Lecture 6: Systems with Transport Lag & Block Diagram Reduction

ELEN 472: Introduction to Digital Control

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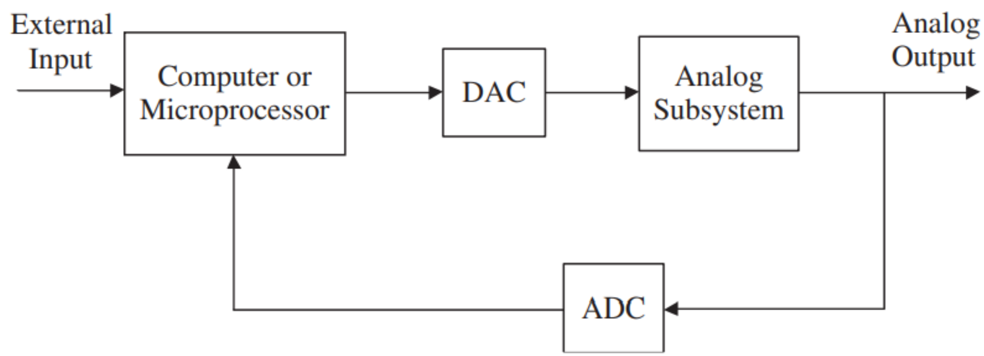
Review

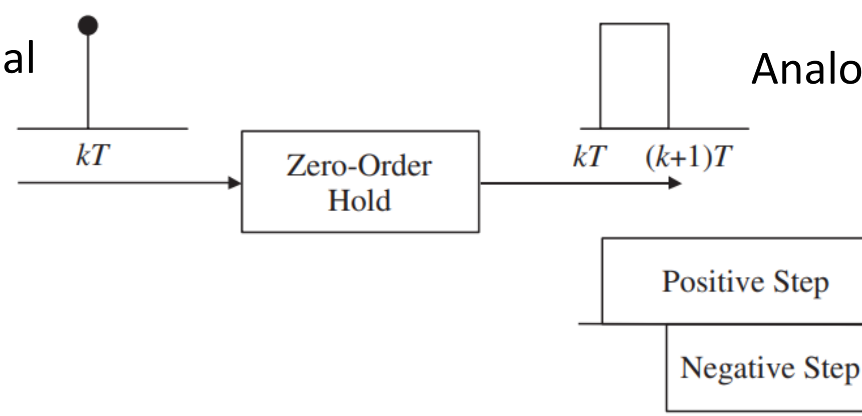
- **General Block Diagram for Digital Control Systems:**



- **DAC**: **D**igital-to-**A**nalog **C**onverter
  - **Z**ero-**O**rder **H**old (ZOH)

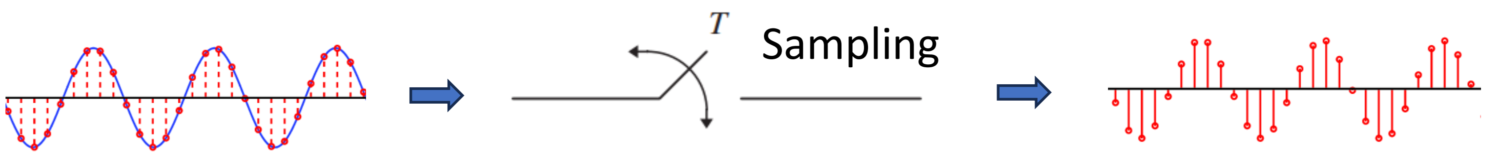
Digital Signal

Analog Signal

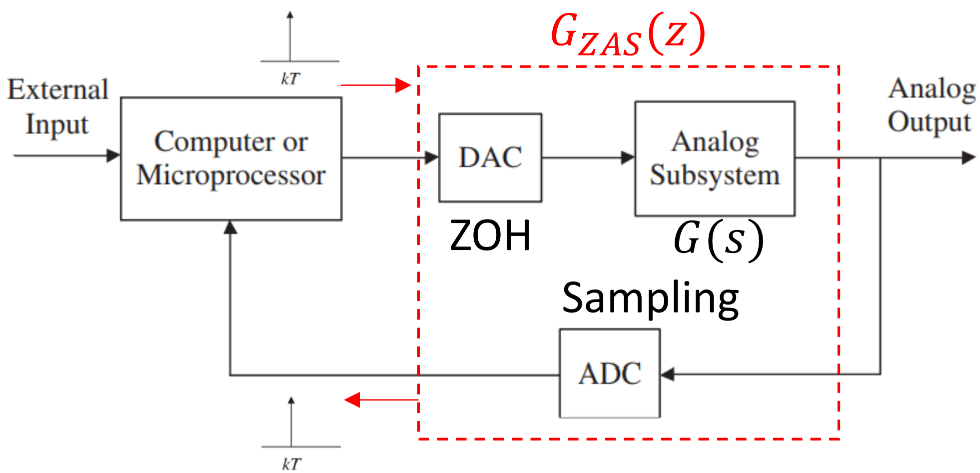


Review

- **ADC: A**nalog-to-**D**igital **C**onverter



- **Transfer Function of DAC, Analog Subsystem, and ADC:**



$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$
$$= \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

Introduction

- Many physical system models include a **time lag** or **delay** in their transfer functions.
  - These include chemical processes, automotive engines, sensors, digital systems, and so on.
  - For instance, when you are driving, you press the brake pad, but it takes a while for the vehicle to react to this input signal.
- The **Transport Lag** is the **delay** between **the time an input signal is applied to a system** and **the time the system reacts to that input signal**.

Modeling Transport Lag

- In continuous-time systems, the transfer function for systems with a transport delay is of the form

$$G(s) = G_a(s)e^{-T_d s}$$

  - Where T_d is the transport delay and can be rewritten as:

$$T_d = lT - mT, \quad 0 \leq m < 1$$

  - T is the **sampling period.**
  - l is a **positive integer.**
  - m is a **fractional number.**
  - $G_a(s)$ is the **analog system**.
- **For example**, a time delay of 3.1 s with a sampling period T of 1 s corresponds to
  - $T_d = 3.1, T = 1,$
  - Thus, $l = 4$ and $m = 0.9$
- **Practice Question:**
  - $T_d = 0.32, T = 0.1$, what are l and m?
  - $l = 4$ and $m = 0.8$

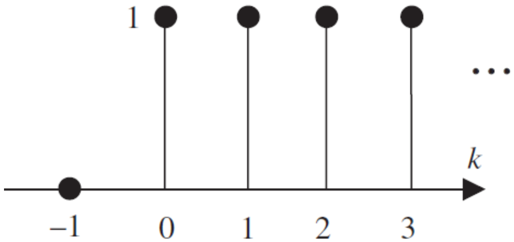
The Modified z-Transform

- The **modified z-transform** (also called **advanced z-transform**) is an extension of z-transform.
  - **Modified z-transform** was designed to incorporate **time delays** that are not multiples of the sampling time.
- **Modified z-transform Formula:**

$$\begin{aligned} Y(z,m) &= \boxed{\mathcal{Z}_m}\{y(kT)\} && \text{Modified z-transform} \\ &= z^{-1}\boxed{\mathcal{Z}}\{y(kT+mT)\} && \text{Normal z-transform} \end{aligned}$$

- **Example**: Step Function
  - The step function has fixed amplitude for all time instances.
  - Thus, shifting or delaying it does not change the sampled values.
  - The modified z-transform of step function is z^{-1} times the original z-transform result of step function:

$$\begin{aligned} Y(z,m) &= z^{-1} \cdot \frac{z}{z-1} \\ &= \frac{1}{z-1} \end{aligned}$$



Example 2: Exponential Function

- Find the modified z-transform of an exponential function $y(t) = e^{-pt}$.
- **Solution:**
  - Based on the modified z-transform formula:

$$\begin{aligned} Y(z,m) &= \mathcal{Z}_m\{y(kT)\} \\ &= z^{-1}\mathcal{Z}\{y(kT+mT)\} \end{aligned}$$

  - We have:

$$\begin{aligned} y(kT+mT) &= e^{-p(kT+mT)} \\ &= e^{-pkT}e^{-pmT} \qquad k = 0, 1, 2 \dots \end{aligned}$$

  - Then, find $\mathcal{Z}\{y(kT+mT)\}$

$$\begin{aligned} \mathcal{Z}\{y(kT+mT)\} &= \mathcal{Z}\left\{e^{-pkT}e^{-pmT}\right\} \\ &= e^{-pmT}\mathcal{Z}\left\{e^{-pkT}\right\} \\ &= e^{-pmT}\frac{z}{z-e^{-pT}} \end{aligned}$$

p, m and T are constants, can be factor outside of the z-transform

Pair 7 at z-transform table (Lecture 2 – Page 17)

  - Final Step: $z^{-1}\mathcal{Z}\{y(kT+mT)\}$

$$Y(z,m) = \frac{e^{-pmT}}{z-e^{-pT}}$$

Example 2: Exponential Function (Continued)

$$Y(z,m) = \frac{e^{-pmT}}{z - e^{-pT}}$$

- **For instance**, if $p = 4$ and $T = 0.2$ s, delay time $T_d = 0.14$ s

$$T_d = (l - m) \cdot T$$
$$0.14 = (l - m) \cdot 0.2$$
$$0.7 = l - m$$

- Thus,

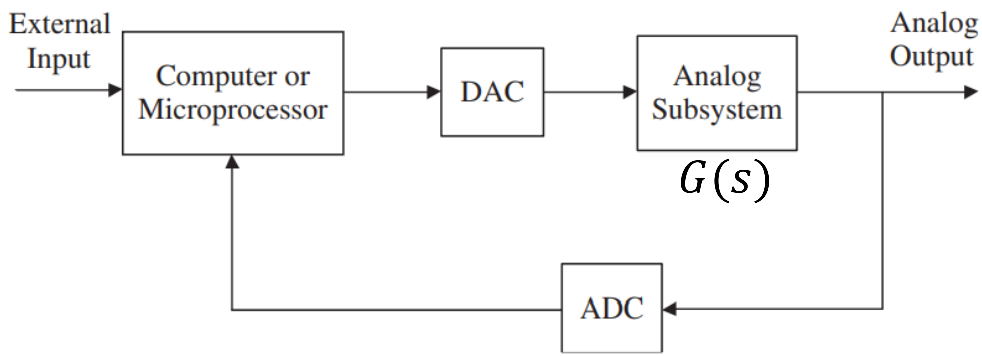
$$l = 1, m = 0.3$$

- Thus,

$$Y(z, 0.3) = \frac{e^{-4 \cdot 0.3 \cdot 0.2}}{z - e^{-4 \cdot 0.2}}$$
$$= \frac{0.787}{z - 0.449}$$

Systems with Transport Lag

- For a digital control system, the block diagram is



- The transfer function of DAC, Analog System, and ADC is

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}$$

  - Where $G(s)$ is the transfer function of Analog System.

Systems with Transport Lag (Continued)

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}$$

- Now, replacing $G(s)$ with $G_a(s)e^{-T(l-m)s}$ (system with delay), we have:

* Indicates sampling

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{G_a(s)e^{-T(l-m)s}}{s}\right]^*\right\}$$

$$= z^{-l}(1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{G_a(s)e^{mTs}}{s}\right]^*\right\}$$

Time delay theorem in Laplace Transform

- Further define $G_s(s) = \frac{G_a(s)}{s}$, we can rewrite the above equation as:

$$G_{ZAS}(z) = z^{-l}(1 - z^{-1})\mathcal{Z}\{\mathcal{L}^{-1}[G_s(s)e^{mTs}]^*\}$$

Time advance theorem in Laplace Transform

$$G_{ZAS}(z) = z^{-l}(1 - z^{-1})\mathcal{Z}\{g_s^*(t + mT)\}$$

$$G_{ZAS}(z) = z^{-l}(1 - z^{-1})\mathscr{Z}\{g_s^*(t + mT)\}$$

- We can replace t as kT to discretize the g_s

$$G_{ZAS}(z) = z^{-l}(1 - z^{-1})\mathscr{Z}\{g_s(kT + mT)\}$$

- Finally, we express the z-transfer function in terms of **the modified z-transform**:
  - We can rewrite $G_{ZAS}(z)$ as:

$$G_{ZAS}(z) = z^{-(l-1)}(1 - z^{-1})\underbrace{z^{-1}\mathscr{Z}\{g_s(kT + mT)\}}_{\text{Modified z-transform}}$$

$$\mathscr{Z}_m\{y(kT)\} = z^{-1}\mathscr{Z}\{y(kT + mT)\}$$

  - Thus,

$$\boxed{G_{ZAS}(z) = \left(\frac{z-1}{z^l}\right)\mathscr{Z}_m\{g_s(kT)\}}$$

The Transform Function for systems with Transport Lag.

Example

- If the sampling period is 0.1 s, determine the z-transfer function $G_{ZAS}(z)$ for the system:

$$G(s) = \frac{3e^{-0.31s}}{s+3}$$

- **Solution:**
  - First, write the delay in terms of the sampling period as $0.31 = 3.1 \times 0.1 = (4 - 0.9) \times 0.1$. Thus, $l = 4$ and $m = 0.9$.
  - Next, obtain the partial fraction expansion

$$G_s(s) = \frac{3}{s(s+3)} = \frac{1}{s} - \frac{1}{s+3}$$

  - Using $G_{ZAS}(z) = \left(\frac{z-1}{z^l}\right)\mathscr{Z}_m\{g_s(kT)\}$ and

$$\mathscr{Z}_m\{1(kT)\} = \frac{1}{z-1}$$

$$\mathscr{Z}_m\{e^{-pkT}\} = \frac{e^{-mpT}}{z-e^{-pT}}$$

  - We have:

$$G_{ZAS}(z) = \left(\frac{z-1}{z^4}\right)\left\{\frac{1}{z-1} - \frac{e^{-0.3\times 0.9}}{z-e^{-0.3}}\right\}$$

$$= z^{-4}\left\{\frac{z-0.741-0.763(z-1)}{z-0.741}\right\} = \frac{0.237z+0.022}{z^4(z-0.741)}$$

Practice Question

- For a system $G(s) = \frac{se^{-1.4s}}{s+1}$ with sampling period $T = 1s$, find the z-transform function $G_{ZAS}(z)$.
- **Solution:**
  - First, $T_d = (l - m) \times T = (2 - 0.6) \times 1$
    - Thus, $l = 2$ and $m = 0.6$
  - Second, $G_s(s) = \frac{G_a(s)}{s} = \frac{s}{s+1} \times \frac{1}{s} = \frac{1}{s+1}$
  - Third,

$$G_{ZAS}(z) = \left(\frac{z-1}{z^l}\right) \mathscr{Z}_m\{g_s(kT)\}$$

    - We have $G_{ZAS}(z) = \left(\frac{z-1}{z^2}\right) \mathcal{Z}_{0.6}\{g_s(kT)\}$
    - $g_s(kT) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-kT}$
      - Thus, $\mathcal{Z}_{0.6}\{g_s(kT)\} = \frac{e^{-0.6}}{z-e^{-1}}$

$$\boxed{\mathscr{Z}_m\{e^{-pkT}\} = \frac{e^{-mpT}}{z - e^{-pT}}}$$

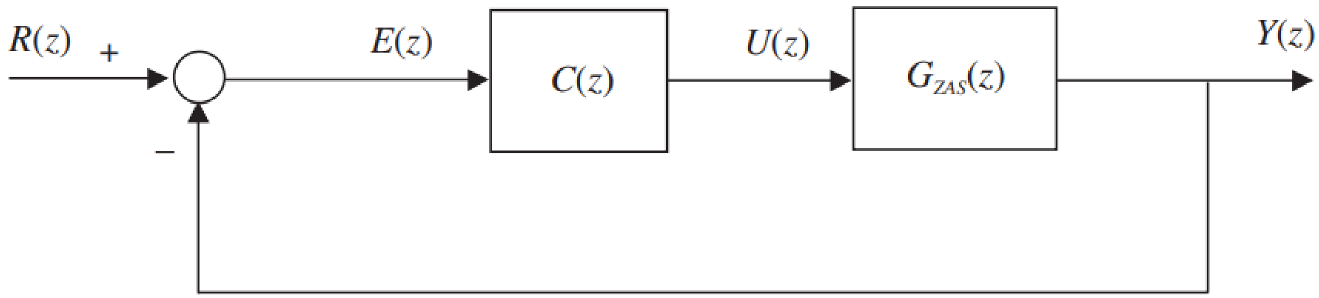
  - Thus,

$$G_{ZAS}(z) = \left(\frac{z-1}{z^2}\right) \times \frac{e^{-0.6}}{z - e^{-1}} = \frac{0.55(z-1)}{z^2(z-0.37)}$$

Block Diagram Reduction in Closed-loop Digital Control Systems

The Closed-loop Transfer Function

- The block diagram is identical to those commonly encountered in s-domain analysis of analog systems, with the variable *s* replaced by z.



- Hence, the closed-loop transfer function for the system is given by

$$G_{cl}(z) = \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)}$$

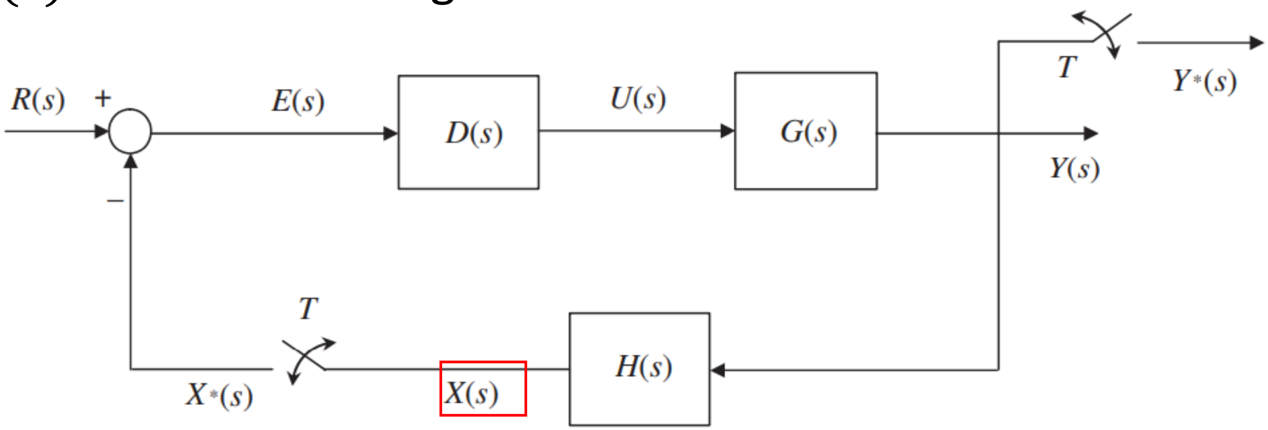
- The closed-loop characteristic equation is

$$1 + C(z)G_{ZAS}(z) = 0$$

- The roots of the closed-loop characteristic equation are poles of the closed-loop system.

Example

- Find the Laplace transform of the analog $Y(s)$ and sampled output $Y^*(s)$ for the block diagram



- **Solution:**
  - The analog variable $x(t)$ has the Laplace transform

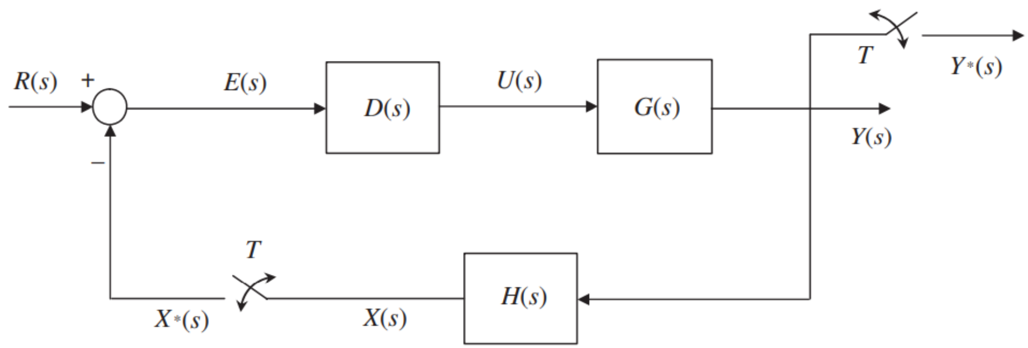
$$X(s) = H(s)G(s)D(s)E(s)$$

  - From the block diagram

$$E(s) = R(s) - X^*(s)$$

Plug back in

$$X(s) = H(s)G(s)D(s)[R(s) - X^*(s)]$$



$$X(s) = H(s)G(s)D(s)[R(s) - X^*(s)]$$

Sampling

$$X^*(s) = (HGDR)^*(s) - (HGD)^*(s)X^*(s)$$

- Next, we solve for $X^*(s)$

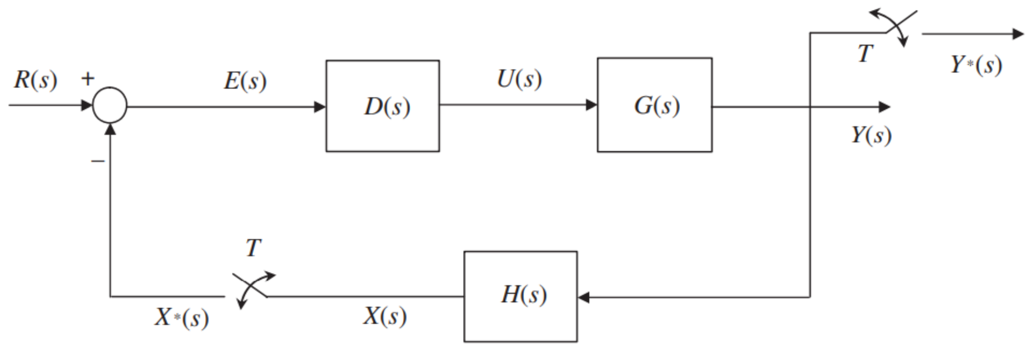
$$X^*(s) = \frac{(HGDR)^*(s)}{1 + (HGD)^*(s)}$$

- Recall $E(s) = R(s) - X^*(s)$

$$E(s) = R(s) - \frac{(HGDR)^*(s)}{1 + (HGD)^*(s)}$$

- From the block diagram, the Laplace transform of the output is $Y(s) = G(s)D(s)E(s)$. Substituting for E(s) gives

$$Y(s) = G(s)D(s)\left[R(s) - \frac{(HGDR)^*(s)}{1 + (HGD)^*(s)}\right]$$



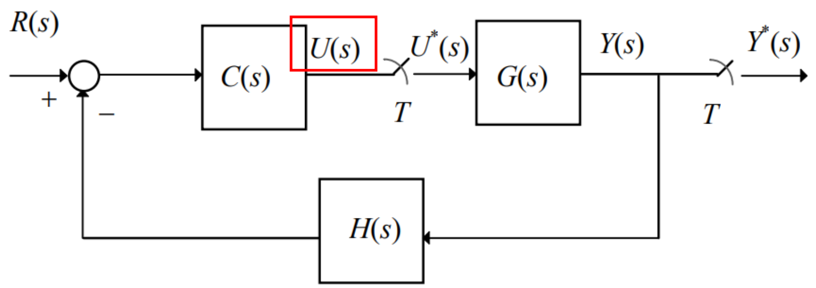
$$Y(s) = G(s)D(s)\left[R(s) - \frac{(HGDR)^*(s)}{1 + (HGD)^*(s)}\right]$$

- Thus, the sampled output is

$$Y^*(s) = (GDR)^*(s) - (GD)^*(s)\frac{(HGDR)^*(s)}{1 + (HGD)^*(s)}$$

Practice

- Obtain expressions for the analog and sampled outputs from the following block diagram:



- **Solution:**

From the block diagram $U(s) = C(s)R(s) - C(s)H(s)G(s)U^*(s)$

Then sampling gives $U^*(s) = (CR)^*(s) - (CHG)^*(s)U^*(s)$

Solving for $U^*(s)$, we obtain $U^*(s) = \dfrac{(CR)^*(s)}{1+(CHG)^*(s)}$

The analog output is $Y(s) = G(s)U^*(s) = \dfrac{G(s)(CR)^*(s)}{1+(CHG)^*(s)}$

The sampled output is $Y(s) = \dfrac{G^*(s)(CR)^*(s)}{1+(CHG)^*(s)}$