Solved by experts

Exercise 33, Linear Algebra: A Modern Introduction, 4th Edition

NEXT QUESTION

Linear Algebra: A Modern Introduction, 4th Edition, Chapter 6: Vector Spaces

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Exercise 33 Answer

Step by step explanation

HIDE ALL

Tip

- In this type of questions, we need to find eigenvectors and eigenvalues.

Explanation

- We will take β as the standard basis.
- We will then find eigenvalues of $[T]_\beta$
- After eigenvalues we will find eigenvector corresponding to eigenvalues.

- We will find that C is linearly independent . Therefore , $[T]_\beta$ exists.

Step 1 of 3

Let, $\beta = \{1, x\}$ be the standard basis of P_1

Then,

$T(1) = 4 + x$

$T(x) = 2 + 3x$

Thus,

$[T]_\beta = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

Eigenvectors of $[T]_\beta$:

$$\det([T]_\beta - \lambda I) = \det(\begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix})$$
$$= \lambda^2 - 7\lambda + 10$$
$$= (\lambda - 5)(\lambda - 2)$$

The eigenvector of $[T]_\beta$ is 5 and 2 .

Step 2 of 3

Eigenvector corresponding to eigenvalue 2 :

Let,

$v_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ be the vector such that

$$[T]_\beta v_1 = 2.v_1$$
$$\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$
$$\begin{bmatrix} 4x + 2y \\ x + 3y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$x = -y$

Thus,

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 2.

Eigenvector corresponding to the eigenvalue 5 :

Let,

$V_2 = \begin{bmatrix} x \\ y \end{bmatrix}$ be vector such that,

$$[T]_\beta v_2 = 5.v_2$$
$$\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix}$$

$\begin{bmatrix} 4x+2y \\ x+3y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

x= 2y

Thus,

$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 5.

Step 3 of 3

We get, $V_1 = [1-x]_\beta$

$v_2 = [2+x]_\beta$

Then, C = $\{1-x, 2$ + x $\}$

Let, α, β be scalar such that

$\alpha(1-x)+\beta(2+x) = 0$

$\alpha, \beta = 0$

Therefore C is linearly independent.

It implies that T is diagonalizable and

$$[T]_c = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Final Answer

C = { 1 -x , 2+ x}

$[T]_c = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$