Lecture 8-2: Nyquist Criterion, Phase Margin, and Gain Margin

ELEN 472: Introduction to Digital Control

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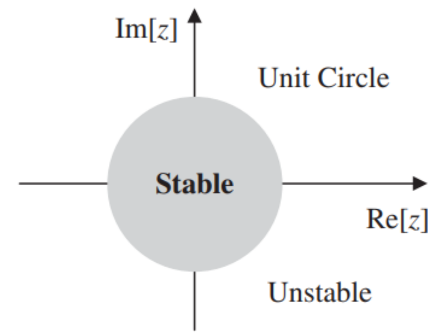
Review

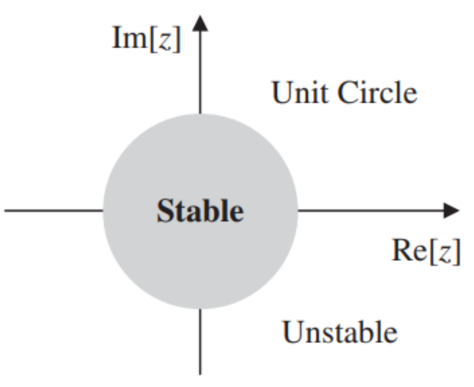


- **Stability of Discrete-time Systems**
  - **Definition:** Bounded input Bounded output
- **Stability Check**
  - **Manual Check:** If all poles lies inside the closed unit circle (|poles|<1)
  - **For High-order Systems:** Jury Test
- **Internal Stability**
  - **Definition:** For a closed-loop system, internal stability requires all signals in the system are bounded.
  - **Two Conditions** for Internal Stability

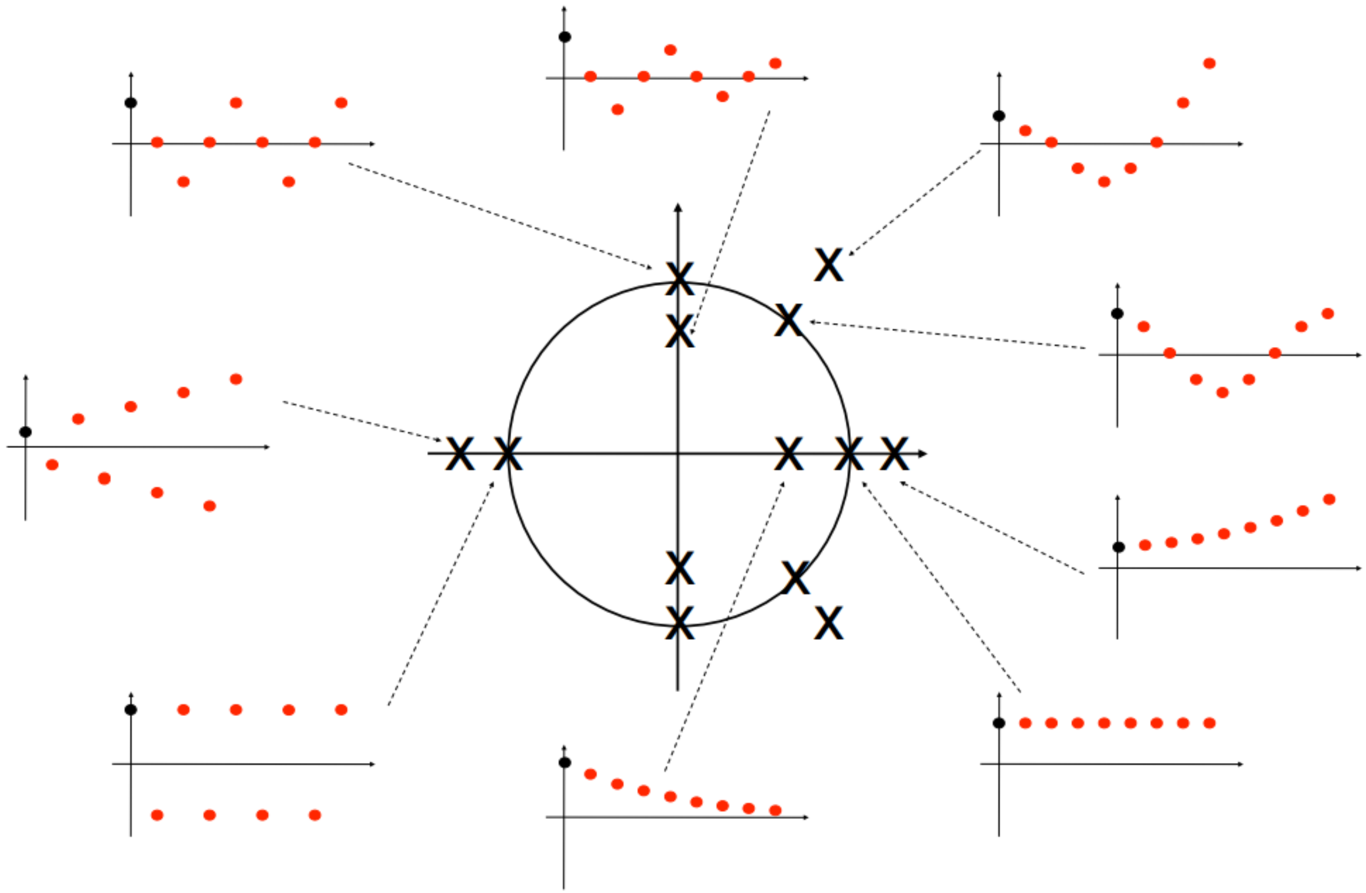
- A closed-loop system is **internal stable** if the following two conditions hold:
  - The characteristic Polynomial $1 + C(z)G_{ZAS}(z)$ has no zeros **on** or **outside** the unit circle.
  - The loop gain $C(z)G_{ZAS}(z)$ has no pole-zero cancellation **on** or **outside** the unit circle.

More Details in Manual Check

- BIBO Stability Definition:
  - For a **stable** discrete-time system, if a **bounded** input signal is provided, the output will also be **bounded**.
  - Bounded Input Bounded Output – BIBO definition.
- How to determine whether a discrete-time system is stable?
  - Depending on system's poles.
    - If all poles are **within** the unit circle -> System is **stable**
    - If one or more (not repeated) poles are **on** the unit circle -> **Marginally Stable**
    - If **repeated** poles **on** the unit circle -> **Unstable**
    - If **one** pole is outside of the unit circle -> **Unstable**



Review of Stability for Discrete-time Systems

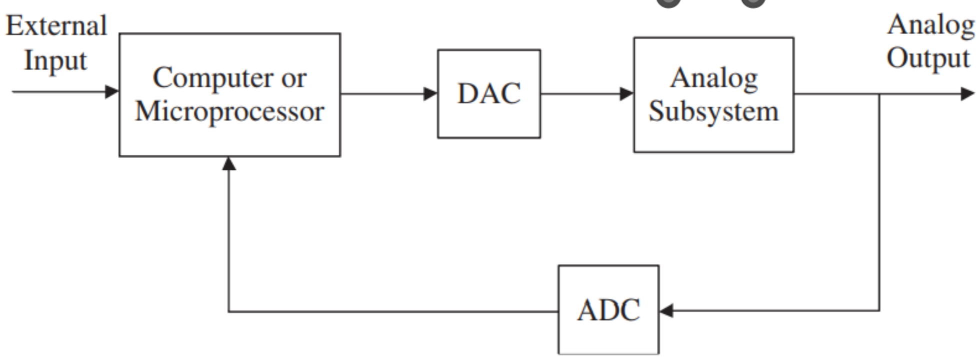


Time responses as a function of the poles location

Example

- Find the **stable** range of the gain K for the unity feedback digital cruise control system with the analog transfer function (the sampling time is 0.02 s):

$$G(s) = \frac{K}{s+3}$$



- **Solution:**
    - First, we obtain the overall z-domain transfer function DAC, Analog Subsystem, and ADC (i.e., $G_{ZAS}(z)$)

$$G_{ZAS}(z) = (1 - z^{-1})\mathscr{Z}\left\{\mathscr{L}^{-1}\left[\frac{G(s)}{s}\right]\right\}$$

$$= (1 - z^{-1})\mathscr{Z}\left\{\mathscr{L}^{-1}\left[\frac{K}{s(s+3)}\right]\right\}$$

Example (Continued)

- **Solution (Continued):**

$$G_{ZAS}(z) = (1 - z^{-1})\mathscr{Z}\left\{\mathscr{L}^{-1}\left[\frac{G(s)}{s}\right]\right\}$$
$$= (1 - z^{-1})\mathscr{Z}\left\{\mathscr{L}^{-1}\left[\frac{K}{s(s+3)}\right]\right\}$$

  - Using the partial fraction expansion:

$$\frac{K}{s(s+3)} = \frac{K}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

  - We obtain the transfer function:

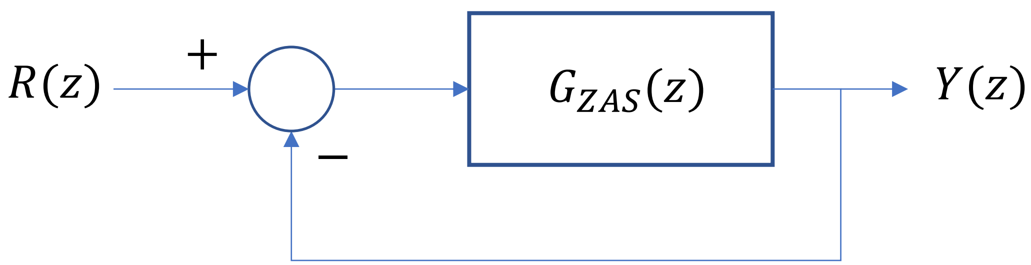
$$G_{ZAS}(z) = \frac{1.9412 \times 10^{-2}\, K}{z - 0.9418}$$

Note: Z-transform pairs

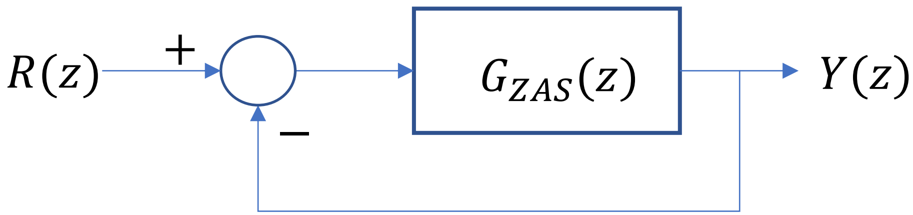
$$\frac{1}{s} \to \frac{1}{1 - z^{-1}}$$
$$\frac{1}{s+a} \to \frac{1}{1 - e^{-aT} z^{-1}}$$

($T = 0.02$ for this question)

  - Thus, the original system diagram can be simplified as:



Example (Continued)



$$G_{ZAS}(z) = \frac{1.9412 \times 10^{-2}\, K}{z - 0.9418}$$

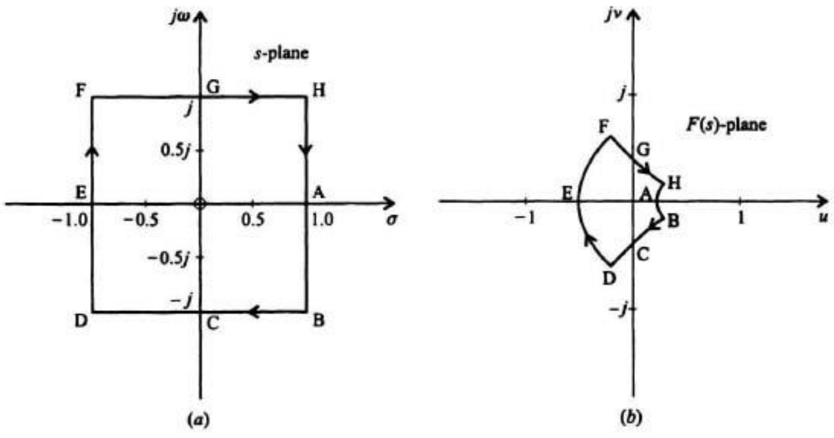
- **Solution (Continued)**
  - For the above unity feedback systems, the closed-loop characteristic equation is
  $$1 + G_{ZAS}(z) = 0$$
  - Which can be simplified to:
  $$z - 0.9418 + 1.9412 \times 10^{-2} \quad K = 0$$
  - From the above equation, we have a pole at
  $$z = 0.98418 - 1.9412 \times 10^{-2} K$$
  - To make the system stable, we need to make sure that this pole is **within** the **unit circle**, i.e., $|z| < 1$ or $-1 < z < 1$.
  - Thus, the stability conditions are
  $$0.9418 - 1.9412 \times 10^{-2} \quad K < 1$$
  $$-0.9418 + 1.9412 \times 10^{-2} \quad K < 1$$
  - Thus, the stable range of K is
  $$-3 < K < 100.03$$

Nyquist Criterion Introduction

- The Nyquist Criterion is a tool that solves two questions:
  - Does the system have closed-loop **poles outside the unit circle**?
  - If yes, **how many poles** are outside the unit circle?
- The answer of the first question can be used to determine the system's stability.
  - If the system have closed-loop poles outside the unit circle (no matter how many) -> The system is unstable.
- Before we introduce the Nyquist Criterion, we need to understand the concept of **Contour**.

Definition of Contour:
A Contour is a **closed directed** simple curve.

A contour must be
- Closed
- Have a Direction (either clockwise or counter-clockwise)



Nyquist Curve

- **Nyquist Curve:**
  - It is a **contour plot** of the **frequency response** of a system.
  - For a continuous-time system $L(s)$, the **frequency response** is $L(j\omega), \omega \in [0, \infty)$.

$|L(j\omega)|$ and $\angle L(j\omega)$ → **Continuous-time System frequency response**

  - The **graphical contour** of **frequency response** of $L(s)$ -> **Nyquist Curve**.

- For instance, let $L(s) = \frac{100}{(s+1)(\frac{1}{10}s+1)}$,
  - We can calculate $|L(j\omega)|$ and $\angle L(j\omega)$ with different ω values.

ω	0	0.1	0.76	1	2	10	20	100	∞
$\lvert L(j\omega)\rvert$	100	96	79.6	70.7	50.2	6.8	2.24	0.10	0
$\angle L(j\omega)$ (degrees)	0	−5.7	−41.5	−50.7	−74.7	−129.3	−150.5	−173.7	−180

  - Then, make a diagram for $|L(j\omega)|$ and $\angle L(j\omega)$, we will have

Nyquist Curve

ω	0	0.1	0.76	1	2	10	20	100	∞
$\lvert L(j\omega)\rvert$	100	96	79.6	70.7	50.2	6.8	2.24	0.10	0
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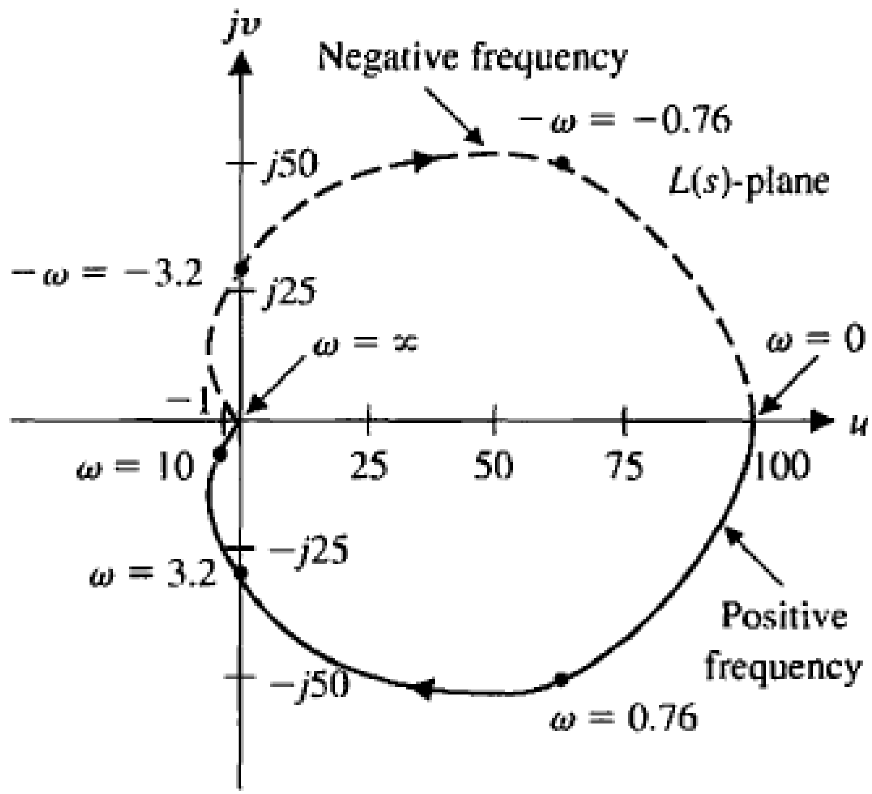


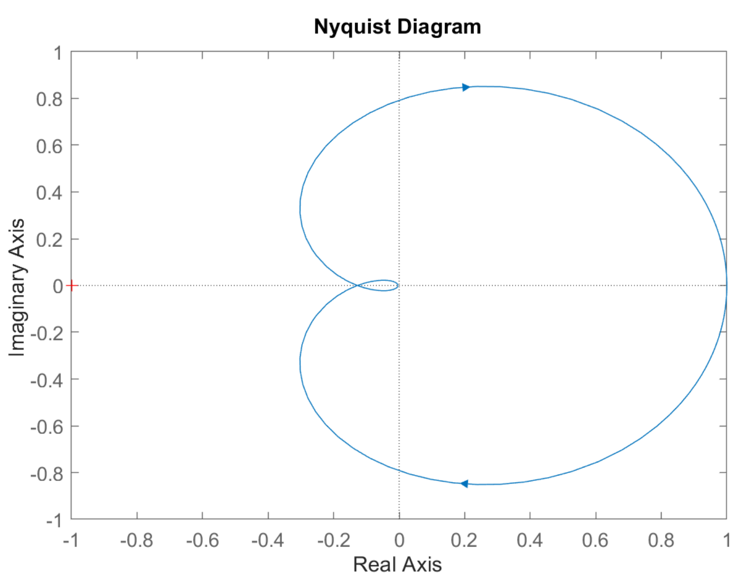
Nyquist Curve

Nyquist Curve

- For a **discrete-time** system $G(z)$, its **frequency response** include: $|G(e^{j\omega T})|\angle G(e^{j\omega T})$
  - With different values of ω, we can also generate a **Nyquist Curve**.
- For instance, if

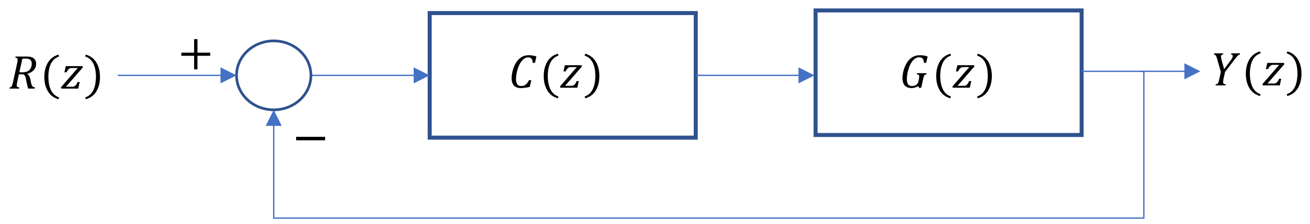
$$G(z) = \frac{0.066z + 0.055}{z^2 - 1.45z + 0.571}$$

  - Assume sample period $T = 0.4$, the **Nyquist Curve** can be plotted as:



An Important Observation

- Consider a closed-loop discrete-time control system as shown below:



  - The TF of the above system is

$$H(z) = \frac{C(z)G(z)}{1 + C(z)G(z)}$$

  - Thus, the characteristic equation is

$$1 + C(z)G(z) = 0$$

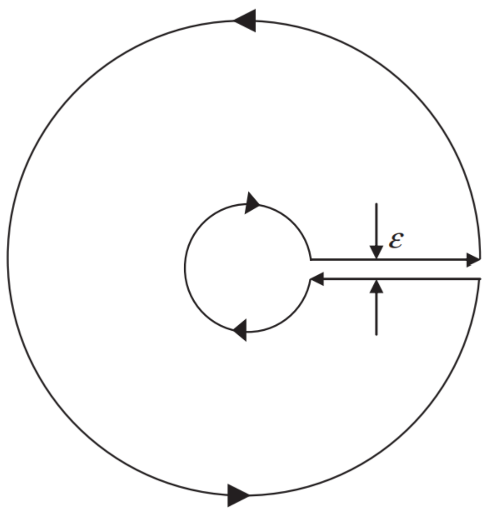
- Important Observation:
  - The **Zeros** of $1 + C(z)G(z) = 0$ are equal to the **Poles** of $H(z)$
  - We can quickly verify it by assuming $C(z) = 1$ and $G(z) = \frac{z}{z-1}$

An Important Observation (Continued)

The **Zeros** of $1 + C(z)G(z) = 0$ are equal to the **Poles** of $H(z)$

- Thus, to check the stability of a closed-loop system (i.e., $H(z)$), we can find **Zeros** of $1 + C(z)G(z) = 0$
  - If all **Zeros** of $1 + C(z)G(z) = 0$ are **within** the unit circle -> Closed-loop System is **Stable**
  - If any **Zeros** of $1 + C(z)G(z) = 0$ is **outside** the unit circle -> Closed-loop System is **Unstable**
- Unstable Region: Outside of the unit circle.
  - We can present the Unstable Region as a contour:
    - The inner circle is the unit circle.
    - The outer circle includes everything outside.

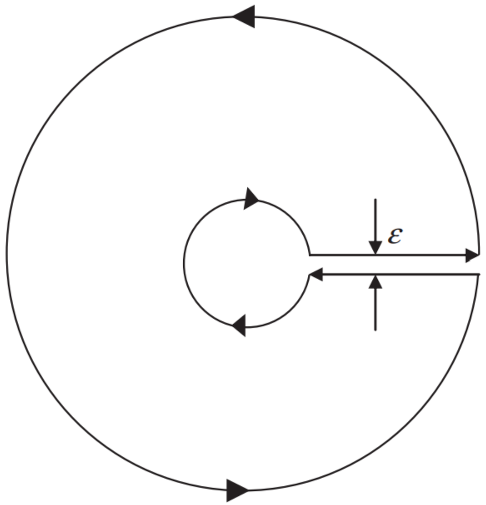
If any **Zeros** of $1 + C(z)G(z) = 0$ lie within **Nyquist Contour** -> The Closed-loop System is Unstable



Nyquist Contour

An Important Observation (Continued)

If any **Zeros** of $1 + C(z)G(z) = 0$ lie within **Nyquist Contour** -> The Closed-loop System is Unstable



Nyquist Contour

- Therefore, for a stable closed-loop system, we don't want any Zeros of $1 + C(z)G(z) = 0$ lie within **Nyquist Contour.**
  - Denote the **number** of Zeros of 1+C(z)G(z) inside this Contour as Z
  - For a Stable System, $Z = 0$
  - For an Unstable System, $Z \neq 0$
- How to determine the value of Z?
  - **Nyquist Criterion**

Nyquist Criterion

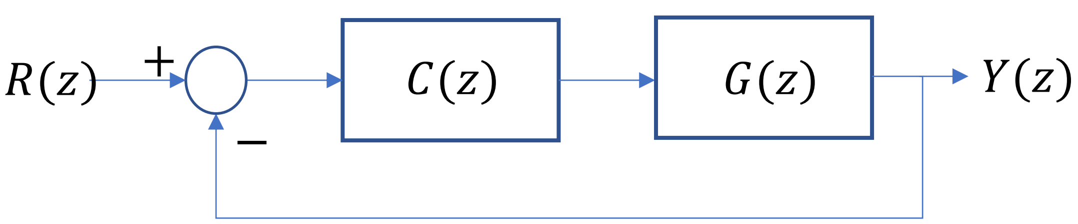
Nyquist Criterion

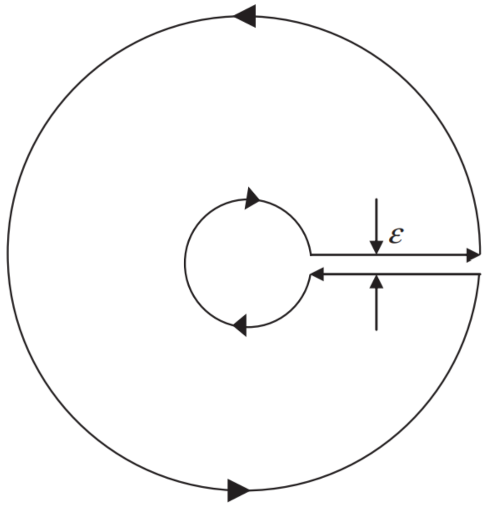
A Closed-Loop system has Z poles **inside** the Nyquist Contour with

$$Z = P - N$$

P is the number of Poles of $L(z)$ **inside** the Nyquist Contour;
N is the number of **anticlockwise contour encirclement** of (-1, 0) given the **Nyquist Curve** of $L(z)$.

$$L(z) = C(z)G(z)$$





Nyquist Contour

Nyquist Criterion Explanation

Nyquist Criterion

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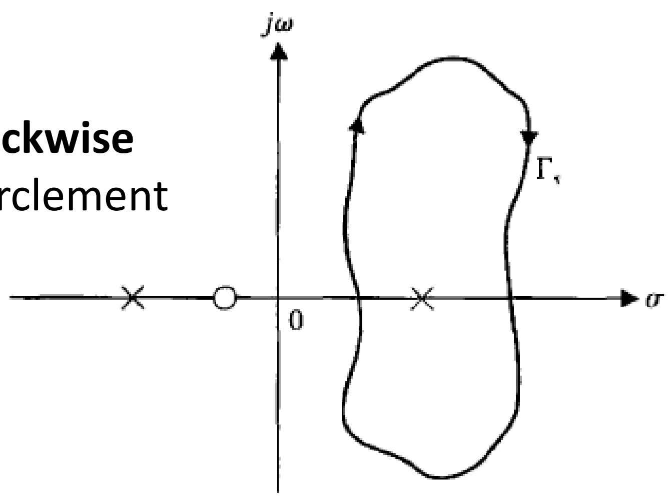
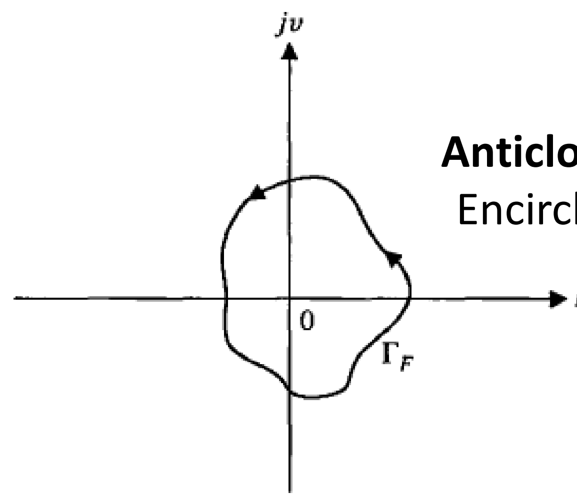
- Examples of encirclement in **different directions**:

Clockwise Encirclement





Anticlockwise Encirclement

Nyquist Criterion Explanation

Nyquist Criterion

A Closed-Loop system has Z poles **inside** the Nyquist Contour with

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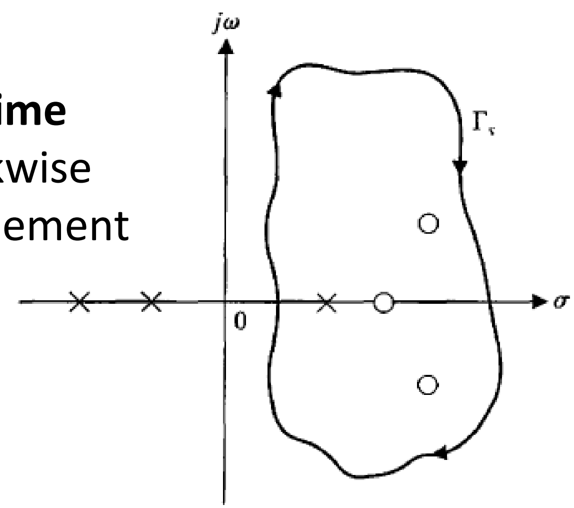
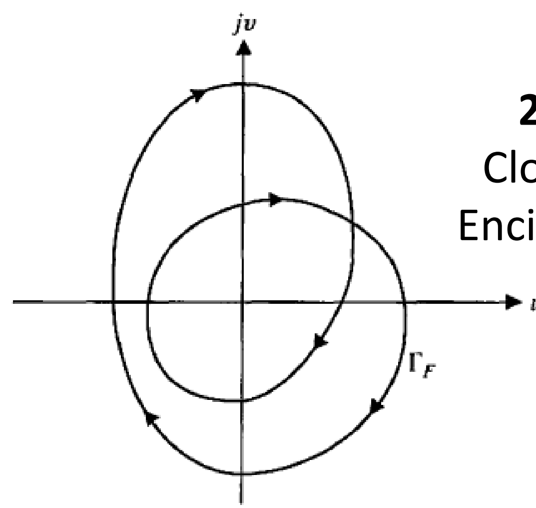
N is the number of **anticlockwise contour encirclement** of (-1, 0) given the **Nyquist Curve** of $L(z)$.

- Examples of encirclement in **different times**:

1-time
Clockwise
Encirclement





2-time
Clockwise
Encirclement

Nyquist Criterion Explanation

Nyquist Criterion

A Closed-Loop system has Z poles **inside** the Nyquist Contour with

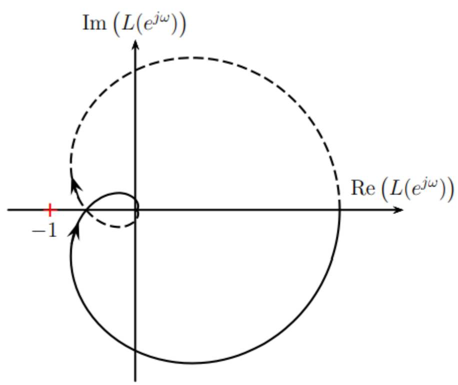
$$Z = P - N$$

P is the number of Poles of $L(z)$ **inside** the Nyquist Contour;
N is the number of **anticlockwise contour encirclement** of (-1, 0) given the **Nyquist Curve** of $L(z)$.

- The Nyquist Criterion is simple to use:
  - If $L(z)$ has no poles inside the Nyquist Contour, i.e., $P = 0$
  - Then, for a closed-loop system to be **stable** (i.e., $Z = 0$), $\boldsymbol{N}$ **must be** $\mathbf{0}$
    - Since $N = P - Z = 0$
    - In other words, the Nyquist Curve of $L(z)$ must not encircle (-1, 0)

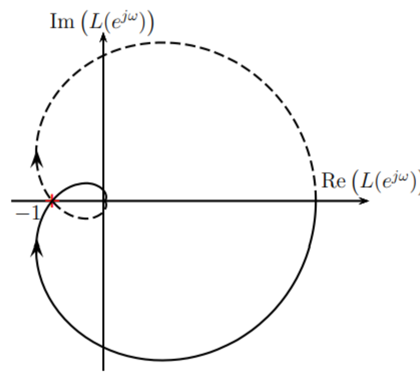
Examples

- Given the following Nyquist Curves, determine the closed-loop system's stability.
  - Assume $P = 0$ for all cases.



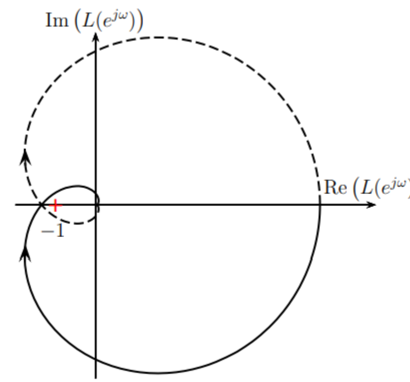
$$\Rightarrow \frac{L(z)}{1+L(z)}$$

Stable



$$\Rightarrow \frac{L(z)}{1+L(z)}$$

Marginally Stable

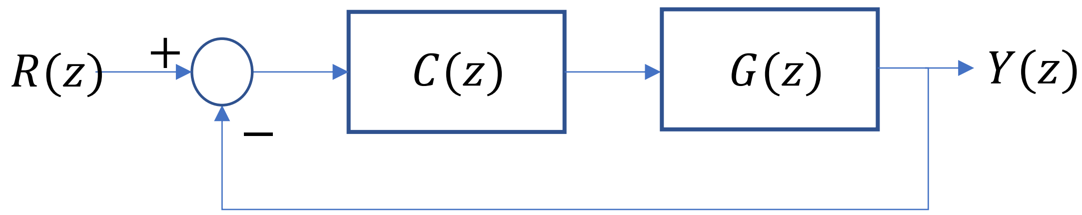


$$\Rightarrow \frac{L(z)}{1+L(z)}$$

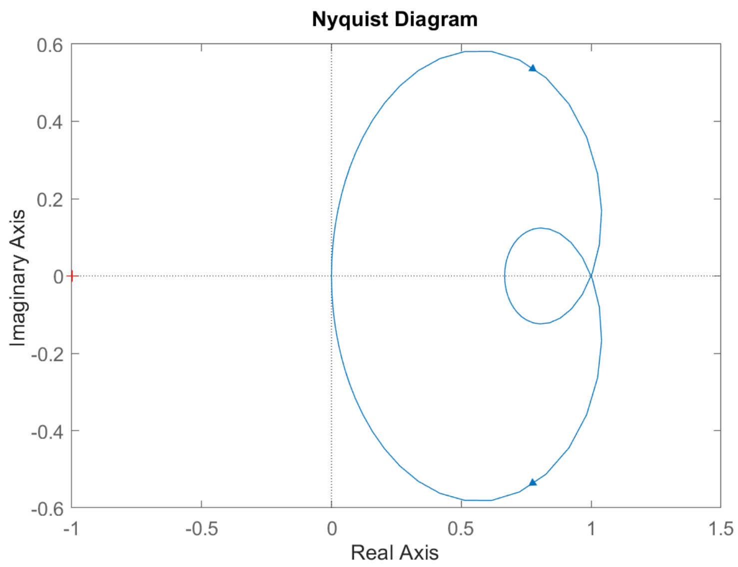
Unstable

Example 2

- Consider the following closed-loop discrete-time



  - Where $C(z) = 1$ and $G(z) = \frac{5z+4}{(z-2)(z-3)}$.
  - Given the Nyquist Curve of $G(z)$, determine the stability of this closed-loop system.

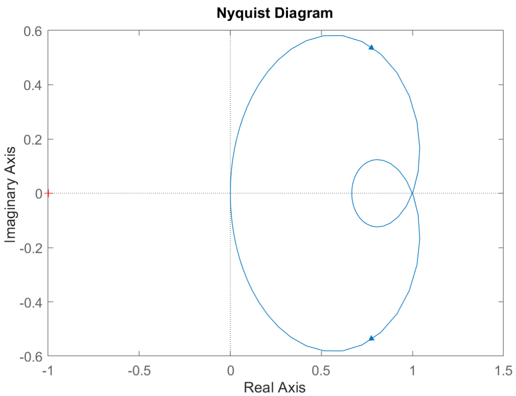


Example 2 Solution

- **Solution:**
  - We first need to determine the value of P.
    - P is the number of poles of **loop gain** outside the unit circle.
    - In this work, **loop gain** is

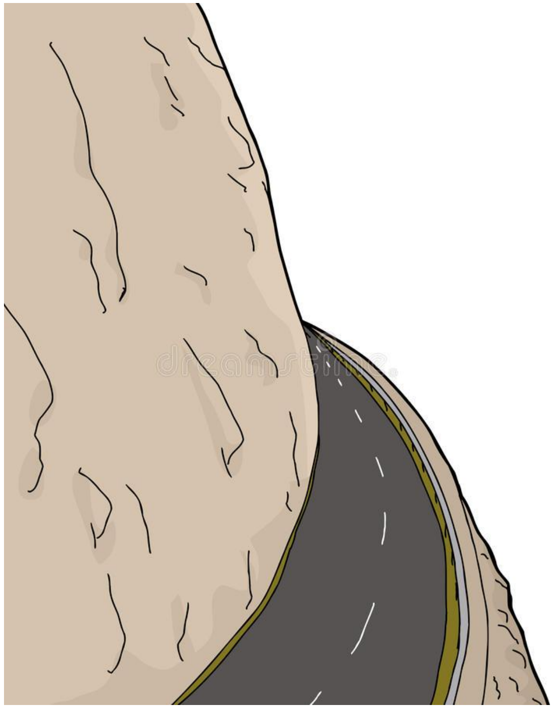
$$G(z) = \frac{5z + 4}{(z - 2)(z - 3)}$$

    - Two poles are at 2 and 3, which are both outside the unit circle.
      - Thus, $P = 2$
  - For a stable system, $Z = P - N$ should be 0
    - Here $P = 2$, thus, $Z = 2 - N = 0$
    - N is the encirclement of (-1, 0) for the Nyquist curve. However, in this diagram, the Nyquist curve does not encircle (-1, 0).
    - Thus, the system is **not stable**.



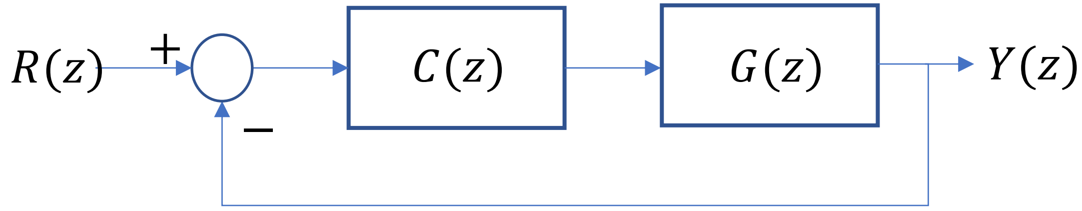
Gain Margin and Phase Margin

- Stability Criteria – Gain Margin and Phase Margin
  - Think of both of these as ***safety margins*** for an open-loop system which you would like to make closed-loop.
- Take a real-world example
  - Think if you are walking next to a cliff, you want a **positive space** or “**margin**” of safety between you and a big disaster.
  - So that **positive margins** indicate there is still a safety margin before instability.
  - Conversely, **negative margins** in an open-loop system indicate instability issues if you try to close this loop.



Closing the Loop

- Consider a Closed-loop Control System:



  - Assume $C(z) = 1$

- The closed-loop TF is: $G_{cl}(z) = \frac{G(z)}{1+G(z)}$
  - Replacing z with $e^{j\omega T}$, we have the frequency response (assume $T = 1$):

$$G_{cl}\left(e^{j\omega}\right) = \frac{G\left(e^{j\omega}\right)}{1 + G\left(e^{j\omega}\right)}$$

- For this closed-loop system, the **instability** occurs when:

$$1 + G\left(e^{j\omega}\right) = 0$$

- For this to happen, we need $G\left(e^{j\omega}\right) = -1$, which means:
  - $\left|G\left(e^{j\omega}\right)\right| = 1$ and $\angle G\left(e^{j\omega}\right) = -180^o$.

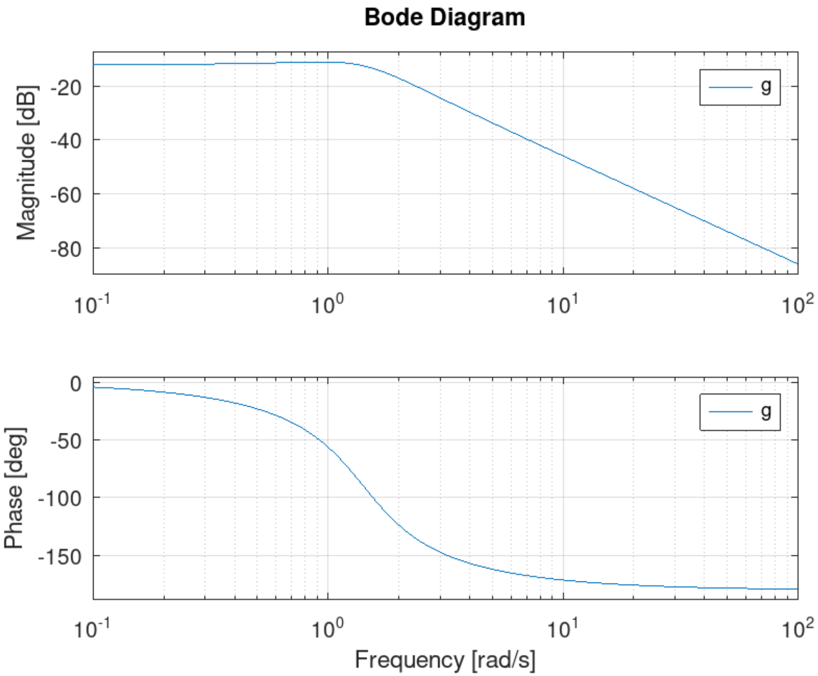
Gain Margin and Phase Margin

- Stability Margins, including **Gain Margin** and **Phase Margin**, measure how far we are from the point |G|=1 and ∠G=-180^o
- We can find Gain Margin and Phase Margin using **Bode Plot**.
  - Bode Plot is a diagram that demonstrate a system's frequency response.
  - Bode Plot includes two sub-diagrams:

Magnitude Plot
Shows ω vs. $|G(e^{j\omega T})|$
Usually, we convert magnitude in to decibel via

$$db = 20\log_{10}|G(e^{j\omega T})|$$

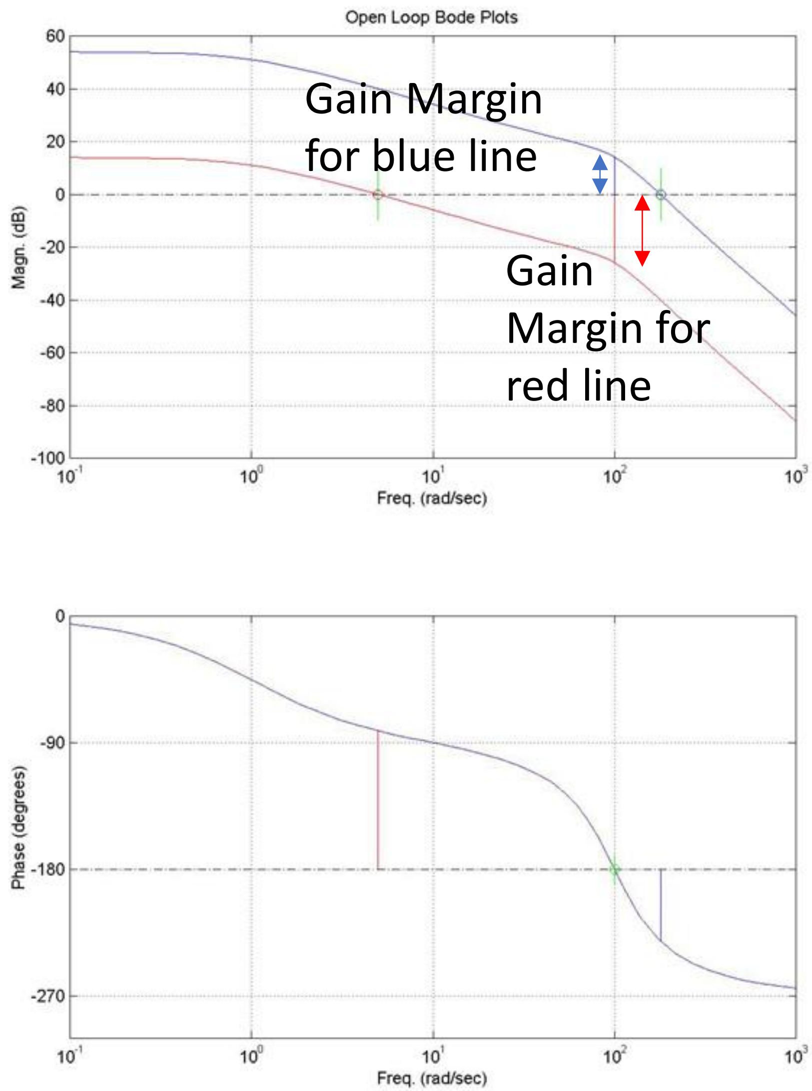
Phase Plot
Shows ω vs. $\angle G\left(e^{j\omega}\right)$
Usually, the unit for phase angle is degree.



Frequency ω

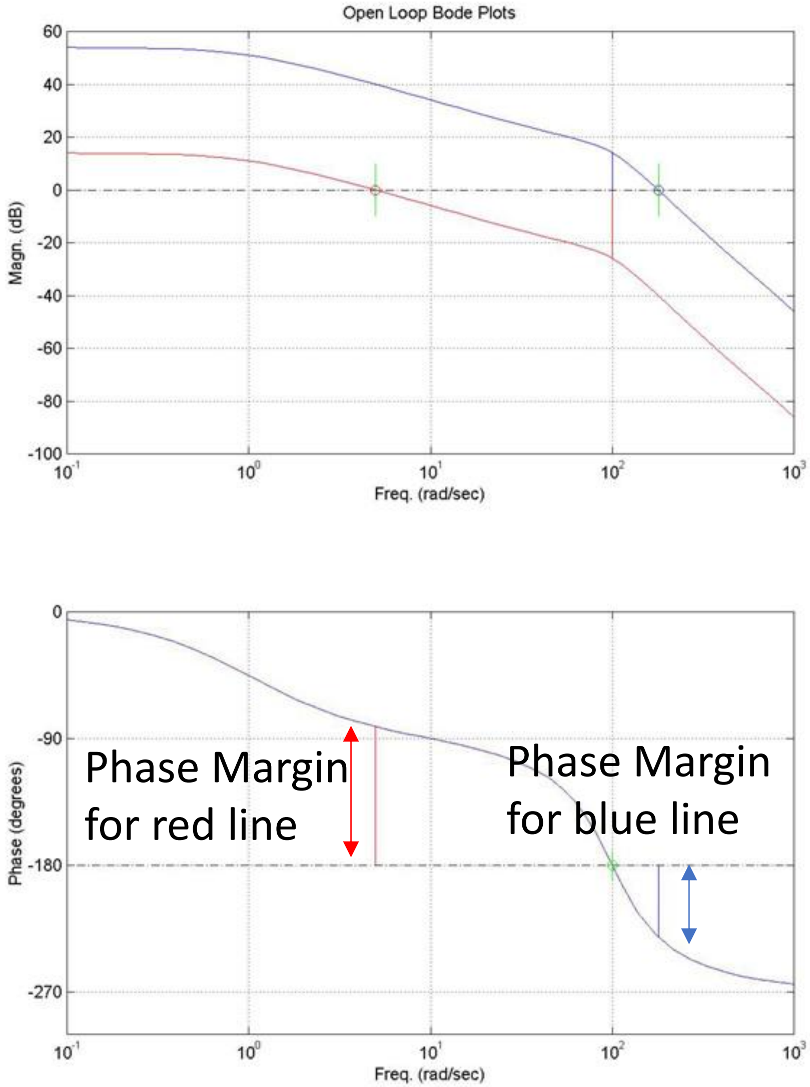
Gain Margin

- **Gain margin (GM)** is defined as the reciprocal of the gain at the frequency at which the phase angle reaches -180^o.
- Here is an example of finding Gain Margin:
    1. Find the frequency where the PHASE becomes -180 degrees.
        - In the pic, this frequency is 100 rad/sec.
    2. Find the GAIN, G (in dB), at the same frequency (from the upper plot).
        - For blue line, $G = 14.1$ dB
        - For red line, $G = -25.9$ dB
    3. **Gain Margin** = $0 - G$ dB
        - For blue line, GM = -14.1 dB
        - For red line, GM = 25.9 dB



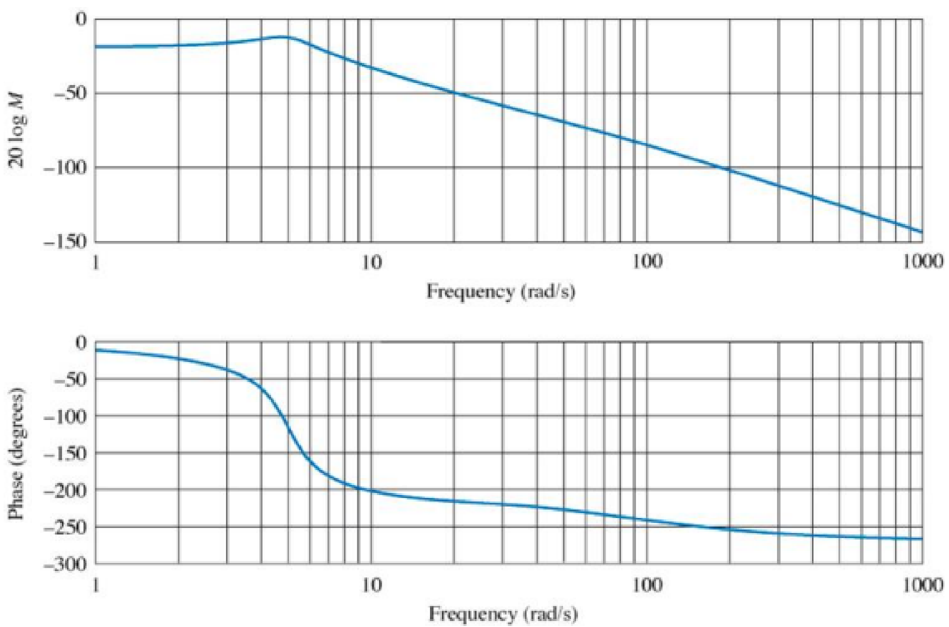
Phase Margin

- **Phase Margin** refers to the amount of phase, which can be increased or decreased without making the system unstable.
- To find Phase Margin
  1. Find the frequency where the GAIN is 0 dB.
     - For the blue Bode plot, the 0 dB crossover occurs at 181 rad/sec; For the red Body plot, this happens at 5 rad/sec.
  2. Find the PHASE, P (in degrees), at the same frequency (look at the lower plot)
  3. Phase Margin = $P + 180$ degrees
     - For blue line, $P =$-231 degrees. Thus, PM = -50.0 degrees
     - For red line, $P =$ -81.3 degrees. Thus, PM = 98.7 degrees.



Undefined Stability Margins

- Note that sometimes the margins are undefined
  - When there is no crossover at 0 dB
  - When there is no crossover at 180^o



Example

- Given the following Bode Plot of a closed-loop discrete-time system, determine the Gain Margin and Phase Margin.
- Solution:
    - Gain Margin: 0 – 7.75 = -7.75 dB
    - Phase Margin: -133 + 180 = 47 degrees