Chap. 1. Preliminaries

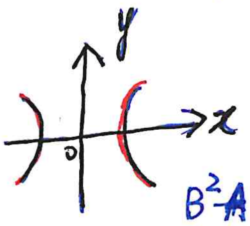
§1.1 Introduction

1. Partial Differential Eqs.

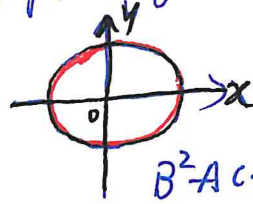
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0$$

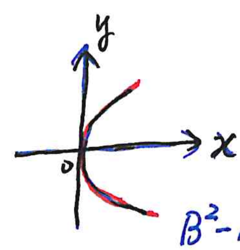
$$B^2 - AC = \begin{cases} > 0, & \text{Hyperbolic eq.} \\ = 0, & \text{Parabolic eq.} \\ < 0. & \text{Elliptic eq.} \end{cases} \qquad \boxed{Ax^2 + 2Bxy + Cy^2 = 0}$$



$B^2 - AC > 0$

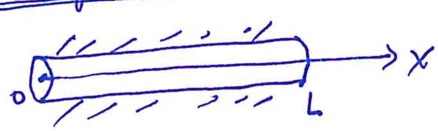


$B^2 - AC < 0$



$B^2 - AC = 0$

2. Heat Eqution



ρ: density $\quad u(x,t)$: temperature at x, t

L: Length, K: conductivity, C: specific heat

Based on the Energy Conservation:

D.E.: $\dfrac{\partial u(x,t)}{\partial t} = k \dfrac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < L, \ t > 0, \ k = K/c\rho.$

B.C.: $u(0,t) = \alpha(t), \ u(L,t) = \beta(t), \ t > 0,$

I.C.: $u(x,0) = f(x), \quad 0 \le x \le L$.

If there is a heat source: $\dfrac{\partial u(x,t)}{\partial t} = k \dfrac{\partial^2 u(x,t)}{\partial x^2} + Q(x,t).$

In 2D case:
$$\frac{\partial u(x,y,t)}{\partial t} = k\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) + Q(x,y,t)$$
$$= k \nabla^2 u + Q.$$

$\nabla^2 u$: called the Laplacian of u.

If it is steady state: $u_t = 0$, $\Rightarrow$ $\nabla^2 u = 0$, Laplace Eq

$\Rightarrow$ $\nabla^2 u = f(x,y)$; Poisson Eq

Some Well-known Equtions:

Wave Eq.: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Convection - Diffusion Eq. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \sigma \frac{\partial^2 u}{\partial x^2}$

Black-scholes Eq. $\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} + F = 0$

Schrödinger Eqs. $i\frac{\partial u}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2 u}{\partial x^2} + \frac{V}{\hbar}u$, $i\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda |u|^{n-1} u$.

Navier- Stokes Eq. $\begin{cases} u_x + v_y = 0 \\ u_t + u u_x + v u_y = \frac{1}{Re}(u_{xx} + u_{yy}) - P_x \\ v_t + u v_x + v v_y = \frac{1}{Re}(v_{xx} + v_{yy}) - P_y \end{cases}$

Maxwell's Eqs. $\frac{\partial \vec{D}}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \vec{\nabla} \times \vec{H}$, $\vec{D}(\omega) = \varepsilon_r^* \vec{E}(\omega)$, $\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\sqrt{\varepsilon_0 \mu}} \vec{\nabla} \times \vec{E}$

§1.2. Fourier Series Method

Example 1 $\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, & 0<x<L, t>0, \\ u(x,0) = u_0(x), & 0 \le x \le L, \\ u(0,t) = 0, \ u(L,t) = 0, & t>0. \end{cases}$

Solution: $u(x,t) = T(t)X(x) \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda \Rightarrow \begin{cases} T' + \lambda k T = 0 \\ X'' + \lambda X = 0, \ X(0) = X(L) = 0 \end{cases}$

$\Rightarrow X_n(x) = \sin(\frac{n\pi x}{L})$, $\lambda_n = (\frac{n\pi}{L})^2$, $T_n(t) = b_n e^{-k\frac{n^2\pi^2}{L^2}t}$

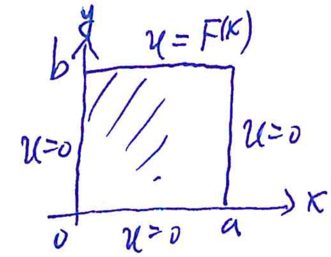
$\Rightarrow u_n(x,t) = b_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin(\frac{n\pi x}{L})$, $n = 1, 2, 3, \cdots$

$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin(\frac{n\pi x}{L})} \Rightarrow u_0(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$

$b_n = \frac{2}{L}\int_0^L u^0(x) \sin(\frac{n\pi x}{L})\,dx$, $n = 1, 2, 3, \cdots$

Example 2. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0, \quad 0<x<a,\ 0<y<b,$

$$\begin{cases} u(x,0)=0,\ u(x,b)=F(x), & 0\le x\le a,\\ u(0,y)=0,\ u(a,y)=0, & 0\le y\le b.\end{cases}$$



Solution: $u(x,y)=X(x)\,Y(y) \Rightarrow \dfrac{X''}{X}=-\dfrac{Y''}{Y}=-\lambda$

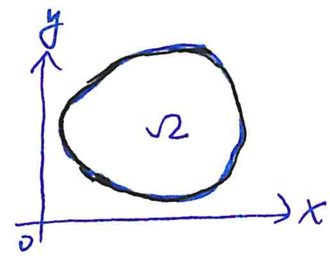
$$\Rightarrow \begin{cases} X''+\lambda X=0,\ X(0)=X(a)=0 & \Rightarrow \lambda_n=\left(\frac{n\pi}{a}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi x}{a}\right),\ n=1,2,3,\cdots\\ Y''+\lambda Y=0,\ Y(0)=0 & \Rightarrow Y_n(y)=c_1e^{\frac{n\pi y}{a}}+c_2e^{-\frac{n\pi y}{a}}\end{cases}$$

$$\Rightarrow Y_n(y)=b_n\sinh\left(\frac{n\pi y}{a}\right). \Rightarrow \boxed{u(x,y)=\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)}$$

$$\Rightarrow F(x)=\sum_{n=1}^{\infty} b_n\sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right),\quad 0<x<a$$

$$\Rightarrow b_n=\frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a F(x)\sin\left(\frac{n\pi x}{a}\right)dx,\quad n=1,2,3,\cdots$$

Challenge: (1) $\dfrac{\partial u}{\partial t}=(x+t)\dfrac{\partial^2 u}{\partial x^2}$. (2) Irregular Geometry



Numerical methods: Finite Difference Method

Finite Element Method.

HW: Ex.1.2.1, P.16