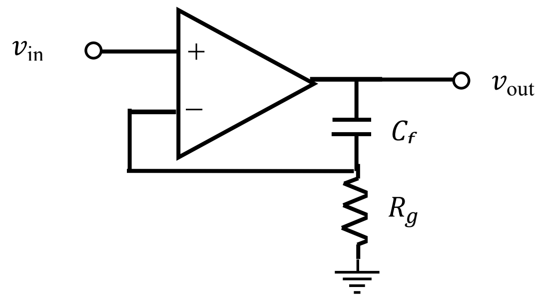
BIEN325 Name: ____________________

Homework 5: Active Filters

1. Find the transfer function for the following circuit.



Answer: This is in the non-inverting amplifier configuration, for which the gain is $G = 1 + \frac{Z_f}{Z_g}$. Since $Z_f = \frac{1}{j\omega C_f}$ and $Z_g = R_g$, the gain is $G = 1 + \frac{1}{j\omega R_g C_f}$.

2. A real diode approximately has a characteristic equation $I = I_s\left(e^{\frac{V}{V_0}} - 1\right)$, where V is the voltage across the diode and I_s and V_0 are constants inherent to the diode specifications. From this equation, find the relationship between output voltage and input voltage for the following configuration. (Hint: First find the current through R_i, which is the same as I, then solve for V, which will be $-v_{\text{out}}$.

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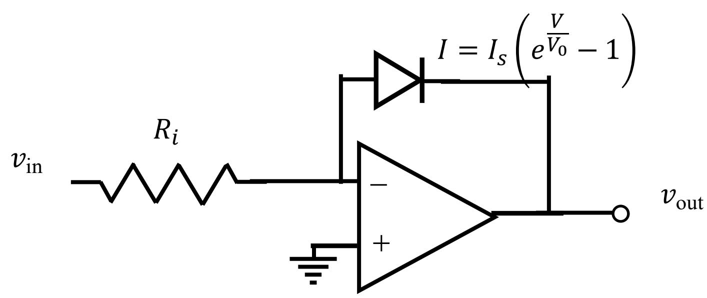


Answer:

The input current is $\frac{v_{\text{in}}}{R_i}$. This current must pass through the diode, and the voltage (V) across the diode is $-v_{\text{out}}$

$$\frac{v_{\text{in}}}{R_i} = I_s\left(e^{\frac{0 - v_{\text{out}}}{V_0}} - 1\right)$$

Solve for v_{out} to get

$$-V_0 \ln\left(\frac{v_{\text{in}}}{R_i I_s} + 1\right) = v_{\text{out}}$$

With some modification, the circuit can therefore take the logarithm of the input voltage.

3. A charge amplifier is designed for a piezoelectric sensor. Calculate the dc offset voltage if a 100 MΩ bypass resistor is used in the feedback loop assuming:
a. The operational amplifier is an LM741, with input bias current of 500 nA.
b. The operational amplifier is an OP07 with input bias current of 4 nA.
c. The operational amplifier is an LF356 jfet with input bias current of 30 pA.

Comment on whether or not any of these offsets could pose a problem with the overall circuit function.

Answer: The offset voltage can only come through the feedback resistor, so it is simply the input bias current multiplied by the feedback resistance.

Case a: $v = (500 \times 10^{-9})(10^8) = 50$ Volts

Case b: $v = (4 \times 10^{-9})(10^8) = 0.4$ Volts

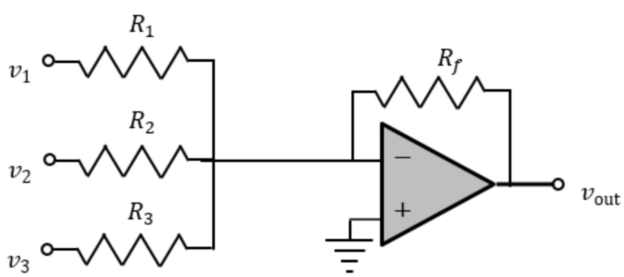
Case c: $v = (30 \times 10^{-12})(10^8) = 3$ mV

Case a will certainly cause problems in that the bias current will throw the circuit into saturation before it has a chance to amplify anything.

Case b will have an offset that will slightly alter the dynamic output range of the circuit, but should not be problematic in most cases. Note that the piezoelectric crystal cannot hold the output to a constant stress, so the value of the offset is irrelevant.

Case c provides a small offset that will generally be negligible.

4. The circuit below does a weighted sum on the three inputs, v_1, v_2, and v_3.



a. Show that the output of the circuit is

$$v_{\text{out}} = -\frac{R_f}{R_1}v_1 - \frac{R_f}{R_2}v_2 - \frac{R_f}{R_3}v_3$$

Recall that the inverting input is a virtual ground, so that the currents through resistors R_1, R_2, and R_3 are v_1/R_1, v_2/R_2, and v_3/R_3, respectively. By Kirchov's current law, the sum of these three currents is the current through the feedback resistor. Since the voltage drop across the feedback resistor is iR_f, $v_{\text{out}} = -iR_f = -(i_1 + i_2 + i_3)R_f = \left(\frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3}\right)R_f$, as was to be shown.

b. Show that if R_f is replaced by a capacitor (C_f), the output voltage is

$$v_{\text{out}} = -\frac{1}{C_f}\int_0^t \left(\frac{v_1}{R_1}+\frac{v_2}{R_2}+\frac{v_3}{R_3}\right)\,\mathrm{d}t\,.$$

This element is a summing integrator, which is useful in feedback control and the design of complex transfer functions.

Again, the current through the feedback element (capacitor) is $\left(\frac{v_1}{R_1}+\frac{v_2}{R_2}+\frac{v_3}{R_3}\right)$. From $i = C\ dv/dt$, $-v_{\text{out}} = \frac{1}{C_f}\int_0^t \left(\frac{v_1}{R_1}+\frac{v_2}{R_2}+\frac{v_3}{R_3}\right)\mathrm{d}t$, or

$$v_{\text{out}} = -\frac{1}{C_f}\int_0^t \left(\frac{v_1}{R_1}+\frac{v_2}{R_2}+\frac{v_3}{R_3}\right)\,\mathrm{d}t$$

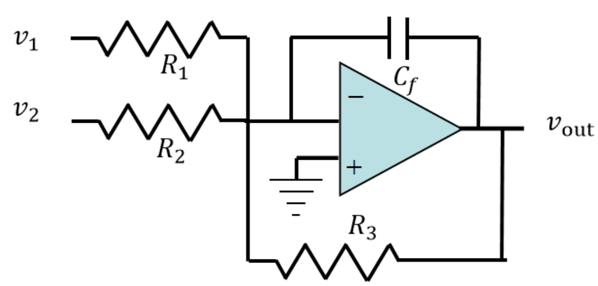
c. It is also possible to feed the output of the circuit back as one of the voltages in the sum. Sketch a system containing a single operational amplifier whose output voltage is given by

$$v_{\text{out}} = -\frac{1}{C_f}\int_0^t \left(\frac{v_1}{R_1}+\frac{v_2}{R_2}+\frac{v_{\text{out}}}{R_3}\right)\,\mathrm{d}t$$

Or, equivalently, that solves the differential equation

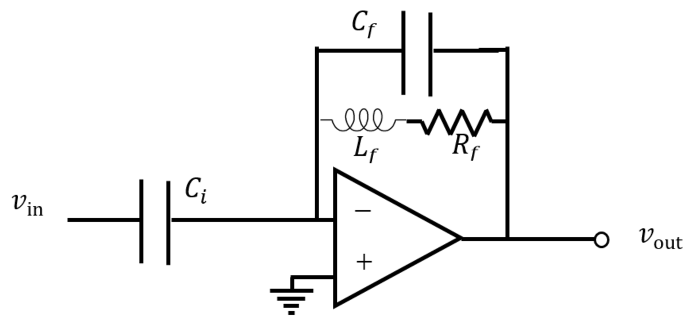
$$\frac{dv_{\text{out}}}{dt}+\frac{v_{\text{out}}}{C_f R_3} = -\frac{v_1}{C_f R_1}-\frac{v_2}{C_f R_1}$$

Answer:



Note: This may seem like a random question, but we will revisit it when we do feedback control in BIEN 403.

5. (Calculation of magnitude and phase) For the following circuit



a. Find the transfer function.

Answer: This is in the standard inverting amplifier configuration, so $\frac{v_{\text{out}}}{v_{\text{in}}} = T(j\omega) = -\frac{Z_f}{Z_i}$

$$Z_i = \frac{1}{j\omega C_i}; Z_f = j\omega L + R_f \text{ in parallel with } \frac{1}{j\omega C_f}$$

$$Z_f = \frac{(j\omega L_f + R_f)/j\omega C_f}{j\omega L_f + R_f + 1/j\omega C_f} = \frac{(j\omega L_f + R_f)}{(j\omega)^2 L_f C_f + j\omega R_f C_f + 1} = \frac{j\omega L_f + R_f}{-\omega^2 L_f C_f + j\omega R_f C_f + 1}$$

$$T(j\omega) = -\frac{(j\omega L_f + R_f) j\omega C_i}{-\omega^2 L_f C_f + j\omega R_f C_f + 1}$$

b. Does the circuit have pathways for the bias currents?

Answer: Yes. The positive input goes directly to ground and the negative input has a pathway from v_{out} through the resistor and inductor.

c. What is the DC gain of the circuit?

Answer: Set $\omega = 0$ to get a dc gain of 0.

d. What is an expression for the magnitude of the transfer function?

Answer:

$$|T(j\omega)| = \frac{\sqrt{(\omega L_f)^2 + R_f^2}\ (\omega C_i)}{\sqrt{(1 - \omega^2 L_f C_f)^2 + \omega^2 R_f^2 C_f^2}}$$

e. What is an expression for the phase of the transfer function.

Answer:

$$\angle T(j\omega) = \arctan\left(\frac{\omega L_f}{R_f}\right) + \arctan(\omega C_1/0) - \arctan\left(\frac{\omega R_f C_f}{1 - \omega^2 L_f C_f}\right)$$

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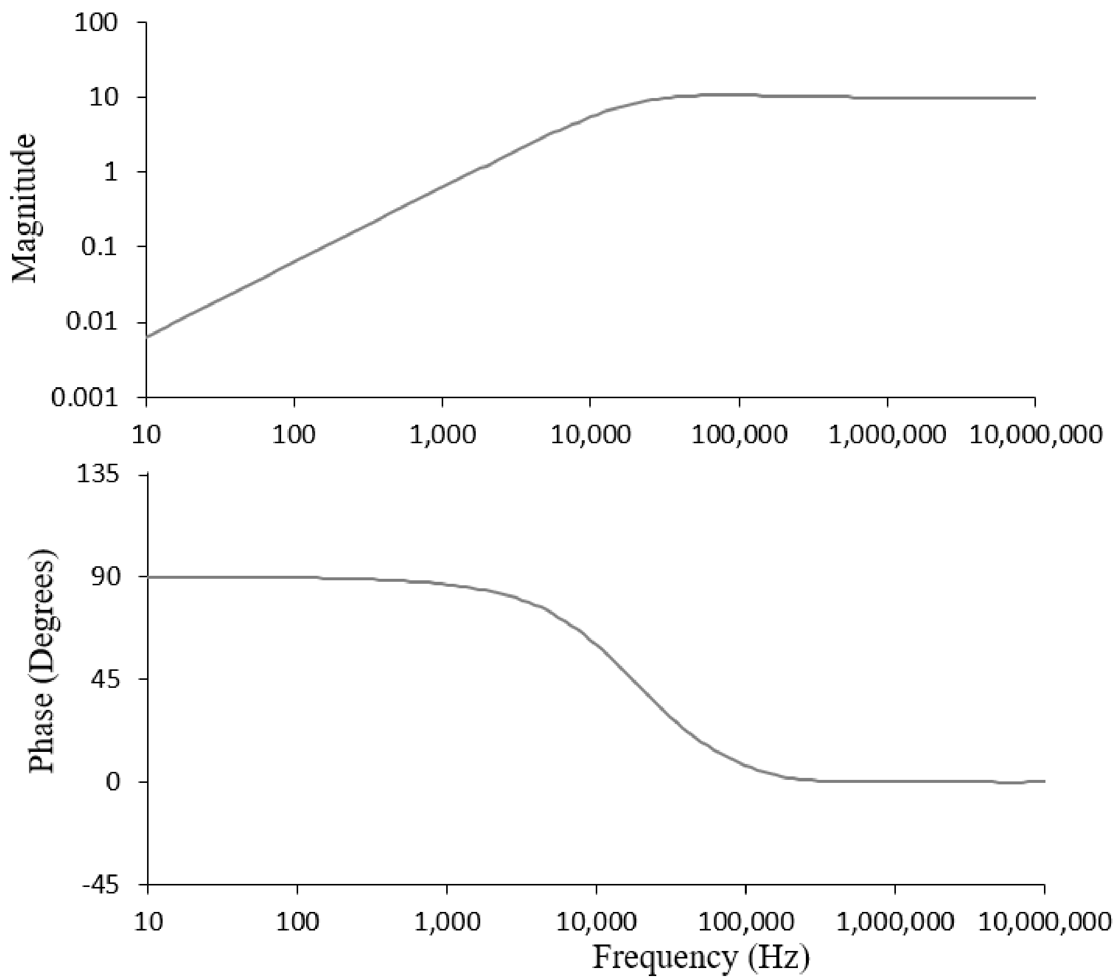
f. Use Excel to plot the magnitude and phase of this transfer function as a function of frequency (i.e. a Bode plot). Use the following parameter values.

$$C_i = 10 \text{ nF}$$
$$C_f = 100 \text{ nF}$$
$$R_f = 1 \text{ kHz}$$
$$L_f = 1 \text{ mH}$$

Answer:



Graduate Content

A piezoelectric sensor plus cable has a 1 nF capacitance and 10 GΩ resistance. The charge generated by the crystal is $q_p = Kx$, where K is the sensitivity and x is the change in height of the crystal. The crystal is therefore modeled as a current source, where $q_p = Kx \Rightarrow i_p = Kdx/dt$. Assume that $K = 5 \times 10^{-9}$ C/µm. A voltage amplifier for this sensor, with a gain of 10, is shown in Figure B.

Sensitivity can also be given in units of voltage per displacement. If the sensitivity is given as 5 V/µm, it means that the output voltage will be $V_{\text{out}} = (5\ \text{V}/\mu\text{m}) \times x$, which means that the charge generated is $q = C_p V_{out} = C_p(5\ \text{V}/\mu\text{m}) \times x$ = $(10^{-9}$ F$)(5\ V/\mu\text{m})x$, where x is in µm.

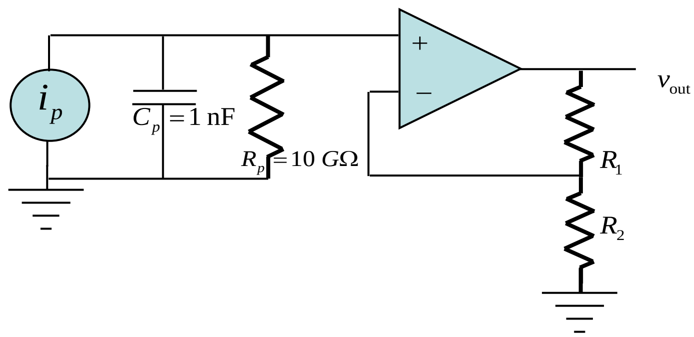


Figure B: Voltage amplifier for a piezoelectric sensor.

1. Assume, initially, that both R_p and the input impedance of the operational amplifier are infinite. What voltage will appear across the capacitor if the piezoelectric crystal is instantaneously compressed by 200 µm, and what value of v_{out} will this charge cause?

Answer: Because the compression is instantaneous, a charge of $Kx = 5 \times 10^{-9}(200) = 1\ \mu C$ will be dumped onto the capacitor. (If the compression is slower, some of the charge drains through R_p while it is being generated, in which case the peak voltage across the capacitor is smaller). For a capacitor, $q = C_p V \Rightarrow\ V = q/C_p = 10^{-6}\ C/10^{-9}\ F = 1000\ V$. Therefore, a gain of 10 implies $v_{\text{out}} = 10(V) = 10{,}000$ Volts. But this value is too high for the voltage supply, so the output will be the supply voltage.

2. We will show later that the input resistance of the non-inverting amplifier is AR_{in}/G, where A is the operational amplifier gain, R_{in} is the operational amplifier input resistance, and G is the amplifier gain. This value can be much larger than R_p. For example, if a FET-input operational amplifier is used, R_{in} will be about 10^9 Ω. With a modest A value of 10^6, and a gain of 10, the input resistance becomes at least 10^{14} Ω. If the piezoelectric crystal is displaced by an amount that causes a charge on the capacitor of 1 μC (which subsequently leaks out through R_p), plot the capacitor voltage as a function of time.

Answer:

$$i_c = C_p \frac{dV_c}{dt}$$
$$i_{R_p} = V_c / R_p$$
$$i_c = -i_{R_p}$$

Therefore,

$$C_p \frac{dV_c}{dt} = -\frac{V_c}{R_p} \Rightarrow C_p \frac{dV_c}{dt} + \frac{V_c}{R_p} = 0 \Rightarrow \frac{dV_c}{dt} + \frac{1}{R_p C_p} V_c = 0$$

The solution to this equation is:

$$V_c = Ae^{-\frac{t}{R_p C_p}}$$

The initial condition, $V_c = 0$ for $t = 0$ requires that $V_c = V_0\, e^{-\frac{t}{R_p C_p}}$.

Here, $R_p C_p = (10^{10}\ \Omega)(10^{-9}\ \mathrm{F}) = 10$ seconds. The plot is shown in Figure B2.

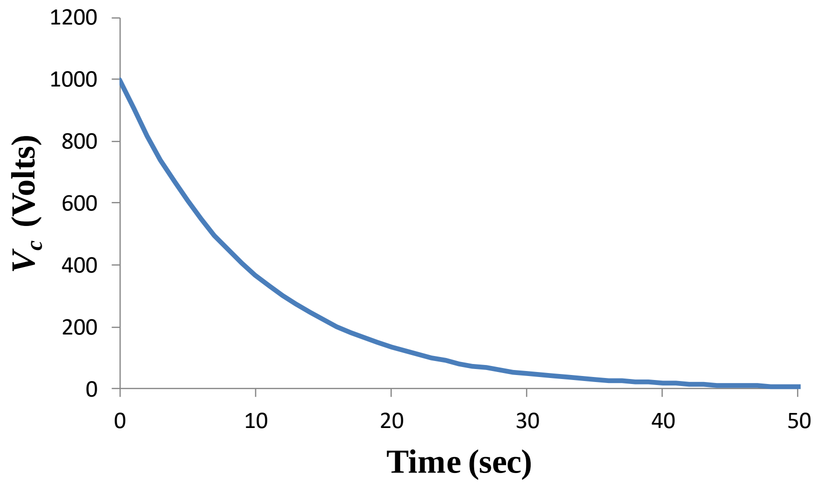


Figure B2: Voltage across the sensor capacitor as a function of time.

3. For the same system described in Part 2 (amplifier resistance → ∞), plot the frequency response of V_c as a function of x.

Answer:

$$i_c = C_p \frac{dV_c}{dt}$$

$$i_{R_p} = \frac{V_c}{R_p}$$

But

$$i_p = i_c + i_{R_p}$$

So

$$i_p = Cp\,\frac{dV_c}{dt} + \frac{V_c}{R_p}$$

And because

$$i_p = K dx/dt$$

$$K\frac{dx}{dt} = C_p\frac{dV_c}{dt} + \frac{V_c}{R_p}$$

Rearrange this equation with the output (V_c) on the left-hand side, and the input (x) on the right-hand side.

$$\frac{dV_c}{dt} + \frac{1}{R_pC_p}V_c = \frac{K}{C_p}\frac{dx}{dt}$$

Take the Laplace transform.

$$s\bar{V}_c + \frac{1}{R_pC_p}\bar{V}_c = \frac{K}{C_p}sX$$

So that the transfer function becomes

$$\frac{\bar{V}_c}{X} = \frac{\frac{Ks}{C_p}}{s + \frac{1}{R_pC_p}} = \frac{KR_ps}{R_pC_ps + 1}$$

Or, in terms of ω,

$$\frac{\bar{V}_c}{X} = \frac{KR_pj\omega}{R_pC_pj\omega + 1}$$

For $\omega \ll 1/R_pC_p$, the sensor acts as a high pass filter. For $\omega \gg 1/R_pC_p$, the sensor has a constant gain of $K/C_p = (5 \times 10^{-9}\ C/\mu m)/(10^{-9}\ F) = 5\ (V/\mu\text{m})$.

The plot is shown in Figure B3.

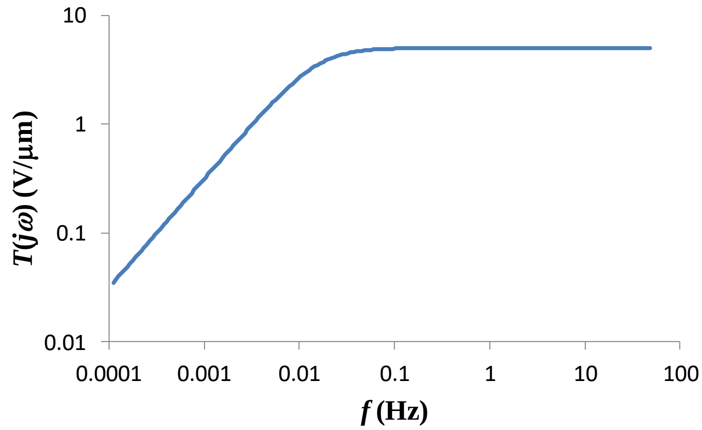


Figure B3: Transfer function between displacement (µm) and capacitor voltage (Volts) for the piezoelectric crystal.

4. A noise source causes a displacement amplitude of 1 µm at 1 kHz. You are not concerned about this signal itself, but you need to ensure that it does not cause the amplifier to go into saturation (i.e. the signal must not cause the output voltage of the amplifier to exceed ± 12 Volts). You therefore modify the system with a low-pass filter, as shown in Figure B4.

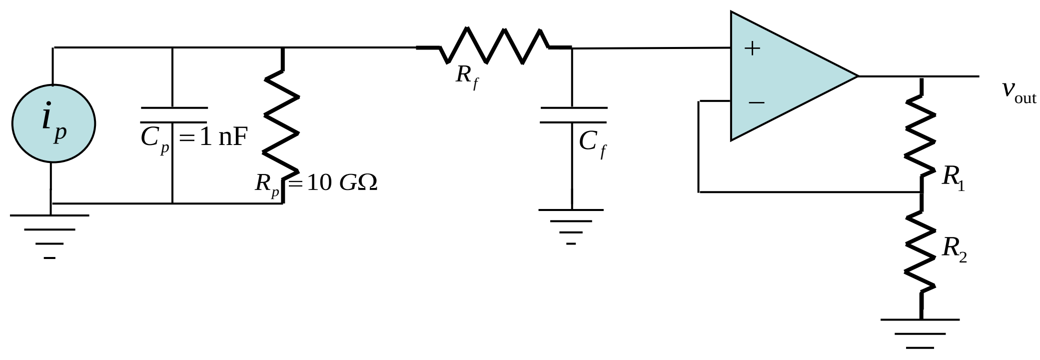


Figure B4: Voltage amplifier modified with a low pass filter to prevent saturation by an 1 kHz signal.

What value must the time constant, $R_f C_f$ have if the 1 kHz noise is to generate an output signal no more than 1/3rd the saturation voltage?

Answer: From the frequency response shown in Figure B3, the 1 µm signal would cause an output of $(1\ \mu\text{m})(5\ \text{V}/\mu\text{m})(10) = 50$ V. The filter must reduce this voltage to 4 V (i.e. it must attenuate the signal at 1 kHz by a factor of 50/4, or 12.5). The low pass filter characteristic is:

$$|H(j\omega)| = \frac{1}{\sqrt{(2\pi RCf)^2 + 1}} \rightarrow \frac{1}{2\pi RCf}$$

Therefore, we need $2\pi R_f C_f f \approx 12.5$, or $R_f C_f \approx 12.5/(2\pi 1000) \approx 0.002$. For example, if R_f is selected to be 10 kΩ, then C_f will need to be $2 \times 10^{-3}/10^4 = 2 \times 10^{-7}$ F (i.e. 200 nF). The cutoff frequency for the filter is $1/\left(2\pi R_f C_f\right) = 1/(2\pi 0.002) =$ 79.6 Hz.

5. How low must the bias current be to prevent the output from going into saturation?

Answer: The bias current to the positive input will need to come through R_p, so it will cause a voltage at the positive input of $i_b R_p$. The bias current for the negative input will come (mostly) through R_2, but since $R_2 \ll R_p$, $i_b R_2 \ll i_b R_p$, so the voltage at the negative input caused by bias current into the negative terminal is negligible. If the power supplies are set up so that the output saturates at 10 Volts, it is necessary that $i_b R_p(G) < 10$ V, so $i_b < 10/(R_p G)$. Thus, $i_b < 10/\left((10^{10})(10)\right) = 10^{-10}$ Amps. This value, 100 pA, can be easily accommodated with an OPA129 Ultra-Low Bias Current DiFET Operational Amplifier, which has a typical bias current of 0.03 pA and a maximum bias current of 0.1 pA. Of course, the higher the bias current, the less voltage range is available for the signal, so for some applications, this large bias current may not be acceptable.

6. An alternative strategy to prevent the bias current from causing saturation is to compensate for the increased voltage at the positive terminal with a resistor leading to the negative input, as shown in Figure B6. The ideal value for R_b is R_p, the sensor leakage resistance, but some error is likely to occur in balancing the two resistances. Show that if R_b is different from R_p by 10%, the output voltage caused by the bias current is a factor of 10 less than the value it would have if R_b were absent.

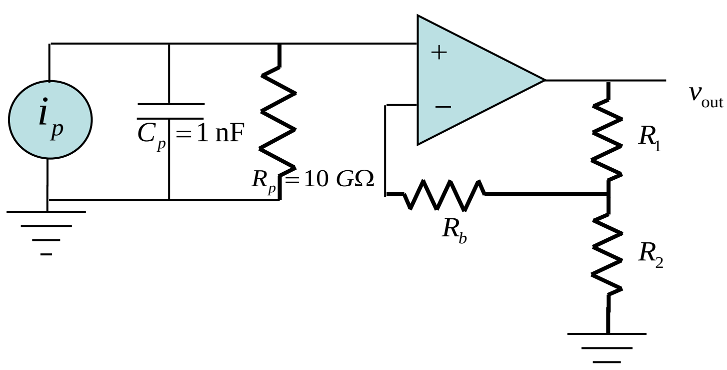


Figure B6: Amplifier modified to compensate for bias current.

Answer: The effect of R_2 can be ignored because $R_2 \ll R_b$. Therefore, the voltage caused by the bias currents is $i_b R_p$ on the positive terminal and $i_b R_b = 0.9 i_b R_p$ on the negative terminal. The difference is $0.1\, i_b R_p$. With $R_b = 0$, the difference would be $i_b R_p$, so the input voltage difference is reduced by a factor of 10, hence the output voltage is reduced by a factor of 10.

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Homework 5: Active Filters

Power Spectra

The MATLAB m file pspect.m calculates the power spectrum for the input signal. The spectrum is properly scaled so that the frequencys are correct and the power in the spectrum integrates to the power in the signal according to:

$$\int_{-f_{\max}}^{f_{\max}} P(f)\,df = \frac{1}{T}\int_{-\infty}^{\infty} s^2(t)\,dt$$

Use this routine to calculate the power spectrum in the following exercises. Please upload all answers and plots in a single Word file. (A handy way to copy the plots into a Word file is to use the snip function of Windows).

7. Generate a series of 25,600 random numbers with the MATLAB rand function. I.e., calculate:

>> x = rand(1,25600) – 0.5;

>> dt = 0.001;

>> npts = 256;

And calculate the power spectrum with:

```
>> [p, farray] = pspect(x,dt,npts)
```

Explain the difference between the averaged spectrum and the individual spectrum

Answer:

Figure 1 is a short segment of the generated random sequence:

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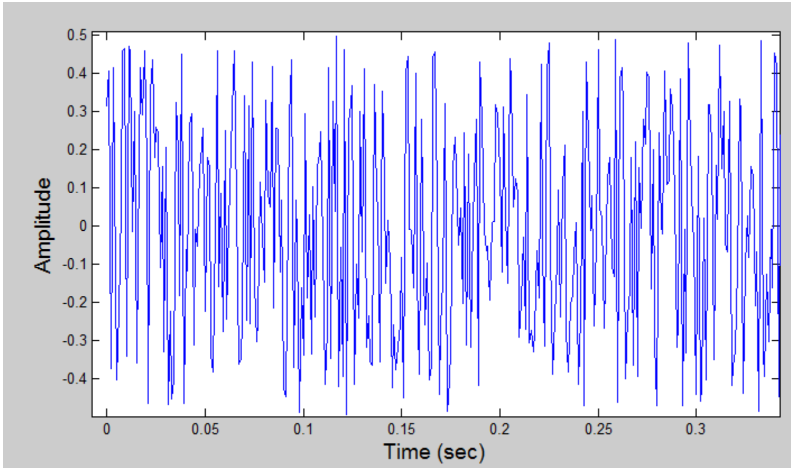

Figure 1: Generated random data.

Figure 2 shows the two spectra. Notice that the frequency range is from −500 Hz to +500 Hz because the maximum frequency is half the sample frequency.

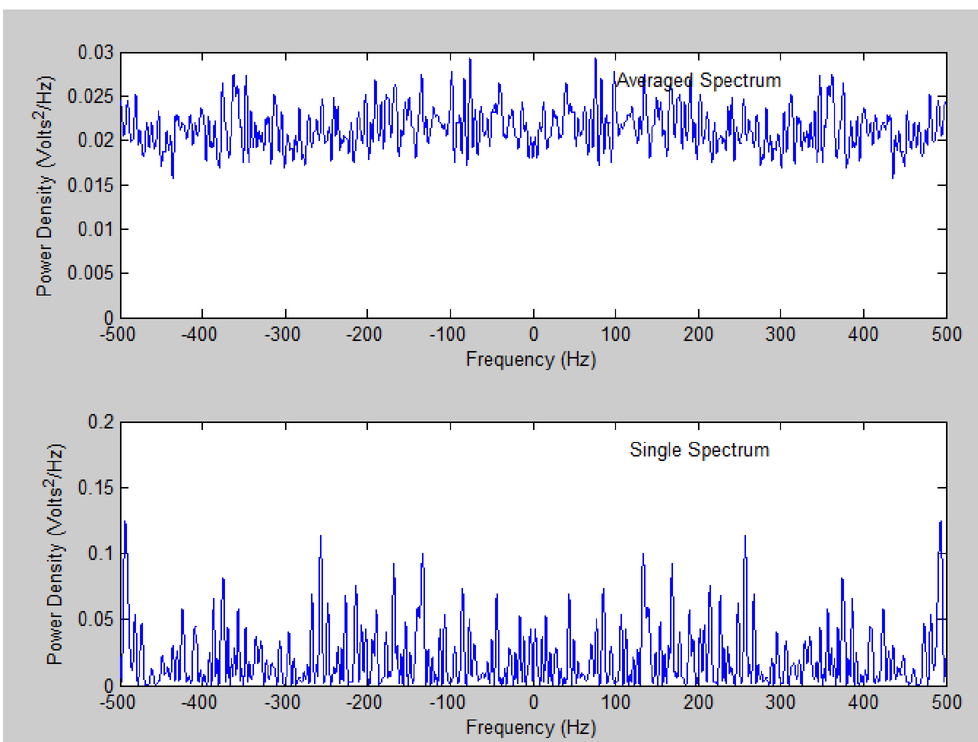

Figure 2: The averaged spectrum (upper) and the individual spectrum (lower).

The averaged spectrum is more converged and looks more like a flat spectrum (corresponding to white noise). The individual spectrum has more randomness to it. Notice that both spectra have a mean of approximately 0.2 Volts^2/Hz, and that the scale of the Power Density axis for the single spectrum is much larger because it has to accommodate the larger amount of randomness.

8. Apply the moving average filter to the signal and then calculate the power spectrum again. The moving average filter is defined as:

$$s(k) = \sum_{l=1}^{n} C(l)x(k+l),$$

which is a discrete version of the convolution integral. To apply this filter, first create the coefficients. For example, if the filter window is rectangular and of length $10dt$, the coefficients are:

```
>> C = [1 1 1 1 1 1 1 1 1 1];
```

The filter can then be applied with:

```
>> s = filter(C,1,x);
```

Explain the difference between the filtered signal and the unfiltered signal. Also explain the difference between the ensemble spectrum of the filtered signal and the ensembled spectrum of the unfiltered signal. Finally, explain the difference between the ensemble spectrum of the the filtered signal and the individual spectrum of the filtered signal.

Answer:

Figure 3 shows the filtered signal, with the same time axis as Figure 1. The filtered signal changes more slowly.

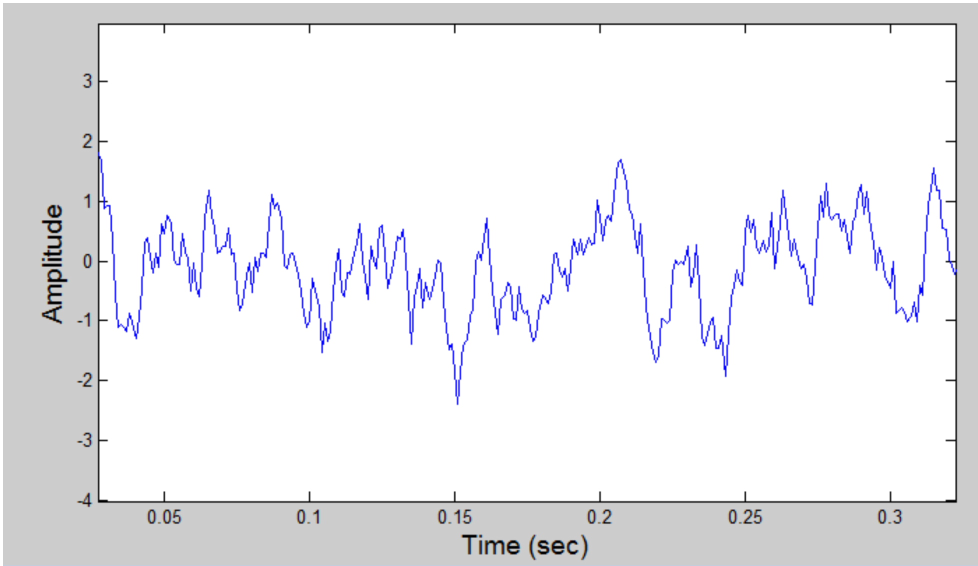

Figure 3: Random noise filtered with a rectangular window.

Figure 4 shows the averaged (upper curve) and single spectrum (lower curve) for the filtered signal. The averaged spectrum has the shape of a sinc^2 function. The individual spectrum has a similar shape, but with random differences. The explanation for the shape of the spectrum is as follows:

1. The filtered signal is the convolution of the original signal with a rectangle.
2. The Fourier transform of the convolution of two signals is the product of the two Fourier transforms, so the power spectrum of the convolution is the product of the two power spectra.
3. The power spectrum of the original signal is constant.
4. The power spectrum of the rectangle is a sinc function.
5. A constant multiplied by a sinc function is still a sinc function.

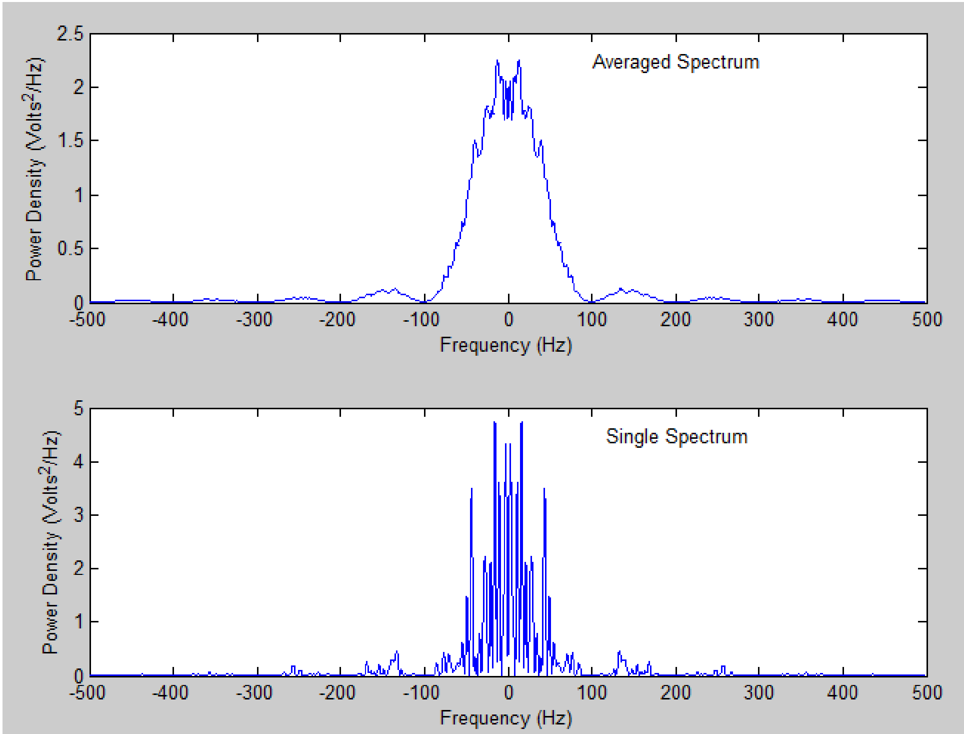

Figure 4: Power spectra of the filtered signal.

9. Find the mathematical description of the magnitude-squared of the Fourier transform of the rectangle (it will have the form of the sinc function). Plot this function on top of the power spectrum of the ensemble spectrum calculated in Exercise 2. Explain the similarities and differences.

The explanation for the shape of the spectrum is as follows:

1. The filtered signal is the convolution of the original signal with a rectangle.
2. The Fourier transform of the convolution of two signals is the product of the two Fourier transforms, so the power spectrum of the convolution is the product of the two power spectra.
3. The power spectrum of the original signal is constant.
4. The power spectrum of the rectangle is a sinc function.
5. A constant multiplied by a sinc function is still a sinc function.

We have used a rectangular window of length 0.01 seconds. Therefore, the power spectrum of this window is:

$$\mathcal{F}\big(w(t)\big) = \frac{\sin^2(\pi f T/2)}{(\pi f T/2)^2}.$$

.

10. Repeat Exercises 3 and 4 with a Hanning window instead of a rectangular window. You can generate a Hanning window of length 10 in MATLAB with:

```
>> win = hanning(10);
```

The equation for the Hanning window of length 10 is:

$$w = \frac{1}{2}\left(1 + \cos\left(\frac{\pi n}{11}\right)\right)$$

Where n ranges from -9 to $+9$ in steps of 2. Use this relationship to determine the Fourier transform of the window, and compare the power spectrum of the window to the ensemble power spectrum of the windowed random signal.

Answer:

The time series function is shown in Figure 5. This waveform is much smoother than the one shown in Figure 1 because the window does not have a discontinuity at the edges.

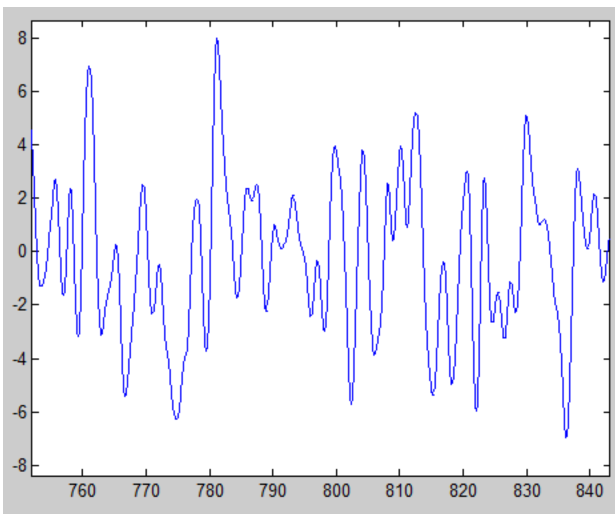

Figure 5: Random signal filtered with a Hanning window shape.

The power spectra are shown in Figure 6. The ensemble spectrum approaches the shape of the transform of $(1 + \cos\omega t)$ squared.

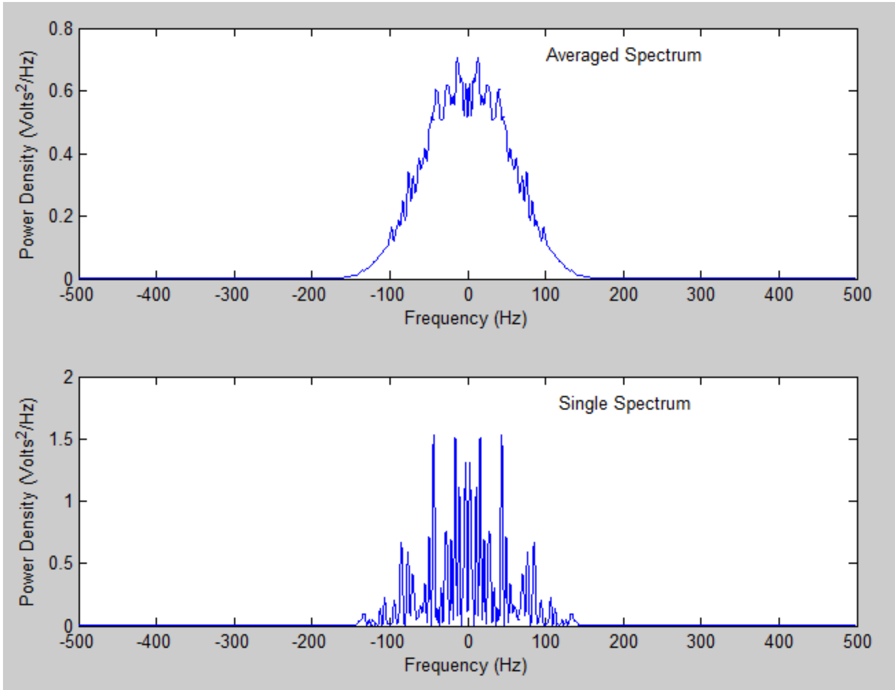

Figure 6: Power spectrum of the random signal filtered with a Hanning window.

11. The derivative of a function is approximated by:

$$\frac{df}{dt} \approx \frac{(f_{i+1} - f_i)}{dt}.$$

Therefore, the following filter approximates the derivative:

```
>> deriv = filter([-1 1],dt,x);
```

Apply this function to the random sequence x. Explain the appearance of the ensembled spectrum. (Hint: Think first about the Fourier transform of the derivative of a function, where:

$$\mathcal{F}\left(\frac{df}{dt}\right) = -j\omega\mathcal{F}(f).$$

In addition, think about the square of the Fourier transform of $-\delta(0) + \delta(dt)$. Is the spectrum of the derivative in any way related to the spectrum of the Hanning window?)

Answer:

The averaged and single spectra are shown in Figure 7. The spectrum has a sinusoidal shape because the pair of delta functions with opposing signs is the transform of a sinusoid. Thus, if we start with delta functions in the time domain, we end up with a sinusoid in the frequency domain.