# Lecture 14: Properties of State-Space Models & State Feedback Control

**ELEN 472: Introduction to Digital Control**

**Lingxiao Wang**, Ph.D.
Assistant Professor of Electrical Engineering
Louisiana Tech University

# A Quick Recap on Previous Lecture

- **Continuous-Time** State Space Models:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$$

- **The Solution** of **Continuous-Time** State Space Models:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \qquad e^{At} \rightarrow \mathscr{L}^{-1}\{[sI_n - A]^{-1}\}$$

- **Discrete-Time** State Space Models:

$$\mathrm{x}[k+1] = \mathrm{A}_d\mathrm{x}[k] + \mathrm{B}_d\mathbf{f}[k]$$
$$\mathrm{y}[k] = \mathrm{C}_d\mathrm{x}[k] + \mathrm{D}_d\mathbf{f}[k]$$

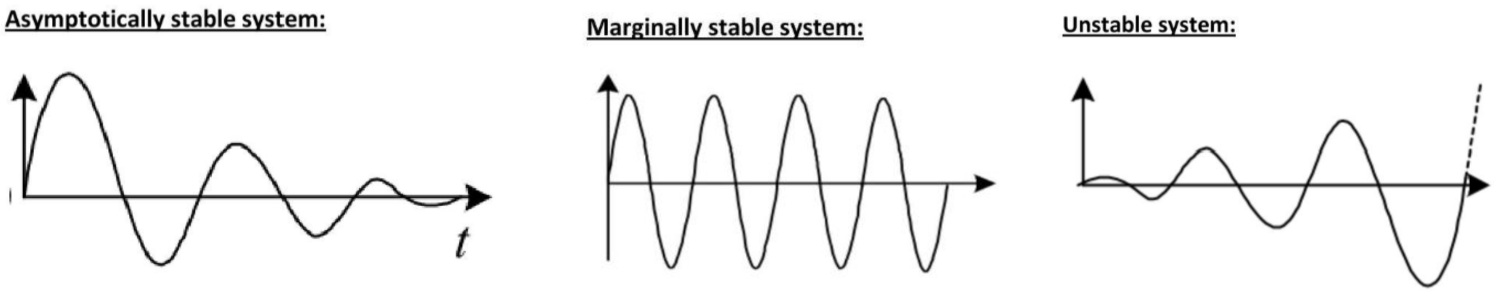
$$\mathrm{A_d} = \mathrm{e}^{AT} \qquad \mathrm{C_d} = \mathrm{C}$$
$$\mathrm{B_d} = (\mathrm{A_d} - \mathrm{I})\mathrm{A}^{-1}\mathrm{B} \qquad \mathrm{D_d} = \mathrm{D}$$

# Topics in this Lecture

- In this lecture, we examine some properties of discrete-time state space models, including
  - **Stability**:
    - Determine output behaviors;
  - **Controllability**:
    - Determine the effectiveness of state feedback control;
  - **Observability**:
    - Determine the possibility of state estimation from the output measurements.

# Stability of State-Space Realizations

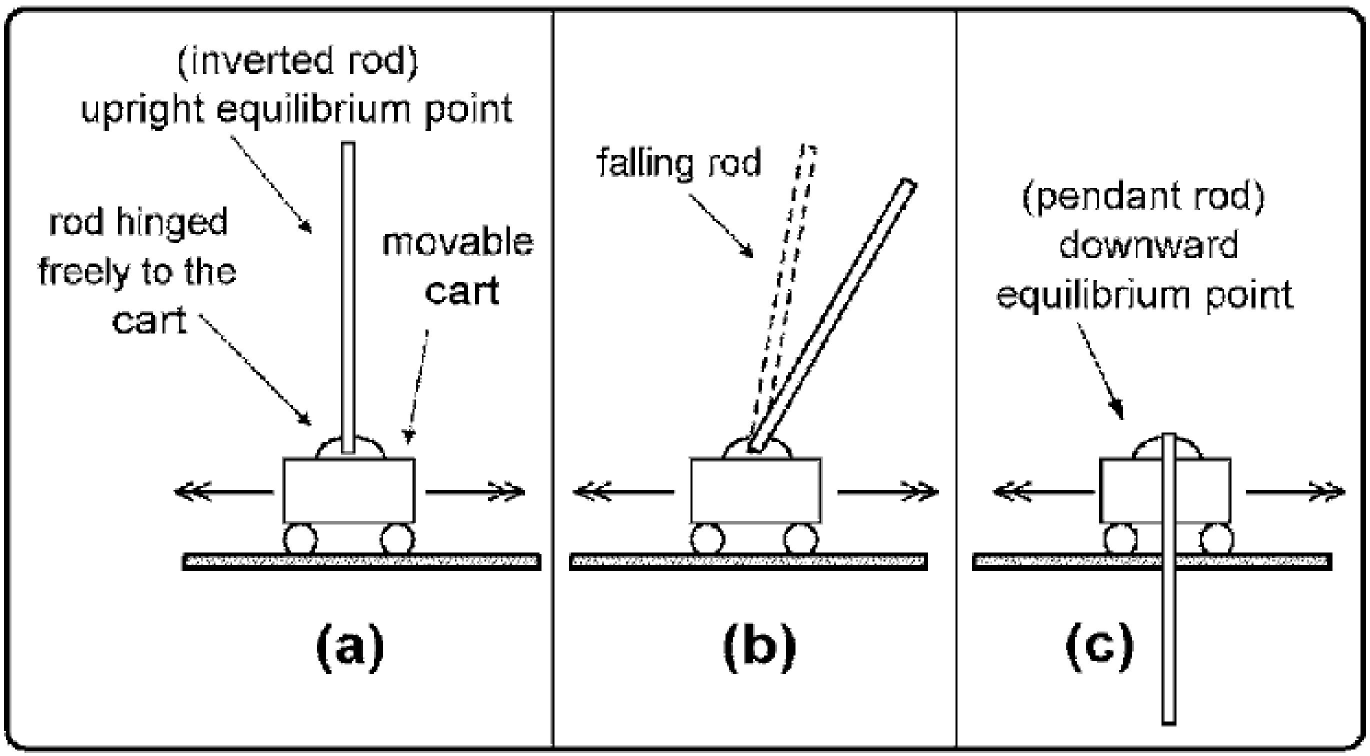
- The natural response of a linear system from its initial conditions may:
  - **Case I**: Converge to a constant state (e.g., 0) -> **Stable**
  - **Case II**: Remain in a bounded region in the vicinity of a constant state (e.g., 0) -> **Marginally Stable**
  - **Case III**: Grow unbounded -> **Unstable**




- Critical to the understanding of stability of both linear and nonlinear systems is the concept of an **equilibrium state**.
  - An **equilibrium point** is an initial state from which the system never departs unless perturbed.

# Equilibrium Point

- An **equilibrium point** is an initial state from which the system never departs unless perturbed.
  - How many equilibrium points for this system?
    - -> 2



# How to Find Equilibrium Points?

- For the state equation

$$\mathbf{x}(k+1) = \mathbf{f}[\mathbf{x}(k)]$$

  - Equilibrium states $\mathbf{x}_e$ satisfy the condition

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{f}[\mathbf{x}(k)] \\ &= \mathbf{f}[\mathbf{x}_\mathbf{e}] = \mathbf{x}_\mathbf{e}\end{aligned}$$

Nonlinear

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) \\ &= A\mathbf{x}_\mathbf{e} = \mathbf{x}_\mathbf{e} \Leftrightarrow [A - I_n]\mathbf{x}_\mathbf{e} = \mathbf{0}\end{aligned}$$

Linear

  - For the equilibrium states of linear systems, if $A - I_n$ is invertible, then $x_e = 0$.

# Example

- Find the equilibrium points of the following system

$$x(k+1) = x(k)[x(k) - 0.5]$$

- **Solution**:
  - At equilibrium, we have $x_e = x_e[x_e - 0.5]$
  - We can rearrange and get $x_e(x_e - 1.5) = 0$
  - Thus, the system has two equilibrium states:

$$x_e = 0 \text{ and } x_e = 1.5$$

# Example 2

- Find the equilibrium points of the following two systems:

$$x(k+1) = 2x(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 1 & 0.9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

- **Solution:**
  - For the first system $x(k+1) = 2x(k)$:
    - The equilibrium condition is $x_e = 2x_e$.
    - Thus, the system has one equilibrium point at $x_e = 0$
  - For the second system:
    - The equilibrium condition is

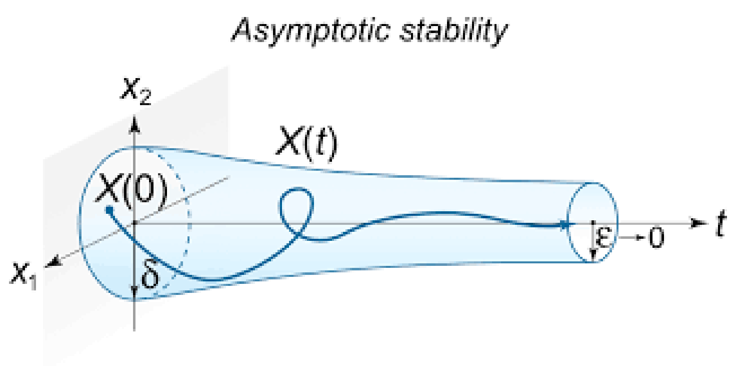
$$\begin{bmatrix} x_{1e}(k) \\ x_{2e}(k) \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 1 & 0.9 \end{bmatrix} \begin{bmatrix} x_{1e}(k) \\ x_{2e}(k) \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0.1-1 & 0 \\ 1 & 0.9-1 \end{bmatrix} \begin{bmatrix} x_{1e}(k) \\ x_{2e}(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

    - The system has a unique equilibrium state at

$$x_e = [x_{1e}, x_{2e}]^T = [0{,}0]^T$$

# Asymptotic Stability

- **Definition:**
  - A Linear System is said to be **Asymptotically Stable** if all its trajectories converges to the **origin (i.e., 0)**.
  - In other words, for any initial state $x(k_0)$, $x(k) \rightarrow 0$ as $k \rightarrow \infty$.




- How to Check Asymptotic Stability of A System?
  - A discrete-time linear system is asymptotic stable if and only if all the **eigenvalues** of its state matrix are **inside the unit circle**.
- Relation to BIBO stability
  - If a system is Asymptotically Stable -> The system is also BIBO stable.

# Example

- Determine the Asymptotic Stability of the following system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 1 & 0.9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

- **Solution:**
  - We need to calculate the eigenvalues of the state matrix $A$.

$$A = \begin{bmatrix} 0.1 & 0 \\ 1 & 0.9 \end{bmatrix}$$

  - The eigenvalues of $A$ are

$$det(\lambda I - A) = 0$$

  - Solve this equation we have two possible values for $\lambda$

$$\lambda_1 = 0.1;\ \lambda_2 = 0.9$$

  - Both of them are less than 1 -> System is Asymptotically Stable.

# Practice Question

- Determine the stability of the following systems:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 1 & 0.2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \mathbf{u}(k)$$

- Solution:
  - Two eigenvalues are at 0.1 and 0.2 and both are within the unit circle.
  - Thus, the system is Asymptotically Stable.

# Controllability

- Definition of Controllability:
  - A system is said to be controllable if for any initial state $x(k_0)$ there exists a control sequence $u(k)$, $k = k_0, k_0 + 1, \dots, k_f - 1$, such that an arbitrary final state $x(k_f)$ can be reached in finite $k_f$.
- Controllability Rank Condition:

**THEOREM 8.5: CONTROLLABILITY RANK CONDITION**

A linear time-invariant system is completely controllable if and only if the $n \times m.n$ controllability matrix

$$\mathscr{C} = \left[ B_d \mid A_d B_d \mid \dots \mid A_d^{n-1} B_d \right] \tag{8.15}$$

has rank $n$.

  - $n$ is the size of states.
  - $m$ is the size of inputs.

# Example

- Determine the controllability of the following state equation:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ \hline 0 & -0.4 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{u}(k)$$

- Solution:
  - To test whether a system is controllable, we need to construct the controllability matrix, i.e.,:

$$\mathscr{C} = \left[ B_d \mid A_d B_d \mid A_d^2 B_d \right]$$

$$= \left[\begin{array}{cc|cc|cc} 1 & 0 & -2 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 & -0.5 & -0.9 \\ 1 & 1 & -0.5 & -0.9 & -0.15 & 0.05 \end{array}\right]$$

  - We can see that the matrix has rank 3, which implies that the system is controllable.

# Observability

- Definition of Observability:

**DEFINITION 8.6: OBSERVABILITY**

A system is said to be observable if any initial state $x(k_0)$ can be estimated from the control sequence $u(k)$, $k = k_0, k_0 + 1, \ldots, k_f - 1$ and the measurements $y(k)$, $k = k_0, k_0 + 1, \ldots, k_f$.

- Observability Rank Condition:

**THEOREM 8.9: OBSERVABILITY RANK CONDITION**

A linear time-invariant system is completely observable if and only if the $l.n \times n$ observability matrix

$$\mathscr{O} = \begin{bmatrix} C \\ --- \\ CA_d \\ --- \\ \vdots \\ --- \\ CA_d^{n-1} \end{bmatrix} \tag{8.23}$$

has rank $n$.

  - Where $l$ is the number of outputs and $n$ is the number of states.

# Example

- Determine the observability of the system:

$$A = \begin{bmatrix} \mathbf{0}_{2\times1} & & I_2 \\ 0 & -3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

- Solution:
  - The observability matrix of the system is

$$\mathscr{O} = \begin{bmatrix} C \\ \hline CA_d \\ \hline CA_d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \hline 0 & -3 & 4 \\ \hline 0 & -12 & 13 \end{bmatrix}$$

  - We can see that the matrix rank is 2<3; thus, the system is not observable.