Lecture 3: z-Transform Inversion and Final Value Theorem

ELEN 472: Introduction to Digital Control

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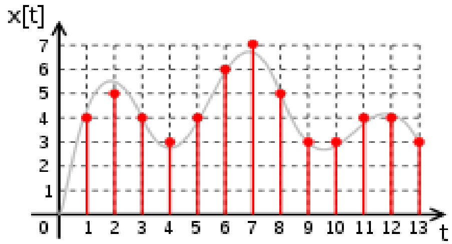
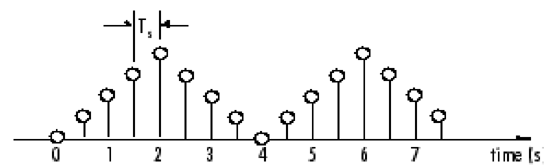
Review

- Discrete-Time Signals:
    - A Sequence of Values that correspond to particular instants in time.



Sampling



Sample Period

- Discrete-Time Systems:
    - Systems operated with Discrete-Time Signals
    - System equation is **Difference Equation**.
- Z-Transform and its properties:

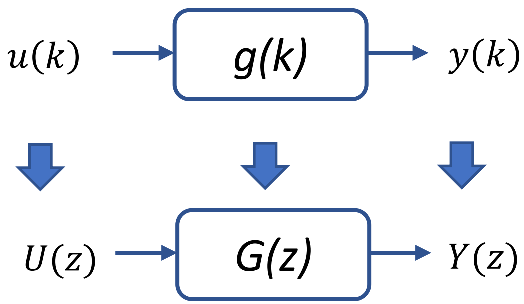
DEFINITION 2.1

Given the causal sequence $\{u_0, u_1, u_2, \ldots, u_k, \ldots\}$, its z-transform is defined as

$$U(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_k z^{-k}$$
$$= \sum_{k=0}^{\infty} u_k z^{-k}$$

Motivation

- Why **z-Transform**?
  - It simplifies the solution of discrete-time problems by converting **Convolution** into **Multiplication**.



$$y(k) = u(k) * g(k)$$
$$= \sum u(n)g(k-n)$$

z-Transform

$$Y(z) = U(z) \cdot G(z)$$

- Why **inverse z-Transform**?
  - After we have the output signal in z-domain, we can use the inverse z-Transform to get the time-domain signals.

$Y(z)$ ➡ *Inverse z-Transform* ➡ $y(k)$

Inversion of the Z-transform

- **Method 1: Long Division**
  - We first use **long division** to obtain as many terms as desired of the z-transform expansion
  - Then we use the coefficients of the expansion to write the time sequence.
- There are **two steps** to get the inverse z-transform of a function $F(z)$.
  - **Step 1**: Using **long division**, expand $F(z)$ as a series to obtain:

$$F_t(z) = f_0 + f_1 z^{-1} + \cdots + f_i z^{-i} = \sum_{k=0}^{i} f_k z^{-k}$$

  - **Step 2**: Write the inverse transform as the sequence:

$$\{f_0, f_1, \ldots, f_i, \ldots\}$$

Example of Inverse Z-transform

- Obtain the inverse z-transform of the function:

$$F(z) = \frac{z+1}{z^2 + 0.2z + 0.1}$$

- **Solution:**
  - Using long division

$$\begin{array}{r l}
 & z^{-1} + 0.8z^{-2} - 0.26z^{-3} + \ldots\ldots \\
\hline
z^2 + 0.2z + 0.1 \,\big) & z + 1 \\
 & z + 0.2 + 0.1z^{-1} \\
\hline
 & 0.8 - 0.10z^{-1} \\
 & 0.8 + 0.16z^{-1} + 0.08z^{-2} \\
\hline
 & -0.26z^{-1} - \ldots
\end{array}$$

    - Thus, $F_t(z) = 0 + z^{-1} + 0.8z^{-2} - 0.26z^{-3} + \cdots$
  - Inverse transformation
    - $\{f_k\} = \{0,1,0.8, -0.26 \ldots\}$

Practice Questions

- Find the inverse z-transform of the following functions:
  - $F(z) = 1 + 3z^{-1} + 4z^{-2}$
  - $F(z) = \frac{z}{z^2+0.3z+0.02}$
- **Solution**:
  - $\{f(k)\} = \{1,3,4,0,0,0 \dots\}$
  - Using Long Division:

$$
\begin{array}{r l}
 & z^{-1} - 0.3z^{-2} + 0.07z^{-3} + \dots\dots \\
z^2 + 0.3z + 0.02 \,\big) & z \\
 & \underline{z + 0.3 + 0.02z^{-1}} \\
 & -0.3 - 0.02z^{-1} \\
 & \underline{-0.3 - 0.09z^{-1} - 0.006z^{-2}} \\
 & 0.07z^{-1} + 0.006z^{-2}
\end{array}
$$

  - Write the inverse z-transform:

$$F(z) = \frac{z}{z^2 + 0.3z + 0.02} = z^{-1} - 0.3z^{-2} + 0.07z^{-3} + \dots\dots \qquad \{f(k)\} = \{0,1,-0.3,0.07,\dots.\}$$

Partial Fraction Expansion

- **Method 2: Partial Fraction Expansion**
  - This method is almost identical to that used in inverting Laplace transforms.
  - However, since most z-transform have the term z in their **numerator,** it is often convenient to expand $F(z)/z$ rather than $F(z)$.
- The procedure for inverse z-transformation is
  - **Step 1**: Find the partial fraction expansion of $F(z)/z$ or $F(z)$
  - **Step 2**: Obtain the inverse transform $f(k)$ using the z-transform tables

No.	Continuous Time	Laplace Transform	Discrete Time	*z*-Transform
1	$\delta(t)$	1	$\delta(k)$	1
2	$1(t)$	$\frac{1}{s}$	$1(k)$	$\frac{z}{z-1}$
3	t	$\frac{1}{s^2}$	kT^{**}	$\frac{zT}{(z-1)^2}$
4	t^2	$\frac{2!}{s^3}$	$(kT)^2$	$\frac{z(z+1)T^2}{(z-1)^3}$
5	t^3	$\frac{3!}{s^4}$	$(kT)^3$	$\frac{z(z^2+4z+1)T^3}{(z-1)^4}$
6	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	$a^{k\;***}$	$\frac{z}{z-a}$

Partial Fraction Expansion

- The most convenient method to obtain the partial fraction expansion of a function with simple real roots is the **method of residues**.
- **Step – 1**: For a z-Transform expression $F(z)$, get $\frac{F(z)}{z}$
- **Step – 2**: Express $F(z)/z$ into the a sum of individual terms

$$\frac{F(z)}{z} = \sum_{i=0}^{n} \frac{A_i}{z - z_i}$$

  - Where A_i is the partial fraction coefficient of the i-th term of the expansion:

$$A_i = (z - z_i)\frac{F(z)}{z}\Big|_{z \to z_i}$$

- **Step – 3**: Restore $F(z)$ via $\frac{F(z)}{z} \times Z$
- **Step – 4**: Get inverse z-Transform of individual terms using z-Transform table (Lecture 2)

Example

- Obtain the inverse z-transform of the function

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

- Solution:
  - **Step-1**: Dividing the function by z, we expand as

$$\frac{F(z)}{z} = \frac{z+1}{z(z^2 + 0.3z + 0.02)}$$
$$= \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2}$$

Factorization:
$$z^2 + 0.3z + 0.02$$
$$= (z+0.1)(z+0.2)$$

  - **Step-2**: The partial fraction coefficients are given by

$$A = z\frac{F(z)}{z}\bigg]_{z=0} = F(0) = \frac{1}{0.02} = 50$$

$$A_i = (z - z_i)F(z)\Big|_{z \to z_i}$$

$$B = (z+0.1)\frac{F(z)}{z}\bigg]_{z=-0.1} = \frac{1-0.1}{(-0.1)(0.1)} = -90$$

$$C = (z+0.2)\frac{F(z)}{z}\bigg]_{z=-0.2} = \frac{1-0.2}{(-0.2)(-0.1)} = 40$$

Example Solution

- **Step-3**: Thus, the partial fraction expansion is

$$F(z) = \frac{50z}{z} - \frac{90z}{z+0.1} + \frac{40z}{z+0.2}$$

- **Step-4:** Table Lookup:

$$f(k) = \begin{cases} 50\delta(k) - 90(-0.1)^k + 40(-0.2)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

- Note that $f(0) = 0$, so the time sequence can be rewritten as:

$$f(k) = \begin{cases} -90(-0.1)^k + 40(-0.2)^k, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

Practice Question

- Find the inverse z-transform of the function

$$F(z) = \frac{z}{(z+0.1)(z+0.2)(z+0.3)}$$

- Solution:
  - **Step-1**: Dividing by z simplifies the numerator and gives the expansion:

$$\frac{F(z)}{z} = \frac{1}{(z+0.1)(z+0.2)(z+0.3)}$$

$$= \frac{A}{z+0.1} + \frac{B}{z+0.2} + \frac{C}{z+0.3}$$

  - **Step-2**: The partial fraction coefficients are

$$A = (z+0.1)\frac{F(z)}{z}\bigg]_{z=-0.1} = \frac{1}{(0.1)(0.2)} = 50$$

$$B = (z+0.2)\frac{F(z)}{z}\bigg]_{z=-0.2} = \frac{1}{(-0.1)(0.1)} = -100$$

$$C = (z+0.3)\frac{F(z)}{z}\bigg]_{z=-0.3} = \frac{1}{(-0.2)(-0.1)} = 50$$

Practice Question Solution

- Step-3: Thus, the partial fraction expansion after multiplying by z is

$$F(z) = \frac{50z}{z+0.1} - \frac{100z}{z+0.2} + \frac{50z}{z+0.3}$$

- Step-4: Table Lookup

$$f(k) = \begin{cases} 50(-0.1)^k - 100(-0.2)^k + 50(-0.3)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

Repeated Roots

- For a function $F(z)$ with a **repeated root** of multiplicity r, r partial fraction coefficients are associated with the repeated root. The partial fraction expansion is of the form:

$$F(z) = \frac{N(z)}{(z-z_1)^r \prod_{j=r+1}^{n} z - z_j} = \sum_{i=1}^{r} \frac{A_{1i}}{(z-z_1)^{r+1-i}} + \sum_{j=r+1}^{n} \frac{A_j}{z - z_j}$$

  - The coefficients for repeated roots are:

$$A_{1,i} = \frac{1}{(i-1)!} \frac{d^{i-1}}{dz^{i-1}} (z-z_1)^r F(z) \Big]_{z \to z_1}, \qquad i = 1, 2, \ldots, r$$

Example

- Obtain the inverse z-transform of the function:

$$G(z) = \frac{1}{z^2(z-0.5)}$$

- **Solution**:
  - Dividing $G(z)$ by z gives

$$F(z) = \frac{G(z)}{z} = \frac{1}{z^3(z-0.5)}$$

  - Compare with the general form:

$$F(z) = \frac{N(z)}{(z-z_1)^r \prod_{j=r+1}^{n} z-z_j} = \sum_{i=1}^{r} \frac{A_{1i}}{(z-z_1)^{r+1-i}} + \sum_{j=r+1}^{n} \frac{A_j}{z-z_j}$$

    - We can see that $r = 3$, $z_1 = 0$, $N(z) = 1$, $i = 1,2,3$. Thus:

$$F(z) = \frac{A_{11}}{z^3} + \frac{A_{12}}{z^2} + \frac{A_{13}}{z^1} + \frac{A_4}{z-0.5}$$

  - where:

$$A_{1,i} = \frac{1}{(i-1)!}\frac{d^{i-1}}{dz^{i-1}}(z-z_1)^r F(z)\Big]_{z\to z_1}, \qquad i = 1,\ 2,\ \ldots,\ r$$

$$\xrightarrow{i=1} A_{11} = z^3 \frac{F(z)}{z}\bigg|_{z=0} = \frac{1}{z-0.5}\bigg|_{z=0} = -2$$

Example Solution

$$A_{1,i} = \frac{1}{(i-1)!}\frac{d^{i-1}}{dz^{i-1}}(z-z_1)^r F(z)\Big]_{z\to z_1}, \qquad i = 1,\ 2, \ldots,\ r \longrightarrow i = 2$$

$$A_{12} = \frac{1}{1!}\frac{d}{dz}z^3\frac{F(z)}{z}\Big|_{z=0} = \frac{d}{dz}\,\frac{1}{z-0.5}\Big|_{z=0} = \frac{-1}{(z-0.5)^2}\Big|_{z=0} = -4$$

$$A_{1,i} = \frac{1}{(i-1)!}\frac{d^{i-1}}{dz^{i-1}}(z-z_1)^r F(z)\Big]_{z\to z_1}, \qquad i = 1,\ 2, \ldots,\ r \longrightarrow i = 3$$

$$A_{13} = \frac{1}{2!}\frac{d^2}{dz^2}z^3\frac{F(z)}{z}\Big|_{z=0}$$

$$= \left(\frac{1}{2}\right)\frac{d}{dz}\,\frac{-1}{(z-0.5)^2}\Big|_{z=0} = \left(\frac{1}{2}\right)\frac{(-1)(-2)}{(z-0.5)^3}\Big|_{z=0} = -8$$

$$A_4 = (z-0.5)\frac{F(z)}{z}\Big|_{z=0.5} = \frac{1}{z^3}\Big|_{z=0.5} = 8$$

$$F(z) = \frac{A_{11}}{z^3} + \frac{A_{12}}{z^2} + \frac{A_{13}}{z^1} + \frac{A_4}{z - 0.5}$$
$$= \frac{-2}{z^3} + \frac{-4}{z^2} + \frac{-8}{z^1} + \frac{8}{z - 0.5}$$

- Since,

$$F(z) = \frac{G(z)}{z} \Rightarrow G(z) = zF(z) = \frac{-2}{z^2} + \frac{-4}{z} - 8 + \frac{8z}{z - 0.5}$$
$$= -2z^{-2} - 4z^{-1} - 8 + \frac{8z}{z - 0.5}$$

- Thus,

$$g(k) = -2\delta(k-2) - 4\delta(k-1) - 8\delta(k) + 8 \cdot (0.5)^k$$

Final Value Theorem

- The **Final Value Theorem (FVT)** allows us to calculate the limit of a sequence as time k tends to infinity.
  - In other words, we can use FVT to find $f(k \to \infty)$
- **Proof of FVT**:
  - From z-transform definition equation, we have:
$$Z[f(k)] = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$
  - Based on the z-transform's time advance property, we have:
$$Z[f(k+1)] = zF(z) - zf(0)$$
  - Also, based on z-transform definition equation, we have
$$Z[f(k+1)] = \sum_{k=0}^{\infty} f(k+1) z^{-k}$$
  - Thus,
$$\begin{aligned} Z[f(k+1)] - Z[f(k)] &= zF(z) - zf(0) - F(z) \\ &= \sum_{k=0}^{\infty} [f(k+1) - f(k)] z^{-k} \end{aligned}$$

Final Value Theorem

$$\begin{aligned} Z[f(k+1)] - Z[f(k)] &= zF(z) - zf(0) - F(z) \\ &= (z-1)F(z) - zf(0) \\ &= \sum_{k=0}^{\infty} [f(k+1) - f(k)]z^{-k} \end{aligned}$$

- Expand the right-hand-side of the equation:

$$(z-1)F(z) - zf(0) =$$
$$[f(1) - f(0)]z^0 + [f(2) - f(1)]z^{-1} + [f(3) - f(2)]z^{-2} + \cdots$$

- Taking limit $z \to 1$ on both sides of the equation, we get:

$$\lim_{z\to 1}[(z-1)F(z) - zf(0)] =$$
$$\lim_{z\to 1}\{[f(1) - f(0)]z^0 + [f(2) - f(1)]z^{-1} + [f(3) - f(2)]z^{-2} + \cdots\}$$

$$\Downarrow$$

$$\lim_{z\to 1}[(z-1)F(z)] - f(0)$$
$$= f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + \cdots + f(\infty) - f(\infty - 1)$$

Final Value Theorem

$$\lim_{z\to 1}[(z-1)F(z)] - f(0)$$
$$= \cancel{f(1)} - f(0) + \cancel{f(2)} - \cancel{f(1)} + \cancel{f(3)} - \cancel{f(2)} + \cdots + f(\infty) - \cancel{f(\infty - 1)}$$
$$= f(\infty) - f(0)$$

$$\lim_{z\to 1}[(z-1)F(z)] - \cancel{f(0)} = f(\infty) - \cancel{f(0)}$$

$$\boxed{f(\infty) = \lim_{z\to 1}[(z-1)F(z)]}$$

Final Value Theorem

Example Question

- Find the final value for the function:

$$F(z) = \frac{z}{z^2 - 1.2z + 0.2}$$

- **Solution**:

$$\begin{aligned} f(\infty) &= \lim_{z\to 1}[(z-1)F(z)] \\ &= \lim_{z\to 1}[(z-1)\cdot\frac{z}{z^2 - 1.2z + 0.2}] \\ &= \lim_{z\to 1}\left[\frac{(z-1)z}{(z-0.2)(z-1)}\right] \\ &= \lim_{z\to 1}\frac{z}{z-0.2} \\ &= \frac{1}{0.8} = 1.25 \end{aligned}$$

Practice Question

- Find the final value for the function:

$$F(z) = \frac{z^2}{z^2 - 1.3z + 0.3}$$

- **Solution**:

$$f(\infty) = \lim_{z \to 1}[(z-1)F(z)]$$
$$= \lim_{z \to 1}\left[(z-1)\frac{z}{(z-1)(z-0.3)}\right]$$
$$= \lim_{z \to 1}\left[\frac{z}{z-0.3}\right]$$
$$= \frac{1}{0.7} = 1.43$$

Z-Transform Solution of Difference Equations

- Recall **linear difference equation expression**:

$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k)$$
$$= b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k)$$

  - Our goal is to solve this equation to **get the expression of $\boldsymbol{y(k)}$**.
  - $u(k)$ will be provided.

- The key to solve this equation is to use **z-Transform** and its **time delay/advance properties.**

Time Delay Property: $\mathscr{Z}\{f(k-n)\} = z^{-n}F(z)$

Time Advance Property: $\mathscr{Z}\{f(k+1)\} = zF(z) - zf(0)$

$$\mathscr{Z}\{f(k+n)\} = z^n F(z) - z^n f(0) - z^{n-1} f(1) - \cdots - zf(n-1)$$

Example

- Solve the linear difference equation

$$x(k+2) - (3/2)x(k+1) + (1/2)x(k) = 1(k)$$

for $x(k)$, with the initial conditions $x(0) = 1, x(1) = 5/2$.

- Solution:
  - z-transform: We begin by z-transforming the difference equation using time advance property to obtain

$$[z^2X(z) - z^2x(0) - zx(1)] - (3/2)[zX(z) - zx(0)] + (1/2)X(z) = z/(z-1)$$

  - Then we substitute the initial conditions and rearrange terms to obtain

$$[z^2 - (3/2)z + (1/2)]X(z) = z/(z-1) + z^2 + (5/2 - 3/2)z$$

$$X(z) = \frac{z[1 + (z+1)(z-1)]}{(z-1)(z-1)(z-0.5)} = \frac{z^3}{(z-1)^2(z-0.5)}$$

  - Partial Fraction Expansion of $X(z)/z$:

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-0.5)} = \frac{A_{11}}{(z-1)^2} + \frac{A_{12}}{z-1} + \frac{A_3}{z-0.5}$$

Example Solution

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-0.5)} = \frac{A_{11}}{(z-1)^2} + \frac{A_{12}}{z-1} + \frac{A_3}{z-0.5}$$

- Where,

$$A_{11} = (z-1)^2 \frac{X(z)}{z}\bigg|_{z=1} = \frac{z^2}{z-0.5}\bigg|_{z=1} = \frac{1}{1-0.5} = 2$$

$$A_{12} = \frac{d}{dz}(z-1)^2 \frac{X(z)}{z}\bigg|_{z=1} = \frac{d}{dz}\frac{z^2}{(z-0.5)}\bigg|_{z=1} = \frac{z^2-z}{(z-0.5)^2}\bigg|_{z=1} = 0$$

$$A_3 = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z^2}{(z-1)^2}\bigg|_{z=0.5} = \frac{(0.5)^2}{(0.5-1)^2} = 1$$

- Thus,

$$X(z) = \frac{2z}{(z-1)^2} + \frac{z}{z-0.5} \quad \xrightarrow[\text{z-transform}]{\text{Inverse}} \quad x(k) = 2k + (0.5)^k$$

Practice Question

- Solve the following difference equations:

$$y(k+1) - 0.8\, y(k) = 0,\ y(0) = 1$$
$$y(k+1) - 0.8\, y(k) = 1(k),\ y(0) = 0$$

- Solution:

(a) $y(k+1) - 0.8\, y(k) = 0, \qquad \mathrm{y}(0) = 1$

z-transform

$$zY(z) - z - 0.8Y(z) = 0 \Rightarrow Y(z) = \frac{z}{z-0.8} \qquad f(k) = (0.8)^k, k = 0,1,2,\ldots$$

(b) $y(k+1) - 0.8\, y(k) = 1(k), \qquad \mathrm{y}(0) = 0$

z-transform

$$(z-0.8)Y(z) = \frac{z}{z-1} \Rightarrow Y(z) = \frac{z}{(z-0.8)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-0.8)(z-1)} = 5\left[\frac{1}{z-1} - \frac{1}{z-0.8}\right] \qquad f(k) = 5\left[1-(0.8)^k\right], k = 0,1,2,\ldots$$