Lecture 2: Discrete Time Systems and Z-Transform

ELEN 472: Introduction to Digital Control

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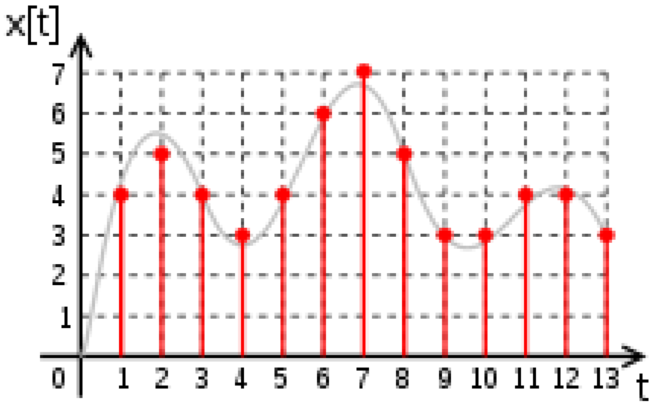
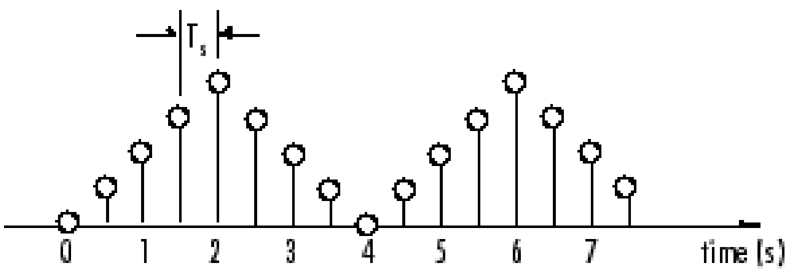
Discrete-Time Signals

- A discrete-time signal is **a sequence of values that correspond to particular instants in time**.
  - By ***sampling***, a continuous-time signal can be converted into a discrete-time signal.
- The time instants at which the signal is defined are the signal's ***sample times***, and the associated signal values are the signal's ***samples.***
- For a periodically sampled signal, the equal interval between any pairs of consecutive sample times is the signal's ***sample period*** T_s.



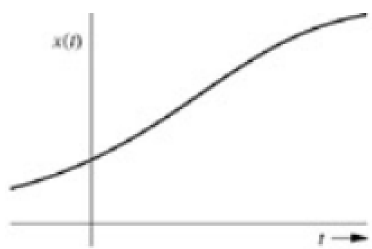
Sampling



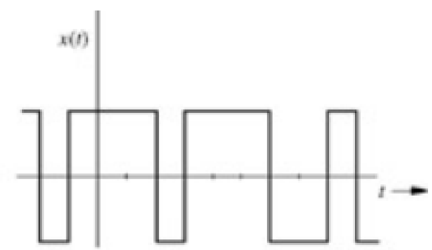
Sample Period

Relationship of Discrete-time/Continuous-time and Digital/Analog Signals

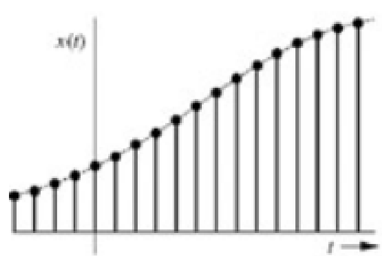
- Discrete-time/Continuous-time describe the **time behaviors** of signals
  - If the signal is **continuous in time** -> Continuous-time Signal
  - If the signal is **discrete in time** -> Discrete-time Signal
- Digital/Analog describe the **amplitude values** of signals
  - If the signal’s amplitude can take **any (infinite) values** -> Analog Signals
  - If the signal’s amplitude only **takes finite values** -> Digital Signals



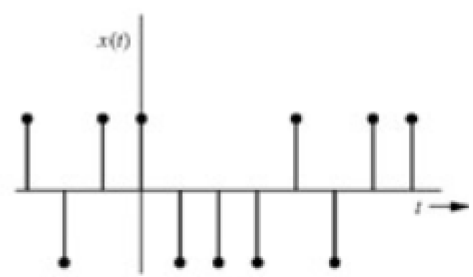
Continuous Time, Analog



Continuous Time, Digital



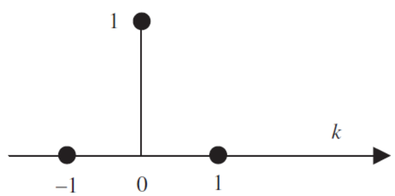
Discrete Time, Analog



Discrete Time, Digital

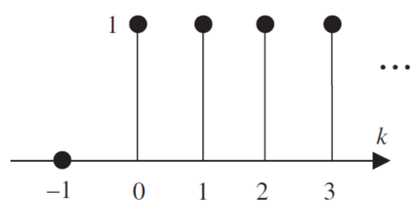
Some Useful Discrete-Time Signals

- Discrete-Time Impulse function $\delta[k]$:



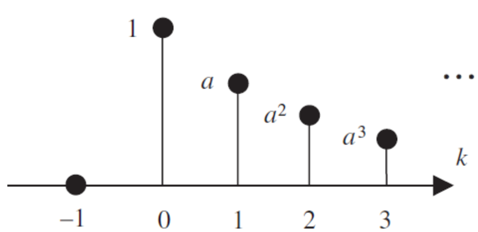
$$u(k) = \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

- Discrete-Time Unit Step Function $u[k]$:



$$u[k] = \begin{cases} 1 & for\ k \geq 0 \\ 0 & for\ k < 0 \end{cases}$$

- Discrete-Time Exponential function $u[k]$:



$$u(k) = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

Discrete-Time Systems

- Systems whose inputs and outputs are **discrete-time signals** are called **discrete-time systems**.

$u[k]$ → **Discrete-Time System** → $y[k]$

- Example Discrete-Time System:
  - **Question**: A person makes a deposit in a bank regularly at an interval of T (say, 1 month). The bank pays a certain interest (r) on the account balance during the period T and mails out the account balance to the depositor. **Find** the equation relating the balance $y[k]$ to the deposit $u[k]$.
  - **Solution:** The balance $y[k]$ is the sum of:
    - The previous balance $y[k-1]$
    - The interest on $y[k-1]$ during the period T
    - The deposit $u[k]$
    - $y[k] = y[k-1] + ry[k-1] + u[k]$
    - $y[k] = (1+r)y[k-1] + u[k] \Longrightarrow y[k] - (1+r)y[k-1] = u[k]$

Difference Equations

- In the previous example, the equation:

$$y[k] - (1 + r)y[k-1] = u[k]$$

is named as **Difference Equation**.

- ***Difference Equations*** relate to ***differential equations*** as discrete mathematics relates to continuous mathematics.

Continuous-Time Systems	**Discrete-Time Systems**
$\dfrac{dy(t)}{dt} = u(t) + \cdots$	$y(k) - y(k-1) = u(k) \ldots$

- General Form of Difference Equations:

$$\begin{aligned} y(k+n) = f[&y(k+n-1),\ y(k+n-2),\ \ldots,\ y(k+1),\ y(k),\ u(k+n), \\ &u(k+n-1),\ \ldots,\ u(k+1),\ u(k)] \end{aligned}$$

Linear Difference Equations

- **Linear Difference Equations:**

$$
\begin{aligned}
&y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k) \\
&\quad = b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k)
\end{aligned}
$$

  - where $a_0, a_1, \dots a_{n-1}$ and $b_0, b_1, \dots, b_n$ are coefficients of $y(.)$ and $u(.)$.

- **Properties:**
  - **System Order**: the difference between the highest and lowest arguments of $y(.)$ and $u(.)$.
  - **Time Invariant**: If these coefficients ($a_0, \dots, a_{n-1}, b_0, \dots, b_n$) are constants, then this difference equation is **T**ime **I**nvariant.
  - **Homogeneous**: If $u(.) = 0$, then this difference equation is **Homogeneous**.

Examples

- For each of the following difference equations, determine the order of the equation. Is the equation (a) linear, (b) time invariant, or (c) homogeneous?

$$y(k+2) + 0.8y(k+1) + 0.07y(k) = u(k)$$
$$y(k+4) + \sin(0.4k)y(k+1) + 0.3y(k) = 0$$
$$y(k+1) = -0.1y^2(k)$$

- **Solution:**
  - System 1:
    - The order is 2.
    - The system is ***linear*** and ***time invariant*** since all coefficients are constant.
    - The system is ***not homogeneous*** due to $u(k)$.
  - The order is 4. The system is ***linear*** but ***time varying*** due to the second coefficient. The system is ***homogeneous***.
  - The order is 1. The right-hand side is a nonlinear function of $y(k)$, but does not include $u(k)$ and coefficients that depend on time explicitly. The system is ***nonlinear***, ***time invariant***, and ***homogeneous***.

Practice Questions

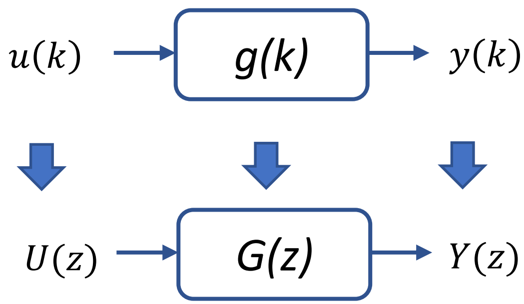
- For each of the following equations, determine the order of the equation and then test it for (1) linearity, (2) time invariance, and (3) homogeneity.

  (a) $y(\mathrm{k}+2) = y(\mathrm{k}+1)\ y(k) + u(k)$
  (b) $y(k+3) + 2\ y(k) = 0$
  (c) $y(k+4) + y(k-1) = u(k)$

- **Solution:**
  - (a): System order: 2; Linear: No, due to $y(k+1)y(k)$; Time-invariant: yes; homogeneous: no, due to $u(k)$.
  - (b): System order: 3; Linear: yes; Time-invariant: yes; homogeneous: yes.
  - (c): System order: 5, $4 - -1 = 5$; Linear: yes; Time-invariant: yes; Homogeneous: no, due to $u(k)$.

The Z-Transform

- The z-Transform is an important tool in the analysis and design of discrete-time systems.
  - It is equivalent to Laplace-Transform in continuous-time systems.
- **Why z-Transform?**
  - It simplifies the solution of discrete-time problems by converting **Convolution** into **Multiplication**.



$$y(k) = u(k) * g(k)$$
$$= \sum u(n)g(k-n)$$

z-Transform

$$Y(z) = U(z) \cdot G(z)$$

- z-Transform is the discrete-time version of Laplace-Transform

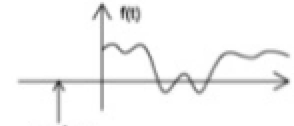
$$z = e^{sT}$$

  - Where T is the sampling period.

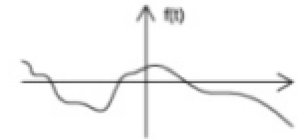
The Z-Transform

Causal signals: Signals with zero values for negative time.

Causal



Noncausal



- The following is the definition of the z-transform:

DEFINITION 2.1

Given the causal sequence $\{u_0, u_1, u_2, \ldots, u_k, \ldots\}$, its z-transform is defined as

$$U(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_k z^{-k}$$
$$= \sum_{k=0}^{\infty} u_k z^{-k}$$

  - The z^{-1} in the above equation can be regarded as a time delay operator.

- Example:
  - Obtain the z-transform of the sequence $\{u_k\}_{k=0}^{\infty} = \{1, 3, 2, 0, 4, 0, 0, 0, 0..\}$
- Solution:
  - Using the z-transform's definition equation, we have:

$$U(z) = 1 + 3z^{-1} + 2z^{-2} + 4z^{-4}$$

Practice Questions

- Find the z-transforms of the following sequences:

  (a) $\{0, 1, 2, 4, 0, 0, \ldots\}$
  (b) $\{0, 0, 0, 1, 1, 1, 0, 0, 0, \ldots\}$
  (c) $\{0, 2^{-0.5}, 1, 2^{-0.5}, 0, 0, 0, \ldots\}$

- Solution:
  - (a) $z^{-1} + 2z^{-2} + 4z^{-3}$
  - (b) $z^{-3} + z^{-4} + z^{-5}$
  - (c) $2^{-0.5}z^{-1} + z^{-2} + 2^{-0.5}z^{-3}$

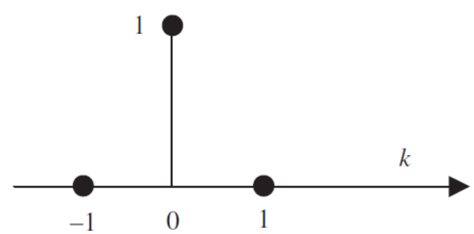
Z-Transforms of Standard Discrete-time Signals: Unit Impulse

- Example: Unit Impulse
  - Consider the discrete-time impulse:

$$u(k) = \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

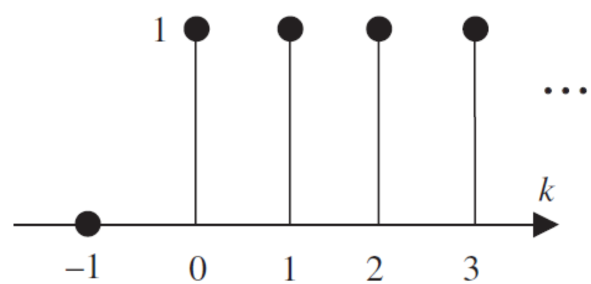
  - Apply the z-transform, we have

$$U(z) = 1$$



Z-Transforms of Standard Discrete-time Signals: Sampled Step

- Consider the sequence $\{u_k\} = \{1, 1, 1, 1, 1, \dots \dots\}$



- Using the z-transform on u_k, we have:

$$U(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-k} + \dots$$
$$= \sum_{k=0}^{\infty} z^{-k}$$

- Since $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \qquad |a| < 1$
  - We have $U(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

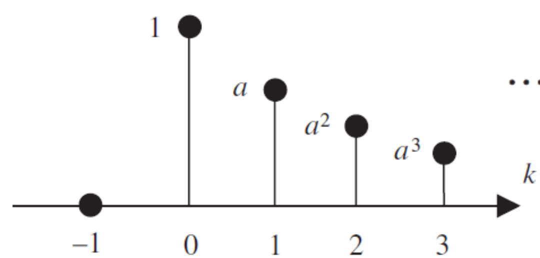
Note that this equation is **only valid** for $|z| < 1$. This implies that the z-transform expression we obtain has a region of convergence outside which is not valid.

Z-Transforms of Standard Discrete-time Signals: Exponential

- Let

$$u(k) = \begin{cases} a^k, k \geq 0 \\ 0, k < 0 \end{cases}$$

  - If $0 < a < 1$, we can plot the $u(k)$ as follows:

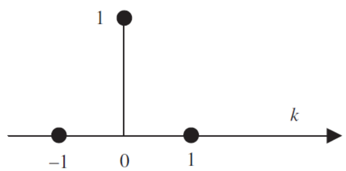
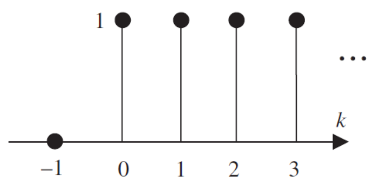
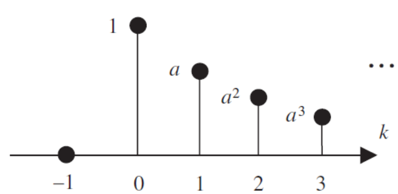


- Using z-transform on $u(k)$, we have

$$U(z) = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$

- Since $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \qquad |a| < 1$
  - We have $U(z) = \frac{1}{1-a/z} = \frac{z}{z-a}$

Z-Transforms of Standard Discrete-time Signals: Summary

Discrete-time Signals	Z-transform Results
Unit Impulse $u(k) = \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$ 	$U(z) = 1$
Sampled Step $\{u_k\} = \{1, 1, 1, 1, 1, \dots \dots\}$ 	$U(z) = \dfrac{1}{1 - z^{-1}} = \dfrac{z}{z - 1}$
Exponential $u(k) = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$ 	$U(z) = \dfrac{1}{1 - a/z} = \dfrac{z}{z - a},$ $\lvert a \rvert < 1$

Z-Transform Pairs

No.	Continuous Time	Laplace Transform	Discrete Time	z-Transform
1	$\delta(t)$	1	$\delta(k)$	1
2	$1(t)$	$\frac{1}{s}$	$1(k)$	$\frac{z}{z-1}$
3	t	$\frac{1}{s^2}$	kT**	$\frac{zT}{(z-1)^2}$
4	t^2	$\frac{2!}{s^3}$	$(kT)^2$	$\frac{z(z+1)T^2}{(z-1)^3}$
5	t^3	$\frac{3!}{s^4}$	$(kT)^3$	$\frac{z(z^2+4z+1)T^3}{(z-1)^4}$
6	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	a^k***	$\frac{z}{z-a}$
7	$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$	$1-a^k$	$\frac{(1-a)z}{(z-1)(z-a)}$
8	$e^{-\alpha t}-e^{-\beta t}$	$\frac{\beta-\alpha}{(s+\alpha)(s+\beta)}$	a^k-b^k	$\frac{(a-b)z}{(z-a)(z-b)}$

***The function $e^{-\alpha kT}$ is obtained by setting $a = e^{-\alpha T}$.

Z-Transform Pairs (Continued)

9	$te^{-\alpha t}$	$\dfrac{1}{(s+\alpha)^2}$	kTa^k	$\dfrac{azT}{(z-a)^2}$
10	$\sin(\omega_n t)$	$\dfrac{\omega_n}{s^2+\omega_n^2}$	$\sin(\omega_n kT)$	$\dfrac{\sin(\omega_n T)z}{z^2-2\cos(\omega_n T)z+1}$
11	$\cos(\omega_n t)$	$\dfrac{s}{s^2+\omega_n^2}$	$\cos(\omega_n kT)$	$\dfrac{z[z-\cos(\omega_n T)]}{z^2-2\cos(\omega_n T)z+1}$
12	$e^{-\zeta\omega_n t}\sin(\omega_d t)$	$\dfrac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$	$e^{-\zeta\omega_n kT}\sin(\omega_d kT)$	$\dfrac{e^{-\zeta\omega_n T}\sin(\omega_d T)z}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$

Properties of the Z-transform

- Linearity:

$$\mathcal{Z}\{\alpha f_1(k) + \beta f_2(k)\} = \alpha F_1(z) + \beta F_2(z)$$

- **Example:**
  - Find the z-transform of the causal sequence:

$$f(k) = 2 \times 1(k) + 4\delta(k), \qquad k = 0,1,2,3, \dots$$

- **Solution:**
  - Using linearity, the transform of the sequence is:

$$F(z) = \mathcal{Z}\{2 \times \underbrace{1(k)}_{\text{Sampled Step Signal}} + 4\underbrace{\delta(k)}_{\text{Unit Impulse}}\} = 2\mathcal{Z}\{1(k)\} + 4\mathcal{Z}\{\delta(k)\} = \frac{2z}{z-1} + 4 = \frac{6z-4}{z-1}$$

Practical Question

- Use the linearity of the z-transform to obtain the transform of the following discrete-time functions:
  - $\sin(k\omega T)$
  - Hint: $\sin(k\omega T) = \frac{e^{jk\omega T} - e^{-jk\omega T}}{2j}$

$$a^k \text{ ***} \Longrightarrow \frac{z}{z-a}$$

- **Solution**

$$Z\{\sin(k\omega T)\} = \frac{1}{2j}\left[Z\{e^{jk\omega T}\} - Z\{e^{-jk\omega T}\}\right]$$

$$= \frac{1}{2j}\left[\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{-j\omega T}}\right]$$

$$= \frac{1}{2j}\left[\frac{\left(e^{j\omega T} - e^{-j\omega T}\right)z}{z^2 - \left(e^{j\omega T} + e^{-j\omega T}\right)z + 1}\right] = \frac{\sin(\omega T)z}{z^2 - 2\cos(\omega T)z + 1}$$

Properties of the Z-transform: Time Delay

- This equation shows the time delay property of z-transform:

$$\mathscr{Z}\{f(k-n)\} = z^{-n}F(z)$$

- Example:
  - Find the z-transform of the causal sequence:

$$f(k) = \begin{cases} 4, & k = 2,\ 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- Solution:
  - The given sequence is a sampled step starting at $k = 2$ rather than $k = 0$ (i.e., it is delayed by two sampling periods). Using the delay property, we have:

$$F(z) = \mathscr{Z}\{4 \times 1(k-2)\} = 4\, z^{-2}\mathscr{Z}\{1(k)\} = z^{-2}\frac{4z}{z-1} = \frac{4}{z(z-1)}$$

Properties of the Z-transform: Time Advance

- The following equations show the time advance property:

$$\mathscr{Z}\{f(k+1)\} = zF(z) - zf(0)$$

$$\mathscr{Z}\{f(k+n)\} = z^n F(z) - z^n f(0) - z^{n-1} f(1) - \cdots - zf(n-1)$$

- **Example:**
  - Using the time advance property, find the z-transform of the causal sequence: $\{f(k)\} = \{4, 8, 16, \dots\}$
- **Solution:**
  - The sequence can be written as $f(k) = 2^{k+2} = g(k+2), k = 0,1,2, \dots$
  - $g(k)$ is the exponential time function, $g(k) = 2^k, k = 0,1,2,3 \dots$
  - Using the time advance property, we can write:

$$F(z) = z^2 G(z) - z^2 g(0) - zg(1) = z^2 \frac{z}{z-2} - z^2 - 2z = \frac{4z}{z-2}$$

Properties of the Z-transform: Multiplication by Exponential

- The following equation presents the multiplication by exponential property:

$$\mathscr{Z}\{a^{-k} f(k)\} = F(az)$$

- **Example:**
  - Find the z-transform of the exponential sequence: $f(k) = e^{-\alpha k T}, k = 0,1,2, \ldots$
- **Solution:**
  - Recall that z-transform of a sampled step signal is $F(z) = (1 - z^{-1})^{-1}$
  - Observe that $f(k)$ can be rewritten as $f(k) = \left(e^{\alpha T}\right)^{-k} \times 1$
  - Then apply the multiplication by exponential property:

$$\mathscr{Z}\left\{ (e^{\alpha T})^{-k} f(k) \right\} = [1 - (e^{\alpha T} z)^{-1}]^{-1} = \frac{z}{z - e^{-\alpha T}}$$

Practice

- Use the multiplication by exponential property to obtain the transforms of the following discrete-time functions.
  - $\mathrm{e}^{-\alpha kT}\sin(k\omega T)$
- **Solution:**
  - The multiplication by exponential property gives:

$$z\left\{e^{-\alpha kT} f(k)\right\} = F(e^{\alpha T} z)$$

$$Z\left\{e^{-\alpha kT}\sin(k\omega T)\right\} = \frac{\sin(\omega T)\left(e^{\alpha T} z\right)}{\left(e^{\alpha T} z\right)^2 - 2\cos(\omega T)\left(e^{\alpha T} z\right) + 1} = \frac{\sin(\omega T) e^{-\alpha T} z}{z^2 - 2\cos(\omega T) e^{-\alpha T} z + e^{-2\alpha T}}$$

Properties of the Z-transform: Complex Differentiation

- The complex differentiation property can be presented by using:

$$\mathscr{Z}\left\{ k^m f(k) \right\} = \left(-z \frac{d}{dz} \right)^m F(z)$$

- **Example:**
  - Find the z-transform of the sampled ramp sequence:

$$f(k) = k, \qquad k = 0,1,2, \ldots$$

- **Solution:**
  - Recall that the z-transform of a sampled step signal is

$$F(z) = \frac{z}{z-1}$$

  And observe that $f(k)$ can be rewritten as $f(k) = k \times 1, \quad k = 0,1,2, \ldots$
  - Then, apply the complex differentiation property to obtain:

$$\mathscr{Z}\{k \times 1\} = \left(-z \frac{d}{dz} \right) \left(\frac{z}{z-1} \right) = (-z) \frac{(z-1) - z}{(z-1)^2} = \frac{z}{(z-1)^2}$$