Lecture 8: Stability of Digital Control Systems

ELEN 472: Introduction to Digital Control

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Review

- **Steady-state Error**:
  - Stead-state error is the **difference** between the input (command) and the output of a system, as time goes to infinity.
  - Equation to calculate steady-state error:

$$e(\infty) = \lim_{z\to 1}[(z-1)E(z)]$$
$$= \lim_{z\to 1}\frac{(z-1)R(z)}{1+L(z)}$$

$R(z)$: Input signal

$L(z) = \frac{N(z)}{(z-1)^n D(z)}$: Loop gain

- **Steady-state Error for Various Input Signals & System Types**

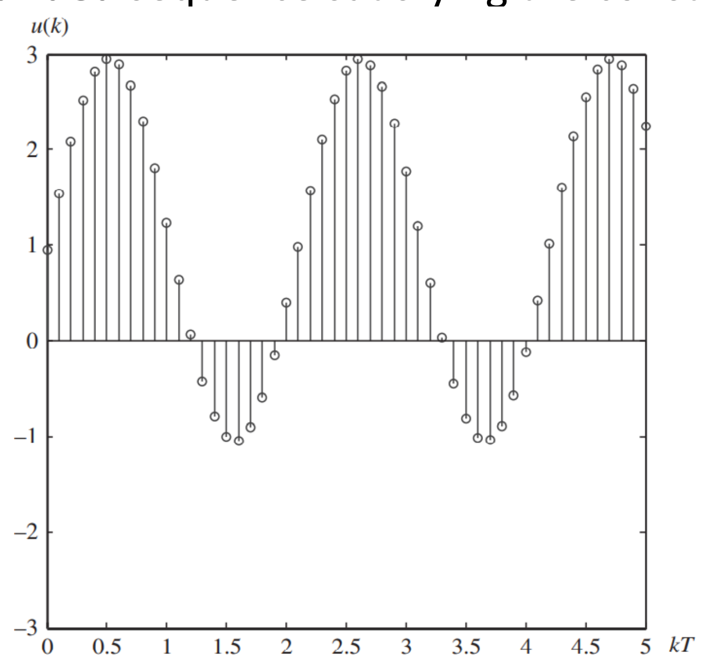
Signal	Type-0	Type-1	Type-2
Sampled step input	$\frac{1}{1+L(1)}$ or $\frac{1}{1+K_p}$	0	0
Sampled ramp input	∞	$\frac{T}{(z-1)L(z)\|_{z=1}}$ or $\frac{1}{K_v}$	0
Sampled parabolic input	∞	∞	$\frac{T^2}{(z-1)^2L(z)\|_{z=1}}$ or $\frac{1}{K_a}$

Definitions of Stability in Discrete-time Systems

- **Stability Definition:**
  - If the steady-state response is **bounded** -> System is **Stable**
  - If the steady-state response is **unbounded** -> System is **Unstable**.
- A **bounded input** satisfies the condition

$$|u(k)| < b_u, \quad k = 0,\ 1,\ 2, \ldots$$
$$0 < b_u < \infty$$

  - For example, a **bounded** sequence satisfying the constraint $|u(k)| < 3$



Stable z-domain Pole Locations

- Consider the sampled exponential function $f(k)$ and its z-transform

$$f(k) = p^k, k = 0, 1, 2, \ldots \quad \Rightarrow \quad F(z) = \frac{z}{z-p}$$

  - where p can be either real or complex numbers.

- Depending on the value of p, the time sequence for large k is given by

$$|p|^k \to \begin{cases} 0, & |p| < 1 \\ 1, & |p| = 1 \\ \infty, & |p| > 1 \end{cases}$$

- Any time sequence can be described by

$$f(k) = \sum_{i=1}^{n} A_i p_i^k, \qquad k = 0,\ 1,\ 2, \ldots$$

  - where A_i are partial fraction coefficients.

Stable z-domain Pole Locations (Continued)

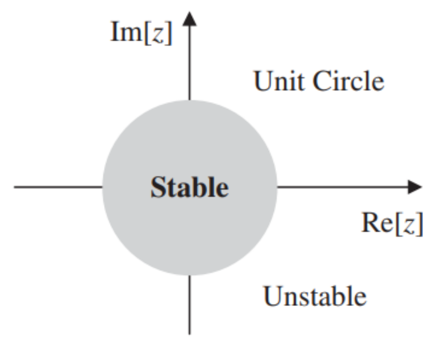
- Apply z-transfer on $f(k)$, we will have:

$$f(k) = \sum_{i=1}^{n} A_i p_i^k, \qquad k = 0,\ 1,\ 2, \ldots \overset{\mathscr{Z}}{\longleftrightarrow} F(z) = \sum_{i=1}^{n} A_i \frac{z}{z - p_i}$$

  - Where $p_i, i = 1,2,3 \ldots n$ are z-domain **poles**.

- Hence, we conclude that the sequence is bounded (i.e., stable) if **its poles lie inside the closed unit disc (i.e., on or inside the unit circle).**

$$|p|^k \rightarrow \begin{cases} 0, & |p| < 1 \\ 1, & |p| = 1 \\ \infty, & |p| > 1 \end{cases}$$



  - **Note:** if there is one pole **on** the unit circle -> **marginally stable**
  - **Note2:** if there are **repeated poles on** the unit circle -> **unstable**

Examples

- Determine the stability of the following systems:

$$H(z) = \frac{5(z - 0.3)}{(z - 0.2)(z - 0.1)}$$

$$H(z) = \frac{8(z - 0.2)}{(z - 0.1)(z - 1)}$$

- **Solution:**
  - System 1: all poles are inside the unit circle -> stable system.
  - System 2: one pole is on the unit circle -> marginally stable system.

Practice Question

- Determine the stability of the following system:

$$y(k+2) - 0.8y(k+1) + 0.07y(k) = 2u(k+1) + 0.2u(k) \quad k = 0, 1, 2, \ldots$$

- **Solution:**
  - First, find the transfer function $G(z) = \frac{Y(z)}{U(z)}$

$$G(z) = \frac{2z + 0.2}{z^2 - 0.8z + 0.07} = \frac{2(z + 0.1)}{(z - 0.7)(z - 0.1)}$$

  - Since |0.7|<1 and |0.1|<1, the system is stable.

Problem with Pole Location-based Stability Test

- A problem with the above pole location-based stability test is to find poles for high-order systems.
  - For instance, it is easy to find poles of low-order systems:

$$G(z) = \frac{2z + 0.2}{z^2 - 0.8z + 0.07} = \frac{2(z + 0.1)}{(z - 0.7)(z - 0.1)}$$

  - However, for high-order systems, it is difficult to find the poles.
    - For instance, to find poles of the following system:

$$G(z) = \frac{z + 1}{z^5 + 4z^4 + 3z^3 + 2z^2 + z + 1}$$

    - We need to solve $z^5 + 4z^4 + 3z^3 + 2z^2 + z + 1 = 0$
    - The calculation is not easy.
- Is there an easy way to determine system stability without calculating the pole locations?
  - **Jury Test**.

Jury test

- It is possible to investigate the stability of z-domain polynomials directly using the **Jury test** for real coefficients.
- **Jury Test**:
  - For the polynomial (the characteristic equation of TF)

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \qquad a_n > 0$$

    - the roots of the polynomial are inside the unit circle if and only if

$$\left.\begin{array}{ll} (1) & F(1) > 0 \\ (2) & (-1)^n F(-1) > 0 \\ (3) & |a_0| < a_n \\ (4) & |b_0| > |b_{n-1}| \\ (5) & |c_0| > |c_{n-2}| \\ & \quad\vdots \\ (n+1) & |r_0| > |r_2| \end{array}\right\}$$

There are $n + 1$ conditions that correspond to $n + 1$ coefficients of $F(z)$

    - where the terms in the $n + 1$ conditions are calculated from the **Jury Table.**

Jury Table

Table 4.1 Jury Table

Row	z^0	z^1	z^2	$\ldots$	z^{n-k}	$\ldots$	z^{n-1}	z^n
1	a_0	a_1	a_2	$\ldots$	a_{n-k}	$\ldots$	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	$\ldots$	a_k	$\ldots$	a_1	a_0
3	b_0	b_1	b_2	$\ldots$	b_{n-k}	$\ldots$	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	$\ldots$	b_k	$\ldots$	b_0	
5	c_0	c_1	c_2	$\ldots$	$\ldots$	c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	$\ldots$	$\ldots$	c_0		
.	.	.	.	$\ldots$	$\ldots$			
.	.	.	.	$\ldots$	$\ldots$			
.	.	.	.	$\ldots$	$\ldots$			
$2n-5$	s_0	s_1	s_2	s_3				
$2n-4$	s_3	s_2	s_1	s_0				
$2n-3$	r_0	r_1	r_2					

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \qquad k = 0, 1, \ldots, n-1$$

$$c_k = \begin{vmatrix} b_0 & b_{n-k-1} \\ b_{n-1} & b_k \end{vmatrix}, \qquad k = 0, 1, \ldots, n-2$$

$$\vdots$$

$$r_0 = \begin{vmatrix} s_0 & s_3 \\ s_3 & s_0 \end{vmatrix}, \quad r_1 = \begin{vmatrix} s_0 & s_2 \\ s_3 & s_1 \end{vmatrix}, \quad r_2 = \begin{vmatrix} s_0 & s_1 \\ s_3 & s_2 \end{vmatrix}$$

Jury Table Explanation

Row 1 is a listing of coefficients of the polynomial $F(z)$ in order of **increasing** power of z

The coefficients of each **even** row are the same as the odd row above it with a **reversed** order.

Table 4.1 Jury Table

Row	z^0	z^1	z^2	...	z^{n-k}	...	z^{n-1}	z^n
1	$\boxed{a_0}$	a_1	a_2	...	a_{n-k}	...	$\boxed{a_{n-1}}$	$\boxed{a_n}$
2	$\boxed{a_n}$	a_{n-1}	a_{n-2}	...	a_k	...	$\boxed{a_1}$	$\boxed{a_0}$
3	$\boxed{b_0}$	b_1	b_2	...	b_{n-k}	...	$\boxed{b_{n-1}}$	
4	$\boxed{b_{n-1}}$	b_{n-2}	b_{n-3}	...	b_k	...	$\boxed{b_0}$	
5	c_0	c_1	c_2	...	...	c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	...	...	c_0		
.	.	.	.	...	...			
.	.	.	.	...	...			
.	.	.	.	...	...			
$2n-5$	s_0	s_1	s_2	s_3				
$2n-4$	s_3	s_2	s_1	s_0				
$2n-3$	r_0	r_1	r_2					

$$b_0 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix} \qquad b_1 = \begin{vmatrix} a_0 & a_{n-1} \\ a_n & a_1 \end{vmatrix}$$

$$c_0 = \begin{vmatrix} b_0 & b_{n-1} \\ b_{n-1} & b_0 \end{vmatrix} \qquad c_1 = \begin{vmatrix} b_0 & b_{n-2} \\ b_{n-1} & b_1 \end{vmatrix}$$

Jury test Review

- For the polynomial

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \qquad a_n > 0$$

  - The roots of the polynomial are inside the unit circle if and only if

$$\left.\begin{array}{ll} (1) & F(1) > 0 \\ (2) & (-1)^n F(-1) > 0 \\ (3) & |a_0| < a_n \\ (4) & |b_0| > |b_{n-1}| \\ (5) & |c_0| > |c_{n-2}| \\ & \quad \vdots \\ (n+1) & |r_0| > |r_2| \end{array}\right\}$$

There are $n + 1$ conditions that correspond to $n + 1$ coefficients of $F(z)$

  - Condition (1) and (2) are calculated from $F(z)$ directly.
    - If one of these two conditions is violated -> the system is unstable
  - Conditions (3) → $(n + 1)$ are calculated using the coefficient of the first column of the Jury table together with the last coefficient of the preceding row.

Example

- Test the stability of the polynomial:

$$F(z) = z^5 + 2.6z^4 - 0.56z^3 - 2.05z^2 + 0.0775z + 0.35 = 0$$

- **Solution:**
  - First compare $F(z)$ with the general format to determine coefficients a_0, a_1, a_2 ....

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \qquad a_n > 0$$

  - Exam the Jury Test one by one:
    - $F(1) = 1.4175 > 0$
    - $(-1)^5 F(-1) = -0.3825 < 0$
  - Since Condition (2) is unsatisfied -> unstable system.

(1)	$F(1) > 0$
(2)	$(-1)^n F(-1) > 0$
(3)	$\|a_0\| < a_n$
(4)	$\|b_0\| > \|b_{n-1}\|$
(5)	$\|c_0\| > \|c_{n-2}\|$
	.
	.
	.
$(n+1)$	$\|r_0\| > \|r_2\|$

Example 2

- Use the Jury Criterion to determine the stability of the following polynomial:

$$z^5 + 0.2z^4 + z^2 + 0.3z - 0.1 = 0$$

- **Solution:**
  - The system order $n = 5$, thus, we need to verify 6 conditions in Jury Test

(1) $F(1) > 0$
(2) $(-1)^5 F(-1) > 0$
(3) $|a_0| < a_5$
(4) $|b_0| > |b_4|$
(5) $|c_0| > |c_3|$
(6) $|d_0| > |d_2|$

$n = 5$

(1)	$F(1) > 0$
(2)	$(-1)^n F(-1) > 0$
(3)	$\|a_0\| < a_n$
(4)	$\|b_0\| > \|b_{n-1}\|$
(5)	$\|c_0\| > \|c_{n-2}\|$
	.
	.
	.
$(n+1)$	$\|r_0\| > \|r_2\|$

  - Check all conditions one by one. If one of these conditions are **unsatisfied**, then the system is **unstable**.
  - The first 2 conditions:

(1) $F(1) = 1 + 0.2 + 1 + 0.3 - 0.1 = 2.4 > 0$

(2) $(-1)^5 F(-1) - (-1)(-1 + 0.2 + 0.3 - 0.1) - 0.2 > 0$

Example 2 (Continued)

$$F(z) = z^5 + 0.2z^4 + z^2 + 0.3z - 0.1 = 0$$

- Condition (3): $|a_0| < a_5$
  - $a_0 = -0.1; a_5 = 1$
  - Thus, condition (3) satisfied
- Condition (4): $|b_0| > |b_4|$
  - We need to build the Jury Table
  - **Note:** you don't need to calculate all values in Jury Table, just calculate the parameters you need.
  - For the demonstration purpose, here is the complete Jury Table:

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	–0.1	0.3	1	0	0.2	1
2	1	0.2	0	1	0.3	–0.1
3	–0.99	–0.23	–0.1	–1	–0.32	
4	–0.32	–1	–0.1	–0.23	–0.99	
5	0.8777	0.0923	0.0067	0.9164		
6	0.9164	0.0067	0.0923	0.8777		
7	–0.0199	–0.1277	–0.131			

Example 2 (Continued)

$$F(z) = z^5 + 0.2z^4 + z^2 + 0.3z - 0.1 = 0$$

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \qquad k = 0, 1, \ldots, n-1$$

- Condition (4): $|b_0| > |b_4|$
  - To calculate b_0, we have:
    $$b_0 = \begin{vmatrix} a_0 & a_5 \\ a_5 & a_0 \end{vmatrix} = \begin{vmatrix} -0.1 & 1 \\ 1 & -0.1 \end{vmatrix} = -0.99$$
  - To calculate b_4, we have:
    $$b_4 = \begin{vmatrix} a_0 & a_1 \\ a_5 & a_4 \end{vmatrix} = \begin{vmatrix} -0.1 & 0.3 \\ 1 & 0.2 \end{vmatrix} = -0.32$$
  - Condition (4), $|b_0| > |b_4|$, is satisfied.

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	–0.1 a_0	0.3 a_1	1	0	0.2 a_4	1 a_5
2	1	0.2	0	1	0.3	–0.1
3	–0.99 b_0	–0.23	–0.1	–1	–0.32 b_4	
4	–0.32	–1	–0.1	–0.23	–0.99	
5	0.8777	0.0923	0.0067	0.9164		
6	0.9164	0.0067	0.0923	0.8777		
7	–0.0199	–0.1277	–0.131			

Example 2 (Continued)

$$F(z) = z^5 + 0.2z^4 + z^2 + 0.3z - 0.1 = 0$$

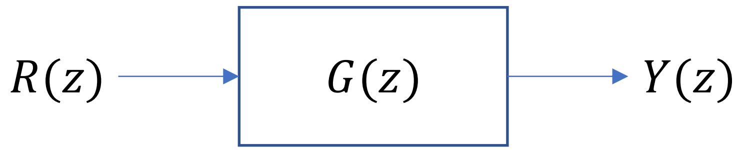
$$c_k = \begin{vmatrix} b_0 & b_{n-k-1} \\ b_{n-1} & b_k \end{vmatrix}, \quad k = 0, 1, \ldots, n-2$$

- Condition (5): $|c_0| > |c_3|$
  - To calculate c_0, we have:
    $$c_0 = \begin{vmatrix} b_0 & b_4 \\ b_4 & b_0 \end{vmatrix} = \begin{vmatrix} -0.99 & -0.32 \\ -0.32 & -0.99 \end{vmatrix} = 0.8777$$
  - To calculate c_3, we have:
    $$c_3 = \begin{vmatrix} b_0 & b_1 \\ b_4 & b_3 \end{vmatrix} = \begin{vmatrix} -0.99 & -0.23 \\ -0.32 & -1 \end{vmatrix} = 0.9164$$
  - Condition (5) is **NOT** satisfied -> System is **NOT** stable.

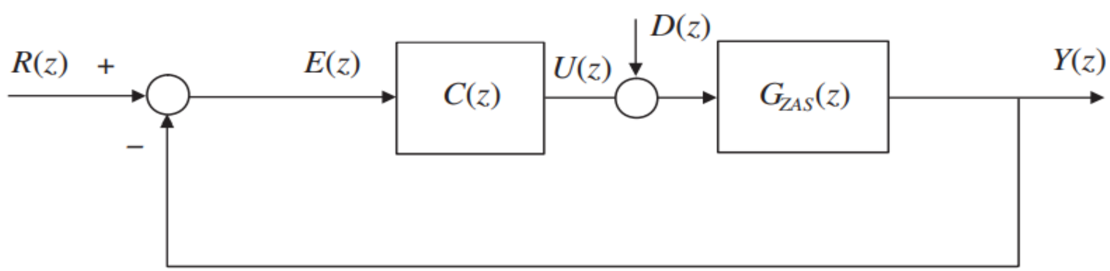
Row	z^0	z^1	z^2	z^3	z^4	z^5
1	–0.1	0.3	1	0	0.2	1
2	1	0.2	0	1	0.3	–0.1
3	–0.99 b_0	–0.23	–0.1	–1	–0.32 b_4	
4	–0.32	–1	–0.1	–0.23	–0.99	
5	0.8777	0.0923	0.0067	0.9164		
6	0.9164	0.0067	0.0923	0.8777		
7	–0.0199	–0.1277	–0.131			

Internal Stability

- The previous definition of stability only considers **open-loop** cases.



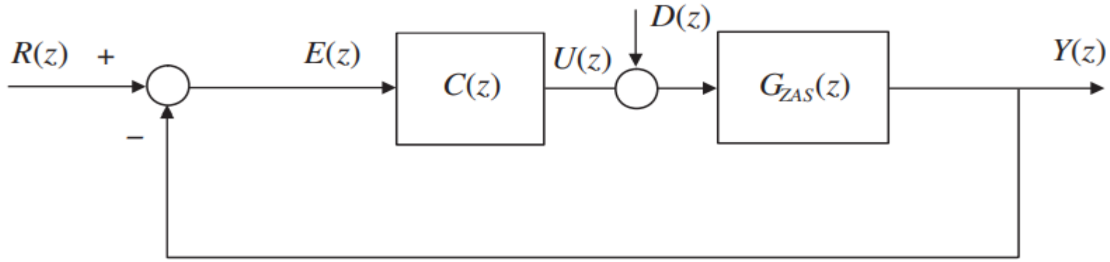
  - $G(z)$ is stable if all poles of $G(z)$ are within the unit circle.
  - OR, the characteristic equation of $G(z)$ satisfies the Jury Test
- Consider a Closed-loop Control System (with external disturbance):



Digital control system with disturbance D(z).

  - The Closed-loop system is stable if **all signals in the loop are bounded.**
    - Include **Internal signals**: $E(z), U(z)$
    - and **External signals**: $R(z), D(z), Y(z)$

Internal Stability (Continued)



Digital control system with disturbance D(z).

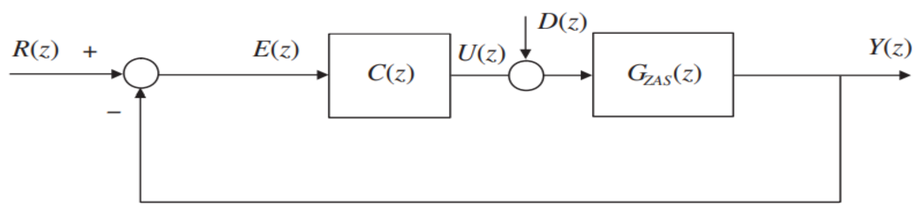
- We consider that system as having two outputs, Y and U, and two inputs, R and D. The transfer function is:

$$\begin{bmatrix} Y(z) \\ U(z) \end{bmatrix} = \begin{bmatrix} \dfrac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} & \dfrac{G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} \\ \dfrac{C(z)}{1 + C(z)G_{ZAS}(z)} & -\dfrac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} \end{bmatrix} \begin{bmatrix} R(z) \\ D(z) \end{bmatrix}$$

  - For **bounded** input R, the output signals Y and U should also be **bounded**, if the system is **stable.**
  - Moreover, when D is applied, Y should be **bounded**.
- This stability is referred to as **Internal Stability**.

Internal Stability Definition

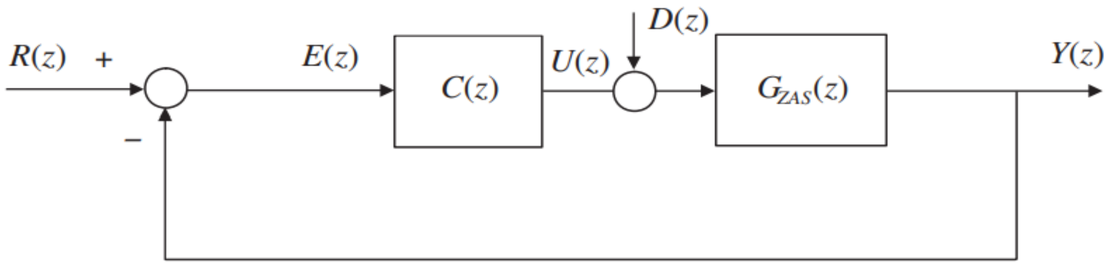
- A closed-loop system is **internally stable** if the following two conditions hold:
  - The characteristic Polynomial $1 + C(z)G_{ZAS}(z)$ has no zeros **on** or **outside** the unit circle.
  - The loop gain $C(z)G_{ZAS}(z)$ has no pole-zero cancellation **on** or **outside** the unit circle.



$$\begin{bmatrix} Y(z) \\ U(z) \end{bmatrix} = \begin{bmatrix} \dfrac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} & \dfrac{G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} \\ \dfrac{C(z)}{1 + C(z)G_{ZAS}(z)} & -\dfrac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} \end{bmatrix} \begin{bmatrix} R(z) \\ D(z) \end{bmatrix}$$

Example Question

- Consider the following closed-loop system:



Digital control system with disturbance D(z).

  - Where $D(z) = 0$; $G_{ZAS}(z) = \frac{-0.07(z-1.334)}{(z-0.81)(z-0.77)}$; $C(z) = \frac{-10(z-0.81)(z-0.77)}{(z-1)(z-1.334)}$
  - **Determine the Internal Stability of this system.**
- **Solution:**
  - The transfer function from the reference input to the system output is given by

$$\frac{Y(z)}{R(z)} = \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} = \frac{0.75997}{z - 0.24}$$

    - The system is stable since all its poles inside the unit circle.

Example Question (Continued)

- **Solution (Continued):**
  - However, the system is not **internally stable** as seen by examining the transfer function:

$$\frac{U(z)}{R(z)} = \frac{C(z)}{1 + C(z)G_{ZAS}(z)} = \frac{-10(z-0.8149)(z-0.7655)}{(z-0.24)(z-1.334)}$$

  - The transfer function has a pole at 1.334, which is outside of the unit circle.
  - Thus, control variable U is unbounded even when the reference input R is bounded.
  - In fact, this system violates **Condition 2** of Internal Stability Definition (Page 9):

> - A closed-loop system is **internal stable** if the following two conditions hold:
>   - The characteristic Polynomial $1 + C(z)G_{ZAS}(z)$ has no zeros **on** or **outside** the unit circle.
>   - The loop gain $C(z)G_{ZAS}(z)$ has no pole-zero cancellation **on** or **outside** the unit circle.

    - The pole at 1.334 cancels in the loop gain **outside** the unit circle:

$$C(z)G_{ZAS}(z) = \frac{-10(z-0.8149)(z-0.7655)}{(z-1)(z-1.334)} \times \frac{-0.075997(z-1.334)}{(z-0.8149)(z-0.7655)}$$

Practice Question

- For a closed-loop system with

$$G_{ZAS}(z) = \frac{-0.1}{z - 1.01}$$

$$C(z) = -\frac{z - 1.01}{z - 1}$$

  - Determine whether the closed-loop system is **internally stable**.

- **Solution:**

  - Not internally stable, due to the pole-zero cancellation on 1.01 in the loop gain.