Lecture 16: Optimal Control

ELEN 472: Introduction to Digital Control

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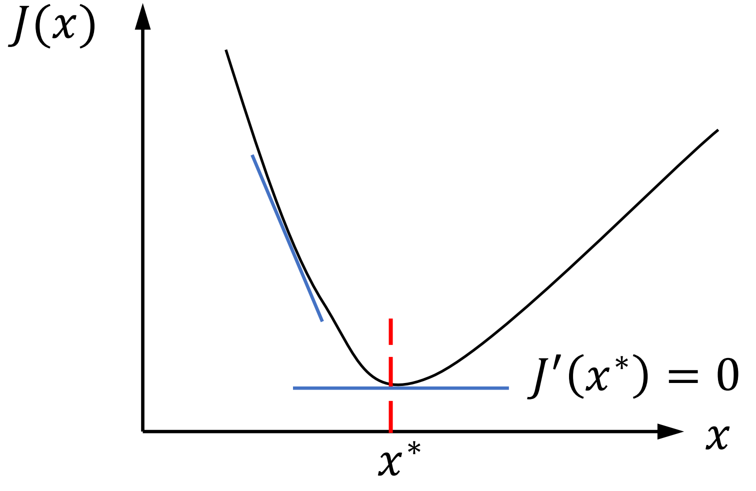
Introduction: Optimization

- Many problems in engineering can be solved by minimizing a **measure of cost (i.e., cost function)**
  - The designer must select a suitable **performance measure** to include the most important performance criteria. -> Define Cost Function.
  - The designer must also select a **mathematical form of the function** that makes solving the optimization problem tractable. -> Define Optimization Method.
- The process of **optimization** is to find a value that results a minimal cost function.
  - At x^*, the gradient of $J(x)$ is 0



Constrained Optimization

- In many practical applications, the parameter vector $\mathbf{x}$ are subject to physical and economic **constraints**.
  - E.g., the speed of a vehicle cannot exceed a limit.
- Assume that our vector of parameters is subject to the constraint:

$$\mathbf{m}(\mathbf{x}) = 0$$

  - We can add the constraint in the optimization problem using the **Lagrangian**:

$$\underbrace{\boxed{L(\mathbf{x})}}_{\text{Lagrangian}} = \underbrace{\boxed{J(\mathbf{x})}}_{\text{Normal Cost Function}} + \underbrace{\boxed{\lambda^T}}_{\text{Constant Parameters}}\mathbf{m}(\mathbf{x})$$

  - We than solve for $\mathbf{x}$ and λ that minimize the Lagrangian.

Example

- A manufacturer decides the production level of two products based on maximizing profit subject to constraints on production. The manufacturer estimates profit using the simplified measure

$$J(\mathbf{x}) = x_1^{\alpha} x_2^{\beta}$$

  - where x_i is the quantity produced for product i, $i = 1\ or\ 2$, and the parameters (α, β) are determined from sales data.
  - The sum of the quantities of the two products produced can not exceed a fixed level b.
  - Determine the optimum production level for the two products subject to the production constraint $x_1 + x_2 = b$.

Product 1, Quality: x_1

Product 2, Quality: x_2

Profit: $J(\mathbf{x}) = x_1^{\alpha} x_2^{\beta}$

Constrain: $\mathbf{m}(\mathbf{x}) = x_1 + x_2 - b$

Solution

- We use the negative of the profit as the cost function.
  - Minimizing the cost function -> Maximizing the profit.
- Using the Lagrangian,

$$L(\mathbf{x}) = J(\mathbf{x}) + \lambda^T \mathbf{m}(\mathbf{x})$$

Profit: $J(\mathbf{x}) = x_1^{\alpha} x_2^{\beta}$

Constrain: $\mathbf{m}(\mathbf{x}) = x_1 + x_2 - b$

- we have

$$L(\mathbf{x}) = -x_1^{\alpha} x_2^{\beta} + \lambda(x_1 + x_2 - b)$$

- To minimize this cost function $L(\mathbf{x})$, we have:

$$\frac{\partial L}{\partial x_1} = -\alpha x_1^{\alpha-1} x_2^{\beta} + \lambda = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2} = -\beta x_1^{\alpha} x_2^{\beta-1} + \lambda = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 - b = 0 \quad (3)$$

From Eqn (1) and (2):

$$\lambda = \alpha x_1^{\alpha-1} x_2^{\beta} = \beta x_1^{\alpha} x_2^{\beta-1}$$

$$x_2 = \frac{\beta}{\alpha} x_1$$

Substitute in Eqn (3)

$$x_1 = \frac{\alpha b}{\alpha + \beta}$$

$$x_2 = \frac{\beta b}{\alpha + \beta}$$

Optimal Control

- To optimize the performance of a discrete-time dynamic system, we minimize the performance measure

$$J = J_f(\mathbf{x}(k_f),\ k_f) + \sum_{k=k_0}^{k_f-1} L(\mathbf{x}(k), \mathbf{u}(k), k)$$

  - Subject to the constraint $\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k),\ k = k_0, \ldots, k_f - 1$
  - Here,

$J_f(\mathbf{x}(k_f),\ k_f) \Rightarrow$ Terminal Penalty

$\sum_{k=k_0}^{k_f-1} L(\mathbf{x}(k), \mathbf{u}(k), k) \Rightarrow$ Costs at time k

  - The goal of optimal control is to find $\mathbf{u}(k)$ that minimize $\boldsymbol{J}$.

Linear Quadratic Regulator

- A popular choice of the performance measure J is a **quadratic** function of state variable and the control inputs:

$$J = \frac{1}{2}\mathbf{x}^T(k_f)S(k_f)\mathbf{x}(k_f) + \frac{1}{2}\sum_{k=k_0}^{k_f-1}\left(\mathbf{x}^T(k)\boxed{Q(k)}\mathbf{x}(k) + \mathbf{u}^T(k)\boxed{R(k)}\mathbf{u}(k)\right)$$

Constant Matrices, determined by the user

  - Q has the same number of rows and columns of $\mathbf{x}$.
  - Q penalizes the bad performance in system states.
  - R has the same number of rows and columns of $\mathbf{u}$.
  - R penalizes large control efforts.
  - Both Q and R are diagonal matrices.

- The optimal control expression is

$$u^*(k) = -\mathbf{K}(k)\mathbf{x}(k)$$

$$\mathbf{K}(k) = [R(k) + B^T S(k+1)B]^{-1}B^T S(k+1)A$$

  - $S(k)$ can be found iteratively backwards via **Riccati Equation**:

$$S(k) = A_{cl}^T(k)S(k+1)\boxed{A_{cl}(k)} + K^T(k)R(k)K(k) + Q(k)$$

$$\to [A - BK(k)]$$

Example

- A mechanical system can be approximately modeled by the following system:

$$
\begin{aligned}
\mathbf{x}(k+1) &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix}u(k) \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k)
\end{aligned}
$$

  - where $u(k)$ is the applied force, and the sample period is $T = 0.02$ s.
- Design a linear quadratic regulator for the system with terminal weight $S(100) = diag\{10,1\}, Q = diag\{10, 1\}$, and control weight $R = 0.1$.
- Simulate the system response with initial condition $\mathbf{x}(0) = [1, 0]^T$

Solution

See the MATLAB file.