Lecture 4: Time & Frequency Response of Discrete-time System

ELEN 472: Introduction to Digital Control

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Review

- **Inverse z-Transform**:

$$Y(z) \;\Rightarrow\; \boxed{\textit{Inverse z-Transform}} \;\Rightarrow\; y(k)$$

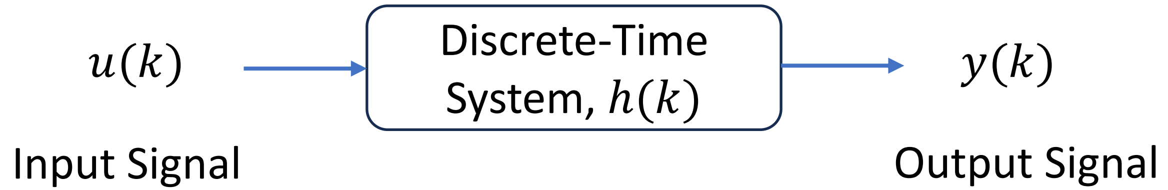
  - Long Division
  - Partial Fraction Expansion (with a special case with repeated roots)
- **Final Value Theorem:**
  - Allows us to find the limit of a sequence as time $k \to \infty$

$$f(\infty) = \lim_{z \to 1}[(z-1)F(z)]$$

- **Z-Transform Solution of Difference Equations:**
  - Use z-Transform to solve difference equations
  - Workflow:
    - Use z-Transform to convert difference equations into z-domain;
    - Find expression of the discrete-time signal;
    - Use inverse z-Transform to get the time-domain signal.

Time Response of Discrete-time Systems

- The **time response** of discrete-time systems is the **output** of the discrete-time system, providing the input.



- To calculate this output signal, we need to use the **convolution**:

$$y(k) = h(k) * u(k) = \sum_{i=0}^{k} h(k-i)u(i)$$

Example

- Obtain the Convolution of two sequences $f\{k\} = \{1,1,1\}$ and $g(k) = \{1,2,3\}$
- Solution:
  - Since $y(k) = h(k) * u(k) = \sum_{i=0}^{k} h(k-i)u(i)$.
    - $y(k) = f\{k\} * g(k) = \sum_{i=0}^{k} f(k-i)g(i)$

$k = 0$ y(0) = f(0).g(0)= 1×1=1

$k = 1$ y(1) = f(1).g(0) + f(0).g(1) = 1×1+1×2=3

$k = 2$ y(2) = f(2).g(0) + f(1).g(1) + f(0).g(2) = 1×1+1×2 + 1×3 = 6

… y(3) = f(2).g(1) + f(1).g(2) = 1×2 + 1×3 = 5

y(4) = f(2).g(2) = 1×3 = 3

y(k) = 0, $k > 4$

Practice

- Find the Convolution of two sequences $\mathrm{f}\{\mathrm{k}\} = \{2,3,4,5\}$ and $\mathrm{g}(\mathrm{k}) = \{1,2,3\}$
- **Solution:**
  - $y(k) = f\{k\} * g(k) = \sum_{i=0}^{k} f(k-i)g(i)$
  - $k = 0$
    - $y(0) = f(0) * g(0) = f(0)g(0) = 2 \times 1 = 2$
  - $k = 1$
    - $y(1) = f(1) * g(1) = \sum_{i=0}^{1} f(1-i)g(i) = f(1)g(0) + f(0)g(1) = 3 \times 1 + 2 \times 2 = 7$
  - $k = 2$
    - $y(2) = f(2) * g(2) = \sum_{i=0}^{2} f(2-i)g(i) = f(2)g(0) + f(1)g(1) + f(0)g(2) = 4 \times 1 + 3 \times 2 + 2 \times 3 = 16$
  - $k = 3$
    - $y(3) = f(3) * g(3) = \sum_{i=0}^{3} f(3-i)g(i) = f(3)g(0) + f(2)g(1) + f(1)g(2) + f(0)g(3) = 5 \times 1 + 4 \times 2 + 3 \times 1 + 2 \times 0 = 16$
  - $k = 4$
    - $y(4) = f(4) * g(4) = \sum_{i=0}^{4} f(4-i)g(i) = f(3)g(1) + f(2)g(2) = 5 \times 2 + 4 \times 3 = 22$
  - $k = 5$
    - $y(5) = f(5) * g(5) = \sum_{i=0}^{5} f(5-i)g(i) = f(3)g(2) = 5 \times 3 = 15$
  - $k \geq 6$
    - $y(6) = 0$

The Convolution Theorem

- The **Convolution** is a fairly complex operation, especially if the output sequence is required over a long time period.
  - Can we avoid this convolution?
- **THE CONVOLUTION THEOREM**
  - The z-transform of the convolution of two time sequences is equal to the product of their z-transforms.
$$Y(z) = H(z)U(z)$$
  - The function $H(z)$ is known as the z-transfer function or simply the **Transfer Function**.
- Applying the convolution allows us to use the z-transform to find the output of a system without using convolution.

Example

Given the discrete-time system

$$y(k+1) - 0.5y(k) = u(k), \quad y(0) = 0$$

find the impulse response of the system $h(k)$:

1. From the difference equation
2. Using z-transformation

Solution

1. Let $u(k) = \delta(k)$. Then

k=0 $\quad y(1) = 1$

k=1 $\quad y(2) = 0.5y(1) = 0.5$

k=2 $\quad y(3) = 0.5y(2) = (0.5)^2$

$$\text{i.e.,} \quad h(i) = \begin{cases} (0.5)^{i-1}, & i = 1, 2, 3, \ldots \\ 0, & i < 1 \end{cases}$$

2. Alternatively, z-transforming the difference equation yields the transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{z - 0.5}$$

Time Advance Property:

$$\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0)$$

$$\mathcal{Z}\{f(k+n)\} = z^n F(z) - z^n f(0) - z^{n-1} f(1) - \cdots - zf(n-1)$$

Inverse-transforming with the delay theorem gives the impulse response

$$h(i) = \begin{cases} (0.5)^{i-1}, & i = 1, 2, 3, \ldots \\ 0, & i < 1 \end{cases}$$

MATLAB Implementation

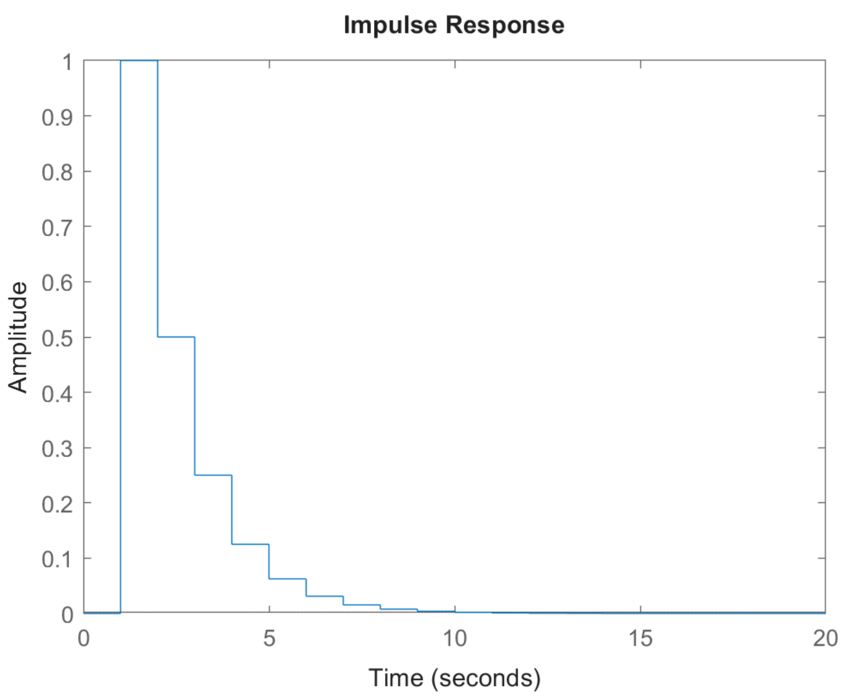
- We can use MATLAB to find and plot the time-domain response of discrete-time systems.

$$H(z) = \frac{1}{z - 0.5}$$

```
% Define the discrete-time system using transfer function tf
% tf(numerator, denominator, sample period)
H = tf(1, [1, -0.5], 1);

% Get the system response to an impulse response
impulse(H)
```

$$h(i) = \begin{cases} (0.5)^{i-1}, & i = 1, 2, 3, \ldots \\ 0, & i < 1 \end{cases}$$



Example 2

- Given the discrete-time system

$$y(k+1) - y(k) = u(k+1)$$

Find the system transfer function and its response to a sampled unit step

- Solution:
  - The transfer function corresponding to the difference equation is

$$zY(z) - Y(z) = zU(z); H(z) = \frac{Y(z)}{U(z)} = \frac{z}{z-1}$$

  - We multiply the transfer function by the sampled unit step's z-transform to obtain

$$Y(z) = \left(\frac{z}{z-1}\right) \times \left(\frac{z}{z-1}\right) = \left(\frac{z}{z-1}\right)^2 = z\frac{z}{(z-1)^2}$$

Note: the z-transform of a unit step signal is $\frac{z}{z-1}$

  - The z-transform of a unit ramp is $F(z) = \frac{z}{(z-1)^2}$
  - Then, using the **time advance property** of the z-transform, we have the inverse transform

$$y(i) = \begin{cases} i+1, & i = 0,\ 1,\ 2,\ 3, \ldots \\ 0, & i < 0 \end{cases}$$

Practice Questions

- **Q1:** Find the transfer function of the following systems:
$$y(k+4) + y(k-1) = u(k)$$
- **Solution:**

$$\text{Z-transform } (z^4 - z^{-1})Y(z) = U(z) \Longrightarrow G(z) = \frac{z}{z^5+1}$$

- **Q2:** Given the discrete-time system
$$y(k+2) - y(k) = 2u(k)$$
  - Find the impulse response of the system
- **Solution:**

Z-transform: $z^2Y(z) - Y(z) = 2U(z) \rightarrow H(z) = \frac{2}{z^2-1} = \frac{1}{z-1} - \frac{1}{z+1}$

$\frac{1}{z-1}$ can be treated as $z^{-1}\frac{z}{z-1}$, and the inverse z-transform is 1^{k-1} for $k > 1$

Thus, the impulse response is $y(x) = \begin{cases} 1^{k-1} - (-1)^{k-1}, & k \geq 1 \\ 0, k < 1 \end{cases}$

Frequency Response of Discrete-time Systems

- Frequency Response of discrete-time systems gives the **magnitude** and **phase response** of the system to the **sinusoids at all frequencies**.

$$u(k) = e^{j\omega k} \longrightarrow \boxed{\text{Discrete-Time System, } h(k)} \longrightarrow y(k) = H(e^{j\omega}) \cdot e^{j\omega k}$$

Input Signal (Sinusoids) — Output Signal (Sinusoids)

- **Proof**:

$$y(k) = h(k) * u(k) = \sum_{i=0}^{k} h(k)u(k-i)$$

  - Now, $u(k) = e^{j\omega k}$. Thus, we have:

$$y(k) = h(k) * u(k) = \sum_{i=0}^{k} h(k)e^{j\omega(k-i)}$$

$$= \underbrace{\boxed{\sum_{i=0}^{k} h(k)e^{-j\omega i}}}_{H(e^{j\omega})} e^{j\omega k}$$

Frequency Response of Discrete-time Systems

$$y(k) = H(e^{j\omega}) \cdot e^{j\omega k}$$

- $H\left(e^{j\omega}\right)$ is define as the **frequency response** of the discrete-time system.
  - $H\left(e^{j\omega}\right)$ is a complex number, which can be represented as:

$$H\left(e^{j\omega}\right) = |H(e^{j\omega})|\angle H(e^{j\omega})$$

$$\Downarrow \quad y(k) = H(e^{j\omega}) \cdot e^{j\omega k}$$

$$y(k) = |H\left(e^{j\omega}\right)| \cdot e^{j\left(\omega k + \angle H\left(e^{j\omega}\right)\right)}$$

- Recall **Euler equation**:

$$e^{j\omega} = \cos(\omega) + j \cdot \sin(\omega)$$

$$y(k) = \boxed{\left|H\left(e^{j\omega}\right)\right| \cdot \cos\left(\omega k + \angle H\left(e^{j\omega}\right)\right)} + j \cdot \boxed{\left|H\left(e^{j\omega}\right)\right| \sin\left(\omega k + \angle H\left(e^{j\omega}\right)\right)}$$

If input $u(t) = \cos(\omega k)$

If input $u(t) = \sin(\omega k)$

Example

- Find the steady-state response of the system

$$H(z) = \frac{1}{(z-0.1)(z-.5)}$$

  - due to the sampled sinusoid $u(k) = 3\cos(0.2\,k)$

- **Solution:**
  - Since the input signal is a cosine signal, the output signal should only contain cosine part, i.e.,

$$y(k) = \left|H\left(e^{j\omega}\right)\right| \cdot 3\cos\left(\omega k + \angle H\left(e^{j\omega}\right)\right)$$

$$\Big\Downarrow \quad \omega = 0.2$$

$$y(k) = \left|H\left(e^{j0.2}\right)\right| \cdot 3\cos\left(0.2k + \angle H\left(e^{j0.2}\right)\right)$$

$$= \left|\frac{1}{(e^{j0.2}-0.1)(e^{j0.2}-0.5)}\right| 3\cos\left(0.2k + \angle\frac{1}{(e^{j0.2}-0.1)(e^{j0.2}-0.5)}\right)$$

$$= 6.4\cos(0.2k - 0.614)$$

Practice Question

- Find the steady-state response of the systems due to the sinusoidal input $u(k) = 0.5\sin(0.4\,k)$

$$H(z) = \frac{z}{z - 0.4}$$

- Solution:

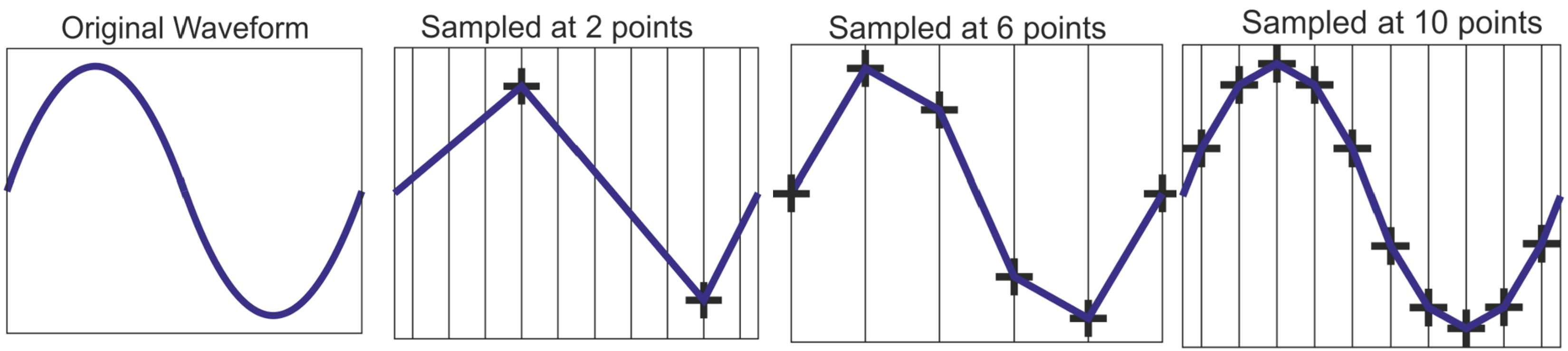
$$H(z) = \frac{z}{z-0.4} = \frac{1}{1-0.4z^{-1}}$$

$$H\left(e^{j0.4}\right) = \frac{1}{1-0.4e^{-j0.4}} = 1.537\angle -0.242$$

$$u(k) = 0.5 \times 1.537\sin(0.4\,k - 0.242) = 0.769\sin(0.4k - 0.242)$$

The Sampling Theorem: Introduction

- Sampling is necessary for the processing of analog data using digital elements.
- Successful digital data processing requires that the samples reflect **the Nature of the Analog Signal,** and that **Analog Signal is Recoverable**.
  - The following figures show different discrete-time signals with varying sampling rate.



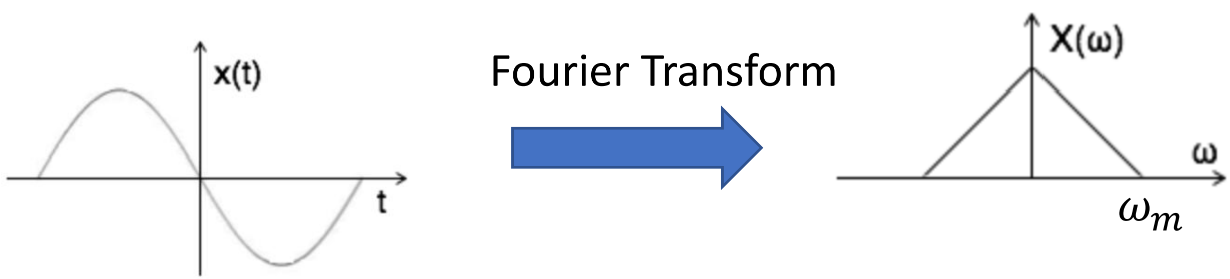
- Obviously, **faster sampling** would produce distinguishable sequences.
  - Thus, it is obvious that **sufficiently fast** sampling is a prerequisite for successful digital data processing.
  - How fast is sufficiently fast? -> The **Sampling Theorem** gives **a lower bound** on the **sampling rate** necessary for a given band-limited signal.

The Sampling Theorem

- The Sampling Theorem:
  - A continuous-time signal can be represented in its samples and can be recovered back when sampling frequency f_s is **greater than or equal to the twice of the highest frequency** component of the signal:

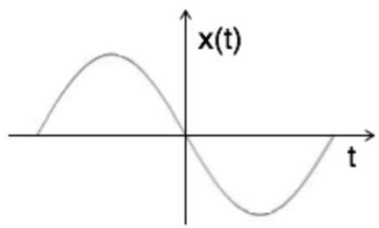
$$f_s \geq 2f_m$$

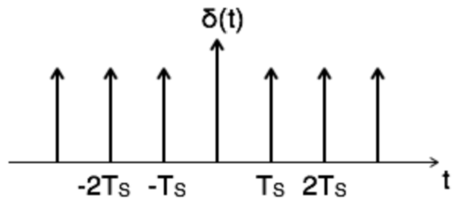
- Example: consider a continuous time signal $x(t)$. The spectrum (Fourier Transform Result) of $x(t)$ is limited to f_m Hz (or ω_m rad/s).

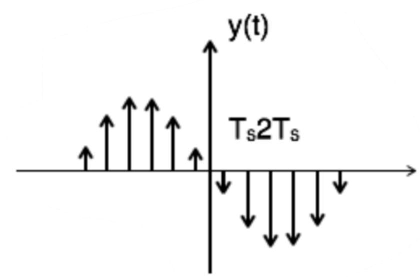


The Sampling Theorem

- The sampling of $x(t)$ can be obtained by multiplying $x(t)$ with an impulse train $\delta(t)$



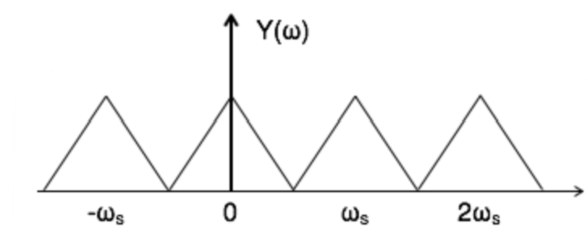




Fourier Transform

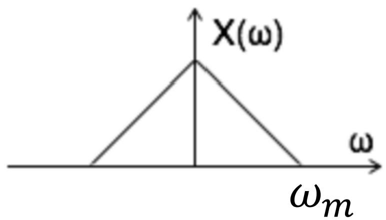
T_s is the sampling period

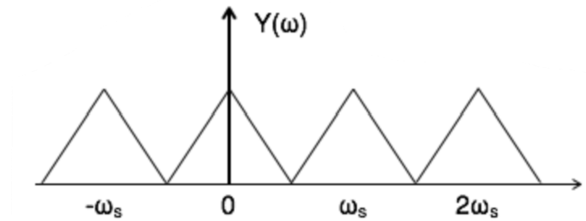
$f_s = \frac{1}{T_s}$ is the sampling frequency



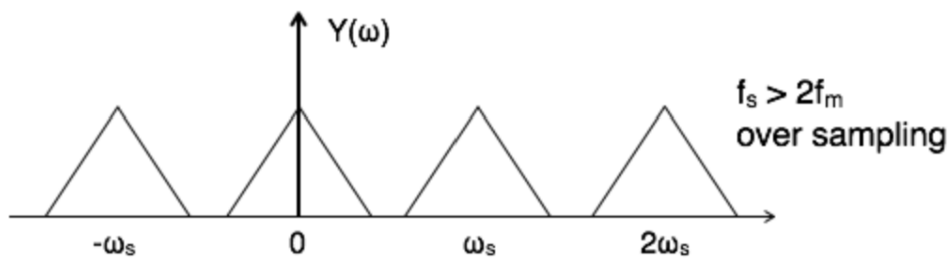
$\omega_s = 2\pi f_S$

The Sampling Theorem

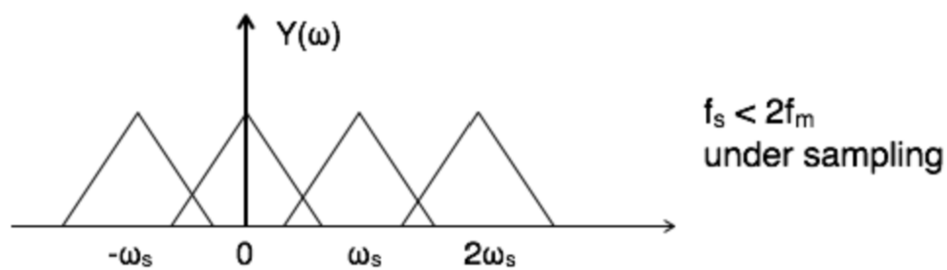




- If $f_s > 2f_m$, i.e., $\omega_s > 2\omega_m$



- If $f_s < 2f_m$, i.e., $\omega_s < 2\omega_m$



Selection of the Sampling Frequency

- In practice, the sampling rate chosen is often **much larger than** the lower bound specified in the sampling theorem.
  - The sampling frequency is chosen as $\omega_s = k\omega_m, 35 \le k \le 70$
- **Example**
  - Given a first-order system of bandwidth 10 rad/s (i.e., $\omega_m = 10$ rad/s), select a suitable sampling frequency and find the corresponding sampling period.
  - Solution:

    (Any values between 35 to 70 is feasible)

    - A suitable choice of sampling frequency is $\omega_s = 60 * \omega_m$ rad/s=$60 * 10 = 600$ rad/s.
    - The corresponding sampling period is approximately $T_s = \frac{2\pi}{\omega_s} = \frac{2\pi}{600} = 0.01$ s.

Selection of the Sampling Frequency: Second-Order Systems

- For a **second-order system**, the bandwidth of the system is approximated by the damped natural frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

  - The **sampling frequency** for a second-order system is

$$\omega_s = k\omega_d, \quad 35 \leq k \leq 70$$

Selection of the Sampling Frequency: Second-Order Systems

- **Example**:
  - A second-order closed-loop control system has a damping ratio 0.7 (i.e., $\zeta = 0.7$) and natural frequency 10 rad/s (i.e., $\omega_n = 10$ rad/s). Select a suitable sampling period for the system.
- **Solution:**

Let the sampling frequency be

$$\begin{aligned}\omega_s &\geq 35\omega_d \\ &= 35\omega_n\sqrt{1-\zeta^2} \\ &= 350\sqrt{1-0.49} \\ &= 249.95 \quad \text{rad/s}\end{aligned}$$

The corresponding sampling period is $T = 2\pi/\omega_s \leq 0.025$ s.

Practice Questions

- **Questions:**
  - For a first-order system, the system bandwidth is 20 rad/s. Select the suitable sampling period.
  - For a second-order system, the system natural frequency is 5 rad/s and the damping ratio is 0.7. Find the suitable sampling period.
- **Solution:**
  - $\omega_m = 20$ rad/s, we can choose $k = 40$, thus, $\omega_s = 800$ rad/s and $T = \frac{2\pi}{\omega_s} = 0.00785 \text{ s} = 7.85\text{ms}$
  - For second order systems, the system bandwidth is $\omega_d = \omega_n\sqrt{1-\zeta^2}$
    - Thus, $\omega_d = 5\sqrt{1-0.7^2} = 3.57$ rad/s
    - We can choose $k = 70$, thus

$$T = \frac{2\pi}{\omega_s} = \frac{2\pi}{70\omega_d} = 0.025s \qquad \text{Let } T = 25 \text{ ms.}$$