdot g)' = f''\cdot g + f'\cdot g' \Rightarrow f''\cdot g = (f'\cdot g)' - f'\cdot g'.$$

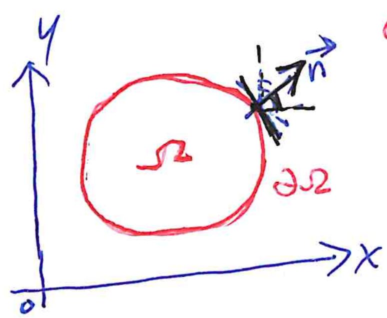
$$\Rightarrow \boxed{\int_a^b f''\cdot g\,dx = f'\cdot g\Big|_a^b - \int_a^b f'\cdot g'\,dx\ , \text{ Green's First formula}}$$

$$(pf'\cdot g)' = (pf')'\cdot g + pf'g'$$

$$\Rightarrow \boxed{\int_a^b (pf')'\cdot g\,dx = pf'\cdot g\Big|_a^b - \int_a^b pf'\cdot g'\,dx\ , \text{ generalized Green's First formula}}$$

2. Gauss's Formulas

2D: $$\iint_\Omega \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) g\,dx\,dy = -\iint_\Omega \left(f_x g_x + f_y\cdot g_y\right) dx\,dy + \int_{\partial\Omega} \frac{\partial f}{\partial \vec{n}}\cdot g\,ds.$$



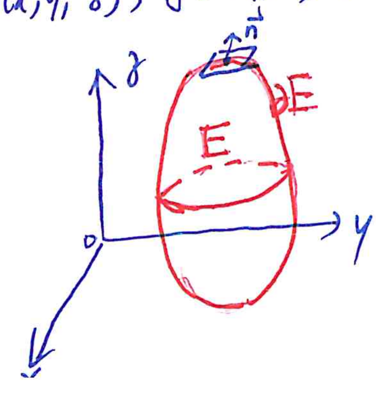
Given $f(x,y)$, $g(x,y)$

$$\frac{\partial f}{\partial \vec{n}} = f_x \cos(\vec{n},\vec{x}) + f_y \cos(\vec{n},\vec{y})$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right),\quad \nabla^2 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv \Delta$$

$$\Rightarrow \boxed{\iint_\Omega \Delta f\cdot g\,dx\,dy = -\iint_\Omega (f_x\cdot g_x + f_y\cdot g_y)\,dx\,dy + \int_{\partial\Omega}\frac{\partial f}{\partial \vec{n}}\cdot g\,ds.}$$

3D: Given $f(x,y,z)$, $g(x,y,z)$ in Ω



$$\Delta f = f_{xx} + f_{yy} + f_{zz}.$$

$$\frac{\partial f}{\partial \vec{n}} = f_x \cos(\vec{n},\vec{x}) + f_y \cos(\vec{n},\vec{y}) + f_z \cos(\vec{n},\vec{z}).$$

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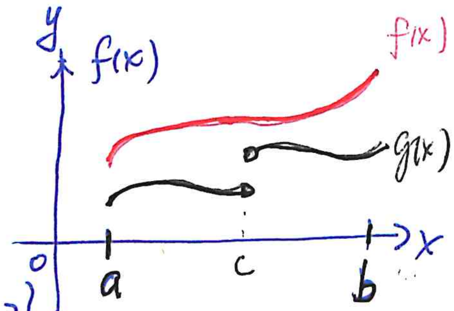
$$\boxed{\iiint\limits_{E} \Delta f \cdot g \, dx\,dy\,dz = -\iiint\limits_{E} (f_x g_x + f_y g_y + f_z g_z)\, dx\,dy\,dz + \iint\limits_{\partial E} \frac{\partial f}{\partial \vec{n}} \cdot g \, d\Omega.}$$

2. Space



(1) C-space

If $f(x)$ is continuous in $[a, b]$, we say

$f(x) \in C^0[a,b] = \{ V(x) \mid V \text{ is continuous in } [a,b] \}.$

$C^1[a,b] = \{ f(x) \mid f, f' \text{ are continuous in } [a,b] \}.$

$C^2[a,b] = \{ f(x) \mid f, f', f'' \text{ are continuous in } [a,b] \}.$

....

$C^\infty[a,b] = \{ f(x) \mid f, \text{ and its all orders of derivative are continuous in } [a,b] \}.$

Example: $f(x) = \sin(x)$, e^x

$C_0^1[a,b] = \{ f \mid f(x) \in C^1[a,b], \text{ and } f(a) = f(b) = 0 \}.$

$C_0^\infty[a,b] = \{ f \mid f(x) \in C^\infty[a,b] \text{ and } f(a) = f(b) = 0 \}.$

(2) H-space (Hilbert space)

$H^1[a,b] = \{ f(x) \mid \int_a^b [f^2 + (f')^2]\, dx < +\infty \}.$

$H^2[a,b] = \{ f(x) \mid \int_a^b [f^2 + (f')^2 + (f'')^2]\, dx < +\infty \}.$

$H_E^1[a,b] = \{ f(x) \mid f \in H^1[a,b],\ f(a) = 0 \}.$

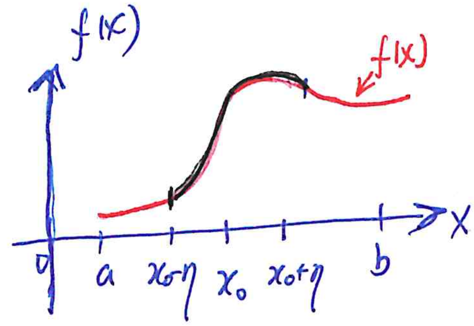
$H_0^1[a,b] = \{ f(x) \mid f \in H^1[a,b],\ f(a) = f(b) = 0 \}.$

$H^1(\Omega) = \{ f(x,y) \mid \iint\limits_{\Omega} [f^2 + (f_x)^2 + (f_y)^2]\, dx\,dy < +\infty \}.$

3. **Lemma**

Assume $f(x) \in C^0[a,b]$ and $\int_a^b f(x)\,\theta(x)\,dx = 0$ for any $\theta(x) \in C_0^\infty[a,b]$,

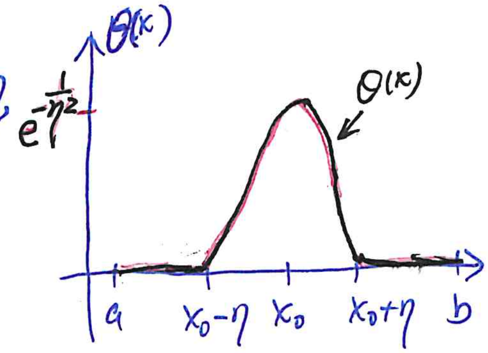
$\Rightarrow f(x) \equiv 0.$



Proof. If $f(x_0) \neq 0$ for some x_0.

Since $f(x)$ is continuous, we can find a small interval $[x_0-\eta, x_0+\eta]$ such that $f(x) > 0$.

Define $$\theta(x) = \begin{cases} e^{-\frac{1}{\eta^2-(x-x_0)^2}}, & x_0-\eta \le x \le x_0+\eta, \\ 0, & \text{otherwise.} \end{cases}$$



$\Rightarrow \theta(x) \in C_0^\infty[a,b]$

$$\Rightarrow\ 0 = \int_a^b f(x)\,\theta(x)\,dx = \int_{x_0-\eta}^{x_0+\eta} f(x)\, e^{-\frac{1}{\eta^2-(x-x_0)^2}}\,dx > 0, \text{ Contradiction!}$$

$\Rightarrow f(x) \equiv 0.$