Exercise 36, Linear Algebra: A Modern Introduction, 4th Edition

Solved by experts

NEXT QUESTION

Linear Algebra: A Modern Introduction, 4th Edition, Chapter 6: Vector Spaces

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Exercise 36 Answer

Step by step explanation

HIDE ALL

Tip

In this question, we will find eigenvalues and eigenvectors.

Explanation

- We will take β as the standard basis.
- We will find eigenvalues and corresponding eigenvectors of $[T]_{\beta}$
- We get that C is linearly independent. Therefore $[T]_{c}$ exists.

Step 1 of 3

Let, $\beta = \left\{1, x, x^2\right\}$ be the standard basis of P_2

Then,

$T(1) = 1$

$T(x) = 3x + 2$

$T(x^2) = (3x+2)^2$

Thus,

$$[T]_\beta = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix}$$

Eigenvalues of $[T]_\beta$:

$$\det([T]_\beta - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 4 \\ 0 & 3-\lambda & 12 \\ 0 & 0 & 9-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(3-\lambda)(9-\lambda)$$

Eigenvalues are 1,3,9.

Step 2 of 3

Eigenvector corresponding to 1 :

Let,

$v_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vector such that ,

$$[T]_\beta v_1 = 1.v_1$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} a+2b+c \\ 3b+12c \\ 9c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$a \in \mathbb{R}$, b = c = 0

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue 1.

Eigenvector corresponding to 3 :

Let,

$v_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the vector such that,

$$[T]_\beta v_2 = 3.v_2$$

$$\begin{bmatrix}1 & 2 & 4\\ 0 & 3 & 12\\ 0 & 0 & 9\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}3a\\ 3b\\ 3c\end{bmatrix}$$

$$\begin{bmatrix}a+2b+c\\ 3b+12c\\ 9c\end{bmatrix} = \begin{bmatrix}3a\\ 3b\\ 3c\end{bmatrix}$$

a = b and c = 0

$v_2 = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$ is an eigenvector corresponding to eigenvalue 3.

Eigenvector corresponding to 9 :

Let,

$v_3 = \begin{bmatrix}a\\ b\\ c\end{bmatrix}$ be vector such that,

$$[T]_\beta v_3 = 9.v_3$$

$$\begin{bmatrix}1 & 2 & 4\\ 0 & 3 & 12\\ 0 & 0 & 9\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}9a\\ 9b\\ 9c\end{bmatrix}$$

$$\begin{bmatrix}a+2b+c\\ 3b+12c\\ 9c\end{bmatrix} = \begin{bmatrix}9a\\ 9b\\ 9c\end{bmatrix}$$

a = c and b = 2c , c $\in \mathbb{R}$

$v_3 = \begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$ is an eigenvector corresponding to eigenvalue 9.

Step 3 of 3

$v_1 = [1]_\beta$

$v_2 = [1+x]_\beta$

$v_3 = [1+2x+x^2]_\beta$

Then, C = $\{1, 1+x, 1+2x+x^2\}$

C is linearly independent thus, C is basis of P_2

Therefore, T is diagonalizable,

$$[T]_c = \begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 9\end{bmatrix}$$

Final Answer