Homework 6 on Biopotentials **Name:________Solutions__**

1. What is the Nernst (equilibrium) potential of a monovalent cation (+ valence) at 37°C given the intracellular ion concentration is 10 mM and the extracellular ion concentration is 80 mM?

$$V = 61.5 \log \frac{C_o}{C_i} = 61.5 \log \frac{80}{10} = 55.5\ mV$$

2. What is the resting membrane potential at 37 °C with relative permeability's of $P_{Na} = 3$, $P_K = 33$ given the table below?

Ion	Extracellular concentration (mM)	Intracellular concentration (mM)
Na^+	145	12
K^+	4	155

Goldman-Hodgkin-Katz $$V_m = \frac{RT}{F} \ln \left(\frac{P_{Na}[Na^+]_o + P_K[K^+]_o}{P_{Na}[Na^+]_i + P_K[K^+]_i} \right)$$

$$V_m = 61.5 \log \frac{3(145) + 33(4)}{3(12) + 33(155)} = 61.5 \log \frac{567}{5151} = -58.9\ mV$$

3. The two voltage wavefroms associated with the electroneurogram in Figure P3A differ in shape and size. The change in voltage shape with distance can be thought of as a low pass filter process, where a given distance along the nerve corresponds to one passage through a low pass filter, as shown in Figure P3B.

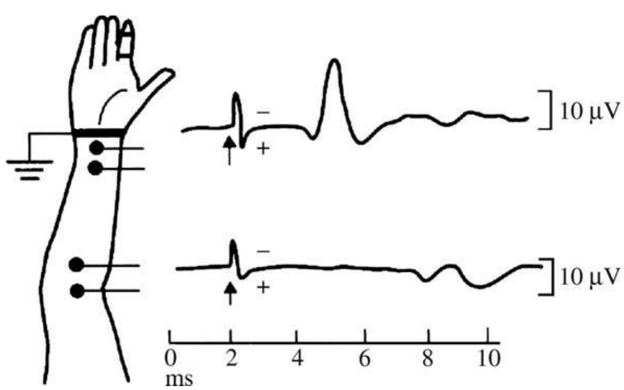
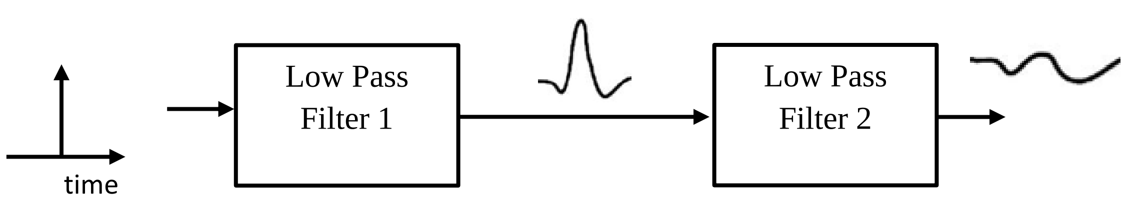


Figure P3A.



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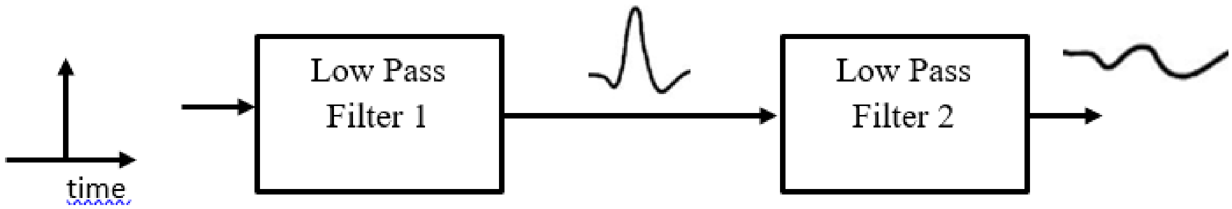


Figure P3B

a. If the initial stimulus is modeled as a delta function in time,

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases}$$

and if Filter 1 has a "boxcar" shape in frequency

$$H(j\omega) = \begin{cases} 1 & \text{if } \left|\frac{\omega}{2\pi}\right| \geq 1000 \ \text{ Hz} \\ 0 & \text{if } \left|\frac{\omega}{2\pi}\right| > 1000 \ \text{ Hz} \end{cases}$$

what is a mathematical form for the output of the first filter? (Hint: You will need to review filter impulse responses from BIEN 225.)

Answer: Because the input is a delta function (an impulse), the output is the impulse response, which is the inverse Fourier transform of the filter's response.

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}\ d\omega$$

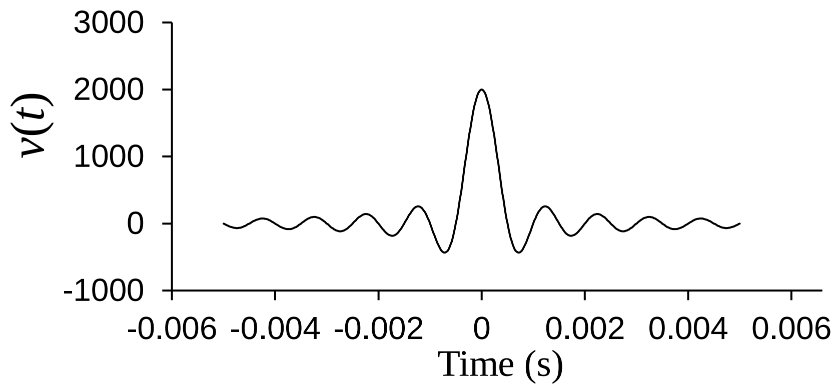
$$v(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega t}\ d\omega$$

$$v(t) = \frac{1}{2\pi}\left(\frac{e^{\omega_0 t}}{jt} - \frac{e^{-\omega_0 t}}{jt}\right) = \frac{1}{\pi t}\sin(\omega_0 t) = \frac{\omega_0}{\pi}\mathrm{sinc}(\omega_0 t)$$

b. Plot the output of the first filter as a function of time.

Answer:

The plot bears some resemblance to the output shown in Figure P3B. The sharp delta function has been changed into a smoother curve, and the width of the central peaks are similar in time. However, our simplification led to side lobes on the voltage curve that are not prevalent in the curve of Figure P3B.



c. If the second filter has a shape $H(j\omega)$ and the input to this filter is $v(t)$, what is the mathematical relationship that you would need to evaluate to find the filter's output?

Answer: If the time waveform is $v(t)$ and the transfer function is $H_2(j\omega)$, we simply need to multiply the Fourier transform of $v(t)$ by $H_2(j\omega)$ and then take the inverse Fourier transform.

$$v_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left\{\int_{-\infty}^{\infty} v(t)\, e^{-j\omega t}\, dt\right\} H_2(j\omega)e^{j\omega t}\right] d\omega$$

Graduate Content

FFT and Fourier Series

I. In MATLAB perform the following:

```
a=[0:1:200];

wt=2.*3.14159.*a/200;

z=1 + cos(10*wt) + 0.5*cos(20*wt) + 0.25*cos(30*wt) + 0.125*cos(40*wt);

figure(1);

plot(wt,z);

b=fft(z);

c=abs(b);

figure(2);
```

```
plot(c);
```

Answer the following questions:

1. What is the meaning of the variable “wt.” Why is it an appropriate variable to use in this program?

Frequency times time. Declaring this variable prevents the multiplication from being done unnecessarily for each cosine argument.

2. What is the time average of the signal z?

$\bar{z} = 1$ (the DC component)

3. What does the line b=fft(z) do?

Takes the Fourier transform of z

4. Why is it necessary to take abs(b)?

Power spectrum is magnitude squared, and b is a complex array. Need to convert b to a real variable to get something that you can plot.

5. What is the x-axis in the plot of c?

An arbitray default axis. Each data point increments in steps of 1.

6. The plot of c looks like a series of delta functions. Why?

Because the power spectrum of a sum of cosines is a series of delta functions at the cosines’ frequencies.

7. What is the true frequency that corresponds to the x-axis value of 200 on the plot?

The input has 200 data points, so the output should have 200 data points. The first output point is zero Hz, the next 99 are positive frequencies, and the last 100 are negative frequencies. Therefore, the last data point corresponds to $-\Delta f$, where Δf is $1/T$ and T is the total time of the signal. Because the code does not separate frequency from time in ωt, it is not possible to give a numerical value for Δf.

8. What is the true frequency that corresponds to the x-axis value of 180 on the plot?

$$-21\Delta f$$

9. Find the component that corresponds to the third term in the expression for z. What is the value of the peak of this component on the plot?

Although time and frequency are not separated, it is possible to determine which of the four cosines corresponds to that peak. It does not depend on the specific value of ω. Arbitrarily assume that a corresponds to time. Then the sample frequency is 1 Hz, and $\omega = 2\pi/200$. The frequency of the first cosine is then $10\omega/(2\pi) = 0.05$ Hz, the second frequency is 0.1 Hz, the third is 0.15 Hz, and the fourth is 0.2 Hz. Δf is $1/200 = 0.005$ Hz. Therefore, the component that corresponds to the third term, $0.5\cos(20\omega t)$, occurs at 0.1/0.005, or the 20th point.

(Note: It's ok if you interpreted the question as referring to the third cosine, in which case your answer should be the 30th point).

10. Why is the value of the peak of the third term peak not equal to 0.5?

The power corresponding to this peak should correspond to the power in the cosine. The power is $P(\omega)\Delta f$, where (according to the interpretation in Problem 9) $\Delta f = 0.005$ Hz. The power in the cosine is $(0.5)^2/2 = 0.125$. To get the power from the FFT, we need to first normalize it correctly:

```
cnorm = c/npts/(sqrt(deltaf);
```

then square it (to get the power spectrum), and then multiply by Δf. The value of cnorm is $50/200/\sqrt{0.005} = 12.5$. The power is therefore $12.5\times 0.005 = 0.0625$. But remember that the cosine consists of peaks at both positive and negative frequency, so we need to multiply this result by 2 to get 0.125.

Now run the following Matlab code:

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```
a=[0:1:400];

wt=2.*3.14159.*a/200;

z1=1 + cos(10*wt)+ 0.5*cos(20*wt) + 0.25*cos(30*wt) + 0.125*cos(40*wt);

z2 = 1+zeros(1,401);

a2 = [0:1:801];

wt2=2*3.14159*a2/400;

h1 = figure(1);

z = [z1 z2];

plot(wt2,z);

xlabel('Time')

ylabel('Signal')

b=fft(z);

c=abs(b);

h2 = figure(2);

plot(c);

xlabel('Frequency');

ylabel('Amplitude');
```

11. At what x-axis value is the peak corresponding to the third term of z1 located?

$x = 80$

12. Explain why the value is not the same as the value obtained with the first set of code.

The input data were zero padded, so the time duration of the signal is twice as long, which makes Δf half as large.

13. What is the peak value of the peak corresponding to the third term of z1?

200

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14. Explain why the peak value is not the same as it was with the first set of code.

The power spectrum needs to include Δf in the normalization.

15. Zoom in on the first non-DC peak (corresponding to cos(10*wt)) and describe its shape.

It has a central peak with multiple side lobes. (Mathematically, it is a sinc^2).

16. Explain why this peak has the shape that it does.

The signal has been windowed with a rectangle of duration T/2. The spectrum is the convolution of the signal spectrum with the window spectrum, and the window spectrum is the sinc function.

17. Find a mathematical expression that describes the spacing (in frequency) between the secondary peaks around the peak.

The Fourier transform of the rectangle with duration T is:

$$\mathcal{F}(\omega) = \frac{2\sin(\omega T/2)}{\pi\omega}$$

Therefore, the peaks correspond to places where the sin is 1, or where the argument of the sin is $\omega T/2 = (2n+1)\pi/2,\ n = 0, 1, 2, 3, \ldots..$ (The central peak, corresponding to $\omega = 0$ is an exception). The difference between peaks corresponding to n_k and n_{k-1} is

$$\Delta\omega = \left(\frac{2}{T}\right)\left(\frac{(2n_k+1)\pi}{2}\right) - \left(\frac{2}{T}\right)\left(\frac{(2n_{k-1}+1)\pi}{2}\right)$$

$$\Delta\omega = \left(\frac{2\pi}{T}\right)(n_k - n_{k-1})$$

For example, if $n_k = 1$ and $n_{k-1} = 0$,

$$\Delta\omega = \frac{2\pi}{T}$$

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18. Find a mathematical expression for the peak values of the secondary peaks.
From the transform of the rectangle,

$$\mathcal{F}(\omega) = \frac{2\sin(\omega T/2)}{\pi\omega}.$$

The peaks occur when the sin portion is 1, so:

$$f(\omega_k) = \frac{2}{\pi\omega_k}$$

Thus, they fall off hyperbolically with ω_k.

Print out the plots of (wt, z) and the plot of c, and turn these in with the answers to the above questions.

The plots for the first section of code are shown in Figure 1.

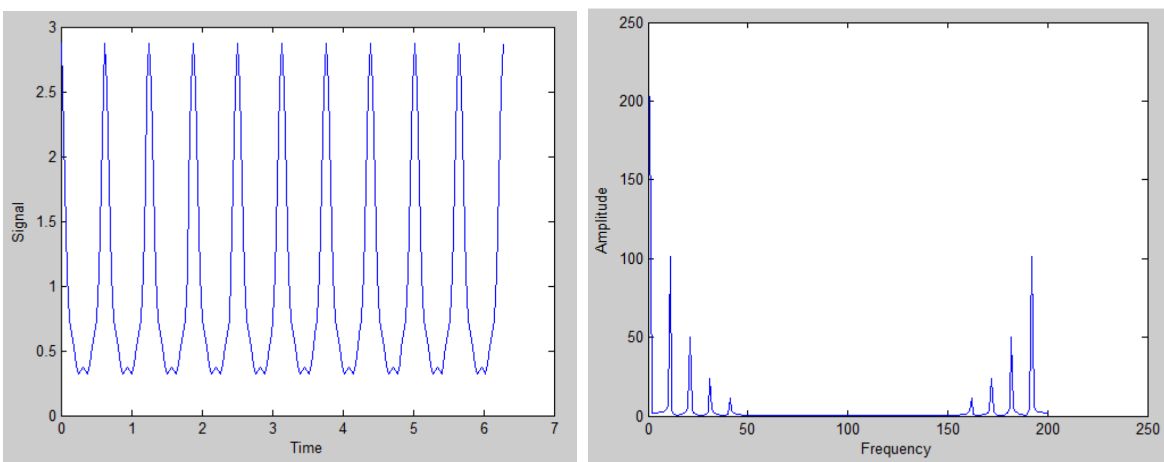

Figure 1: Signal (left) and power spectrum (right) for the first section of code.

The plots for the second section of code are shown in Figure 2.

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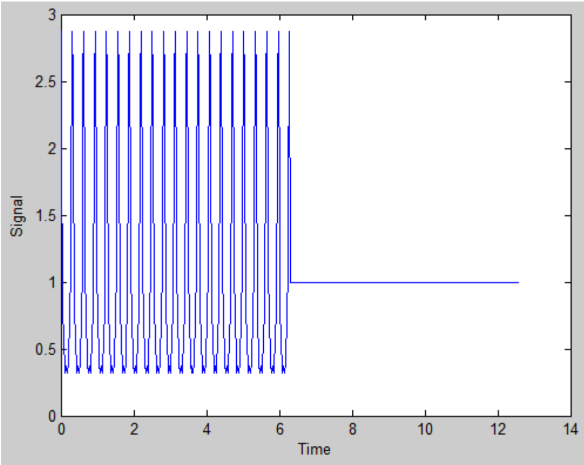

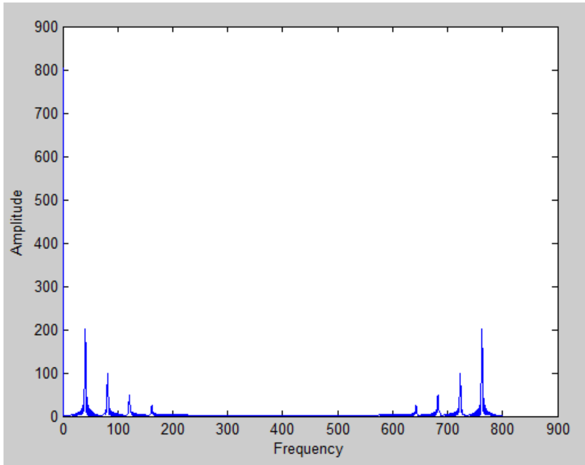

Figure 1: Signal (left) and power spectrum (right) for the first section of code.

II. Use the Fourier series from the book that represents a square wave, and write a program that **computes** and **plots** the first n terms of this expansion (where n is a variable that you will input when you call the function). You should be able to call this routine with the line:

>> squarewave(n);

```
function squaresig = squarewave(n)
% Construct a square wave (squaresig) from the first n terms of the
Fourier series


npts = 1000.;
totaltime = 1.0; %seconds
dt  = totaltime/npts;
t = 0:dt:totaltime-dt;
f = 1.0/totaltime;
wt  = 2*pi*f*t;
a0 = 2/pi;
```

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```
squaresig = zeros(1,npts);
for k = 1:2:n
    a = a0/k;
    squaresig = squaresig + a*sin(k*wt);
end
plot(t,squaresig);

end
```

With this routine, calculate and plot the n-term Fourier expansion of the square wave for n=1, n=2, n=10, n=100, and n=1000.

I've generated the curves for all cases on one graph in Figure 3.

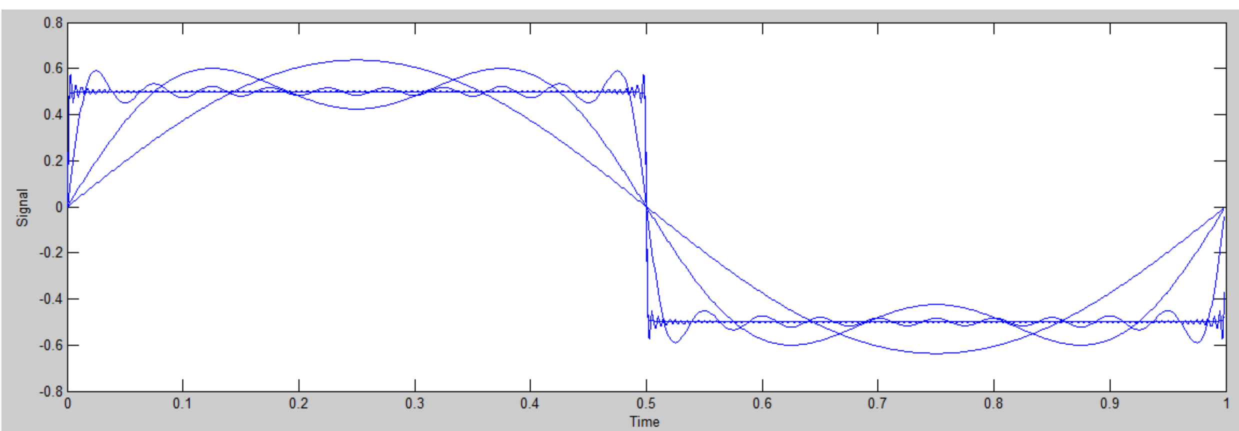

Figure 3: Fourier approximation of the square wave with 1, 2, 10, 100, and 1000 terms.

The case for $n = 1000$ does not show up well on the single graph, so it is shown by itself in Figure 4. The large number of terms appears to have eliminated the problem seen for $n = 100$, where ringing appears at the edges. However, the ringing is eliminated only because the sample rate is low. If the sample rate is increased, (here, I've changed it from 1000 points per cycle to 100,000 points per cycle), the ringing does appear, as shown in Figure 5.

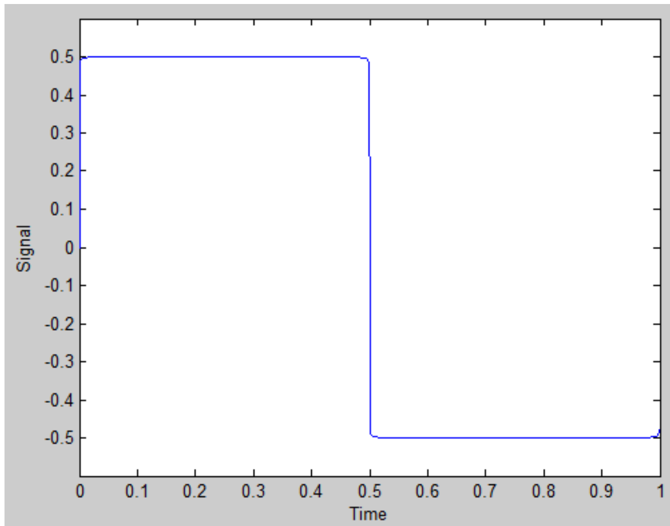

Figure 4: Fourier approximation of the square wave with 1000 terms.

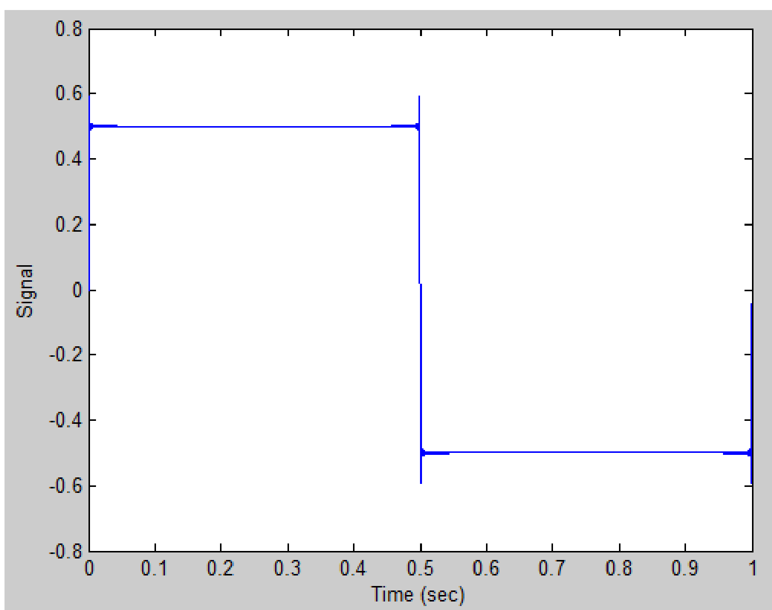

Figure 5: Fourier approximation of the square wave with 1000 terms and 100,000 time samples. The ringing at the edges is present, and does not disappear, regardless of the number of terms used.

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Turn in the plots for these five cases.

How many terms are needed to obtain a reasonable approximation of the square wave.

III. Modify the program you wrote in II above to calculate a triangle wave instead of a square wave.

The triangle wave is the integral of the square wave, so the equation is:

$$s(t) = \sum_{k=1}^{2N} \frac{1}{k^2} \cos(k\omega t)\,, \qquad k \text{ odd}$$

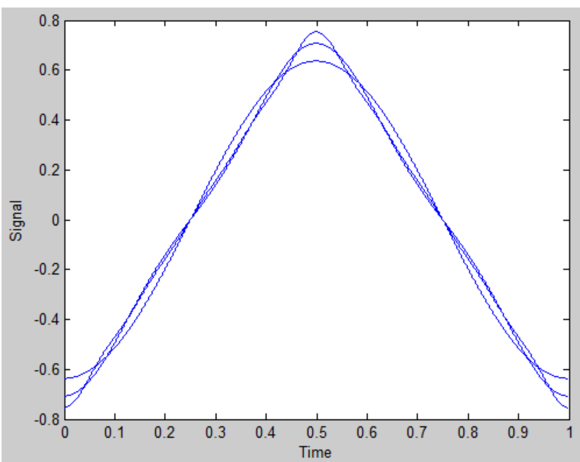
And the program becomes:

```
function trianglesig = trianglewave(n)

% Construct a square wave (squaresig) from the first n terms of the Fourier series


npts = 1000.;

totaltime = 1.0; %seconds

dt  = totaltime/npts;

t = 0:dt:totaltime-dt;

f = 1.0/totaltime;

wt  = 2*pi*f*t;

a0 = 2/pi;


trianglesig = zeros(1,npts);

for k = 1:2:2*n

   a = -a0/(k*k);
```

```
    trianglesig = trianglesig + a*cos(k*wt);
end
plot(t,trianglesig);
end
```

Now how many terms are required to obtain a reasonable approximation of the triangle wave?

Figure 6 shows the result for n values of 1, 2, and 5. Even for these small numbers of terms, the triangle wave is fairly good.



Why is this number so much smaller than the number of terms required for the square wave?

The triangle wave does not have the vertical lines that are present in the square wave. Given that a sinusoid does not have any region where its derivative is infinite, it cannot accurately approximate the square wave unless the number of terms is infinite. Even then, the only reason that the ringing can disappear is because the width of each cycle goes to zero, not because the amplitude of the ringing goes to zero.