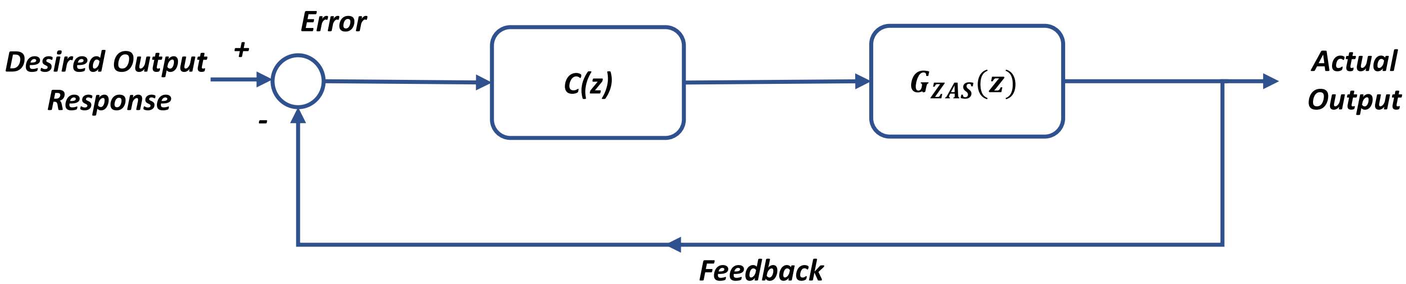
Lecture 10: Digital Control System Design I

ELEN 472: Introduction to Digital Control

Lingxiao Wang, Ph.D.
Assistant Professor of Electrical Engineering
Louisiana Tech University

Introduction

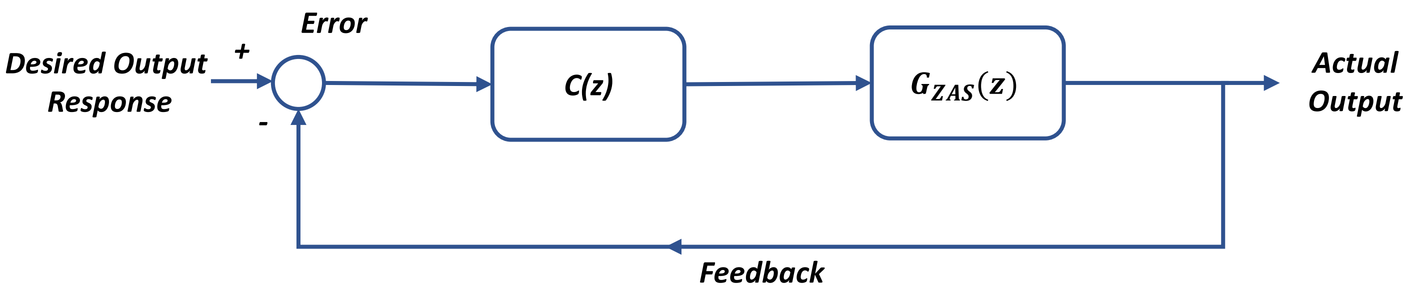
- A closed-loop digital control system can be presented as:



- To design a **digital control system**, our goal is to find
  - The **z-domain transfer function** of **the controller** that meets given design specifications.
- There are two ways to get the desired controller $C(z)$:
  - Design the **digital** controller that meets the design requirement **directly**.

  OR
  - Design an **analog** controller that meets the design requirement, then convert it into **digital** domain.

z-Domain Root Locus



- For a closed-loop digital control system (as shown above), the Transfer Function is

$$H(z) = \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)}$$

- Thus, the characteristic equation is

$$q(s) = 1 + C(z)G_{ZAS}(z) = 0$$

- If $C(z) = K$, then, the above equation becomes:

$$q(s) = 1 + KG_{ZAS}(z) = 0$$

  - Changing the value of K, we can get different **poles locations**.
- We can plot all poles locations (associated with different K values) in a diagram -> **Root Locus Diagram**.

z-Domain Root Locus (Continued)

- z-Domain Root Locus diagram can be obtained via **MATLAB**
- **Example:**
  - Obtain the root locus diagram for the first-order type 1 system with open-loop TF with sampling time $T_s = 0.1\ s$:

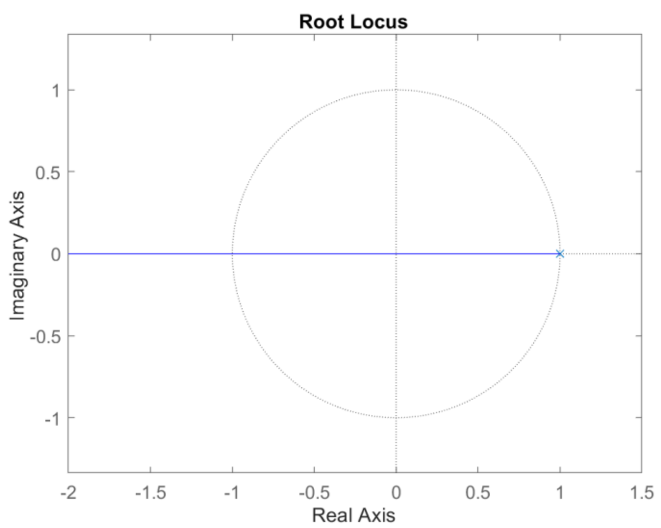
$$G_{ZAS}(z) = \frac{1}{z-1}$$

- **Solution**:
  - The root locus diagram for digital system can be obtained via MATLAB command ‘**rlocus**’.
  - Here is the complete code and the result:

```
numerator = 1;
denominator = [1, -1];
ts = 0.1;

sys = tf(numerator, denominator, ts)

rlocus(sys)
axis equal
```



Critical Gain

- **Critical Gain** K_{cr} is the gain that puts the system in the limit between stable and unstable.
  - For discrete time systems, the system is **stable** <u>if the magnitude of all poles are inside the unit circle</u>, i.e., |z|<1
  - Thus, the **critical gain** can be found at the **intersection** between **root locus plot** and the **unit circle**.
- For the **first order systems**, you can simply plugin $z = -1$ in the **characteristic equation** to find K_{cr}.
- **Example**:
  - Find the critical gain of the following discrete time system:

$$G_{ZAS}(z) = \frac{1}{z-1}$$

- **Solution:**
  - This system is a first-order system, and the characteristic equation is:

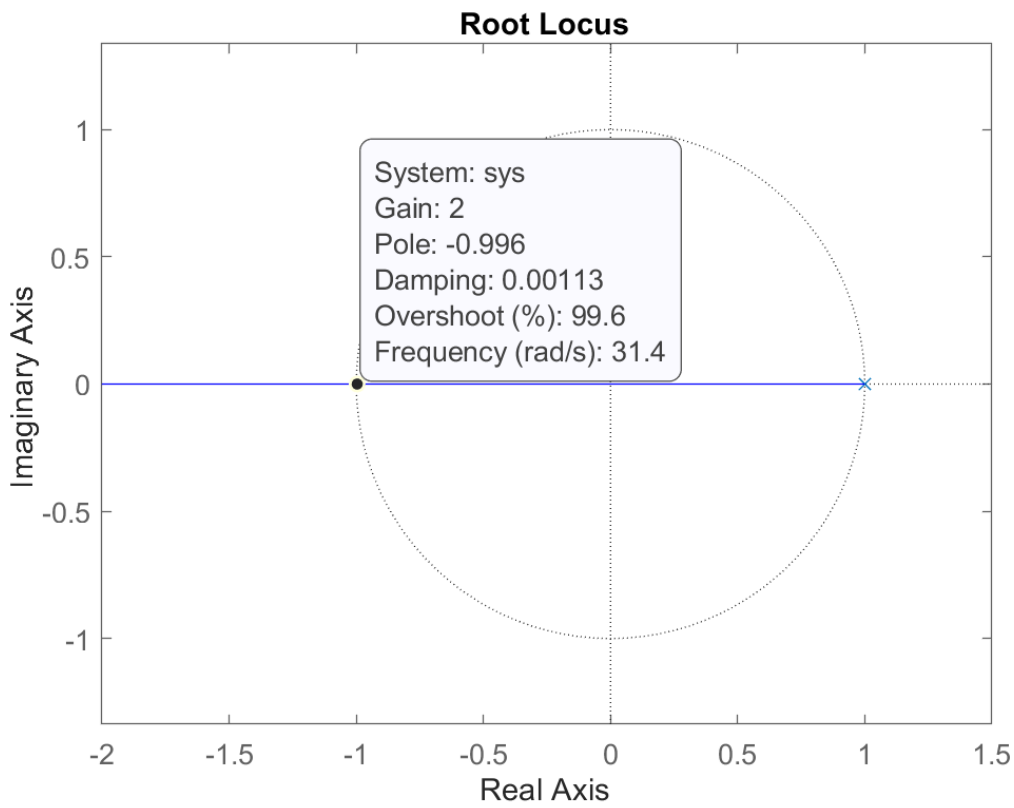
$$1 + KG_{ZAS}(z) = 0$$

$$1 + K_{cr}\left.\frac{1}{z-1}\right|_{z=-1} = 0$$

$$K_{cr} = 2$$

Critical Gain (Continued)

- On root locus diagram, we can find the **critical gain** at the intersection of **root locus branch** and the **unit circle**.
- In the previous example, we can find critical gain at:



Critical Gain (Continued)

- For the **second and higher order systems**, you need to use the **Jury test** (detailed in Lecture 6) and make all conditions satisfied to find K_{cr}.
  - Jury Test Conditions:

(1)	$F(1) > 0$
(2)	$(-1)^n F(-1) > 0$
(3)	$\lvert a_0 \rvert < a_n$
(4)	$\lvert b_0 \rvert > \lvert b_{n-1} \rvert$
(5)	$\lvert c_0 \rvert > \lvert c_{n-2} \rvert$
	.
	.
	.
$(n+1)$	$\lvert r_0 \rvert > \lvert r_2 \rvert$

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \qquad a_n > 0$$

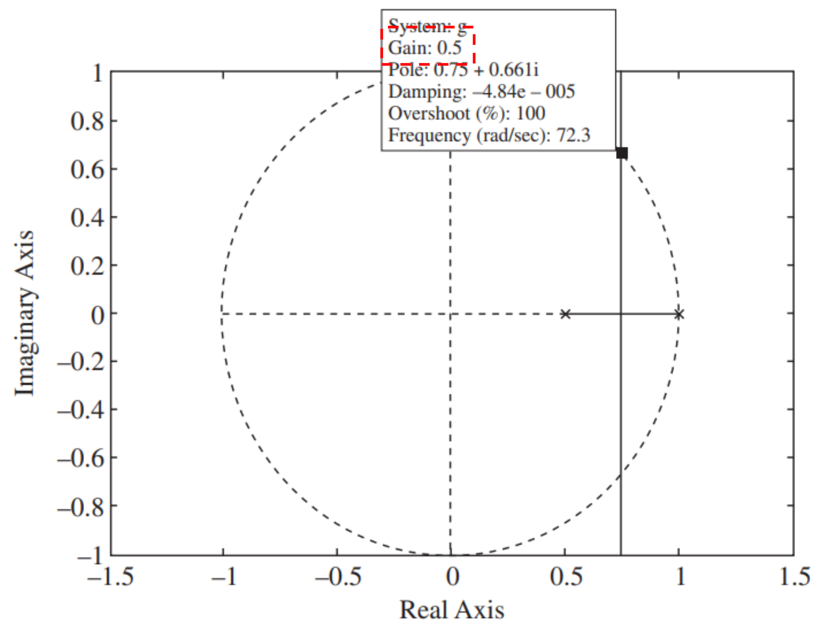
This is closed-loop characteristic equation

Example 2

- Obtain the root locus plot and the critical gain for the second-order type 1 system with loop gain:

$$L(z) = \frac{1}{(z-1)(z-0.5)}$$

- **Solution**:
  - From MATLAB, we can get the critical gain as 0.5:



- To obtain the critical gain manually, we first write the closed-loop characteristic equation:

$(z-1)(z-0.5)+K = z^2 - 1.5z + K + 0.5 = 0$

- Using Jury Test, we can get the critical gain, i.e., K_{cr}
  - For second-order systems, the Jury Test has 3 conditions:
    - $F(1) > 0, F(-1) > 0, |a_0| < a_2$
  - The first and second conditions are satisfied.
  - For the last one, $a_0 = K + 0.5 = 1, K_{cr} = 0.5$

Practice Question

- Get the critical gain of the following system:

$$G(z) = \frac{z}{(z-0.2)(z-1)}$$

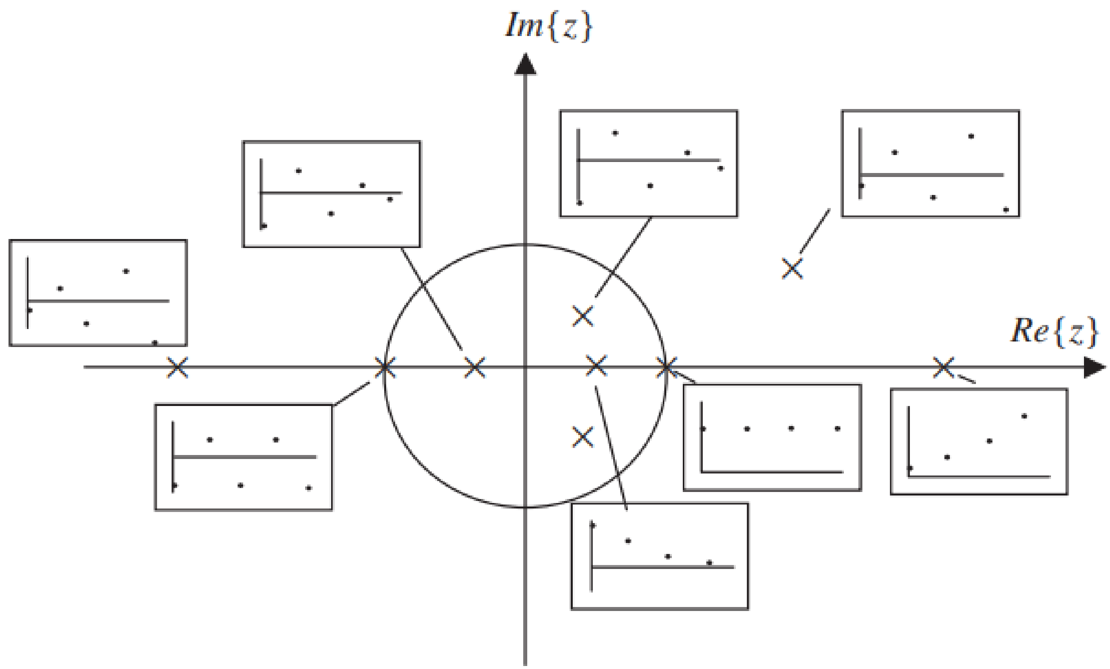
- **Solution:**
  - First, we need to get the characteristic equation $1 + KG(z) = 0$

$$K\,z + z^2 - 1.2\,z + 0.2 = 0.$$

  - Use Jury Test:
    - $F(1) = K + 1 - 1.2 + 0.2 = K > 0 \Rightarrow$ Satisfied
    - $F(-1) = -K + 1 + 1.2 + 0.2 = -K + 2.4 > 0 \Rightarrow \boxed{K_{cr} = 2.4}$
    - $|a_0| < a_2 \Rightarrow 0.2 < 1 \Rightarrow$ Satisfied

Z-Domain Digital Control System Design

- Select the **poles** of the z-domain transfer function to characterize the **system response**.
- This figure shows z-domain **pole locations** and the associated **system response**.



- The **exponential decay** for poles **inside** the unit circle and **exponential increase** for poles **outside** it.
- Poles **on the unit circle** are associated with a response of **constant magnitude**.
- **Negative poles** are associated with **alternating signs**.
- Poles with **imaginary** parts is **oscillating**.

Second Order Systems

If the Laplace transform $F(s)$ has a **pole** $\boldsymbol{p_s}$, then the z-transform $F(z)$ has a **pole** at $\boldsymbol{p_z = e^{p_s T}}$, where T is the sampling period.

- The s-domain characteristic polynomial for a **second-order system** is

$$(s - p_1)(s - p_2)$$

  - Where $p_1 = -\zeta\omega_n + j\omega_d$ and $p_2 = -\zeta\omega_n - j\omega_d$ $\quad \omega_d = \omega_n\sqrt{1 - \zeta^2}$
  - ω_n and ζ are natural frequency and damping ratio, respectively

- In z-domain, we can write the second-order characteristic polynomial as:

$$(z - e^{p_1 T})(z - e^{p_2 T})$$

$$\Downarrow$$

$$(z - e^{(-\zeta\omega_n + j\omega_d)T})(z - e^{(-\zeta\omega_n - j\omega_d)T}) = z^2 - 2\cos(\omega_d T)e^{-\zeta\omega_n T} z + e^{-2\zeta\omega_n T}$$

- Hence, the poles of the system are given by:

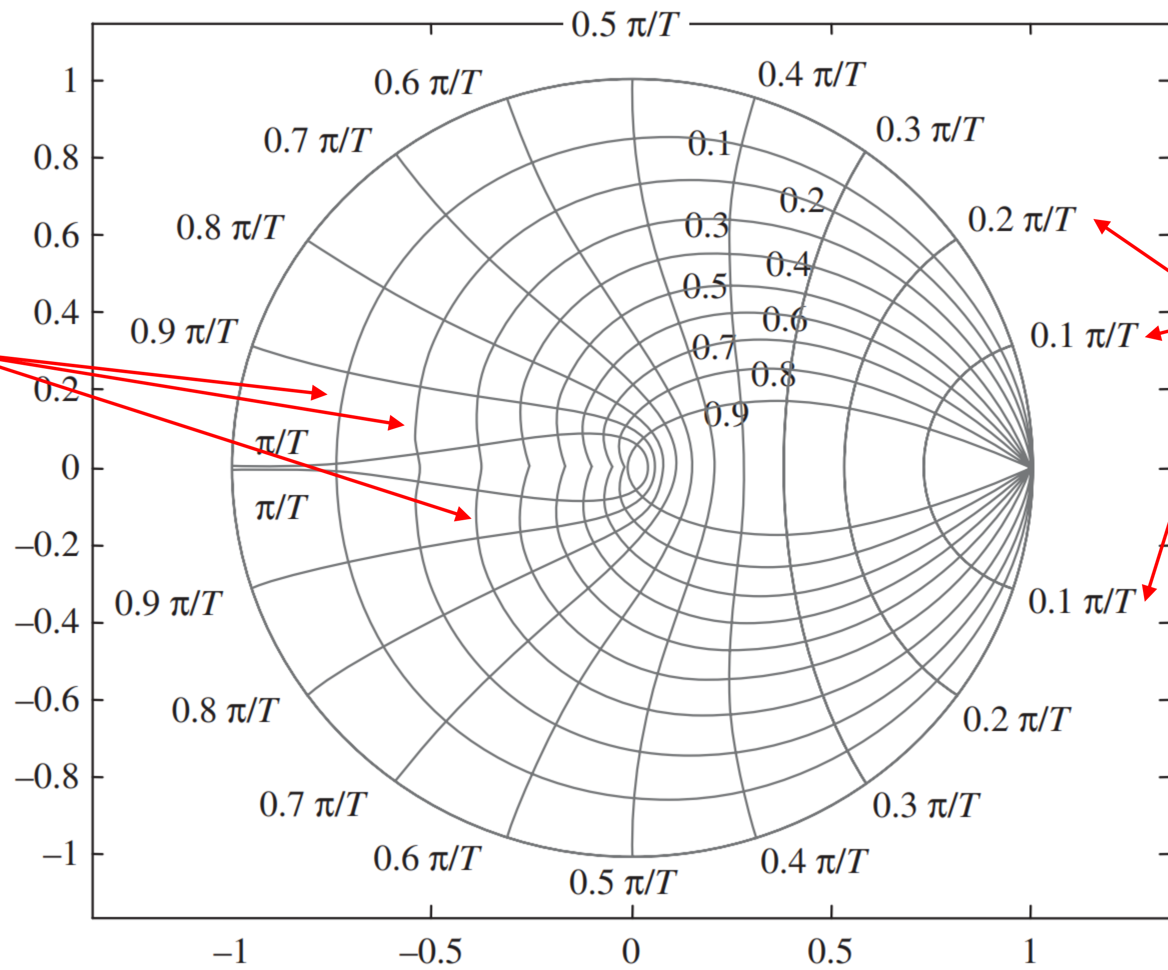
$$z_1 = e^{p_1 T} \text{ and } z_2 = e^{p_2 T}$$

$$z_{1,2} = e^{-\zeta\omega_n T} \angle \pm\omega_d \boldsymbol{T}$$

  - Varying the values of ζ and ω_n, we can have a plot of all poles.

Z-Domain Contours in Second Order Systems

Fix ω_n and Vary ζ -> **Circles**



Fix ζ and Vary ω_n -> **Radial Lines**

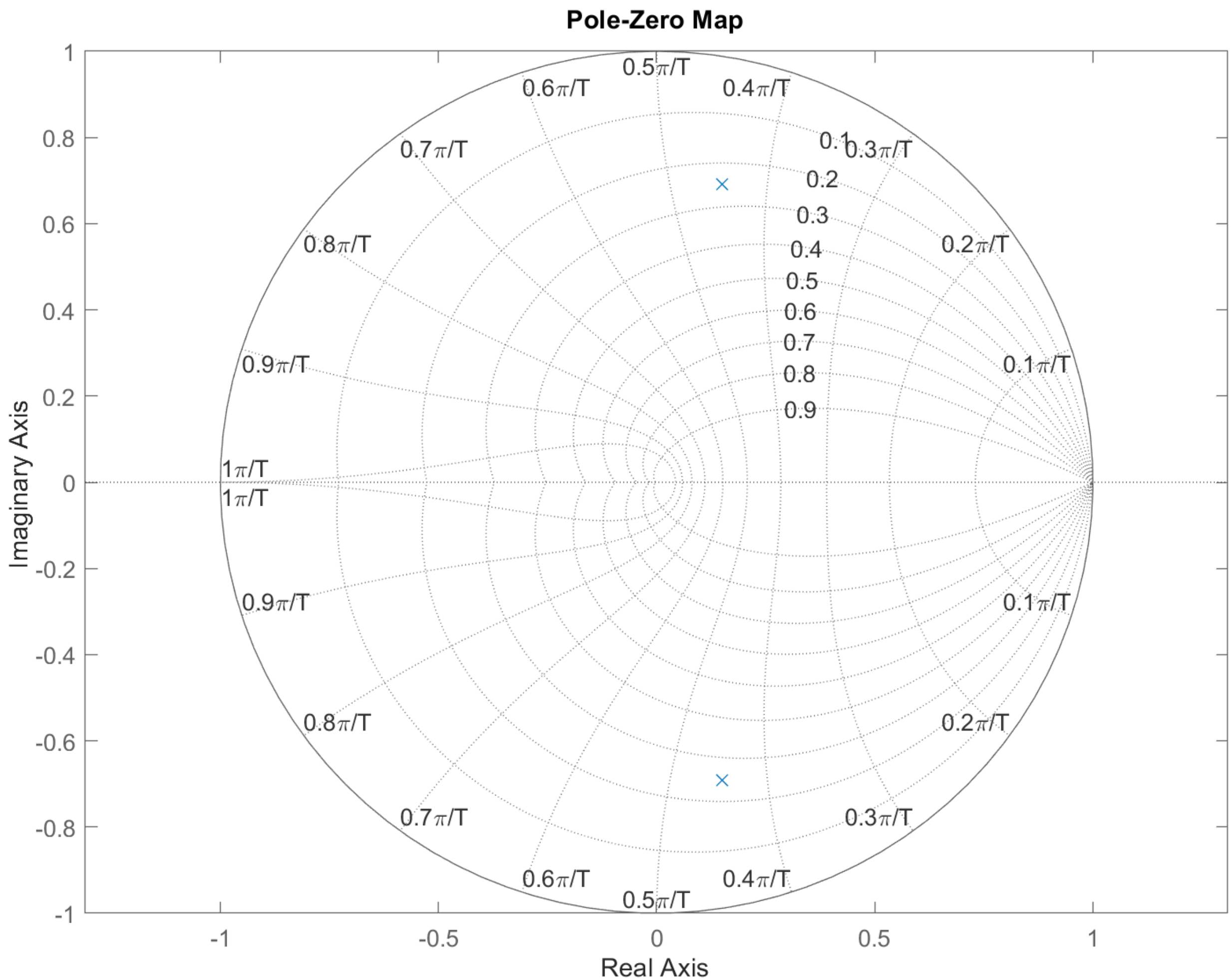
Z-Domain Contours in Second Order Systems (Continued)

- We can use the above diagram to **quickly determine ζ and ω_n values for a given second order TF z-domain function.**
- **Example**:
  - Suppose we have the following discrete transfer function

$$G(z) = \frac{1}{z^2 - 0.3z + 0.5}$$

  - Determine ζ and ω_n of the above system.
- **Solution:**
  - By solving the characteristic equation, we have

$$z_{1,2} = 0.15 \pm 0.69j$$



$$\omega_n = \frac{0.45\pi}{T}$$
$$\zeta = 0.25$$

Time Specifications for Z-Domain

- The specifications for z-domain design are similar to those for s-domain design. Typical design specifications are as follows:
  - *Time constant*: This is the time constant of exponential decay for the continuous envelope of the sampled waveform.

$$\tau = \frac{1}{\zeta \omega_n}$$

  - *Settling time:* The settling time is defined as the period after which the envelope of the sampled waveform stays within a specified percentage (usually 2%) of the final value.

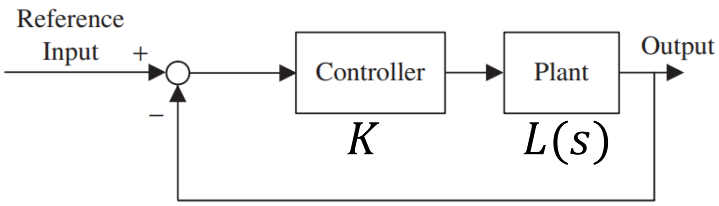
$$T_s = \frac{4}{\zeta \omega_n}$$

Example

- Design a proportional controller for the digital system

$$L(z) = \frac{1}{(z-1)(z-0.5)}$$

  With a sampling period $T = 0.1$ s to obtain
  - The frequency of oscillation $\omega_d = 5$ rad/s
  - The time constant $\tau = 0.5$ s
  - The damping ratio $\zeta = 0.7$



- **Solution:**
  - We first need to determine the closed-loop characteristic equation, i.e., $1 + KL(z) = 0$

$$(z-1)(z-0.5) + K = z^2 - 1.5\,z + K + 0.5 = 0$$

  - The general form of second-order polynomial in z-domain is:

$$z^2 - 2\cos(\omega_d T)e^{-\zeta\omega_n T} z + e^{-2\zeta\omega_n T}$$

  - Equating the coefficients of the above two equations, we have

$$z^1: \quad 1.5 = 2\cos(\omega_d T)e^{-\zeta\omega_n T}$$
$$z^0: \quad K + 0.5 = e^{-2\zeta\omega_n T}$$

Example Solution

$$z^1: \quad 1.5 = 2\cos(\omega_d T)e^{-\zeta\omega_n T}$$
$$z^0: \quad K + 0.5 = e^{-2\zeta\omega_n T}$$

The frequency of oscillation $\omega_d = 5$ rad/s
The time constant $\tau = 0.5$ s
The damping ratio $\zeta = 0.7$

- **Case 1**: Find K to obtain $\omega_d = 5$
  - From the z^1 equation

$$\zeta\omega_n = \frac{1}{T}\ln\left(\frac{2\cos(\omega_d T)}{1.5}\right) = 10\ln\left(\frac{2\cos(0.5)}{1.5}\right) = 1.571$$

  - Since $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\omega_d = 5$

$$\omega_d^2 = \omega_n^2(1-\zeta^2) = 25$$

  - Hence, we obtain the ratio

$$\frac{\omega_d^2}{(\zeta\omega_n)^2} = \frac{1-\zeta^2}{\zeta^2} = \frac{25}{(1.571)^2}$$

  - This gives a damping ratio $\zeta = 0.3$ and $\omega_n = 5.24$ rad/s
  - Finally, the z^0 equation gives the gain:

$$K = e^{-2\zeta\omega_n T} - 0.5 = e^{-2\times1.571\times0.1} - 0.5 = 0.23$$

Example Solution

$$z^1: \quad 1.5 = 2\cos(\omega_d T)e^{-\zeta\omega_n T}$$
$$z^0: \quad K + 0.5 = e^{-2\zeta\omega_n T}$$

The frequency of oscillation $\omega_d = 5$ rad/s
The time constant $\tau = 0.5$ s
The damping ratio $\zeta = 0.7$

- **Case 2**: Find K for $\tau = 0.5$ s:
  - From the definition of time constant $\tau = \frac{1}{\zeta\omega_n}$ and z^1 equation:

$$\zeta\omega_n = \frac{1}{\tau} = \frac{1}{0.5} = 2 \ \text{rad/s}$$
$$\omega_d = \frac{1}{T}\cos^{-1}\left(\frac{1.5e^{\zeta\omega_n T}}{2}\right) = 10\cos^{-1}(0.75e^{0.2}) = 4.127 \text{ rad/s}$$

  - Similar to the previous case, we have

$$\frac{\omega_d^2}{(\zeta\omega_n)^2} = \frac{1-\zeta^2}{\zeta^2} = \frac{(4.127)^2}{2^2}$$

  - This equation gives $\zeta = 0.436$ and $\omega_n = 4.586$ rad/s
  - Finally, the gain K can be obtained via z^0

$$K = e^{-2\zeta\omega_n T} - 0.5 = e^{-2\times 2\times 0.1} - 0.5 = 0.17$$

Example Solution

$$z^1: \quad 1.5 = 2\cos(\omega_d T)e^{-\zeta\omega_n T}$$
$$z^0: \quad K + 0.5 = e^{-2\zeta\omega_n T}$$

The frequency of oscillation $\omega_d = 5$ rad/s
The time constant $\tau = 0.5$ s
The damping ratio $\zeta = 0.7$

- **Case 3**: Find K for $\zeta = 0.7$
  - Since $\omega_d = \omega_n\sqrt{1-\zeta^2}$, equation z^1 can be rewritten as:

$$1.5 = 2\cos(0.0714\omega_n)e^{-0.07\omega_n}$$

  - Solve this equation we have

$$\omega_n = 3.63 \text{ rad/s}$$

  - Thus, the gain K can be obtained from z^0 as

$$K = e^{-0.14\times 3.63} - 0.5 = 0.10$$

A Comparison of Different Controllers in the Example

- (a) $K = 0.23$
- (b) $K = 0.17$
- (c) $K = 0.10$

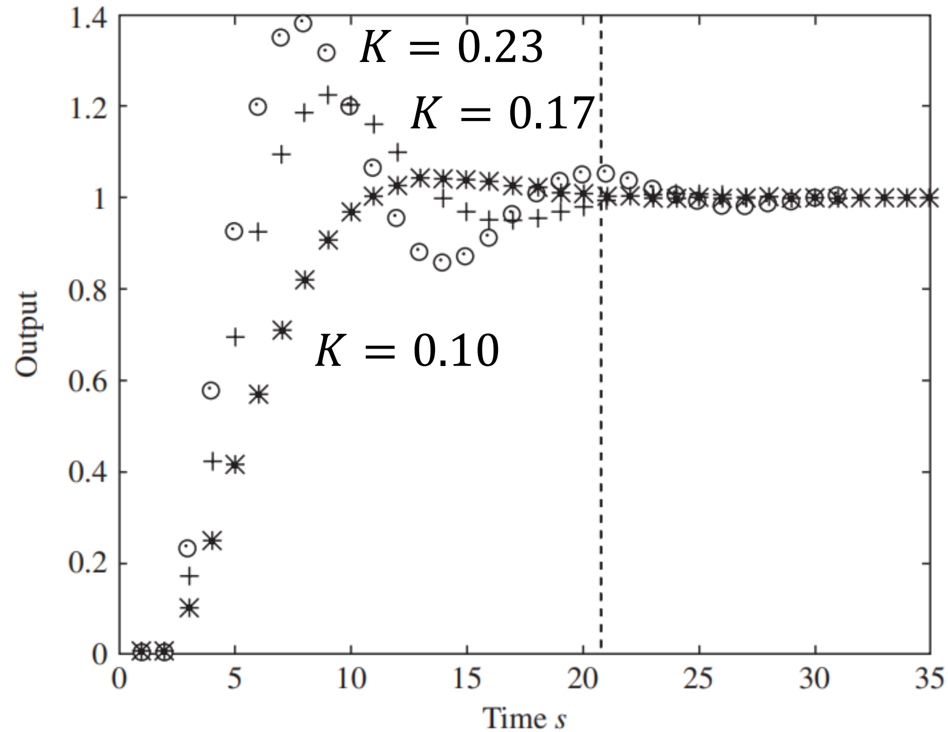


FIGURE 6.8

Time response for the designs of Table 6.4: (a) ⊙, (b) +, (c) *.

Practice

- Consider the vehicle position control system with the transfer function in the s-domain:

$$G(s) = \frac{1}{s(s+5)}$$

  Design a proportional controller for the unity feedback digital control system with analog process and a sampling period $T = 0.04$ s to obtain:
    - A steady-state error of 10% due to a ramp input

- **Solution**:
    - First, we need to convert $G(s)$ into z-domain using the equation:

$$G_{ZAS}(z) = (1 - z^{-1})Z\{\frac{G(s)}{s}\}$$

$$G_{ZAS}(z) = \frac{7.4923 \times 10^{-4}(z + 0.9355)}{(z-1)(z-0.8187)}$$

Practice Question Solution

1. A steady-state error of 10% due to a ramp input

- From $G_{ZAS}(z)$, we can see that the system is type 1

$$G_{ZAS}(z) = \frac{7.4923 \times 10^{-4}(z + 0.9355)}{(z-1)(z-0.8187)}$$

- Thus, the steady-state error can be calculated by K_v, i.e., the velocity error. (check Lecture 7)

$$\begin{aligned} K_v &= \frac{1}{T}\frac{z-1}{z}KG(z)\bigg|_{z=1} \\ &= K\frac{7.4923 \times 10^{-4}(1 + 0.9355)}{(0.04)(1-0.8187)} \\ &= \frac{K}{5} \end{aligned}$$

- Thus, in order to make the steady-state error be 10%, we have:

$$\begin{aligned} \frac{K}{5} = K_v &= \frac{100}{e(\infty)\%} \\ &= \frac{100}{10} = 10 \end{aligned} \Rightarrow K = 50.$$

Practice Question Solution

- Using MATLAB, we can get the system response with $K = 50$