Lecture 5: Modeling of Digital Control Systems

ELEN 472: Introduction to Digital Control

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Review

- **Time Response of Discrete-Time Systems**
  - The output of the discrete-time system.

$$u(k) \longrightarrow \boxed{\begin{array}{c}\text{Discrete-Time}\\ \text{System, } h(k)\end{array}} \longrightarrow y(k)$$

$$y(k) = h(k) * u(k) = \sum_{i=0}^{k} h(k-i)u(i)$$

- **Frequency Response of Discrete-Time Systems**

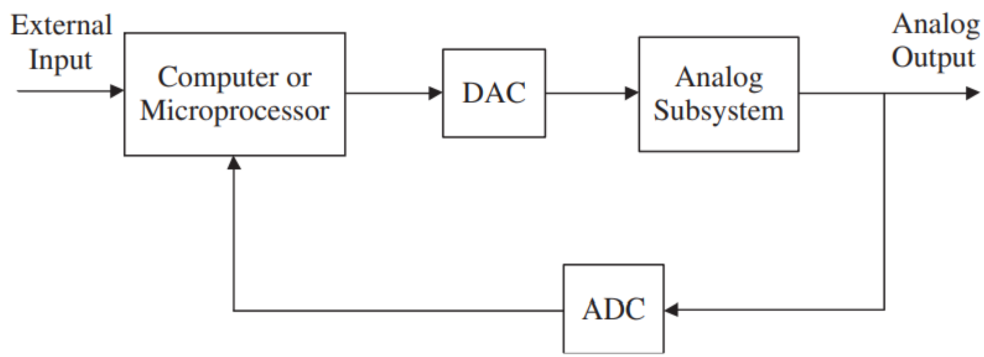
$$u(k) = e^{j\omega k} \longrightarrow \boxed{\begin{array}{c}\text{Discrete-Time}\\ \text{System, } h(k)\end{array}} \longrightarrow y(k) = H(e^{j\omega}) \cdot e^{j\omega k}$$

$$H\left(e^{j\omega}\right) = |H(e^{j\omega})| \angle H(e^{j\omega})$$

- **Sampling Theorem**
  - First-order system: $\omega_s = k\omega_m$, $35 \le k \le 70$
  - Second-order system: $\omega_s = k\omega_d, \quad 35 \le k \le 70$ $\qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$

Introduction

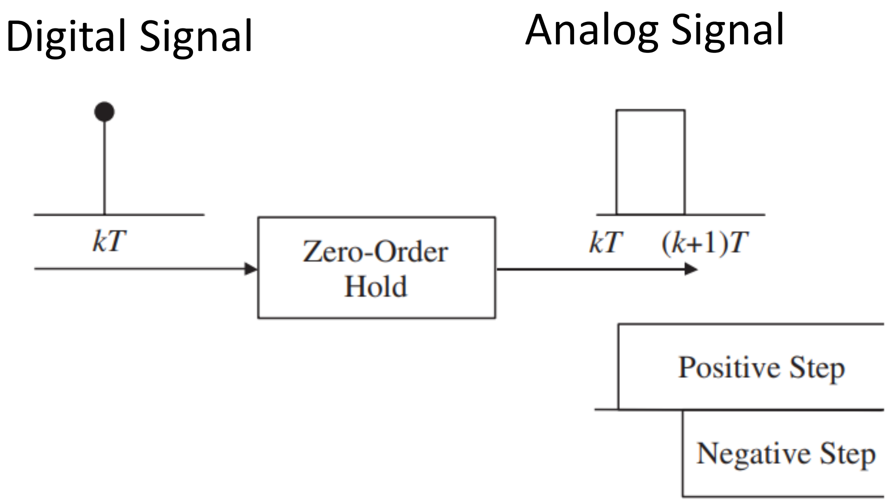
- A common configuration for closed-loop feedback **Digital Control Systems** is shown below:



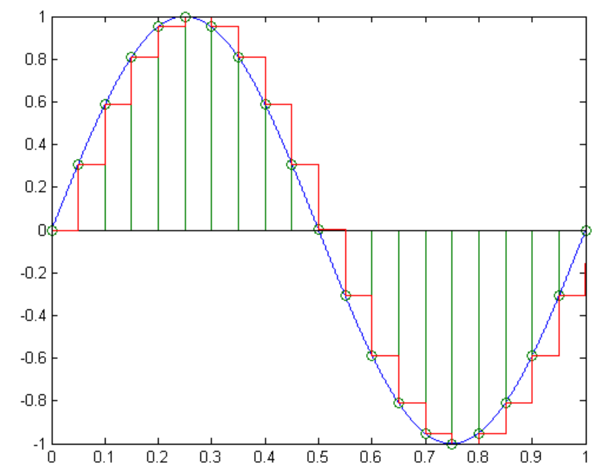
- Key components in this configuration are
  - **D**igital-to-**A**nalog **C**onverter (DAC),
  - An Analog Subsystem,
  - **A**nalog-to-**D**igital **C**onverter (ADC).

DAC Model

- DAC converts **Digital** signals into **Analog** signals.
- A common DAC method is **Z**ero-**O**rder **H**old (ZOH)



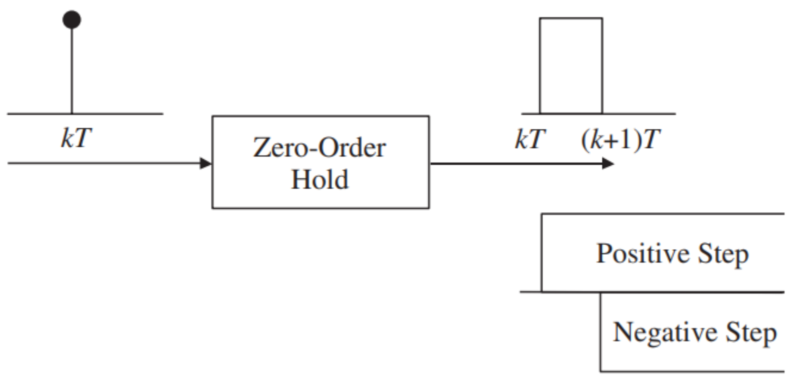
ZOH in impulse signal



ZOH in sine wave

The Transfer Function of the ZOH

- As shown in the figure, the **impulse response** of ZOH is a unit pulse of width T.



  - A pulse can be represented as a **positive step** at time zero followed by a **negative step** at time T.

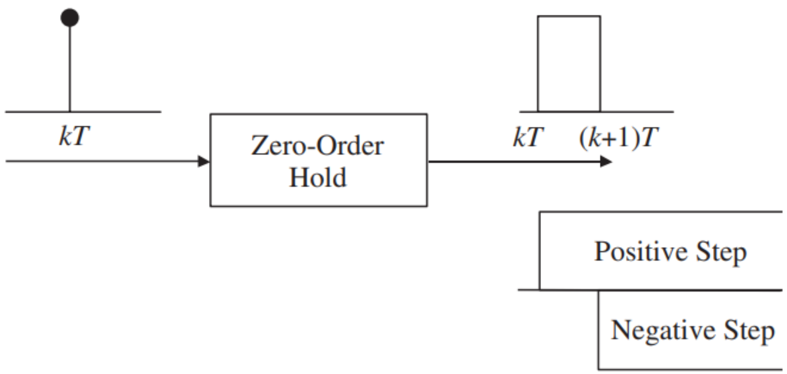
- Using the Laplace transform of a unit step and the time delay theorem for Laplace transforms,

$$\mathscr{L}\{\mathbf{1}(t)\} = \frac{1}{s}$$

$$\mathscr{L}\{\mathbf{1}(t-T)\} = \frac{e^{-sT}}{s}$$

The Transfer Function of the ZOH (Continued)

- Thus, the transfer function of the ZOH is



$$\mathscr{L}\{\mathbf{1}(t)\} = \frac{1}{s}$$

$$\mathscr{L}\{\mathbf{1}(t-T)\} = \frac{e^{-sT}}{s}$$

Subtract

$$G_{ZOH}(s) = \frac{1-e^{-sT}}{s}$$

- Next, we consider the **frequency response** of the ZOH:

$$s = j\omega$$

$$G_{ZOH}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega}$$

Frequency Response of ZOH

$$G_{ZOH}(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega}$$

- We rewrite the frequency response in the form

$$G_{ZOH}(j\omega) = \frac{e^{-j\omega\frac{T}{2}}}{\omega}\left(\frac{e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{j}\right)$$

$$= \frac{e^{-j\omega\frac{T}{2}}}{\omega}\left(2\sin\left(\omega\frac{T}{2}\right)\right) = Te^{-j\omega\frac{T}{2}}\frac{\sin\left(\omega\frac{T}{2}\right)}{\omega\frac{T}{2}}$$

- We now have the magnitude and phase angle of the ZOH:

$$\boxed{\left|\boldsymbol{G_{ZOH}}(j\omega)\right| \angle G_{ZOH}(j\omega) = T\left|\operatorname{sinc}\left(\frac{\omega T}{2}\right)\right| \angle -\frac{\omega T}{2},\ -\frac{2\pi}{T} < \omega < \frac{2\pi}{T}}$$

Example

- Find the magnitude and phase at frequency $\omega = 1$ rad/s of a zero-order hold with sampling time $T = 0.1$ s.
- Solution:

The magnitude results from the expression

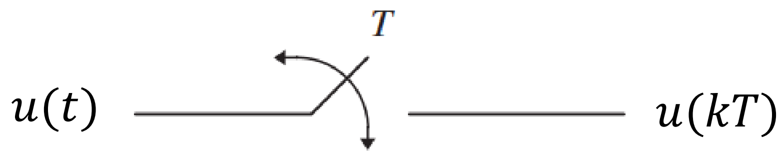
$$|G_{ZOH}(j\omega)| = T\left|\operatorname{sin}c\left(\frac{\omega T}{2}\right)\right| = T\frac{\sin\left(\omega\frac{T}{2}\right)}{\omega\frac{T}{2}} = 0.1\frac{\sin\left(1\frac{0.1}{2}\right)}{1\frac{0.1}{2}} = 0.1$$

while the phase is given by

$$\angle G_{ZOH}(j\omega) = -\omega\frac{T}{2} = -1\frac{0.1}{2} = -0.05 \text{ rad}$$

ADC Model

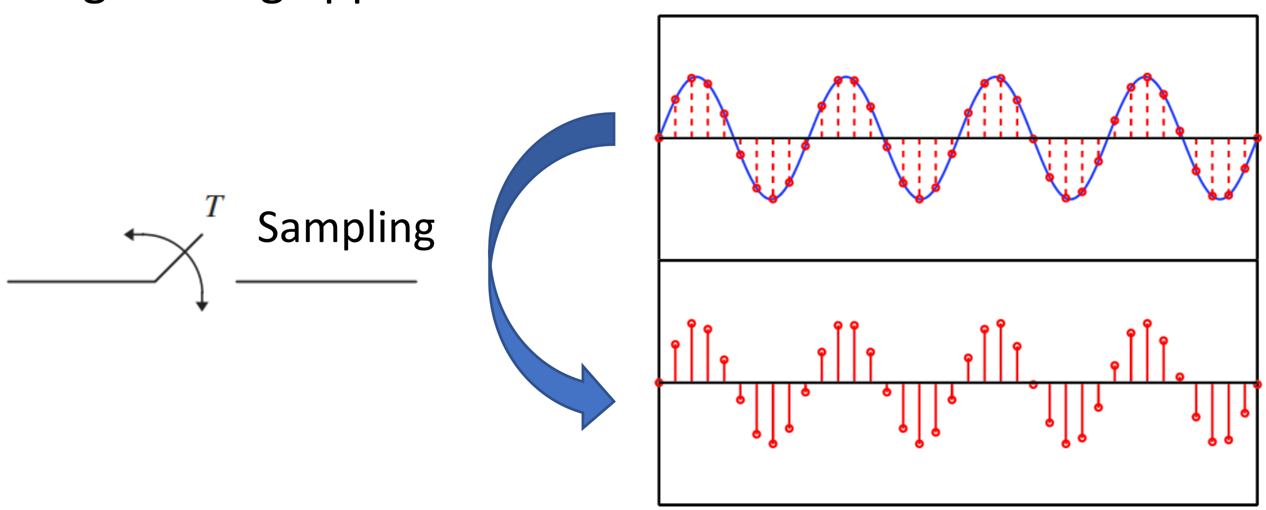
- The ADC can be modeled as an ideal sampler with sampling period T as shown below:



$$t \rightarrow kT$$

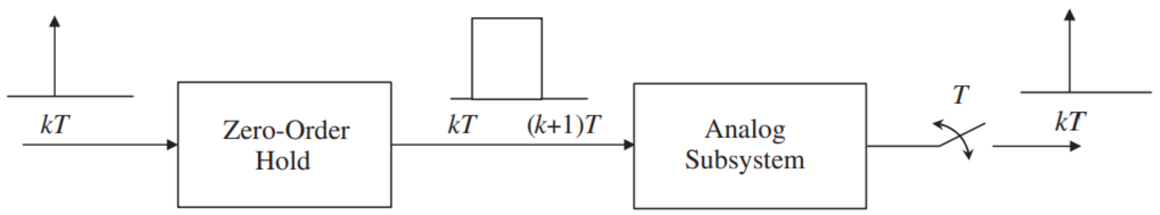
  - This is equivalent to a switch, while it will close and open at every T seconds.

- This is called an ideal sampler (or ADC), which is acceptable for most engineering applications.

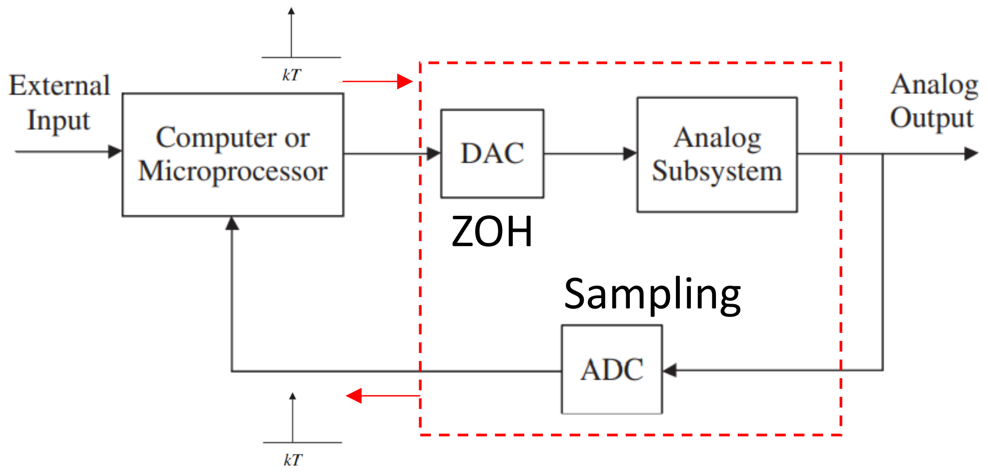


DAC, Analog subsystem, and ADC Combination Transfer Function

- The cascade of a DAC, analog subsystem, and ADC, shown in the figure below, appears frequently in digital control systems.



- Because both the input and the output of the cascade are **discrete signals**, it is possible to obtain its **z-domain transfer function** of the whole system.



DAC, Analog subsystem, and ADC Combination Transfer Function

- Assuming that the transfer function of the analog subsystem is $G(s)$, the transfer function of **ZOH** and **Analog Subsystem** cascade is:

$$\begin{aligned} G_{ZA}(s) &= G(s)G_{ZOH}(s) \\ &= (1 - e^{-sT})\frac{G(s)}{s} \end{aligned}$$

- Apply **inverse Laplace transfer** to get $g_{ZA}(t)$

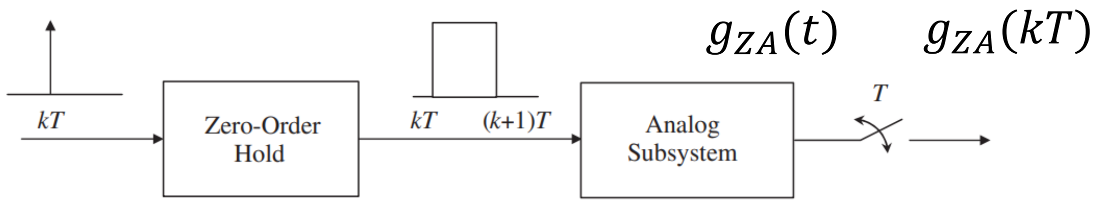
$$g_{ZA}(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-sT}\frac{G(s)}{s}\right\}$$

  - Denote $\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = g_s(t)$, thus $\mathcal{L}^{-1}\left\{e^{-sT}\frac{G(s)}{s}\right\} = g_s(t - T)$

$$g_{ZA}(t) = g_s(t) - g_s(t - T)$$

  - We can see that $g_{ZA}(t)$ is $g_s(t)$ minus $g_s(t)$ delayed by T

DAC, Analog subsystem, and ADC Combination Transfer Function (Continued)



- The analog output $g_{ZA}(t)$ is sampled to give the sampled impulse response:

$$g_{ZA}(t) = g_s(t) - g_s(t-T) \xrightarrow{t = kT} g_{ZA}(kT) = g_s(kT) - g_s(kT-T)$$

- By z-transforming $g_{ZA}(kT)$, we obtain the z-transfer function of the DAC, Analog subsystem, and ADC Combination:

$$\begin{aligned}
G_{ZAS}(z) &= \mathcal{Z}\{g_s(kT)\} - \mathcal{Z}\{g_s(kT-T)\} \\
&= \mathcal{Z}\{g_s(kT)\} - z^{-1}\mathcal{Z}\{g_s(kT)\} \\
&= (1-z^{-1})\mathcal{Z}\{g_s(kT)\} \\
&= (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}
\end{aligned}$$

DAC, Analog subsystem, and ADC Combination Transfer Function (Continued)

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}$$

- Procedures to get $G_{ZAS}(z)$:
  - **Step 1**: get $\frac{G(s)}{s}$
  - **Step 2**: check z-transform pairs (in **Lecture 2 Page 19**) to get $\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}$
    - **Note:** here, we can omit the $\mathcal{L}^{-1}$ notation to make the equation more concise. $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$
  - **Step 3**: multiple $(1 - z^{-1})$ with $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$ to get $G_{ZAS}(z)$

$$\boxed{\begin{aligned} G_{ZAS}(z) &= (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\ &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \end{aligned}}$$

Z-Transform Pairs

$\frac{G(s)}{s}$

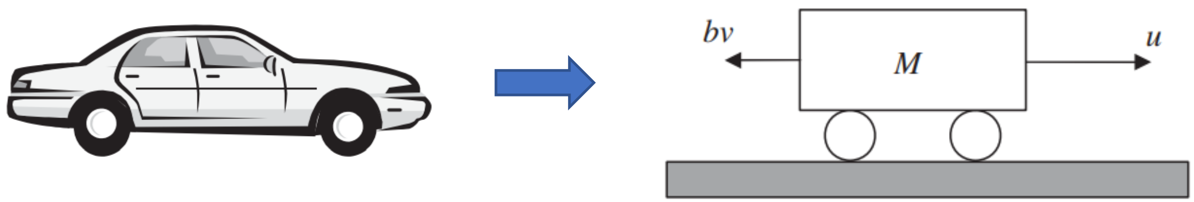
$\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}$

No.	Continuous Time	Laplace Transform	Discrete Time	z-Transform
1	$\delta(t)$	1	$\delta(k)$	1
2	$1(t)$	$\frac{1}{s}$	$1(k)$	$\frac{z}{z-1}$
3	t	$\frac{1}{s^2}$	kT**	$\frac{zT}{(z-1)^2}$
4	t^2	$\frac{2!}{s^3}$	$(kT)^2$	$\frac{z(z+1)T^2}{(z-1)^3}$
5	t^3	$\frac{3!}{s^4}$	$(kT)^3$	$\frac{z(z^2+4z+1)T^3}{(z-1)^4}$
6	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	a^k***	$\frac{z}{z-a}$
7	$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$	$1-a^k$	$\frac{(1-a)z}{(z-1)(z-a)}$
8	$e^{-\alpha t}-e^{-\beta t}$	$\frac{\beta-\alpha}{(s+\alpha)(s+\beta)}$	a^k-b^k	$\frac{(a-b)z}{(z-a)(z-b)}$

***The function $e^{-\alpha kT}$ is obtained by setting $a=e^{-\alpha T}$.

Example

- For the cruise control system of the vehicle, u is the input force, v is the velocity of the car, and b is the viscous friction coefficient.



  - Find $G_{ZAS}(z)$ with $M = 1$, $b = 1$, and $T = 1$
- **Solution:**
  - First, we find the system's dynamic equation using Newton's Second law:

$$M\dot{v}(t) + bv(t) = u(t)$$

  - Take the Laplace transform of the above equation, we have:

$$G(s) = \frac{V(s)}{U(s)} = \frac{1}{Ms + b}$$

  - Plugin all known values into $G(s)$, we have:

$$G(s) = \frac{1}{s+1}$$

Example Solution

- Now, we can find $G_{ZAS}(z)$ using 3 steps in Page 16:
  - **Step 1**: Find $\frac{G(s)}{s}$

$$G(s) = \frac{1}{s+1} \Rightarrow \frac{G(s)}{s} = \frac{1}{s(s+1)}$$

  - **Step 2**: Find $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$
    - From z-transform table in page 17, we find a match of $\frac{1}{s(s+1)}$ on row **7.**

$$\frac{\alpha}{s(s+\alpha)} \xrightarrow[\mathcal{Z}]{} \frac{(1-a)z}{(z-1)(z-a)} \qquad a = e^{-\alpha T}.$$

    - In our case, $\alpha = 1$ and $a = e^{-1\times 1} = e^{-1}$. Thus, we have

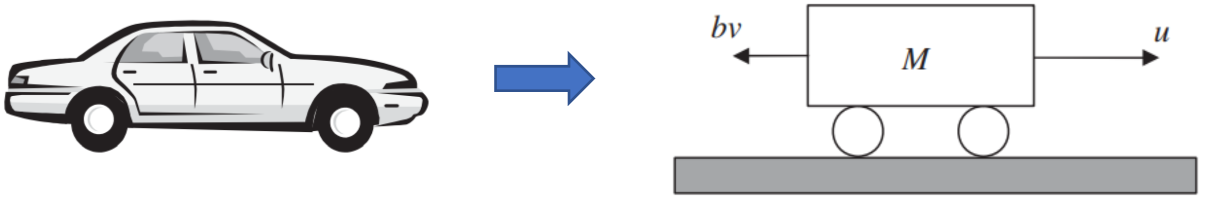
$$\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{(1-e^{-1})z}{(z-1)(z-e^{-1})}$$

  - **Step 3:** Multiple $(1 - z^{-1})$ with $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$

$$G_{ZAS}(z) = (1-z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z} \times \frac{(1-e^{-1})z}{(z-1)(z-e^{-1})} = \frac{1-e^{-1}}{z-e^{-1}}$$

Example 2

- For the **position** control system of the vehicle, u is the input force, y is the velocity of the car, and b is the viscous friction coefficient.



  - Find $G_{ZAS}(z)$ with $M = 1$, $b = 1$, and $T = 1$

- **Solution**:
  - Using Newton's second law, we have:

$$M\ddot{y}(t) + b\dot{y}(t) = u(t)$$

  - The corresponding transfer function is:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(Ms + b)}$$

  - Plugin all known values:

$$G(s) = \frac{1}{s(s+1)}$$

Example 2 Solution

- Now, we can find $G_{ZAS}(z)$ using 3 steps in Page 16:
  - **Step 1**: Find $\frac{G(s)}{s}$

$$G(s) = \frac{1}{s(s+1)} \quad \Rightarrow \quad \frac{G(s)}{s} = \frac{1}{s^2(s+1)}$$

  - **Step 2**: Find $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$
    - No exiting pairs on z-Transform table match $\frac{G(s)}{s}$.
    - In this case, we need to use **partial fraction expansion** to decode $\frac{G(s)}{s}$

$$\frac{G(s)}{s} = \frac{1}{s^2(s+1)} = \frac{A_{11}}{s^2} + \frac{A_{12}}{s} + \frac{A_2}{s+1}$$

$$A_{11} = s^2 \frac{G(s)}{s}\bigg|_{s=0} = \frac{1}{1} = 1$$

$$A_{12} = \frac{d}{ds} s^2 \frac{G(s)}{s}\bigg|_{s=0} = \frac{d}{ds}\frac{1}{(s+1)} = -1 \quad \Rightarrow \quad \frac{G(s)}{s} = \frac{1}{s^2} + \frac{1}{s} + \frac{1}{s+1}$$

$$A_2 = (s+1)\frac{G(s)}{s}\bigg|_{s=-1} = \frac{1}{s^2} = 1$$

Example 2 Solution

- Now, for each term in $\frac{G(s)}{s}$, we can find the corresponding z-transform result:

$$\frac{G(s)}{s} = \frac{1}{s^2} + \frac{1}{s} + \frac{1}{s+1}$$

$$\Downarrow \mathcal{Z}$$

$$\frac{z}{(z-1)^2} \qquad \frac{z}{z-1} \qquad \frac{z}{z-e^{-1}}$$

$$\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{z}{(z-1)^2} + \frac{z}{z-1} + \frac{z}{z-e^{-1}}$$

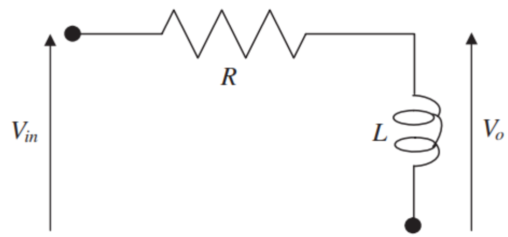
- **Step 3:** Multiple $(1 - z^{-1})$ with $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$

$$G_{ZAS}(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z} \times \left\{\frac{z}{(z-1)^2} + \frac{z}{z-1} + \frac{z}{z-e^{-1}}\right\}$$

$$= \frac{1}{z-1} + 1 + \frac{z-1}{z-e^{-1}}$$

Practice Question

- For an RLC circuit as shown below



$R = 2$ and $L = 1$ and $T = 1$

  - Find $G_{ZAS}(z)$ for $\frac{V_0}{V_{in}}$

- **Solution:**
  - Using voltage division law:

$$\frac{V_o}{V_{in}} = \frac{Ls}{R + Ls} \Longrightarrow G(s) = \frac{V_o}{V_{in}} = \frac{s}{2 + s}$$

  - **Step 1**: Find $\frac{G(s)}{s}$ $\Longrightarrow$ $\frac{G(s)}{s} = \frac{1}{s+2}$
  - **Step 2**: Find $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$

$$\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{z}{z - e^{-2}}$$

Practice Question Solution

- **Step 3:** Multiple $(1 - z^{-1})$ with $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$

$$
\begin{aligned}
G_{ZAS}(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\
&= \frac{z-1}{z}\frac{z}{z-e^{-2}} \\
&= \frac{z-1}{z-e^{-2}}
\end{aligned}
$$