Lecture 15: State Feedback Control

ELEN 472: Introduction to Digital Control

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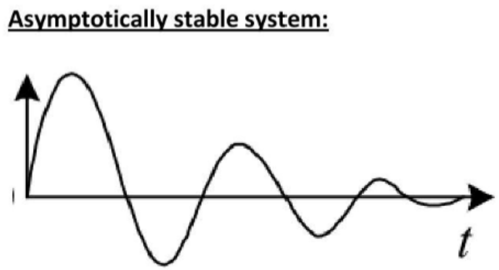
Review

- **Stability of Discrete-Time Control System**
  - **Asymptotically Stable**: the final system response is 0.



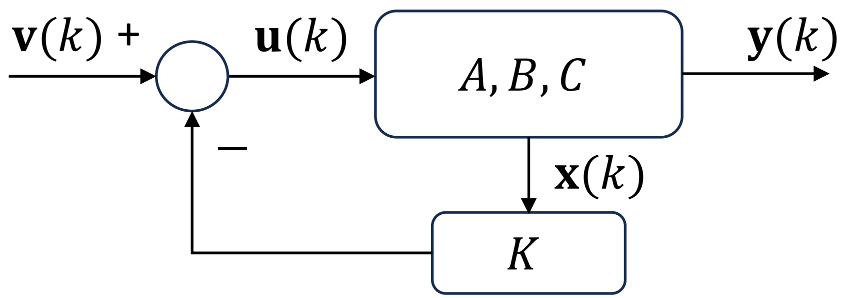
  - **How to check?** -> all eigenvalues of A < 1
- **Controllability**
  - **Definition:** the ability of a given input to steer a system from any initial state to another state within finite time.
  - **How to check?** -> Controllability Matrix is full rank $\mathscr{C} = \left[B_d \mid A_d B_d \mid A_d^2 B_d \right]$
- **Observability**
  - **Definition:** the ability to observe the system state. If the internal state of the system is determined using the input/output signals -> observable.
  - **How to check?** -> Observability Matrix is full rank

$$\mathscr{O} = \begin{bmatrix} C \\ \hline CA_d \\ \hline CA_d^2 \end{bmatrix}$$

Introduction of State Feedback Control

- State feedback control involves using the **state vector** to compute the **control action**.
- For a discrete-time state space model (assume $D = 0$):

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k)$$
$$\mathbf{y}(k) = C\mathbf{x}(k)$$



  - The control action $\mathbf{u}(k)$ can be represented as:

$$\mathbf{u}(k) = -K\mathbf{x}(k) + \mathbf{v}(k)$$

    - $\mathbf{v}(k)$ is the reference input.
    - K is the control gain, and it is a constant.

Introduction

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k)$$
$$\mathbf{y}(k) = C\mathbf{x}(k)$$

$$\mathbf{u}(k) = -K\mathbf{x}(k) + \mathbf{v}(k)$$

- These two equations can be combined to yield the closed-loop state equation:

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B[-K\mathbf{x}(k) + \mathbf{v}(k)] \\ &= [A - BK]\mathbf{x}(k) + B\mathbf{v}(k)\end{aligned}$$

  - We define the **closed-loop state matrix** as

$$A_{cl} = A - BK$$

- Thus, the original state space model can be written as

$$\mathbf{x}(k+1) = A_{cl}\mathbf{x}(k) + B\mathbf{v}(k)$$
$$\mathbf{y}(k) = C\mathbf{x}(k)$$

Pole Placement

- Using State Feedback Control, the poles (or eigenvalues) of the system can be assigned at desired locations.
  - This is known as **Pole Placement**.
- **Pole Placement Theorem:**
  - If the pair (A, B) is **controllable**, then there exists a feedback gain K that arbitrarily assigns the system poles to any set $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$.
- **Pole Placement Procedures**:
  - **Step 1**: Evaluate the desired characteristic polynomial from the desired poles (or eigenvalues) $\lambda_1, \lambda_2, \ldots \lambda_n$
  $$\Delta(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$$
  - **Step 2**: Evaluate the closed-loop characteristic polynomial using the expression:
  $$\det\{\lambda I_n - (A - BK)\}$$
  - **Step 3**: Equating the coefficients of two polynomials to obtain n equations to be solved for K.

Example

- Assign the eigenvalues $\{0.3 \pm 0.2j\}$ to the pair

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- **Solution:**
  - First, we check the controllability of the system

$$P_C = [B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$$

  - Since the controllability matrix P_C is full rank, the system is controllable -> we can use the pole placement.
    - **Step 1**: The desired characteristic equation is

$$(\lambda - 0.3 - j0.2)(\lambda - 0.3 + j0.2) = \boxed{\lambda^2 - 0.6\lambda + 0.13} \quad \text{Eqn. 1}$$

    - **Step 2:** The closed-loop characteristic equation is

$$\det\{\lambda I_n - (A - BK)\} \Rightarrow (A - BK) = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \Rightarrow \det \begin{bmatrix} \lambda & -1 \\ -(3 - k_1) & \lambda - (4 - k_2) \end{bmatrix}$$

$$= \boxed{\lambda^2 - (4 - k_2)\lambda - (3 - k_1)} \quad \text{Eqn. 2}$$

    - **Step 3:** Equating coefficients of **Eqn. 1** and **Eqn. 2**, we have:

$$\begin{aligned} 4 - k_2 &= 0.6 \Rightarrow k_2 = 3.4 \\ -3 + k_1 &= 0.13 \Rightarrow k_1 = 3.13 \end{aligned} \Rightarrow K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 3.13 & 3.4 \end{bmatrix}$$

Practice Question

- Consider the following system with A and B as

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  - Find the gain K to make the poles at {1, 2}
- **Solution:**
  - First, check the controllability of the system:

$$P_C = [B, AB] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

  - The system is not controllable. Thus, pole placement cannot be used.

Ackermann's Formula

- In the Pole Placement Procedures, computing the characteristic polynomial and choosing a suitable K could be a challenging task for high-order systems.
- Ackermann's Formula provides an alternative way to find K:
  - **Step 1:** Compute Controllability Matrix $P_C = [B, AB, \ldots A^{n-1}B]$
  - **Step 2:** Compute the characteristic equation for the closed-loop poles, replacing λ with $A, \text{i.e.}, \Delta(A)$
  - **Step 3:** Computer K via

$$K = [0, 0, \ldots, 1]P_C^{-1}\Delta(A)$$

Example

- Revise our first example. Find K using Ackermann's Formula.
  - Assign the eigenvalues $\{0.3 \pm 0.2j\}$ to the pair
$$A = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- **Solution:**
  - **Step 1**: Compute $P_c = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$
  - **Step 2**: Compute $\Delta(A)$
    - We know $\Delta(\lambda)$ is $(\lambda - 0.3 - j0.2)(\lambda - 0.3 + j0.2) = \lambda^2 - 0.6\lambda + 0.13$
    - Replace λ using A.

$$\begin{aligned} \Delta(A) &= A^2 - 0.6A + 0.13I \\ &= \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} - 0.6\begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0.13 & 0 \\ 0 & 0.13 \end{bmatrix} \\ &= \begin{bmatrix} 3.13 & 3.4 \\ 10.2 & 16.73 \end{bmatrix} \end{aligned}$$

  - **Step 3:** Compute K

$$K = [0, 1]P_C^{-1}\Delta(A) = [3.13 \quad 3.4]$$

Practice

- Assign poles at {-5, -6} given the pair

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  - Using the Ackermann's Formula to find the gain K.

- **Solution**
  - **Step 1:** Find P_C

$$P_C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

  - **Step 2:** Find $\Delta(A)$
    - $\Delta(\lambda) = (\lambda + 5)(\lambda + 6) = \lambda^2 + 11\lambda + 30$
    - Thus,

$$\Delta(A) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^2 + 11 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 30I = \begin{bmatrix} 43 & 14 \\ 14 & 57 \end{bmatrix}$$

  - **Step 3:** Compute K

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 43 & 14 \\ 14 & 57 \end{bmatrix} = \begin{bmatrix} 14 & 57 \end{bmatrix}$$

Example 2

- Design a feedback controller for the pair

$$A = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix}$$

  - To obtain the eigenvalues $\{0.1, 0.4 \pm 0.4j\}$
- **Solution:**
  - In MATLAB, the pole placement command is

    K = place(A, B, p)

    - A and B are system matrices;
    - p is the desired place location

```
%% Example 2
A = [0.1 0 0.1;
0 0.5 0.2;
0.2 0 0.4];
B = [0.01; 0; 0.005];

p = [0.1, 0.4+0.4*1i, 0.4-0.4*1i];

K = place(A, B, p)
```

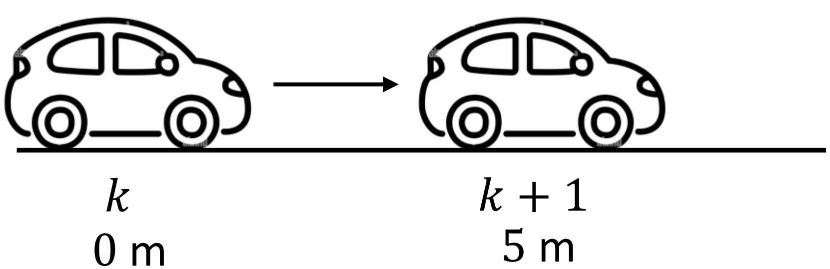
```
K =

  -10.0000   85.0000   40.0000
```

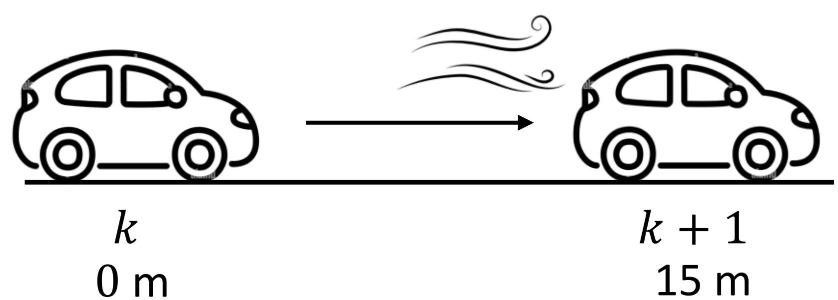
State Estimation

- In most applications, measuring the entire state vector is impossible or prohibitively expensive.
  - To implement state feedback control, an estimate of state $\hat{\mathbf{x}}(k)$ can be used.
  - The state vector can be estimated from the input and output measurements by using a **state estimator** or **observer**.
- **Open-Loop State Estimation**
  - The states can be estimated based on our knowledge to the system model:

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + B\mathbf{u}(k)$$

  - For instance, you can estimate the next position of a car based on its current position and speed.
  - However, this estimation assumes perfect knowledge of the system dynamics and lacks the feedback needed to correct the error (e.g., a strong wind pushes the car).





Full-Order Estimator

- A practical alternative is to feed back the difference between the measured and estimated output of the system.
  - Observer state equation:

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) + L[\mathbf{y}(k) - C\hat{\mathbf{x}}(k)]$$

    - $\mathbf{y}(k)$ is the measurement
    - L is the feedback gain, which is a constant matrix
- Subtracting the observer state equation with the system dynamics, we have the error dynamics:

$$\tilde{\mathbf{x}}(k+1) = (A - LC)\tilde{\mathbf{x}}(k)$$

  - where $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$
- Note that, we can modify the eigenvalues of the error dynamics by proper selection of the gain L.

State Estimation Theory

- **State Estimation Theory**
  - **I**: If the pair (A, C) is **observable**, then there exists a feedback gain matrix L that arbitrarily assigns the observer poles (or eigenvalues) to any set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.
- **Example**
  - Determine the observer gain matrix L with the observer eigenvalues selected as $\{-2, -3\}$.

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -0.2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- **Solution:**
  - **Step 1:** Check observability P_O

$$P_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

    - P_O is full rank -> System is observable
  - **Step 2:** Compute $\Delta(\lambda)$

$$\Delta(\lambda) = (\lambda + 2)(\lambda + 3) = \lambda^2 + 5\lambda + 6$$

Example Solution

- **Step 3:** Compute $\det(\lambda I - (A - LC))$
  - First, let's get $A - LC$

$$A - LC = \begin{bmatrix} 0 & 1 \\ -4 & -0.2 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -L_1 & 1 \\ -L_2 - 4 & -\frac{1}{5} \end{bmatrix}$$

  - Then, $\lambda I - (A - LC)$

$$\lambda I - (A - LC) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -L_1 & 1 \\ -L_2 - 4 & -\frac{1}{5} \end{bmatrix}$$
$$= \begin{bmatrix} \lambda + L_1 & -1 \\ L_2 + 4 & \lambda + \frac{1}{5} \end{bmatrix}$$

  - Finally, $\det(\lambda I - (A - LC))$

$$\det\big(\lambda I - (A - LC)\big) = \lambda^2 + \left(L_1 + \frac{1}{5}\right)\lambda + 4 + \frac{L_1}{5} + L_2$$

Example Solution

- **Step 4:** Equating the coefficients

$$\det\big(\lambda I - (A - LC)\big) = \lambda^2 + \left(L_1 + \frac{1}{5}\right)\lambda + 4 + \frac{L_1}{5} + L_2$$

$$\Delta(\lambda) = \lambda^2 + 5\lambda + 6$$

$$L_1 + \frac{1}{5} = 5$$
$$4 + \frac{L_1}{5} + L_2 = 6$$

$$\Rightarrow \quad L_1 = 4.8, \quad L_2 = 1.04 \quad \Rightarrow \quad L = \begin{bmatrix} 4.8 \\ 1.04 \end{bmatrix}$$

A Quicker Way

- **State Estimation Theory**
  - **II:** The system (A, C) is **observable** if and only if (A^T, C^T) is **controllable**.
- **Revise Example:**
  - Determine the observer gain matrix L with the observer eigenvalues selected as $\{-2, -3\}$.

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -0.2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- **Solution:**
  - The finding of L becomes the finding of K for (A^T, C^T).

```
A = [0 1; -4 -0.2];
C = [1, 0];

L = place(A', C', [-2, -3])'
```

```
L =

    4.8000
    1.0400
```