Final Exam — 2020

1.

$L_i(P) = S_i/S$

$L_j(P) = S_j/S$

$L_k(P) = S_k/S$.

$S = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$.

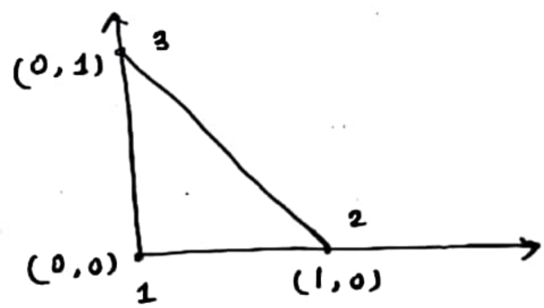
$S_j = \frac{1}{2} \times \frac{1}{3} \times 1 = \frac{1}{6}$

$S_i = \frac{1}{2} \times (1 - \frac{1}{3}) \times 1 = \frac{1}{2} \times \frac{2}{3} = 1/3$.

$L_i(P) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3} \times \frac{2}{1} = 2/3$.

$L_j(P) = \frac{1}{6} \times \frac{2}{1} = \frac{1}{3}$.

$L_k(P) = 1 - L_i(P) - L_j(P) = 1 - \frac{2}{3} - \frac{1}{3} = \frac{3-2-1}{3} = 0$.



2. $K_{ij} = \frac{1}{4S}(b_i b_j + c_i c_j)$, $i \neq j$

$K_{ii} = \frac{1}{4S}(b_i^2 + c_i^2)$, $i = j$

$S = \frac{1}{2} \times 1 \times 1 = 1/2$

$x_1 = 0$, $x_2 = 1$, $x_3 = 0$

$y_1 = 0$, $y_2 = 0$, $y_3 = 1$.

$b_1 = -1$, $b_2 = 1$, $b_3 = 0$, $c_1 = -1$, $c_2 = 0$, $c_3 = 1$.

$K_{11} = \frac{1}{2}(1+1) = 1$. $K_{22} = \frac{1}{2}(1) = \frac{1}{2}$, $K_{33} = \frac{1}{2}$

$K_{12} = \frac{1}{2}(-1) = -\frac{1}{2} = K_{21}$, $K_{13} = \frac{1}{2}(-1) = -\frac{1}{2} = K_{31}$

$K_{23} = \frac{1}{2}(0) = 0 = K_{32}$

3. $K_{11,11} = a(\varphi_{ij}, \varphi_{ij})$

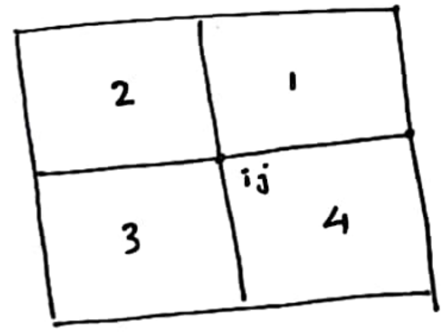
Zone - I

$x: \; x_i \to x_{i+1}$

$y: \; y_j \to y_{j+1}$

$x > x_i$

$y > y_j$



$$\varphi_{ij} = \left(1 - \frac{x - x_i}{\Delta x}\right)\left(1 - \frac{y - y_j}{\Delta y}\right). \qquad \begin{array}{|l} x > x_i \\ y > y_j \end{array}$$

$$K_{11,11} = \cancel{a}($$

$$= \iint_{R'} \left[\left(\frac{\partial}{\partial x}\varphi_{ij}\right)^2 + \left(\frac{\partial}{\partial y}\varphi_{ij}\right)^2\right] dx\,dy$$

$$= \iint_{R'} \left[\left(-\frac{1}{\Delta x}\right)^2\left(1 - \frac{y - y_j}{\Delta y}\right)^2 + \left(-\frac{1}{\Delta y}\right)^2\left(1 - \frac{x - x_i}{\Delta x}\right)^2\right] dx\,dy$$

$$= \frac{1}{\Delta x^2}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}\left(1 - \frac{y - y_j}{\Delta y}\right)^2 dx\,dy + \frac{1}{\Delta y^2}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}\left(1 - \frac{x - x_i}{\Delta x}\right)^2 dx\,dy$$

$\frac{y - y_j}{\Delta y} = u.$ $\qquad$ $\frac{x - x_i}{\Delta x} = v$

$\Rightarrow du = \frac{dy}{\Delta y}$ $\qquad$ $\Rightarrow dx = \Delta x\, dv.$

$\Rightarrow dy = \Delta y\, du$ $\qquad$ $y_{j+1} = y_j + \Delta y$

$$= \frac{\Delta y}{\Delta x}\int_0^1 (1-u)^2 du + \frac{\Delta x}{\Delta y}\int_0^1 (1-v)^2 dv$$

$$= \left[u - u^2 + \frac{u^3}{3}\right]_0^1 + \left[v - v^2 + \frac{v^3}{2}\right]_0^1$$

$$= 1 - 1 + \frac{1}{3} + \frac{1}{3} = 2/3.$$

Zone - II

$x: \quad x_{i-1} \to x_i$ | $x < x_i$

$y: \quad y_j \to y_{j+1}$ | $y > y_i$

$$\varphi_{ij} = \left(1 + \frac{x - x_i}{\Delta x}\right)\left(1 - \frac{y - y_j}{\Delta y}\right).$$

$$K_{11,11} = \iint_{R^2} \left[\left(\frac{1}{\Delta x}\right)^2 \left(1 - \frac{y - y_i}{\Delta y}\right)^2 + \left(-\frac{1}{\Delta y}\right)^2 \left(1 + \frac{x - x_i}{\Delta x}\right)^2 \right] dx\,dy$$

$$= \int_0^1 (1-u)^2\,du + \int_{-1}^0 (1+v)^2\,dv$$

$$= \left[u - u^2 + u^3/3\right]_0^1 + \left[v + v^2 + v^3/3\right]_{-1}^0$$

$$= 2/3.$$

$$\int_{x_{i-1}}^{x_i}$$

$v = \dfrac{x - x_i}{\Delta x} = 0$, $x = x_i$

$v = \dfrac{x_{i-1} - x_i}{\Delta x} = \dfrac{-\Delta x}{\Delta x} = -1.$

$x_i = x_{i-1} + \Delta x$

$\Rightarrow x_{i-1} - x_i = -\Delta x$

Zone - III

$x: \quad x_{i-1} \to x_i$

$y: \quad y_{j-1} \to y_j$

$x < x_i$

$y < y_j$

$$\varphi_{ij} = \left(1 + \frac{x - x_i}{\Delta x}\right)\left(1 + \frac{y - y_i}{\Delta y}\right)$$

$$K_{11,11} = 2/3.$$

Zone - IV

$x: \quad x_i \to x_{i+1}$

$y: \quad y_{j-1} \to y_j$

$x > x_i$

$y < y_j$

$$\varphi_{ij} = \left(1 - \frac{x - x_i}{\Delta x}\right)\left(1 + \frac{y - y_j}{\Delta y}\right)$$

$$K_{11,11} = 2/3.$$

$(K_{11,11}) = 4 \times 2/3$

$= 8/3.$ (Ans.)

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4.

$$-2u'' + u = 1, \quad \underline{0 < x < 1.}$$

$$u(0) = 2, \quad u(1) = 0$$

Non - homogeneous problem,

Let. $u = U + G$. such that. $G(0) = 2$, $G(1) = 0$.

$$-2(U+G)'' + (U+G) = 1$$

$$\Rightarrow -2U'' - 2G'' + U + G = 1$$

$$\Rightarrow -2U'' + U = 1 + 2G'' - G. \quad \boxed{\begin{array}{l} U(0) = 0 \\ U(1) = 0 \end{array}}$$

<u>Weak form</u>, here, $v(x) \in H_0^1[0,1]$.

$$-\int_0^1 2U''v\,dx + \int_0^1 Uv\,dx = \int_0^1 v\,dx + \int_0^1 (2G'' - G)v\,dx$$

$$\Rightarrow \int_0^1 2U'v'\,dx + \int_0^1 Uv\,dx = \int_0^1 v\,dx + \int_0^1 (2G'' - G)v\,dx.$$

$$a(U,v) = (f,v) - a(G,v)$$

Where,

$$a(U,v) = \int_0^1 (2U'v' + Uv)\,dx$$

$$(f,v) = \int_0^1 v\,dx$$

$$a(G,v) = \int_0^1 (G - 2G'')v\,dx.$$

For any, $v(x) \in H_0^1[0,1]$.

Linear system :

$$\sum_{i=1}^{n-1} a(\varphi_i, \varphi_j)\, u_j = (f, \varphi_j) - 2a(\varphi_0, \varphi_j)$$

Tri-diagonal linear system :

$$\begin{bmatrix} a(\varphi_i, \varphi_j) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} (f, \varphi_1) - 2a(\varphi_0, \underline{\varphi_1}) \\ (f, \varphi_2) \\ \vdots \\ (f, \varphi_{n-1}) \end{bmatrix}$$

if, $|i-j| \geq 2$, then, $a(\varphi_i, \varphi_j) = 0$.

we need to find,

✓ $a(\varphi_0, \varphi_1)$, $\underline{a(\varphi_i, \varphi_i)}$, $a(\varphi_i, \varphi_{i+1})$.

$\underline{a(\varphi_{n-1}, \varphi_{n-1})}$, (f, φ_j), $(f, \underline{\varphi_{n-1}})$.

$$a(\varphi_i, \varphi_i) = \int_0^1 \underline{\left(2(\varphi_i')^2 + \varphi_i^2\right)}\, dx.$$

$$\varphi_i = \begin{cases} 1 + \dfrac{x - x_i}{h}, & x_{i-1} \leq x \leq \underline{x_i} \\ 1 - \dfrac{x - x_i}{h}, & x_i \leq x \leq x_{i+1} \end{cases}$$

$$\varphi_i' = \begin{cases} \dfrac{1}{h} \\ -\dfrac{1}{h} \end{cases}$$

$$a(\varphi_i, \varphi_i) = \int_{x_{i-1}}^{x_i} \left[2\left(\frac{1}{h}\right)^2 + \left(1 + \frac{x - x_i}{h}\right) \right] dx$$

$$+ \int_{x_i}^{x_{i+1}} \left[2\left(-\frac{1}{h}\right)^2 + \left(1 - \frac{x - x_i}{h}\right) \right] dx .$$

$$\frac{x - x_i}{h} = u$$

$$\Rightarrow \quad du = \frac{dx}{h}$$

$$\Rightarrow \quad \underline{dx = h\,du} .$$

$$\frac{0}{2} - \frac{(-1)^2}{2}$$

$$0 - \frac{1}{2}$$

$$\Rightarrow a(\varphi_i, \varphi_i) = \frac{2}{h} + \int_{-1}^{0} (1+u)\, h\, du + \frac{2}{h} + \int_{0}^{1} (1-u)\, h\, du .$$

$$= \frac{4}{h} + h \left[u + \frac{u^2}{2} \right]_{-1}^{0} + h \left[u - \frac{u^2}{2} \right]_{0}^{1}$$

$$= \frac{4}{h} + h\left[1 - \frac{1}{2}\right] + h\left[1 - \frac{1}{2}\right] .$$

$$a(\varphi_i, \varphi_i) = \boxed{\frac{4}{h} + h}$$

" $a(\varphi_i, \varphi_{i+1})$

$$\varphi_i = 1 - \frac{x - x_i}{h} \qquad x_i \le x \le x_{i+1}$$

$$\varphi_{i+1} = 1 + \frac{x - x_{i+1}}{h} \qquad x_i \le x \le x_{i+1}$$

$$\varphi_i' = (-1/h) .$$

$$\varphi_{i+1}' = (1/h)$$

$a(\varphi_i, \varphi_{i+1})$

$$= \int_0^1 (2\varphi_i' \varphi_{i+1}' + \varphi_i \varphi_{i+1})\, dx.$$

$$= 2\int_{x_i}^{x_{i+1}} \left(-\frac{1}{h^2}\right) dx + \int_{x_i}^{x_{i+1}} \left(1 - \frac{x-x_i}{h}\right)\left(1 + \frac{x - x_{i+1}}{h}\right) dx.$$

$$= -\frac{2}{h^2}(\underline{x_{i+1} - x_i}) + \int_0^1 (1-u)(1+u-1)\, h\, du$$

Let, $\frac{x-x_i}{h} = u$

$\Rightarrow du = \frac{dx}{h}$

$\Rightarrow dx = h\, du$

$\frac{x - (x_i + h)}{h}$

$= \frac{x - x_i - h}{h} = (u-1).$

$$= -\frac{2}{h} + \int_0^1 (u-u^2)\, h\, du = -\frac{2}{h} + h\left[\frac{u^2}{2} - \frac{u^3}{3}\right]_0^1$$

$$= -\frac{2}{h} + h\left(\frac{1}{2} - \frac{1}{3}\right) = \boxed{\frac{h}{6} - \frac{2}{h}.}$$

Now, (f, φ_j).

$$\varphi_j = \begin{cases} 1 + \frac{x-x_j}{h}, & x_{j-1} \le x \le x_j \\ 1 - \frac{x-x_j}{h}, & x_j \le x \le x_{j+1} \end{cases}$$

$$(f, \varphi_j) = \int_0^1 \varphi_j\, dx. = \int_{x_{j-1}}^{x_j} \left(1 + \frac{x-x_j}{h}\right) dx + \int_{x_j}^{x_{j+1}} \left(1 - \frac{x-x_j}{h}\right) dx.$$

$$= \int_{-1}^0 (1+u)\, h\, du + \int_0^1 (1-u)\, h\, du.$$

$$= h\left[u + \frac{u^2}{2}\right]_{-1}^0 + h\left[u - \frac{u^2}{2}\right]_0^1 = h\left(1 - \frac{1}{2}\right) + h\left(1 - \frac{1}{2}\right)$$

$$= \boxed{h}$$

$$a(\varphi_1,\varphi_1) = \frac{4}{h} + h = 12 + \frac{1}{3} = \frac{36+1}{3} = 37/3. = a(\varphi_1,\varphi_1)$$
$$= a(\varphi_2,\varphi_2)$$
$$= a(\varphi_3,\varphi_3)$$
$$\vdots$$

$$a(\varphi_1,\varphi_2) = \frac{h}{6} - \frac{2}{h}$$
$$= \frac{\frac{1}{3}}{6} - \frac{2}{1/3} = \cancel{\frac{1}{3} \times \frac{6}{1}}$$
$$= \frac{1}{3} \times \frac{1}{6} - 2 \times \frac{3}{1} = \frac{1}{18} - 6 = -\frac{107}{18}.$$

$$a(\varphi_1,\varphi_2) = a(\varphi_2,\varphi_1) = a(\varphi_0,\varphi_1) = a(\varphi_2,\varphi_3)$$
$$= -\frac{107}{18}.$$

$$(f,\varphi_1) = \frac{1}{3}.$$

$$\begin{bmatrix} \frac{37}{3} & -\frac{107}{18} & & & 0 \\ -\frac{107}{18} & \frac{37}{3} & -\frac{107}{18} & & \\ & -\frac{107}{18} & \frac{37}{3} & -\frac{107}{18} & \\ \vdots & & \ddots & \ddots & \\ 0 & & & & \frac{37}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} + 2\left(\frac{107}{18}\right) \\ \frac{1}{3} \\ \vdots \\ 1/3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{37}{3} & -\frac{107}{18} & 0 \\ -\frac{107}{18} & \frac{37}{3} & -\frac{107}{18} \\ 0 & -\frac{107}{18} & \frac{37}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + (2 \times \frac{107}{18}) \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$c_1 = 1.457$

$c_2 = 0.9669$

$c_3 = 0.493$.

$$u(x) = \sum_{i=1}^{n-1} c_i \, \varphi_i(x)$$

$$u(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x) + c_3 \varphi_3(x) + \cdots$$

$$= 1.457 \varphi_1(x) + 0.9669 \varphi_2(x) + 0.493 \varphi_3(x) + \cdots$$