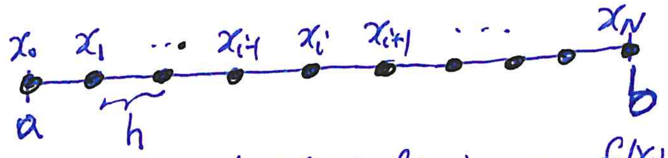
# §1.3 FDM

## 1. Taylor Series expansion:

$$f(x \pm h) = f(x) \pm h f'(x) + \frac{h^2}{2!} f''(x) + \cdots + (-1)^n \frac{h^n}{n!} f^{(n)}(x) + o(h^{n+1}).$$

Use the Taylor Series expansion to find the derivatives of $f(x)$




$f(x_0), f(x_1), \quad f(x_{i-1}), f(x_i), f(x_{i+1}), \cdots f(x_N)$

: $x_i = x_0 + ih$, $i = 1, \cdots N$.

grid size $h$, grid point $x_i$.

**Find $f'(x_i)$**

$$f(x_{i+1}) = f(x_i + h) = f(x_i) + h f'(x_i) + \frac{h^2}{2!} f''(x_i) + O(h^3)$$

$$\Rightarrow f'(x_i) = \frac{1}{h}[f(x_{i+1}) - f(x_i)] - \frac{h}{2!} f''(x_i) + o(h^2)$$

$$\approx \frac{1}{h}[f(x_{i+1}) - f(x_i)], \quad o(h) \text{ called truncation error.}$$

$$f(x_{i-1}) = f(x_i - h) = f(x_i) - h f'(x_i) + \frac{h^2}{2!} f''(x_i) + o(h^3).$$

$$\Rightarrow f'(x_i) = \frac{1}{2h}[f(x_{i+1}) - f(x_{i-1})] + o(h^2) \approx \frac{1}{2h}[f(x_{i+1}) - f(x_{i-1})], \quad o(h^2)$$

**Find $f''(x_i)$**

$$f''(x_i) = \frac{1}{h^2}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})] + o(h^2)$$

$$\approx \frac{1}{h^2}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})] \cancel{+} o(h^2)$$

## 2. General way:

$$f''(x_i) = a f(x_{i-1}) + b f(x_i) + c f(x_{i+1}) + o(h^n)$$

$$= a\{f(x_i) - h f'(x_i) + \frac{h^2}{2!} f''(x_i) - \frac{h^3}{3!} f'''(x_i) + \frac{h^4}{4!} f^{(4)}(x_i) - \frac{h^5}{5!} f^{(5)}(x_i)] + o(h^6)]$$

$$+ b f(x_i)$$

$$+ c\left[f(x_i) + hf'(x_i) + \frac{h^2}{2!}f''(x_i) + \frac{h^3}{3!}f'''(x_i) + \frac{h^4}{4!}f^{(4)}(x_i) + \frac{h^5}{5!}f^{(5)}(x_i) + o(h^6)\right]$$

$$+ o(h^n)$$

$$= (a+b+c)f(x_i) + h(a-c)f'(x_i) + \frac{h^2}{2!}(a+c)f''(x_i) + \frac{h^3}{3!}(a-c)f'''(x_i) + \frac{h^4}{4!}(a+c)f^{(4)}(x_i)$$

$$+ \frac{h^5}{5!}(a-c)f^{(5)}(x_i) + o(h^6) + o(h^n).$$

$$\Rightarrow \begin{cases} a+b+c=0, \\ a-c=0, \\ \frac{h^2}{2!}(a+c)=1. \end{cases} \Rightarrow a=c=\frac{1}{h^2},\ b=-\frac{2}{h^2},\ \frac{h^4}{4!}(a+c)f^{(4)}(x_i) = \frac{h^2}{12}f^{(4)}(x_i)$$

$$\Rightarrow o(h^n) = o(h^2),\ n=2.$$

$$\Rightarrow f''(x_i) \approx \frac{1}{h^2}\left[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right],\ o(h^2)$$

For higher-order scheme:

(1) $f''(x_i) = a f(x_{i-2}) + b f(x_{i-1}) + c f(x_i) + d f(x_{i+1}) + e f(x_{i+2}) + o(h^n).$

(2) $\alpha f''(x_{i-1}) + f''(x_i) + \alpha f''(x_{i+1}) = \frac{a}{h^2}\left[f(x_{i-1}) - 2f(x_i) + f(x_{i+1})\right] + o(h^n).$

$\Rightarrow$ Match both sides of Taylor series expansion.

$\Rightarrow \alpha = \frac{1}{10},\ a = \frac{12}{10},\ n=4.$

$\Rightarrow \frac{1}{10}f''(x_{i-1}) + f''(x_i) + \frac{1}{10}f''(x_{i+1}) = \frac{6}{5h^2}\left[f(x_{i-1}) - 2f(x_i) + f(x_{i+1})\right] + o(h^4).$

Padé scheme.

Ex1.3.1. P.19.

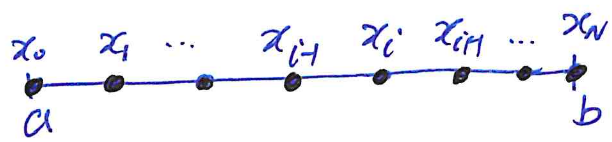
## §1.4. Numerical Solution using FDM

Example 1. 1D problem

$$\begin{cases} -u_{xx} + u = g(x), & a < x < b, \\ u(a) = u_a, \ u(b) = u_b. \end{cases}$$

Derive FD scheme.

grid points




Solution. (1) Design a mesh.

$x_i = a + ih, \ i = 1, 2, \cdots, N.$

$h$: grid size, $h = \dfrac{b-a}{N}$.

(2) use the FD formula

$$\boxed{f''(x_i) = \frac{1}{h^2}\left[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right] + O(h^2).}$$

Find $u(x_i)$, $i = 1, 2, \cdots, N-1$.

Let $-u_{xx}(x_i) + u(x_i) = g(x_i)$.

$$\Rightarrow -\frac{1}{h^2}\left[u(x_{i+1}) - 2u(x_i) + u(x_{i-1})\right] + u(x_i) + O(h^2) = g(x_i),$$

Drop $O(h^2)$, let $U_i \approx u(x_i)$

$$\Rightarrow -\frac{1}{h^2}\left[U_{i+1} - 2U_i + U_{i-1}\right] + U_i = g(x_i).$$

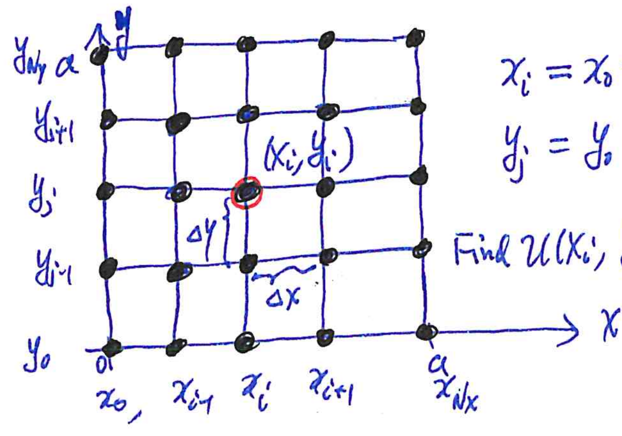
$$\Rightarrow \boxed{\begin{cases} -U_{i-1} + (2+h^2)U_i - U_{i+1} = h^2 g(x_i), & i = 1, 2, \cdots, N-1, \\ U_0 = u_a, \ U_N = u_b. \end{cases}}$$

$$\Rightarrow \begin{bmatrix} 2+h^2 & -1 & & & 0 \\ -1 & 2+h^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2+h^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-2} \\ U_{N-1} \end{bmatrix} = \begin{bmatrix} h^2 g(x_1) + u_a \\ h^2 g(x_2) \\ \vdots \\ h^2 g(x_{N-2}) \\ h^2 g(x_{N-1}) + u_b \end{bmatrix},$$

Called tridiagonal linear system.

Example 2. $\begin{cases} u_{xx} + u_{yy} = -f(x,y), \quad 0 < x, y < a, \\ u(x,0) = 0, \quad u(x,a) = F(x), \\ u(0,y) = 0, \quad u(a,y) = 0. \end{cases}$

Derive FDM scheme.




$x_i = x_0 + i\Delta x, \ i = 1, \cdots N_x$

$y_j = y_0 + j\Delta y, \ j = 1, \cdots, N_y$

Find $u(x_i, y_j)$.

Solution. (1) Mesh.

$$\Delta x = \frac{a}{N_x}, \quad \Delta y = \frac{b}{N_y}.$$

(2) Discretization

$$\boxed{f''(x_i) = \frac{1}{h^2}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})] + o(h^2).}$$

Let $u_{xx}(x_i, y_j) + u_{yy}(x_i, y_j) = -f(x_i, y_j)$.

$$\begin{cases} u_{xx}(x_i, y_j) = \frac{1}{\Delta x^2}[u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)] + o(\Delta x^2), \\ u_{yy}(x_i, y_j) = \frac{1}{\Delta y^2}[u(x_i, y_{j-1}) - 2u(x_i, y_j) + u(x_i, y_{j+1})] + o(\Delta y^2). \end{cases}$$

$$\Rightarrow \frac{1}{\Delta x^2}[u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)]$$
$$+ \frac{1}{\Delta y^2}[u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})] + o(\Delta x^2 + \Delta y^2) = -f(x_i, y_j).$$

Denote $U_{i,j} \approx u(x_i, y_j)$, $\Delta x = \Delta y = h$,

$$\Rightarrow \boxed{\begin{cases} U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} = -h^2 f_{i,j}, \quad i = 1, \cdots N_x - 1, \ j = 1, \cdots, N_y - 1, \\ U_{0,j} = 0, \ U_{N_x,j} = 0, \ U_{i,0} = 0, \ U_{i,N_y} = F(x_i). \end{cases}}$$

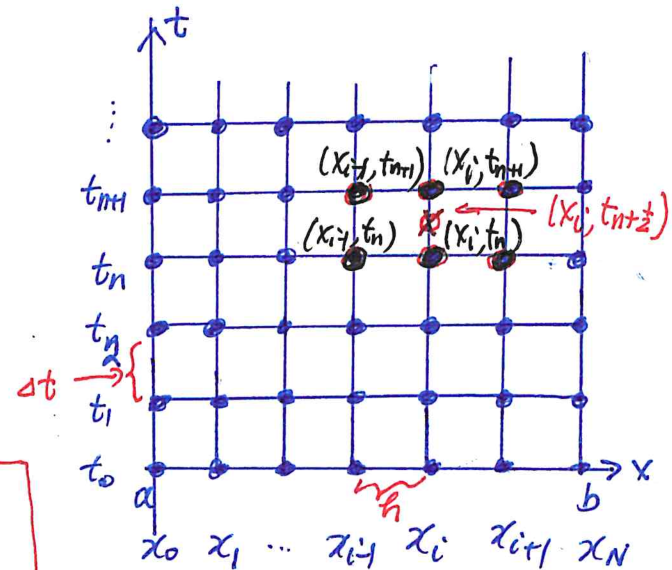
called 5-point scheme.

## Example 3. 1D Heat Conduction problem

$$\begin{cases} u_t = k\,u_{xx} + S(x,t), \quad a<x<b,\ t>0, \\ u(x,0) = f(x), \quad a \le x \le b, \\ u(a,t) = u(b,t) = 0, \quad t>0. \end{cases}$$

Derive FD scheme.

**Solution.** (1) Design a mesh




2D Mesh: $x_i = a + ih,\ i=1,\cdots,N$

$t_n = n\Delta t$, $\Delta t$: time step

$u(x_i, t_n)$, $i=0,1,\cdots,N$,

$n = 0,1,2,\cdots$

(2) Discretization

use the FD formulas:

$$\boxed{\begin{aligned} f'(t_{n+\frac{1}{2}}) &= \frac{1}{\Delta t}\left[f(t_{n+1}) - f(t_n)\right] + O(\Delta t^2), \\ f(t_{n+\frac{1}{2}}) &= \frac{1}{2}\left[f(t_{n+1}) + f(t_n)\right] + O(\Delta t^2), \\ f''(x_i) &= \frac{1}{h^2}\left[f(x_{i-1}) - 2f(x_i) + f(x_{i+1})\right] + O(h^2). \end{aligned}}$$

Let

$$u_t(x_i, t_{n+\frac{1}{2}}) = k\,u_{xx}(x_i, t_{n+\frac{1}{2}}) + S(x_i, t_{n+\frac{1}{2}}).$$

$$\Rightarrow u_t(x_i, t_{n+\frac{1}{2}}) = \frac{1}{\Delta t}\left[u(x_i, t_{n+1}) - u(x_i, t_n)\right] + O(\Delta t^2),$$

$$\begin{aligned} u_{xx}(x_i, t_{n+\frac{1}{2}}) &= \frac{1}{2}\left[u_{xx}(x_i, t_{n+1}) + u_{xx}(x_i, t_n)\right] + O(\Delta t^2) \\ &= \frac{1}{2h^2}\left[u(x_{i-1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i+1}, t_{n+1})\right] \\ &\quad + \frac{1}{2h^2}\left[u(x_{i-1}, t_n) - 2u(x_i, t_n) + u(x_{i+1}, t_n)\right] + O(\Delta t^2 + h^2). \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{1}{\Delta t}\left[u(x_i, t_{n+1}) - u(x_i, t_n)\right] + O(\Delta t^2) &= \frac{k}{2h^2}\left[u(x_{i-1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i+1}, t_{n+1})\right] \\ &\quad + \frac{k}{2h^2}\left[u(x_{i-1}, t_n) - 2u(x_i, t_n) + u(x_{i+1}, t_n)\right] + S(x_i, t_{n+\frac{1}{2}}) \\ &\quad + O(\Delta t^2 + h^2), \end{aligned}$$

Drop $O(\Delta t^2 + h^2)$, Denote $U_i^n \approx u(x_i, t_n)$.

$$\Rightarrow \frac{1}{\Delta t}[U_i^{n+1} - U_i^n] = \frac{k}{2h^2}[U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}] + \frac{k}{2h^2}[U_{i-1}^n - 2U_i^n + U_{i+1}^n] + S_i^{n+\frac{1}{2}}.$$

$i = 1, 2, \cdots, N-1.$

Denote $r = \frac{\Delta t\, k}{2h^2}$

$$\Rightarrow \boxed{-rU_{i-1}^{n+1} + (1+2r)U_i^{n+1} - rU_{i+1}^{n+1} = rU_{i-1}^n + (1-2r)U_i^n + rU_{i+1}^n + \Delta t \cdot S_i^{n+\frac{1}{2}}}$$

$i = 1, 2, \cdots, N-1$

$$\begin{cases} U_i^0 = f(x_i), \ i = 0, 1, \cdots, N, \\ U_0^{n+1} = U_0^n = 0, \ U_N^{n+1} = U_N^n = 0. \ n = 1, 2, 3, \cdots \end{cases}$$

Called Crank - Nicolson Scheme.

Matrix Form:

$$\begin{bmatrix} 1+2r & -r & & & 0 \\ -r & 1+2r & -r & & \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & & -r \\ & & & -r & 1+2r \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{N-2}^{n+1} \\ U_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \end{bmatrix},$$

where $d_i = rU_{i-1}^n + (1-2r)U_i^n + rU_{i+1}^n + \Delta t \cdot S_i^{n+\frac{1}{2}}$. $i = 1, 2, \cdots, N-1$.

Tridiagonal linear system.