§2.2 Weak Form

$\boxed{\text{Objective: Change PDE problems} \Rightarrow \text{Weak Form Problems}}$

Example 1.
$$\begin{cases} -\dfrac{d}{dx}\left(p(x)\dfrac{du}{dx}\right) + q(x)\,u = f(x), & a<x<b,\ p(x)\geq p_{min}>0,\ q\geq 0. \\ u(a)=0,\ u'(b)=0. \end{cases}$$

Find the weak Form.

Solution. Let $Lu \equiv -\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu \Rightarrow Lu = f.$

$$0 = \int_a^b (Lu-f)\cdot v\,dx = \int_a^b \left(p\frac{du}{dx}\cdot\frac{dv}{dx} + quv\right)dx - \int_a^b f\cdot v\,dx - p\frac{du}{dx}\cdot v\Big|_a^b.$$

$$\left[\mp\int_a^b (pf')'\,g\,dx = \mp pf'\cdot g\Big|_a^b \pm \int_a^b pf'\cdot g'\,dx\right]$$

$$= \int_a^b \left(p\frac{du}{dx}\frac{dv}{dx} + quv\right)dx - p(b)\,\cancel{u'(b)}^{\,0}\,v(b) + p(a)\,u'(a)\,\cancel{v(a)}_{\,0} - \int_a^b f\cdot v\,dx.$$

Choose $v(x)\in H^1[a,b]$ and $v(a)=0 \Rightarrow v(x)\in H^1_E[a,b]$.

$$\Rightarrow \underbrace{\int_a^b \left(p\frac{du}{dx}\frac{dv}{dx} + quv\right)dx}_{a(u,v)} = \underbrace{\int_a^b f v\,dx}_{(f,v)},$$

where (Functional)
$$\begin{cases} a(u,v) = \int_a^b \left(p\frac{du}{dx}\frac{dv}{dx} + quv\right)dx, \\ (f,v) = \int_a^b f v\,dx. \end{cases}$$

$a(u,v) = (f,v)$, called weak Form

$\Rightarrow$ **Weak Form Problem.**
Find u such that $a(u,v) = (f,v)$ for any $v\in H^1_E[a,b]$.

Properties of $a(u,v)$

(1) Symmetry $a(u,v) = a(v,u)$.

(2) Bilinear. $a(c_1u_1 + c_2u_2, v) = c_1 a(u_1,v) + c_2 a(u_2,v)$.

$$a(u, c_1 v_1 + c_2 v) = c_1 a(u, v_1) + c_2 a(u, v_2).$$

(3) Positive $\quad a(u,u) = \int_a^b \{p(\frac{du}{dx})^2 + q u^2] dx > 0.$

* PDE problem and Weak Form problem are equivalent if the solution $u_*(x) \in C^2[a,b]$ and $H_E^1[a,b]$.

Proof. $\Rightarrow \quad L u_* - f = 0 \quad \Rightarrow \quad a(u_*, v) - (f, v) = 0.$

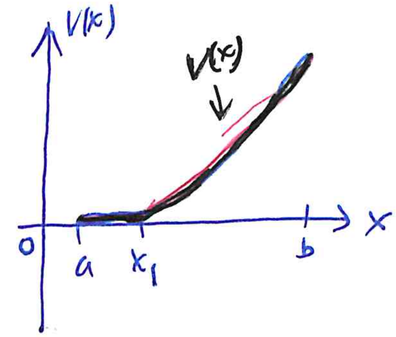
$\Leftarrow \quad 0 = a(u_*, v) - (f, v), \qquad v \in H_E^1(a,b], \ u_*(a) = 0, \ v(a) = 0$

$$= \int_a^b (p u_*' v)' dx - \int_a^b (p u_*')' v dx + \int_a^b q u_* v dx - \int_a^b f v dx$$

$$= p(b) u_*'(b) v(b) - p(a) u_*'(a) \underbrace{v(a)}_{0} + \int_a^b (L u_* - f) v dx.$$

Need to show $u_*'(b) = 0, \ L u_* - f = 0$ from $p(b) u_*'(b) v(b) = 0, \ \int_a^b (L u_* - f) v dx = 0$

choose $v(x) = \begin{cases} (x - x_1)^2, & a < x_1 \le x < b, \\ 0, & \text{others.} \end{cases}$

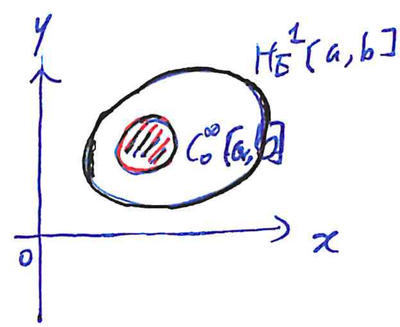


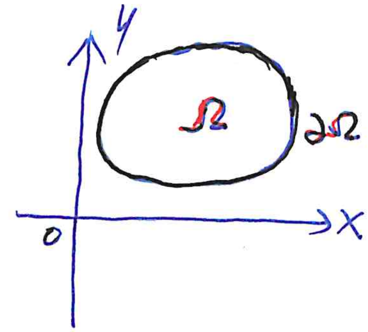
$\Rightarrow v(x) \in H_E^1(a, b], \ v(b) = (b - x_1)^2 > 0$

$\Rightarrow u_*'(b) = 0.$

Choose $v(x) \in C_0^\infty[a,b] \subset H_E^1(a,b]$

$$\int_a^b (L u_* - f) v dx = 0 \xRightarrow{\text{Lemma}} L u_* - f = 0.$$





Example 2. $\begin{cases} -(u_{xx}+u_{yy}) = f(x,y), & (x,y)\in \Omega, \\ u(x,y)\big|_{\partial\Omega} = 0. \end{cases}$

Find the weak Form.

Solution. $-\Delta u = f$

$$\Rightarrow 0 = \iint_{\Omega} (-\Delta u - f)\, v\, dx dy$$

$$= \iint_{\Omega} (u_x v_x + u_y v_y)\, dx dy - \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} v\, ds - \iint_{\Omega} f v\, dx dy$$

$$\underline{\underline{v|_{\partial\Omega}=0}} \quad \underbrace{\iint_{\Omega} (u_x v_x + u_y v_y)\, dx dy}_{a(u,v)} - \underbrace{\iint_{\Omega} f v\, dx dy}_{(f,v)}, \quad v \in H_0^1(\Omega).$$

$\Rightarrow$ $\boxed{\begin{array}{l} \text{Find } u \text{ s.t. } a(u,v) - (f,v) = 0, \text{ for any } v\in H_0^1(\Omega), \\ \text{where } a(u,v) = \iint_{\Omega} (u_x v_x + u_y v_y)\, dx dy, \quad (f,v) = \iint_{\Omega} f v\, dx dy. \end{array}}$

* $a(u,v)$ has to satisfy three properties: symmetry, bilinear, positive.

Example 3. $\begin{cases} -\left[\frac{\partial}{\partial x}\left(a_1 \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(a_1 \frac{\partial u}{\partial y}\right)\right] + a_0 u = f, & (x,y)\in\Omega, \\ u(x,y)\big|_{\partial\Omega} = 0. \end{cases}$

Find the weak Form

Solution. $-\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu = f$ in Example 1, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$\nabla\cdot(a_1 \nabla u) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot\left(a_1\frac{\partial u}{\partial x}, a_1\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(a_1\frac{\partial u}{\partial y}\right)$. dot product

$$\Rightarrow -\nabla\cdot(a_1\nabla u) + a_0 u = f.$$

(Gauss's Formula. $\iint_{\Omega} \nabla\cdot(a_1\nabla u)\, v\, dx dy = -\iint_{\Omega} a_1 \nabla u \cdot \nabla v\, dx dy + \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} v\, ds$)

$$\Rightarrow 0 = \iint_{\Omega} [-\nabla\cdot(a_1\nabla u) + a_0 u - f] v \, dx\, dy$$

$$= \iint_{\Omega} a_1 \nabla u\cdot\nabla v \, dx\, dy - \int_{\partial\Omega} a_1 \frac{\partial u}{\partial n} v \, ds + \iint_{\Omega} a_0 u v \, dx\, dy - \iint_{\Omega} f v \, dx\, dy$$

$$\underline{\underline{v|_{\partial\Omega}=0}} \; \underbrace{\iint_{\Omega} (a_1 \nabla u\cdot\nabla v + a_0 u v) dx\, dy}_{a(u,v)} - \underbrace{\iint_{\Omega} f v \, dx\, dy}_{(f,v)}.$$

$$\Rightarrow \boxed{\begin{array}{l}\text{Find } u \text{ s.t. } a(u,v) - (f,v) = 0, \text{ for any } v \in H_0^1(\Omega), \\ \text{where } a(u,v) = \iint_{\Omega} (a_1 \nabla u\cdot\nabla v + a_0 u v) dx\, dy, \ (f,v) = \iint_{\Omega} f v \, dx\, dy.\end{array}}$$

$$\iint_{\Omega} a_1 \nabla u\cdot\nabla v \, dx\, dy = \iint_{\Omega} a_1 \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)\cdot\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) dx\, dy = \iint_{\Omega} a_1 \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dx\, dy$$

__Example 4.__
$$\begin{cases} -\nabla\cdot(a_1\nabla u) + a_0 u = f, & (x,y)\in\Omega, \\ u(x,y)|_{\partial\Omega} = g(x,y), & (x,y)\in\partial\Omega. \end{cases}$$ Dirichlet Boundary Condition

Find the weak Form.

__Solution.__ Let $u = U + G$ where $G|_{\partial\Omega} = g$ (G is known) $\Rightarrow U|_{\partial\Omega} = 0$.

$$\Rightarrow -\nabla\cdot(a_1\nabla(U+G)) + a_0(U+G) = f.$$

$$\Rightarrow \begin{cases} -\nabla\cdot(a_1\nabla U) + a_0 U = \underbrace{f + \nabla\cdot(a_1\nabla G) - a_0 G}_{F}, \\ U|_{\partial\Omega} = 0. \end{cases}$$

Example 3

$$\Rightarrow \boxed{\begin{array}{l}\text{Find } U \text{ s.t. } a(U,v) - (F,v) = 0, \text{ for any } v \in H_0^1(\Omega), \\ \text{where } a(U,v) = \iint_{\Omega} a_1\left(\frac{\partial U}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial v}{\partial y}\right) dx\, dy, \ (F,v) = \iint_{\Omega} F v \, dx\, dy. \\ \qquad = \iint_{\Omega} a_1 \nabla U\cdot\nabla v \, dx\, dy\end{array}}$$

Example 5.

$$\begin{cases} -\nabla\cdot(a_1\nabla u)+a_0 u = f, & (x,y)\in\Omega, \\ \dfrac{\partial u}{\partial \vec{n}}\Big|_{\partial\Omega} = g(x,y), & (x,y)\in\partial\Omega. \end{cases}$$ Neumann Boundary Condition

Find the weak Form.

Solution.

$$0 = -\iint_\Omega (\nabla\cdot a_1\nabla u)\, v\,dxdy + \iint_\Omega a_0 u v\,dxdy - \iint_\Omega f v\,dxdy$$

$$= \iint_\Omega a_1\nabla u\cdot\nabla v\,dxdy - \int_{\partial\Omega} a_1\frac{\partial u}{\partial\vec{n}} v\,ds + \iint_\Omega a_0 u v\,dxdy - \iint_\Omega f v\,dxdy$$

$$\overset{\frac{\partial u}{\partial \vec{n}}|_{\partial\Omega}=g}{=} \underbrace{\iint_\Omega (a_1\nabla u\cdot\nabla v + a_0 u v)\,dxdy}_{a(u,v)} \underbrace{- \int_{\partial\Omega} a_1 g v\,ds - \iint_\Omega f v\,dxdy}_{l(v)}.$$

Find u s.t. $a(u,v)-l(v)=0$, for any $v\in H^1(\Omega)$,

where $a(u,v)=\iint_\Omega (a_1\nabla u\cdot\nabla v + a_0 u v)\,dxdy$, $l(v)=\int_{\partial\Omega} a_1 g v\,ds + \iint_\Omega f v\,dxdy.$

Example 6.

$$\begin{cases} -\nabla\cdot(a_1\nabla u)+a_0 u = f, & (x,y)\in\Omega, \\ c_1 u + c_2\dfrac{\partial u}{\partial\vec{n}}\Big|_{\partial\Omega} = g, & (x,y)\in\partial\Omega. \end{cases}$$ Robin Boundary Condition

Find the weak Form.

Solution.

$$0 = \iint_\Omega a_1\nabla u\cdot\nabla v\,dxdy - \int_{\partial\Omega} a_1\frac{\partial u}{\partial\vec{n}} v\,ds + \iint_\Omega a_0 u v\,dxdy - \iint_\Omega f v\,dxdy$$

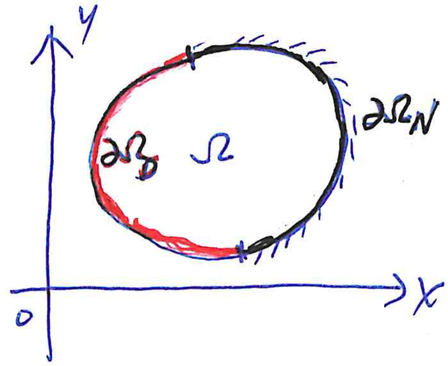
$$\overset{\frac{\partial u}{\partial\vec{n}}|_{\partial\Omega}=\frac{1}{c_2}(g-c_1u)}{=} \iint_\Omega a_1\nabla u\cdot\nabla v\,dxdy + \iint_\Omega a_0 u v\,dxdy + \int_{\partial\Omega}\frac{a_1c_1}{c_2} u v\,ds$$

$$- \int_{\partial\Omega}\frac{a_1 g}{c_2} v\,ds - \iint_\Omega f v\,dxdy$$

$$= a(u,v) - l(v), \quad v\in H^1(\Omega).$$

where $a(u,v)=\iint_\Omega a_1\nabla u\cdot\nabla v\,dxdy + \iint_\Omega a_0 u v\,dxdy + \int_{\partial\Omega}\frac{a_1c_1}{c_2}uv\,ds$, $l(v)=\int_{\partial\Omega}\frac{a_1 g}{c_2} v\,ds + \iint_\Omega f v\,dxdy.$

Example 7.
$$\begin{cases} -\nabla\cdot(a_1\nabla u)+a_0u=f, & (x,y)\in\Omega, \\ \frac{\partial u}{\partial n}\big|_{\partial\Omega_N}=g_N, & u\big|_{\partial\Omega_D}=g_D. \end{cases}$$



Find the weak Form.

Solution. Find $G\big|_{\partial\Omega_D}=g_D$. Let $u=U+G$

$$\Rightarrow -\nabla\cdot(a_1\nabla U)+a_0U=\nabla\cdot(a_1\nabla G)-a_0G+f\equiv F,$$

$$\left\{ \frac{\partial U}{\partial n}\Big|_{\partial\Omega_N}=g_N-\frac{\partial G}{\partial n}, \quad U\big|_{\partial\Omega_D}=0. \right.$$

$$\Rightarrow 0=\underbrace{\iint_\Omega(a_1\nabla U\cdot\nabla v+a_0Uv)\,dxdy}_{a(U,v)}-\underbrace{\int_{\partial\Omega_N}a_1\left(g_N-\frac{\partial G}{\partial n}\right)v\,ds-\iint_\Omega Fv\,dxdy}_{\ell(v)}.$$

$$\boxed{\text{For } v\in\{v\mid v\in H^1(\Omega) \text{ and } v\big|_{\partial\Omega_D}=0\},\ a(U,v)-\ell(v)=0.}$$

Example 8.
$$\begin{cases} \frac{\partial u}{\partial t}=k\Delta u+f(x,y,t), & (x,y)\in\Omega,\ t>0, \quad (\Delta u=u_{xx}+u_{yy}) \\ u(x,y,0)=\psi(x,y), & (x,y)\in\Omega, \\ u(x,y,t)\big|_{\partial\Omega}=0. \end{cases}$$

Find the weak Form.

Solution.
$$0=\iint_\Omega\left(\frac{\partial u}{\partial t}-k\Delta u-f\right)v\,dxdy=\iint_\Omega\frac{\partial u}{\partial t}v\,dxdy-k\iint_\Omega\Delta u\cdot v\,dxdy-\iint_\Omega fv\,dxdy$$

$$\underset{\frac{\partial uv}{\partial t}}{\overset{v(x,y)}{=\!=}}\ \frac{d}{dt}\iint_\Omega uv\,dxdy+k\iint_\Omega(u_xv_x+u_yv_y)\,dxdy-\iint_\Omega fv\,dxdy.$$

$$\Rightarrow\boxed{\begin{cases} \frac{d}{dt}\iint_\Omega uv\,dxdy+a(u,v)-(f,v)=0, & v\in H_0^1(\Omega). \\ \iint_\Omega u(x,y,0)v(x,y)\,dxdy=\iint_\Omega\psi(x,y)v(x,y)\,dxdy. \end{cases}}$$

HW. Ex. 2.2.2, Ex. 2.2.4. on page 63