Lecture 12: Digital Control System Design II

ELEN 472: Introduction to Digital Control

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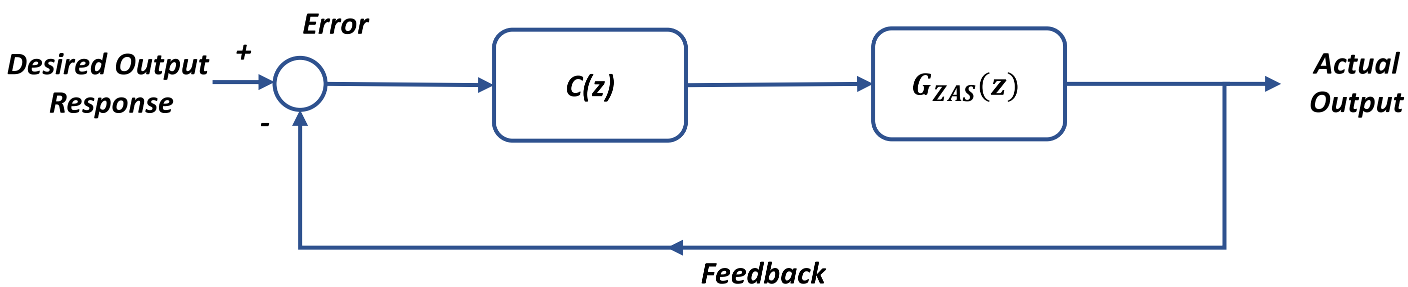
A Quick Recap

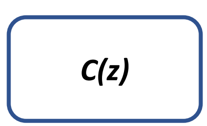
- A closed-loop digital control system can be presented as:

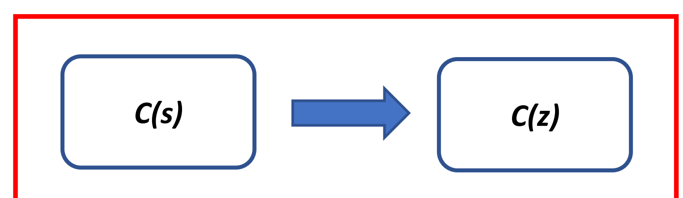


- To design a digital control system, we aim to find
  - The **z-domain transfer function** of **the controller** that meets given design specifications.
- There are two ways to get the desired controller $C(z)$:
  - Design the **digital** controller that meets the design requirement **directly**.

  OR

  - Design an **analog** controller that meets the design requirement, then convert it into **digital** domain.

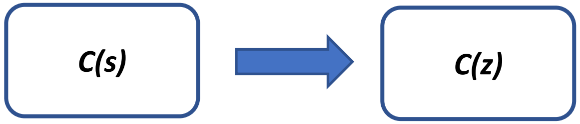




Today

Digital Implementation of Analog Controller Design

- This section introduces an **indirect** approach to digital controller design.
  - The approach is based on designing an **analog controller** for the analog subsystem and then obtaining an **equivalent digital controller** and using it to digitally implement the desired control.



- General procedures to obtain a **digital controller** using **analog design**:
  1. Design a controller $C_a(s)$ for the analog subsystem to meet the desired design specifications.
  2. Map the analog controller to a digital controller $C(z)$ using a **suitable transformation**.

Differencing Methods

$$C(s) \xrightarrow{?} C(z)$$

- We first introduce two approximations methods based on differencing:
  - **Forward differencing** and
  - **Backward differencing**
- **Forward Difference:**
  - Forward Differencing approximates the following equation:

$$\dot{y}(k) \cong \frac{1}{T}[y(k+1) - y(k)]$$

Derivative (continuous) ≈ Difference (discrete)

Laplace Transform

Z-Transform

$$sY(s) \rightarrow \frac{1}{T}[z-1]Y(z)$$

Simplify

$$\boxed{s \rightarrow \frac{z-1}{T}}$$

Example

- Apply the **forward difference** approximation to the second-order analog controller with $T = 0.1$ s:

$$C_a(s) = \frac{5}{s+5}$$

- **Solution**:
    - We can obtain the transfer function of the **digital controller** using the simple **forward difference** transformation:

$$s \to \frac{z-1}{T}$$

    - Thus, we have:

$$C(z) = \frac{5}{\frac{z-1}{T}+5} = \frac{0.5}{z-1+0.5} = \frac{0.5}{z-0.5}$$

Backward Differencing

- The backward differencing approximation of the derivative is

$$\dot{y}(k) \cong \frac{1}{T}[y(k) - y(k-1)]$$

$$\boxed{\text{Derivative (continuous)} \approx \text{Difference (discrete)}}$$

Laplace Transform ⇓ ⇓ Z-Transform

$$sY(s) \rightarrow \frac{1}{T}\left[Y(z) - \frac{1}{z}Y(z)\right]$$

⇓ Simplify

$$\boxed{s \rightarrow \frac{z-1}{zT}}$$

Example Question

- Apply the **backward difference** approximation of the derivative to the second-order analog controller with $T = 0.1$ s:

$$C_a(s) = \frac{5}{s+5}$$

- **Solution:**
  - We obtain the transfer function of the **digital controller** using $s \rightarrow \frac{z-1}{zT}$

$$C(z) = \frac{5}{\frac{z-1}{zT} + 5} = \frac{0.5z}{z - 1 + 0.5z} = \frac{0.5z}{1.5z - 1}$$

Pole-zero Matching

- Another way to convert $C(s) \rightarrow C(z)$ is **Pole-zero Matching.**
  - An s-plane pole/zero at p_s can be mapped to a z-plane pole/zero p_z via:

$$p_z = e^{p_s T}$$

- If an **Analog Controller** has n poles and m zeros:

$$G_a(s) = K \frac{\prod_{i=1}^{m}(s - a_i)}{\prod_{j=1}^{n}(s - b_j)}$$

- The corresponding **Digital Controller** based on **Pole-zero Matching** is:

Adding $n - m - 1$ zeros at $z = -1$

$$G(z) = \alpha K \frac{\boxed{(z+1)^{n-m-1}} \prod_{i=1}^{m}(z - e^{a_i T})}{\prod_{j=1}^{m}(z - e^{b_j T})}$$

  - where α is a constant, selected for equal filter gains.
  - To determine α, use $G_a(0) = G(1)$

Example

- Find a **Pole-zero Matched Digital Controller** approximation for the **Analog Controller**

$$G_a(s) = \frac{5}{s+5}$$

  - Determine the transfer function of the digital filter for a sampling period of T =0.1 s.

- **Solution:**
  - The Analog Controller has none zero, i.e., $m = 0$, and one pole, i.e., $n = 1$
    - The pole is at $p = -5$
  - We apply the **pole-zero matching** transformation to obtain

$$G_a(s) = K\frac{\prod_{i=1}^{m}(s-a_i)}{\prod_{j=1}^{n}(s-b_j)} \Rightarrow G(z) = \alpha K \frac{(z+1)^{n-m-1}\prod_{i=1}^{m}(z-e^{a_iT})}{\prod_{j=1}^{m}(z-e^{b_jT})}$$

$$G_a(s) = 5\frac{1}{(s--5)} \Rightarrow G(z) = \alpha \times 5 \times \frac{(z+1)^{1-0-1}}{(z-e^{p_1T})}$$

Example Solution

$$G(z) = \alpha \times 5 \times \frac{(z+1)^0}{(z - e^{pT})}$$

$$p = -5, T = 0.1$$

$$G(z) = \alpha \times 5 \times \frac{1}{(z - 0.607)} = \frac{5\alpha}{(z - 0.607)}$$

- To determine the value of α, we have:

$$G(z = 1) = G_a(s = 0)$$

$$G(z) = \frac{5\alpha}{(1 - 0.607)} = \frac{5\alpha}{0.393}$$

$$G_a(s) = \frac{5}{(0+5)} = 1$$

- Thus, $\frac{5\alpha}{0.393} = 1 \Rightarrow \alpha = 0.079 \Rightarrow G(z) = \frac{0.393}{(z - 0.607)}$

MATLAB Verification

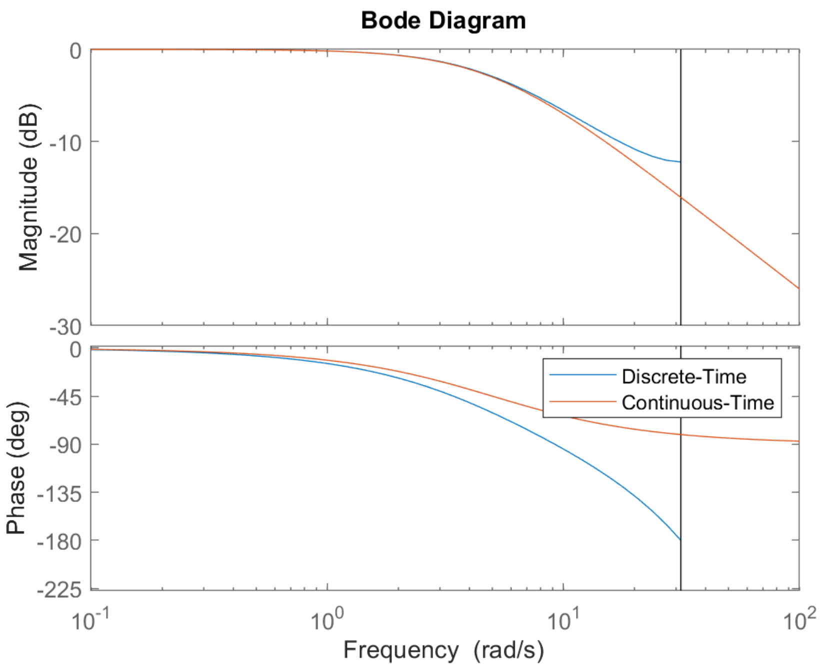
- In MATLAB, you can use the following code to perform **Pole-zero matching**

```
sys = tf([5], [1, 5])

sys_d = c2d(sys, 0.1, 'matched')
```

```
sys =
  5
 -----
 s + 5
```

```
sys_d =
   0.3935
 ----------
 z - 0.6065
```



The frequency responses are almost identical in the **low frequency** range but become different at **high frequencies**.

Bilinear Transformation

- Another way to convert $C_a(s) \rightarrow C(z)$ is **Bilinear Transformation:**

$$s = c\,\frac{z-1}{z+1}$$

  - c is a constant, usually $c = 2/T$, T is the sampling time.

- A digital filter $C(z)$ is obtained from an analog filter $C_a(s)$ by the substitution:

$$C(z) = C_a(s)\,\Big|_{s=c\left[\frac{z-1}{z+1}\right]}$$

- **Example:**
  - Design a digital filter by applying the bilinear transformation to the analog filter:

$$C_a(s) = \frac{1}{0.1s+1}$$

    - With $T = 0.1$ s

- **Solution**:
  - By applying the bilinear transformation, $s = c\,\frac{z-1}{z+1}$, $c = 2/T$, we obtain:

$$C(z) = \frac{1}{0.1\,\frac{2}{0.1}\frac{z-1}{z+1}+1} = \frac{z+1}{3z-1}$$

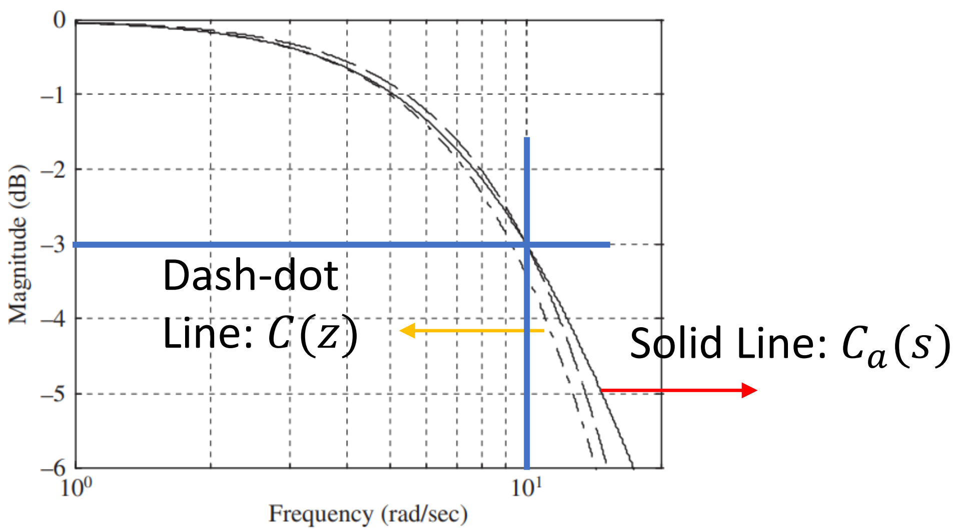
Bilinear Transformation With Pre-wrapping

- When we compare the frequency response between $C_a(s)$ and $C(z)$

$$C_a(s) = \frac{1}{0.1s+1} \quad \Longrightarrow \quad C(z) = \frac{1}{0.1\,\frac{2}{0.1}\frac{z-1}{z+1}+1} = \frac{z+1}{3z-1}$$

Bilinear Transformation:

$$s = c\frac{z-1}{z+1}$$
$$c = 2/T$$



- We can observe that $C_a(s)$ and $C(z)$ are different in frequency response.
- This is called **distortion**.
- The distortion can be mitigated via using **Pre-wrapping**:
  - Simply use $c = \frac{\omega_0}{\tan(\frac{\omega_0 T}{2})}$
  - ω_0 is the $\pm$3-dB frequency

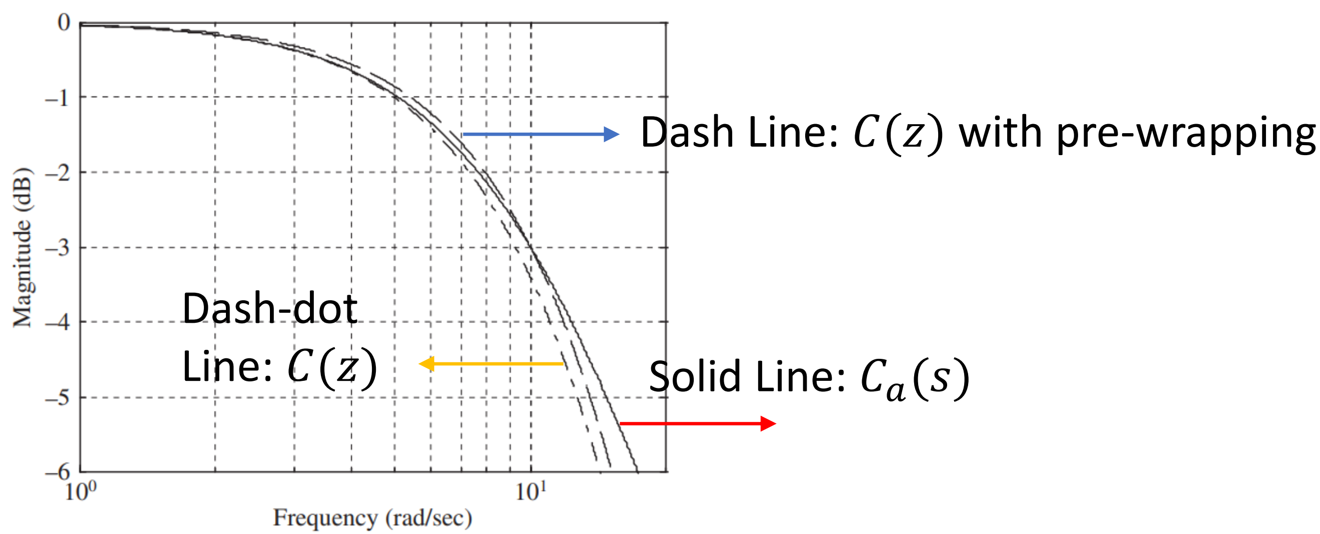
Bilinear Transformation With Pre-wrapping (Continued)

$$C_a(s) = \frac{1}{0.1s + 1} \quad \Longrightarrow \quad C(z) = \frac{1}{0.1\,\frac{10}{\tan(\frac{10\cdot 0.1}{2})}\frac{z-1}{z+1} + 1} \cong \frac{0.35z + 0.35}{z - 0.29}$$

Bilinear Transformation with Pre-wrapping:

$$s = c\,\frac{z-1}{z+1}$$

$$c = \frac{\omega_0}{\tan(\frac{\omega_0 T}{2})}$$



Summary of All Mapping Methods

Mapping Methods	Mapping Equation
Forward Differencing	$s \to \frac{z-1}{T}$
Backward Differencing	$s \to \frac{z-1}{zT}$
Pole-zero Matching	$G_a(s) = K\frac{\prod_{i=1}^{m}(s-a_i)}{\prod_{j=1}^{m}(s-b_j)}$ $\Rightarrow$ $G(z) = \alpha K\frac{(z+1)^{n-m-1}\prod_{i=1}^{m}(z-e^{a_iT})}{\prod_{j=1}^{m}(z-e^{b_jT})}$
Bilinear Transformation	$s = c\frac{z-1}{z+1}$ $c = \frac{2}{T}$, *T is the sampling time*
Bilinear Transformation with Pre-wrapping:	$s = c\frac{z-1}{z+1}$ $c = \frac{\omega_0}{\tan(\frac{\omega_0 T}{2})}$, *$\omega_0$ is the 3-db frequency*

Example

- Design a digital controller for a DC motor speed control system where the analog plant has the transfer function

$$G(s) = \frac{1}{(s+1)(s+10)}$$

To obtain:

- Zero steady-state error due to a unit step
- A damping ratio of 0.7
- A settling time of about 1 s.

- **Solution:**
  - We follow the procedures in page 3 to design a digital controller.
  - **Procedure 1: Design an analog controller to meet the requirements.**
    - The given analog system is a type 0 system, the steady state error for a step input is $\frac{1}{1+K_p}$.
    - Thus, to make the steady-state error zero, the system type must be increased by one.

Example Solution

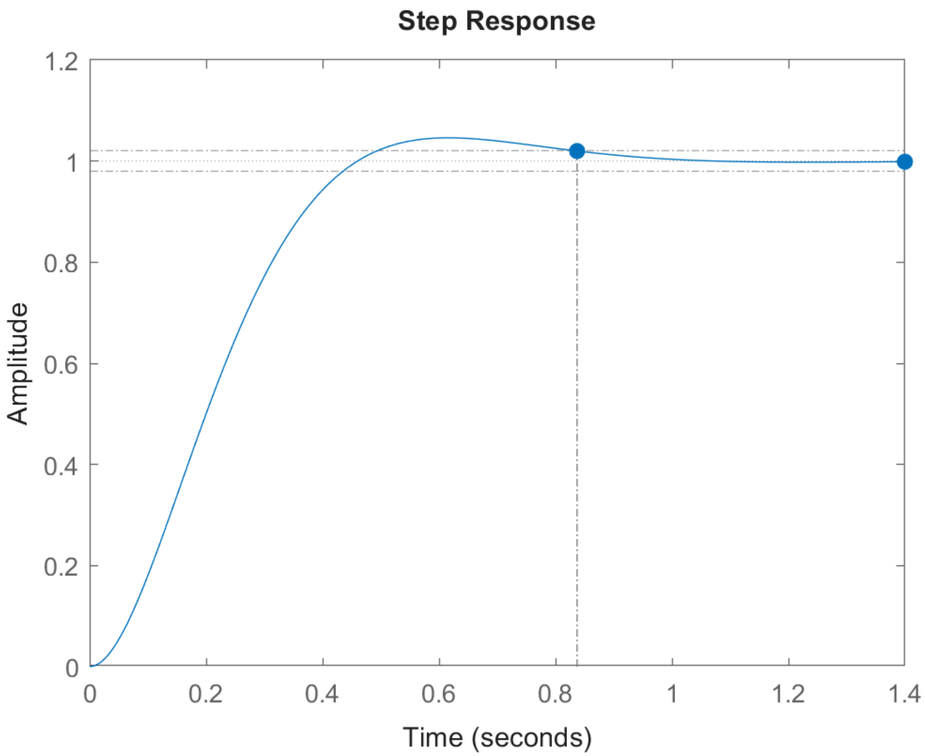
Control Requirements:
- zero steady-state error due to a unit step;
- a damping ratio of 0.7;
- a settling time of about 1 s

- A PI controller can increase system type by 1 (since its transform function contains 1/s)
- The PI controller's transform function is $C_a(s) = K\frac{s+1}{s}$, and the corresponding loop gain is $C_a(s)G(s) = \frac{K}{s(s+10)}$.
- Hence, the closed-loop characteristic equation of the system is:
$$s^2 + 10s + K = s^2 + 2\zeta\omega_n s + \omega_n^2$$
- Equating coefficients gives $\zeta\omega_n = 5$ rad/s and the settling time can be calculated by using
$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{5} = 0.8s$$
  - We can see that T_s is less than the required 1 s.
- Also, from $\zeta\omega_n = 5$ and $\zeta = 0.7$, we can get $\omega_n = \frac{5}{0.7} = 7.142$ rad/s
- Thus, the corresponding analog gain is $K = \omega_n^2 = 51.02$
- We therefore have the analog controller:
$$C_a(s) = 51.02\frac{s+1}{s}$$

Example Solution

$$C_a(s) = 51.02\frac{s+1}{s}$$

- Let's verify this controller using MATLAB to plot the system response.



Control Requirements:

- zero steady-state error due to a unit step -> **final value = 1** ☑
- a damping ratio of 0.7; ☑
- a settling time of about 1 s - > **0.83 s** ☑

Example Solution

- **Procedure 2: Map the analog controller to a digital controller**
  - We first select a suitable sampling period T.
    - According to the **sampling theorem,** the **sampling frequency** should be $\omega_s > k\omega_n$, $35 \leq k \leq 70$.
    - In this example, we can select $k = 40$. In addition, $\omega_n = 7.12$ rad/s based on previous calculation.
      - Thus, $\omega_s = 40 * 7.12$. The **sampling period** $T = \frac{2\pi}{\omega_s} = 0.02$ s
  - Next, we map the analog PI controller TF into the digital PI controller TF using bilinear transform:

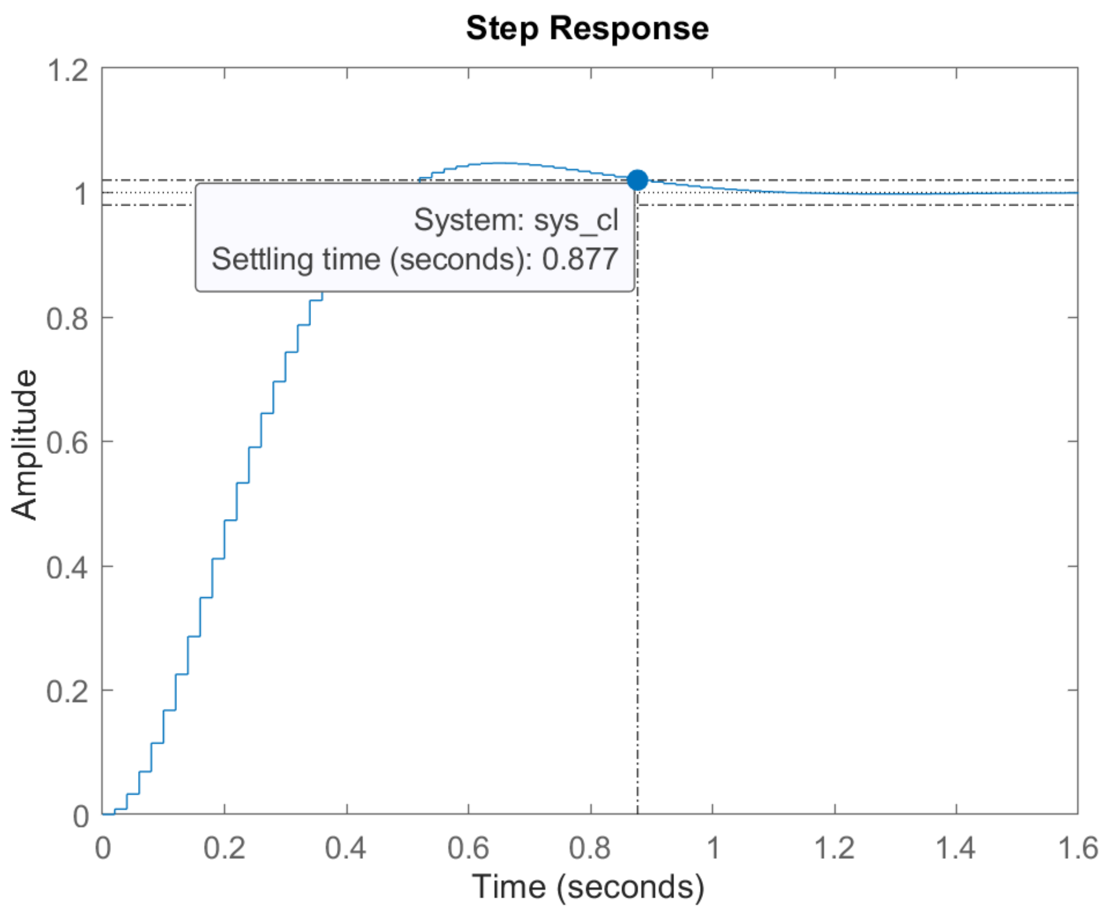
$$s = c\frac{z-1}{z+1}$$

$c = \frac{2}{T}$, *T is the sampling time*

$$C_a(s) = 51.02\frac{s+1}{s} \Rightarrow C(z) = 51.53\frac{z-0.98}{z-1}$$

Example Solution

- MATLAB Verification
  - Generate the step response of the discrete-time closed loop system in MATLAB, we have:



Practice Question

- Design a digital **PD controller** for an analog system

$$G(s) = \frac{1}{s(s+1)(s+10)}$$

To obtain the settling time of about 1 second and a damping ratio of 0.7.

- **Solution:**
  - **Procedure 1**: Since the question provides the controller's type, i.e., PD controller, we can skip procedure 1 here.
  - **Procedure 2**: Convert $C_a(s)$ into $C(z)$
    - Analog PD controller TF: $C_a(s) = K(s+1)$
    - Loog Gain: $C_a(s)G(s) = \frac{K}{s(s+10)}$
    - Characteristic Equation: $1 + C_a(s)G(s) = s(s+10) + K$

$$s^2 + 10s + K = s^2 + 2\zeta\omega_n s + \omega_n^2$$

  - Equating coefficients gives $\zeta\omega_n = 5$ rad/s and the settling time can be calculated by using

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{5} = 0.8s$$

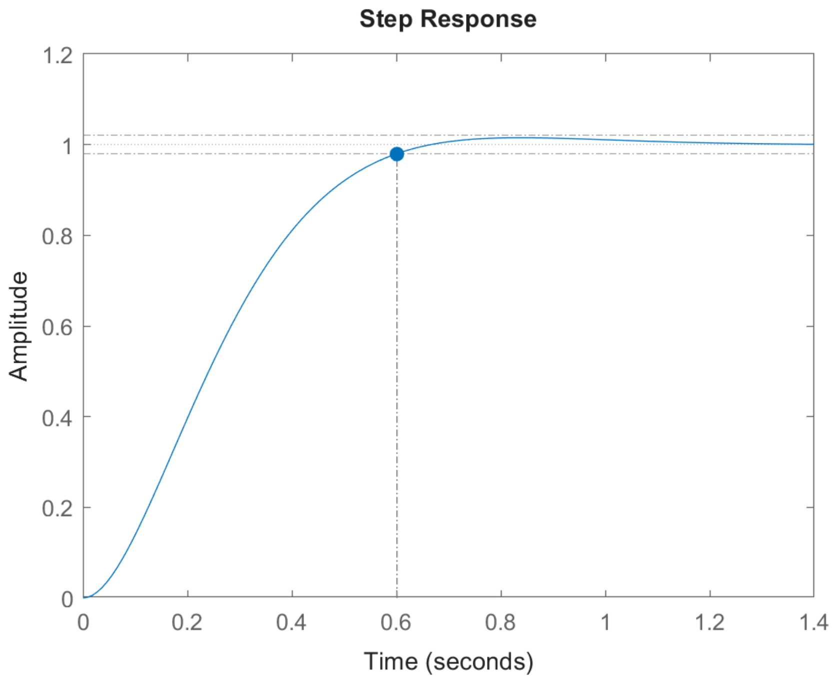
Control Requirements:

- a settling time of about 1 s
- a damping ratio of 0.8

- **Solution:**
  - Also, from $\zeta\omega_n = 5$ and $\zeta = 0.8$, we can get $\omega_n = \frac{5}{0.8} = 6.25$ rad/s.
  - Thus, the corresponding analog gain is $K = \omega_n^2 = 39.06$
  - We have the analog controller:

$$C_a(s) = K(s+1) = 39.06(s+1)$$

  - Let's see its response in MATLAB



Settling Time: 0.6 s

- **Solution:**
  - Then, map this analog controller into the digital controller.
  - First, determine the sampling time: $\omega_n = 6.25$ rad/s, $k = 40$

$$\omega_s = k\omega_n = 250 \text{ rad/s}$$

$$T = \frac{2\pi}{\omega_s} = 0.03 \text{ s}$$

  - Use Bilinear Mapping to get the digital controller

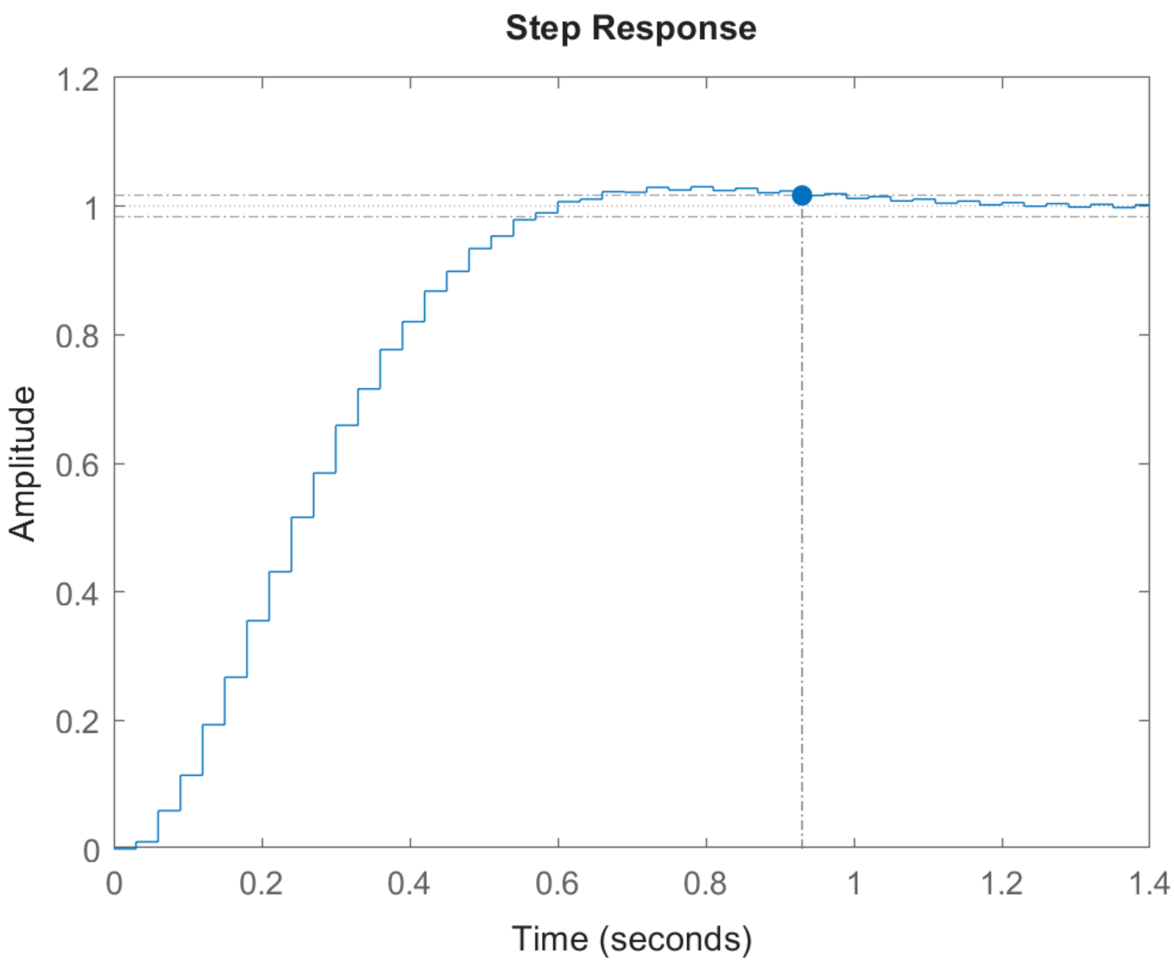
$$s = c\frac{z-1}{z+1}$$

$c = \frac{2}{T}$, *T is the sampling time*

$$C_a(s) = 39.06(s+1)$$

$$\Rightarrow \quad C_z = \frac{2643z - 2565}{z+1}$$

- **Solution:**
  - MATLAB verify:



Settling Time: 0.93 s