Solved by experts

Exercise 31, Linear Algebra: A Modern Introduction, 4th Edition

Exercise 31 Answer

Step by step explanation

HIDE ALL

Tip

- To solve this question we need to find eigenvalues and their eigenvectors.

Explanation

- We will consider β as standard basis of $\mathbb{R}^2$.
- we will find eigenvalues and their eigenvectors.
- Eigenvectors will be basis of C for V.
- with the help of C we get $[T]_c$

Step 1 of 3

Let, $\beta = \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$

$T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(e_2) = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$

Then, $[T]_\beta = \begin{bmatrix} 0 & -4 \\ 1 & 5 \end{bmatrix}$

eigenvalues of $[T]_\beta$:

$$\det([T]_\beta - \lambda I) = \det(\begin{bmatrix} -\lambda & -4 \\ 1 & 5-\lambda \end{bmatrix}$$

$$= \lambda^2 - 5\lambda + 4$$

$$= (\lambda - 4)(\lambda - 1)$$

Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$

step 2 of 3

Eigenvectors:

Let,

$v_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

Then, Eigenvector corresponding to eigenvalue 1 :

$$[T]_{\beta v_1} = 1.\, v_1$$

$$\begin{bmatrix} -4y \\ x+5y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow -4y = x$$

$v_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ be the eigenvector corresponding to eigenvalue 1.

Let,

$v_2 = \begin{bmatrix} z \\ w \end{bmatrix}$

Then, Eigenvector corresponding to eigenvalue 4 :

$$[T]_{\beta v_2} = 4.\, v_2$$

$$\begin{bmatrix} -4w \\ z+5w \end{bmatrix} = \begin{bmatrix} 4z \\ 4w \end{bmatrix}$$

$$\Rightarrow z = -w$$

$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ be the eigenvector corresponding to 4.

Step 3 of 3

$C = \left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Let, α, β be scalars such that

$\alpha v_1 + \beta v_2 = 0$

Then ,

$\begin{bmatrix} -4\alpha \\ \alpha \end{bmatrix} + \begin{bmatrix} \beta \\ -\beta \end{bmatrix} = 0$

Thus, $\alpha, \beta = 0$

Therefore C is linearly independent,

T is diagonalizable,

$[T]_C = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

Final Answer

$C = \left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$[T]_C = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$