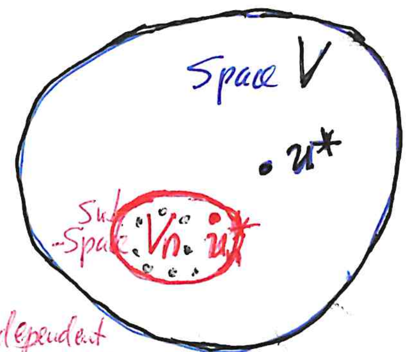
§2.3. Solution of Weak Form Problem

1. Galerkin Method

Recall the weak Form Problem: $\boxed{\begin{cases} \text{Find } u^* \text{ such that } a(u^*, v) - \ell(v) = 0, \\ \text{for any } v \in V \text{ (space } H^1(\Omega) \text{ or } H_0^1(\Omega)). \end{cases}}$

It is difficult!



choose $\{\varphi_1, \cdots, \varphi_n\}$ which are linearly independent. in V.

Set $c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi_n(x) = 0$

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$. called linearly independent.

$V_n = span\{\varphi_1, \varphi_2, \cdots, \varphi_n\} = \{v \mid v = \sum_{i=1}^{n} c_i\varphi_i\}$ called sub-space of V.

$\Rightarrow$ $\boxed{\text{Find } u_n^* = \sum_{i=1}^{n} c_i\varphi_i \text{ such that } a(u_n^*, v) - \ell(v) = 0 \text{ for any } v \in V_n. \text{ Use } u_n^* \text{ to be an approximation of } u^*.}$

$$\Rightarrow 0 = a(u_n^*, v) - \ell(v) = a(u_n^*, \sum_{j=1}^{n} d_j\varphi_j) - \ell(\sum_{j=1}^{n} d_j\varphi_j) = \sum_{j=1}^{n} d_j\left[a(u_n^*, \varphi_j) - \ell(\varphi_j)\right]$$

$$\Rightarrow \quad a(u_n^*, \varphi_j) - \ell(\varphi_j) = 0, \quad j = 1, 2, \cdots, n.$$

Galerkin method: Find $u_n^* = \sum_{i=1}^{n} c_i\varphi_i$ s.t. $a(u_n^*, \varphi_j) = \ell(\varphi_j)$, $j = 1, 2, \cdots, n$.

$$a(u_n^*, \varphi_j) = a(\sum_{i=1}^{n} c_i\varphi_i, \varphi_j) = \boxed{\sum_{i=1}^{n} c_i\, a(\varphi_i, \varphi_j) = \ell(\varphi_j)}, \; j = 1, 2, \cdots, n$$

$$\Rightarrow \begin{bmatrix} a(\varphi_1, \varphi_1) & \cdots & a(\varphi_n, \varphi_1) \\ a(\varphi_1, \varphi_2) & \cdots & a(\varphi_n, \varphi_2) \\ & \cdots\cdots & \\ a(\varphi_1, \varphi_n) & \cdots & a(\varphi_n, \varphi_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow A\vec{c} = \vec{b}.$$

* A is symmetric since $a(\varphi_i,\varphi_j)=a(\varphi_j,\varphi_i)$.
* A is positive definite.

Let $\vec{x}=\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$, $\vec{x}^T A\vec{x}=\sum_{i=1}^n\sum_{j=1}^n a(\varphi_i,\varphi_j)c_ic_j=a\left(\sum_{i=1}^n c_i\varphi_i,\sum_{j=1}^n c_j\varphi_j\right)$

$=a(\vec{x},\vec{x})>0.$

$\Rightarrow$ A^{-1} exists. $\Rightarrow$ $\vec{c}=A^{-1}\vec{b}$.

2. Examples

(1) $\begin{cases}-u''+u=-x, & 0<x<1,\\ u(0)=0,\ u(1)=0.\end{cases}$ Use the Galerkin method to find a solution.

Solution. Weak Form problem:

Find u such that $a(u,v)-(f,v)=0$, where $a(u,v)=\int_0^1(u_xv_x+uv)\,dx$,

$(f,v)=\int_0^1 fv\,dx$, for any $v\in H_0^1[0,1]$.

Choose $\varphi_i(x)=x(1-x)x^{i-1}$, $i=1,\cdots,n$ $\in H_0^1[0,1]$.

or $\varphi_i(x)=\sin(i\pi x)$, $i=1,\cdots,n$

choose $n=2$ for a detailed example. Let $\varphi_1(x)=x(1-x)$, $\varphi_2(x)=x(1-x)x$.

Find $u_2^*=c_1\varphi_1+c_2\varphi_2$ s.t. $a(u_2^*,\varphi_j)=(f,\varphi_j)$, $j=1,2$.

$$\Rightarrow\begin{bmatrix}a(\varphi_1,\varphi_1) & a(\varphi_2,\varphi_1)\\ a(\varphi_1,\varphi_2) & a(\varphi_2,\varphi_2)\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}(f,\varphi_1)\\(f,\varphi_2)\end{bmatrix},$$

where $a(\varphi_1,\varphi_1)=\int_0^1\left[(1-2x)^2+x^2(1-x)^2\right]dx=\frac{11}{30}$, $(f,\varphi_1)=\int_0^1(-x)\cdot x(1-x)\,dx=-\frac{1}{12}$, $\cdots$

$$\Rightarrow\begin{bmatrix}\frac{11}{30} & \frac{11}{60}\\ \frac{11}{60} & \frac{1}{7}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}-\frac{1}{12}\\-\frac{1}{20}\end{bmatrix}\Rightarrow c_1=-\frac{69}{473},\ c_2=-\frac{7}{43}.$$

$$\Rightarrow\boxed{u_2^*=-\frac{69}{473}x(1-x)-\frac{7}{43}x^2(1-x)}.$$

How close to the exact solution?

Here is how to find the exact solution.

$$-u''+u=-x,\quad u(0)=0,\ u(1)=0.$$

Solution. Find u_c. $-u''+u=0$. Let $u=e^{mx}$, $\Rightarrow (-m^2+1)e^{mx}=0$.

$\Rightarrow m^2=1 \Rightarrow m_{1,2}=\pm 1 \Rightarrow u_c=c_1e^{x}+c_2e^{-x}$.

Find $u_p=Ax+B \Rightarrow Ax+B\equiv -x \Rightarrow A=-1,\ B=0 \Rightarrow u_p=-x$

$\Rightarrow \boxed{u^*=u_c+u_p=c_1e^{x}+c_2e^{-x}-x.}$

$\xrightarrow[u(1)=0]{u(0)=0} \begin{cases} c_1+c_2=0, \\ c_1e+c_2e^{-1}-1=0. \end{cases} \Rightarrow c_1=\dfrac{1}{e-e^{-1}}=-c_2$

$=\dfrac{1}{2\sinh(1)}$.

$\Rightarrow \boxed{u^*=\dfrac{1}{2\sinh(1)}(e^{x}-e^{-x})-x}$

Table: Comparison of u^* and u_2^* in 4 digits after decimal point.

x	0.25	0.5	0.75
u^*	-0.0350	-0.0566	-0.0503
u_2^*	-0.0350	-0.0568	-0.0502

(2) $\begin{cases} -u''+u=-x,\ 0<x<1, \\ u(0)=\alpha,\ u(1)=\beta. \end{cases}$

Solution. Choose $G_0(x)=Ax+B$ s.t. $G(0)=\alpha,\ G(1)=\beta \Rightarrow \begin{cases} A=\beta-\alpha, \\ B=\alpha. \end{cases}$

$\Rightarrow G_0(x)=(\beta-\alpha)x+\alpha$. Let $u=U+G_0(x)$.

$\Rightarrow \begin{cases} -U''+U=-x-(\beta-\alpha)x-\alpha, \\ U(0)=0,\ U(1)=0. \end{cases}$ Back to Example (1).

HW: Ex.2.3.2, Ex.2.3.3 on Page 68.