Lecture 7: Steady-state Error and Error Constants

ELEN 472: Introduction to Digital Control

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Review

- **System with Transport Lag**
  - Transport Lag: the **time delay** between the time an input signal is applied and the time the system reacts to the input signal.
  - Time Delay Modeling

$$T_d = lT - mT$$

- **Modified Z-Transform**

Formula:

$$\begin{aligned} Y(z,m) &= \mathcal{Z}_m\{y(kT)\} \\ &= z^{-1}\mathcal{Z}\{y(kT+mT)\} \end{aligned}$$

Commonly Used Pairs:

$$\mathscr{Z}_m\{1(kT)\} = \frac{1}{z-1}$$

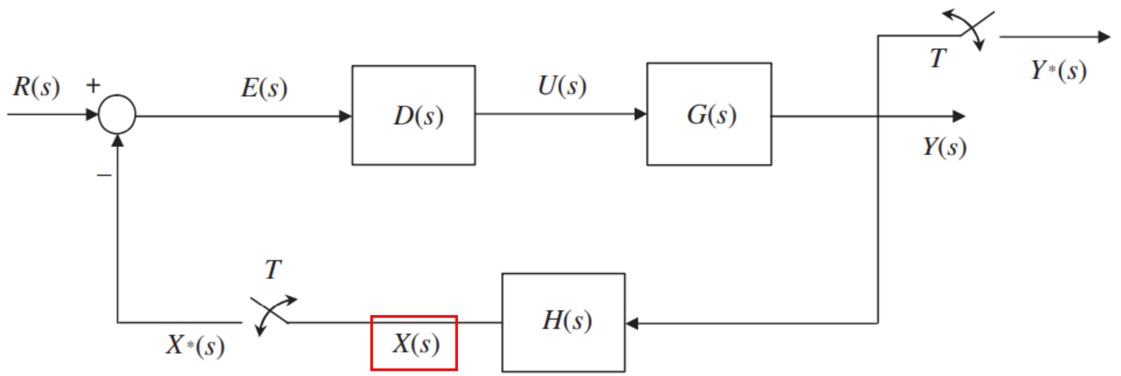
$$\mathscr{Z}_m\{e^{-pkT}\} = \frac{e^{-mpT}}{z-e^{-pT}}$$

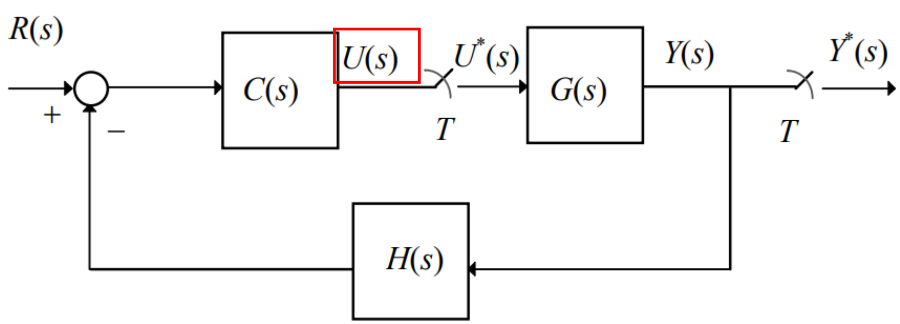
- **Modeling of System with Transport Lag**

$$\boxed{G_{ZAS}(z) = \left(\frac{z-1}{z^l}\right)\mathscr{Z}_m\{g_s(kT)\}}$$

Review

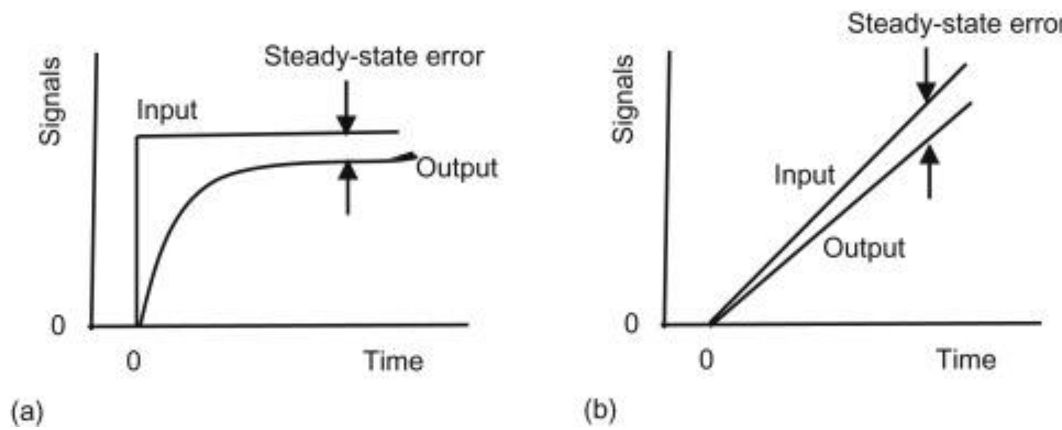
- **Block Diagram Reduction**
  - **Purpose:** Reduce the number of blocks into 1.
  - Start with the signal before the sampling (i.e., the switch) -> Take the sampling on both sides of the equation -> Solve for the signal.





Introduction to Steady-state Error

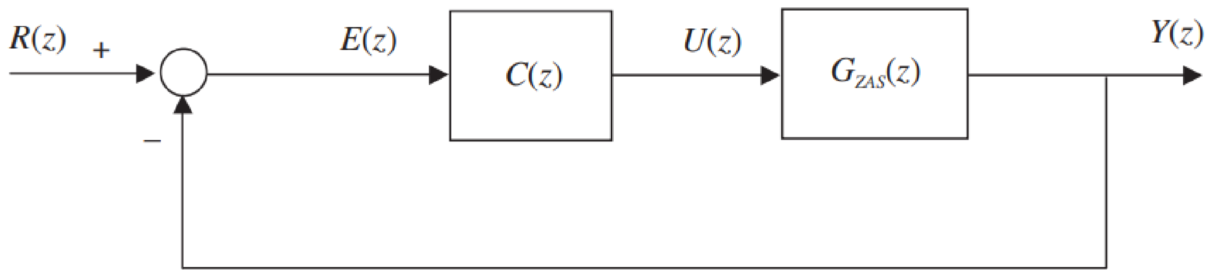
- Steady-state error is defined as **the difference** between the input (command) and the output of a system, as time goes to infinity.



- **Note**: Steady-state error analysis is only useful for **stable** systems.
    - You should always check the **system stability** before performing a steady-state error analysis.
- Steady-state error will depend on the type of inputs (step, ramp, etc.) and the system type (type-0, type-1, or type-2).

Calculating Steady-state Errors

- We consider a unity feedback system subject to standard inputs and determine the associated Steady-state error:



- From the block diagram, the tracking error is given by

$$E(z) = \frac{R(z)}{1 + G_{ZAS}(z)C(z)}$$
$$= \frac{R(z)}{1 + L(z)}$$

  - Where $L(z)$ denotes the **loop gain** of the system.

Calculating Steady-state Errors

- **Review:** Final Value Theorem
  - If a sequence approaches a constant **limit** as time tends to infinity, then this **limit** is given by

$$f(k \to \infty) = \lim_{k \to \infty} f(k)$$
$$= \lim_{z \to 1} [(z-1)F(z)]$$

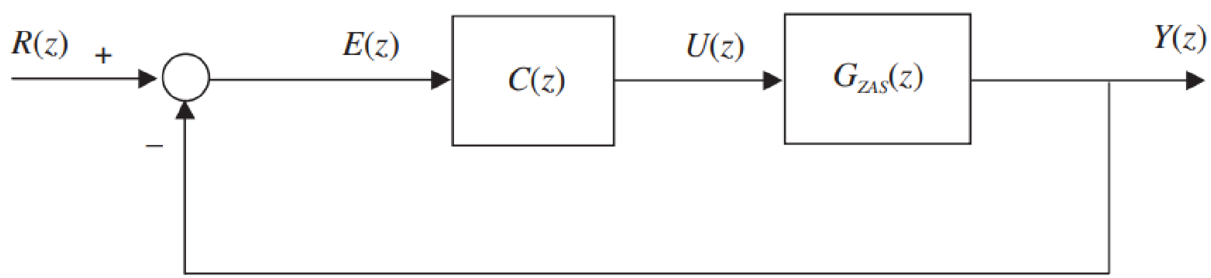
- Applying the final value theorem yields the **steady-state error**:

$$e(\infty) = \lim_{z \to 1} [(z-1)E(z)]$$
$$= \lim_{z \to 1} \frac{(z-1)R(z)}{1+L(z)}$$

  - **Note:** if $z - 1$ cannot be cancelled with the denominator, state-state error does not exist.

Example Question

- For the following system with unity feedback, find the steady-state error for a unit step input.
  - $C(z) = 1$ and $G_{ZAS}(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$



- **Solution**:
  - Since the input is a unity step input, thus $R(z) = \frac{z}{z-1}$ (see Lecture 2 Page 17 for the z-transform table)
  - Since $L(z) = C(z)G_{ZAS}(z) = 1 \times G_{ZAS}(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$

Example Question

- **Solution (Continued):**

$$e(\infty) = \lim_{z \to 1} [(z-1)E(z)]$$
$$= \lim_{z \to 1} \frac{(z-1)R(z)}{1 + L(z)}$$

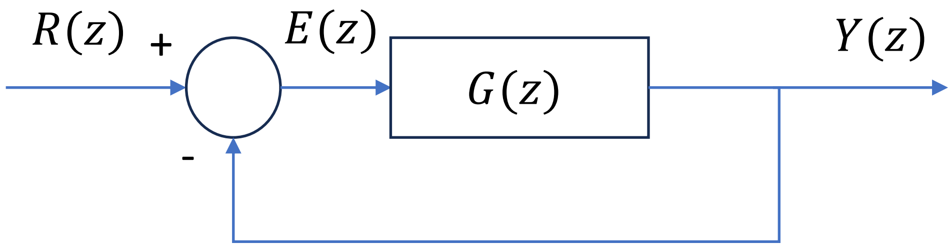
$$R(z) = \frac{z}{z-1}$$

$$L(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$$

$$e(\infty) = \left. \frac{(z-1) \times \frac{z}{z-1}}{\left(1 + \frac{0.4(z+0.2)}{(z-1)(z-0.1)}\right)} \right|_{z=1}$$
$$= \frac{1}{1 + \frac{0.4 \times 1.2}{0}}$$
$$= \frac{1}{1 + \infty}$$
$$= 0$$

Practice Question

- For the following unity feedback control system



$$G(z) = \frac{0.5(z+0.2)}{(z-0.1)(z-0.8)}$$

  - Find the steady-state error with $R(z) = \frac{z}{z-1}$

- **Solution**:
  - Determine the loop gain $L(z)$

$$L(z) = G(z) = \frac{0.5(z+0.2)}{(z-0.1)(z-0.8)}$$

  - Use final value theorem:

$$e(\infty) = \lim_{z\to1}[(z-1)E(z)]$$
$$= \lim_{z\to1}\frac{(z-1)R(z)}{1+L(z)}$$

$$R(z) = \frac{z}{z-1}$$

$$L(z) = \frac{0.5(z+0.2)}{(z-0.1)(z-0.8)}$$

Practice Question Solution

- Solution (Continued)

$$
\begin{aligned}
e(\infty) &= \lim_{z\to 1}[(z-1)E(z)] \\
&= \lim_{z\to 1}\frac{(z-1)R(z)}{1+L(z)} \\
&= \left.\frac{(z-1)\dfrac{z}{z-1}}{1+\dfrac{0.5(z+0.2)}{(z-0.1)(z-0.8)}}\right|_{z\to 1} \\
&= 0.238
\end{aligned}
$$

$$R(z) = \frac{z}{z-1}$$

$$L(z) = \frac{0.5(z+0.2)}{(z-0.1)(z-0.8)}$$

Type Number

- The **type number** of a discrete-time system is the number of unity poles in the system z-transfer function.
- The loop gain $L(z)$ (i.e., $C(z)G_{ZAS}(z)$) can be rewritten as:

$$L(z) = \frac{N(z)}{(z-1)^n D(z)}, \quad n \geq 0$$

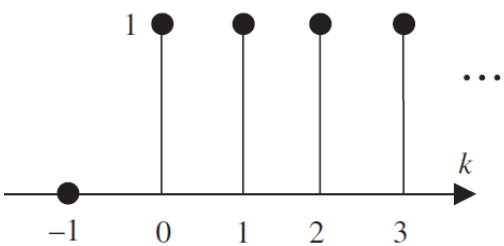
  - $L(z)$ has n poles at unity.
  - The type number is n.

Example Question

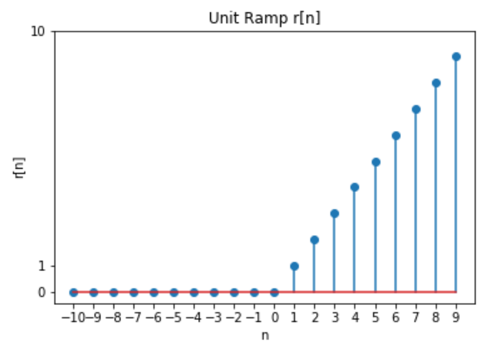
- Determine the Type number of the following systems:
  - $L(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$
  - $L(z) = \frac{0.5}{(z-0.1)(z-0.8)}$
  - $L(z) = \frac{z(z-2)}{(z-1)^3(z-0.5)}$
- **Solution:**
  - The system is type 1 since it has 1 unity pole.
  - The system is type 0 since it does not have unity pole.
  - The system is type 3 since it has 3 unity poles.

Steady-state Error of Standard Input Signals

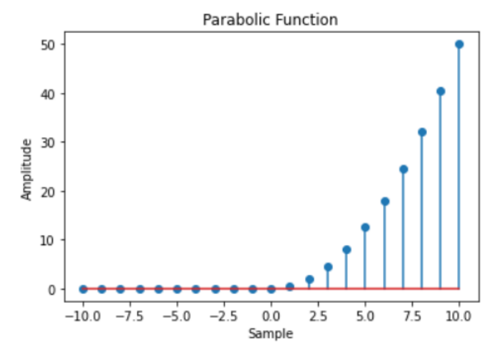
- Here, we discuss the steady-state errors of standard input signals.
  - Standard Input Signals include:
    - Step
    - Ramp
    - Parabolic



$$R(z) = \frac{z}{z-1}$$



$$R(z) = \frac{Tz}{(z-1)^2}$$

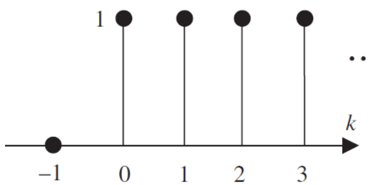


$$R(z) = \frac{z(z+1)T^2}{(z-1)^3}$$

Steady-State Error of Step Input

- The z-transform of a sampled unit step input is

$$R(z) = \frac{z}{z-1}$$



- Using final value theorem, we have:

$$e(\infty) = \left.\frac{1}{1+L(z)}\right|_{z=1}$$

- Define $K_p = L(1)$, thus, the steady-state error can also be written as

$$e(\infty) = \frac{1}{1+K_p}$$

  - K_p is called the **position error constant.**

Steady-State Error of Step Input

- Known that

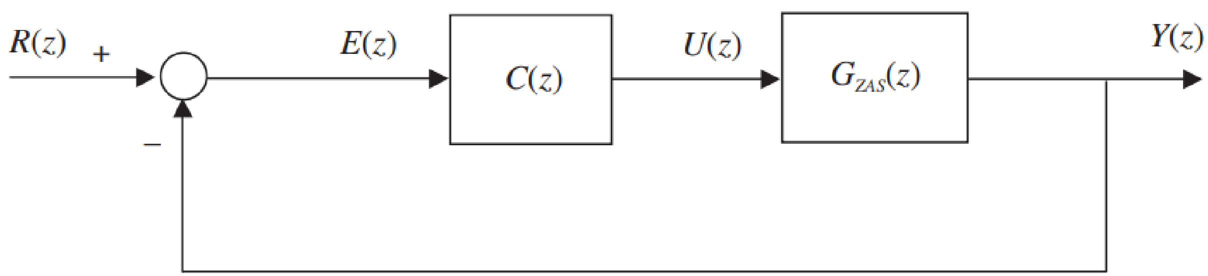
$$L(z) = \frac{N(z)}{(z-1)^n D(z)}, \quad n \geq 0$$

  - For **Type-0** System, i.e., $n = 0$
    - $K_p = L(1) = \frac{N(1)}{D(1)} = constant$
    - Thus, steady-state error is $\frac{\mathbf{1}}{\mathbf{1+K_p}}$.
  - For **Type-1** System, i.e., $n = 1$
    - $K_p = L(1) = \frac{N(1)}{(1-1)D(1)} = \frac{N(1)}{0} = \infty$
    - Thus, steady-state error is $\frac{1}{1+K_p} = \frac{1}{\infty} = \mathbf{0}$.
  - For **Type-2** System, i.e., $n = 2$
    - $K_p = L(1) = \frac{N(1)}{(1-1)^2 D(1)} = \frac{N(1)}{0} = \infty$
    - Thus, steady-state error is $\frac{1}{1+K_p} = \frac{1}{\infty} = \mathbf{0}$.
  - ……

$$\boxed{e(\infty) = \begin{cases} \dfrac{1}{1+L(1)}, & n = 0 \\ 0, & n \geq 1 \end{cases}}$$

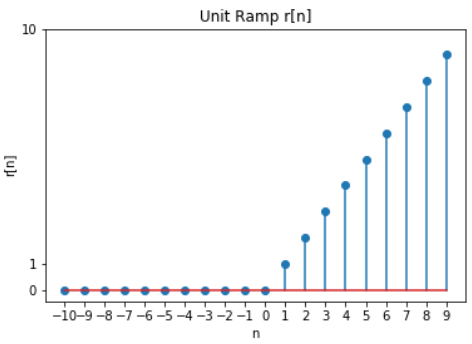
Revise the Previous Example Question

- For the following system with unity feedback, find the steady-state error for a unit step input.
  - $C(z) = 1$ and $G_{ZAS}(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$



- **Solution:**
  - The loop gain $L(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$
    - The system type is **type-1.**
  - For **type-1** system, the steady-state error for unity step input is **0**.

Steady-State Error of Ramp Input



- The z-transform of a sampled unit ramp input is

$$R(z) = \frac{Tz}{(z-1)^2}$$

- Using final value theorem, we have

$$e(\infty) = \left. \frac{T}{[z-1][1+L(z)]} \right|_{z=1}$$

  - Define

$$K_v = \left. \frac{1}{T}(z-1)L(z) \right|_{z=1}$$

  - Thus,

$$e(\infty) = \frac{1}{K_v} \quad \text{with } 1 + L(z) \approx L(z).$$

  - K_v is called **Velocity Error Constant**.

Steady-State Error of Ramp Input (Continued)

$$e(\infty) = \frac{T}{[z-1][1+L(z)]}\bigg|_{z=1} = \frac{1}{K_v} \qquad K_v = \frac{1}{T}(z-1)L(z)\,\bigg|_{z=1}$$

- Using

$$L(z) = \frac{N(z)}{(z-1)^n D(z)}, \quad n \geq 0$$

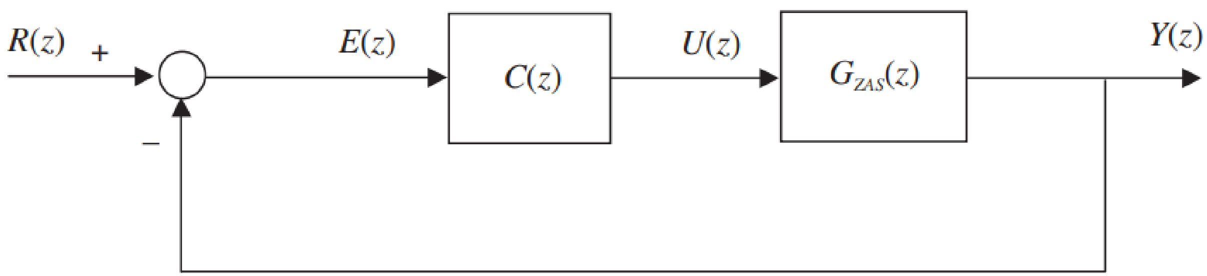
  - For type 0 systems, $K_v = 0$, thus the steady-state error is infinite.
  - For type 1 system, K_v is finite.
  - For type 2 or higher systems, K_v is infinite.

- The corresponding steady-state error:

$$e(\infty) = \begin{cases} \infty, & n = 0 \\ \dfrac{T}{(z-1)L(z)\big|_{z=1}}, & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Example Question

- For the following system with unity feedback, find the steady-state error for a ramp input with sample period $T = 1$ s.
  - $C(z) = 1$ and $G_{ZAS}(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$



- **Solution:**
  - The system type is type-1, thus, according to

$$e(\infty) = \begin{cases} \infty, & n = 0 \\ \dfrac{T}{(z-1)L(z)\big|_{z=1}}, & n = 1 \\ 0 & n \geq 2 \end{cases}$$

$$e(\infty) = \frac{1}{(z-1)\dfrac{0.4(z+0.2)}{(z-1)(z-0.1)}}\Bigg|_{z=1}$$

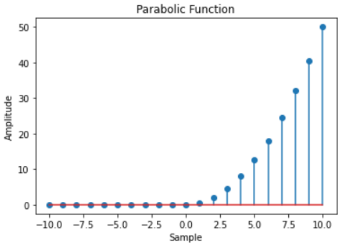
$$= \frac{1}{\dfrac{0.4 \times 1.2}{0.9}} = 1.875$$

Practice Question

- For a system with the loop gain $L(z) = \frac{0.5(z+0.2)}{(z-0.1)(z-0.8)}$
  - Find the steady-state error due to a step input.
  - Find the steady-state error due to a ramp input.
- **Solution:**
  - $\frac{1}{1+L(1)}$
  - Infinite

Steady-State Error of Parabolic Input

- Similar to the previous steps, it can be shown that for a sampled parabolic input, an acceleration error constant given by



$$K_a = \frac{1}{T^2}(z-1)^2 L(z)\Big|_{z=1}$$

- The associated steady-state error is

$$e(\infty) = \begin{cases} \infty, & n \le 1 \\ \dfrac{T^2}{(z-1)^2 L(z)\big|_{z=1}}, & n = 2 \\ 0, & n \ge 3 \end{cases}$$

Summary Table for Error Constants

Position Error Constant, K_p	**Velocity Error Constant,** K_v	**Acceleration Error Constant,** K_a
$K_p = L(1)$	$K_v = \frac{1}{T}(z-1)L(z)\Big\rvert_{z=1}$	$K_a = \frac{1}{T^2}(z-1)^2 L(z)\Big\rvert_{z=1}$

T is the sampling period

$L(z) = G_{ZAS}(z)C(z)$

- $G_{ZAS}(z)$ is the system's transfer function
- $C(z)$ is the controller's transfer function

Summary Table for Steady-State Errors

Signal	Type-0	Type-1	Type-2
Sampled step input	$\frac{1}{1+L(1)}$ or $\frac{1}{1+K_p}$	0	0
Sampled ramp input	∞	$\frac{T}{(z-1)L(z)\|_{z=1}}$ or $\frac{1}{K_v}$	0
Sampled parabolic input	∞	∞	$\frac{T^2}{(z-1)^2L(z)\|_{z=1}}$ or $\frac{1}{K_a}$

$L(z) = G_{ZAS}(z)C(z)$

- $G_{ZAS}(z)$ is the system's transfer function
- $C(z)$ is the controller's transfer function

T is the sampling period

Example

- Find the steady-state position error for the digital position control system with unity feedback and with the transfer functions

$$G_{ZAS}(z) = \frac{K(z+a)}{(z-1)(z-b)}, \quad C(z) = \frac{K_c(z-b)}{z-c}, \quad 0 < a,\ b,\ c < 1$$

  - For a sampled unit step input
  - For a sampled unit ramp input
- **Solution**
  - The loop gain of the system is given by

$$L(z) = C(z)G_{ZAS}(z) = \frac{KK_c(z+a)}{(z-1)(z-c)}$$

  - The system is type 1. Therefore, it has zero steady-state error for a sampled step input
  - For a sampled ramp input given by

$$e(\infty) = \frac{T}{(z-1)L(z)\big|_{z=1}} = \frac{T}{KK_c}\left(\frac{1-c}{1+a}\right)$$

Practice Questions

- For the following systems with unity feedback, find
  - The position error constant.
  - The velocity error constants.
  - The steady state error due to a unit step input.
  - The steady-state error due to a unit ramp input.

$$G(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$$

- **Solution:**
  - The position error constant:
    - The system is Type 1 and has an infinite position error constant.
  - The velocity error constants:

$$K_v = \frac{1}{T}(z-1)G(z)\Big|_{z=1} = \frac{0.4(1+0.2)}{T(1-0.1)} = \frac{0.5333}{T}$$

  - The steady state error due to a unit step input:
    - The system is Type 1 and has zero steady-state error due to step.
  - The steady-state error due to a unit ramp input:
    - $e(\infty) = \frac{1}{K_v} = \frac{T}{0.5333}$