Lecture 10: Analog Control System Design

ELEN 472: Introduction to Digital Control

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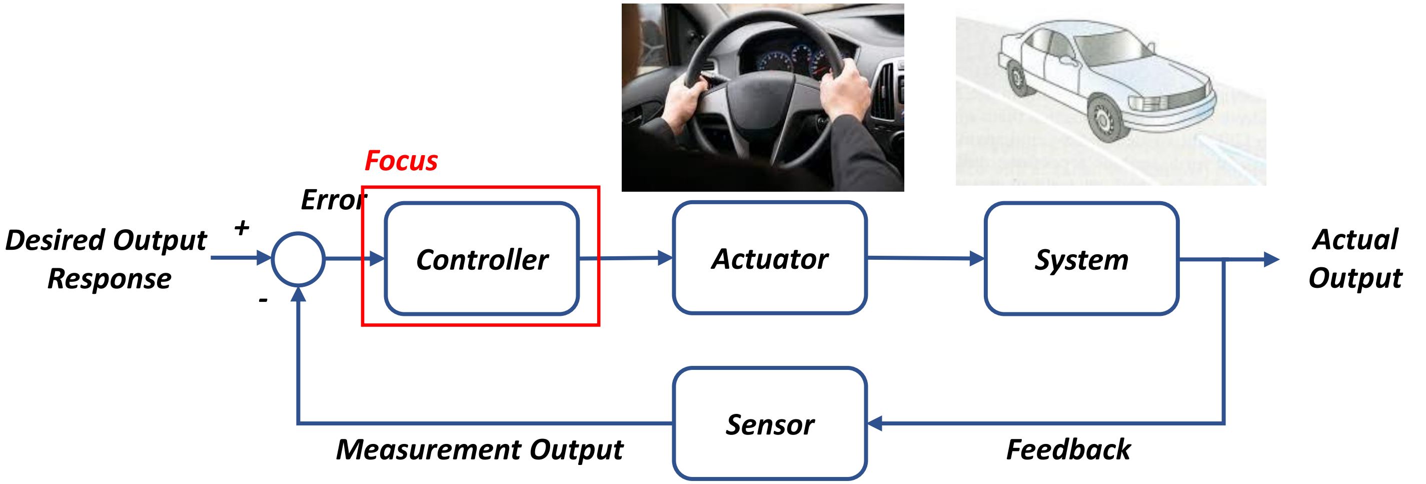
Introduction

- **Purpose:** This lecture reviews the design of analog controllers and prepares us for the digital controller design methods.



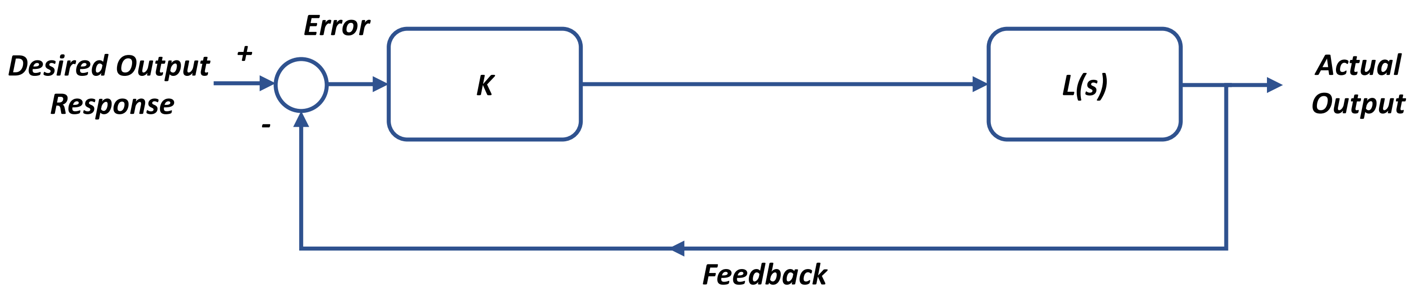
- **Closed-loop Control System:** control system with a feedback link.
  - The feedback link can monitor the actual output.
  - Based on the feedback measurements, the controller adjust its output to **control** the analog system outputs the desired response.

Root locus

- The **root locus** method provides a quick way of predicting the closed-loop behavior of a system based on its open-loop **poles** and **zeros**.
  - The method is based on the properties of the closed-loop **characteristic equation:**

$$1 + KL(s) = 0$$

  - Where K is a design parameter and $L(s)$ is the loop gain of the system.

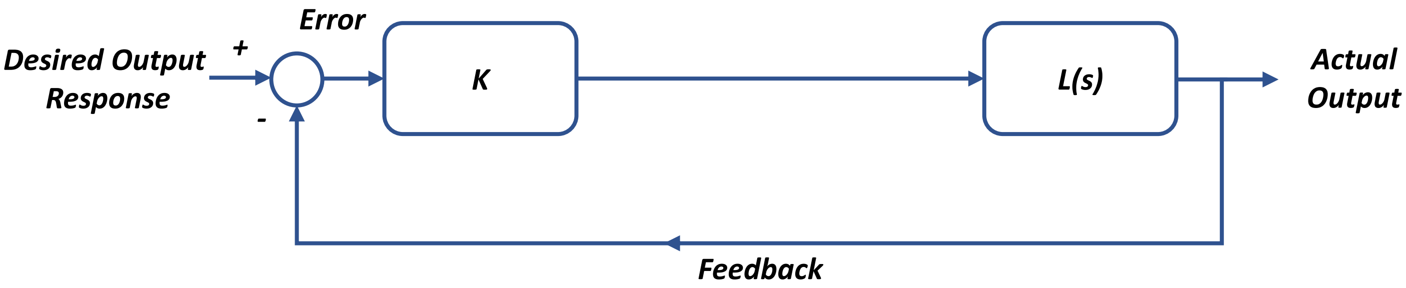


  - Varying the value of K, we can get different **closed-loop poles.**
  - Plot all closed-loop poles together -> **Root Locus Diagram**

An Example

- Consider the system with the transfer function

$$L(s) = \frac{1}{s(s+7)(s+11)}$$



  - where K is the gain of the transfer function, and it is positive.
- The transfer function of the closed-loop system is

$$T(s) = \frac{KL(s)}{1+KL(s)} = \frac{K}{s^3 + 18s^2 + 77s + K}$$

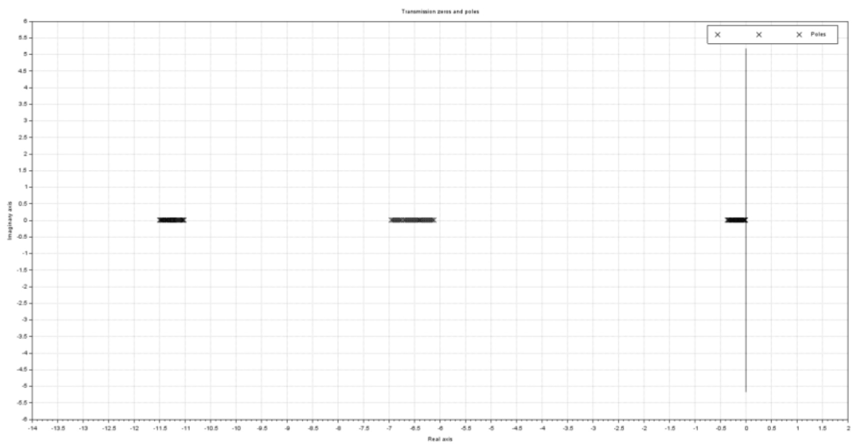
- Thus, the characteristic equation is

$$q(s) = s^3 + 18s^2 + 77s + K = 0$$

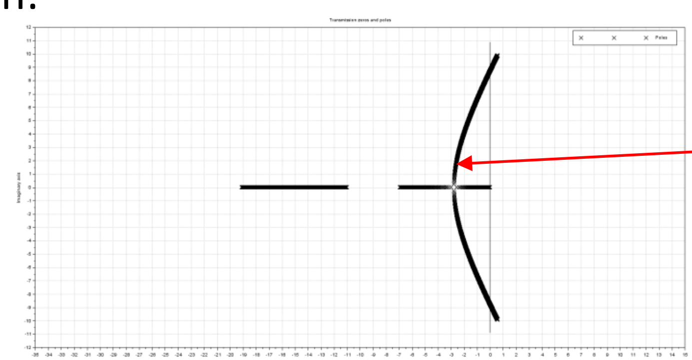
- The constant K here affects the **closed-loop poles**' locations.

An Example

- To check the effect of K, let's assign a few values to K and then see how the locations of poles varies.
  - Let's vary K from 1 to 1000:



    - This cross marks are locations of poles
  - After completion:



Every point on the brank is associated with a value of K

Root locus Rules – Manually Generate Root Locus Diagram

- We can use the following rules for sketching root locus diagrams:
  1. The number of root locus branches is equal to the number of open-loop poles of $L(s)$.
  2. The root locus branches start at the open-loop poles and end at the open-loop zeros or at infinity.
  3. The branches going to infinity asymptotically approach the straight lines defined by the angle:

$$\theta = \frac{(2q+1)180°}{P-Z}, q = 0,1,2,\ldots, P-Z-1$$

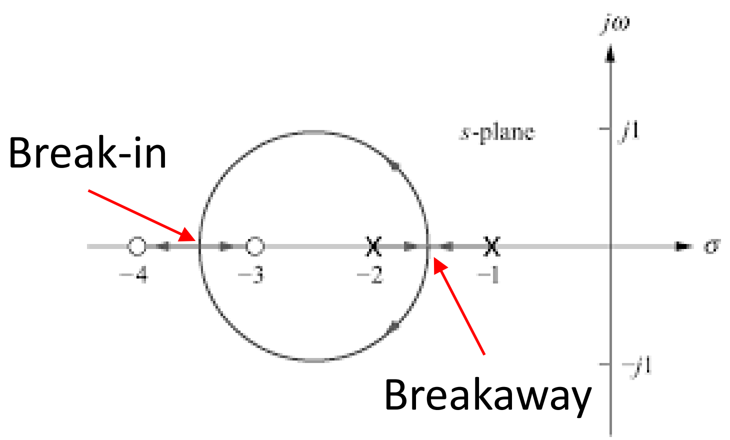
  - And the intercept:

$$\alpha = \frac{\sum Real\ Part\ of\ Poles\ - \sum Real\ Part\ of\ Zeros}{P-Z}$$

  - P is the number of poles of $L(s)$; Z is the number of zeros of $L(s)$

Root locus Rules – Manually Generate Root Locus Diagram

- 4. Find Breakaway/Break-in points:
  - Express K using $L(s)$: $K = -\frac{1}{L(s)}$
  - Breakaway points (points of departure from the real axis) correspond to local maxima of K, i.e., $\frac{dK}{ds} = 0$
  - Break-in points (points of arrival at the real axis) correspond to local minima of K, i.e., $\frac{dK}{ds} = 0$



Example Question

- **Example:**
  - Sketch the root locus plots for the loop gains $L(s) = \frac{1}{(s+1)(s+3)}$
- **Solution:**
  - Using rule 1, the function has two root locus branches; $P = 2$, Z=0
  - By rule 2, the branches start at -1 and -3 and go to infinity.
  - Rule 3 gives the asymptote angles and intercept

$$\theta = \frac{(2q+1)180°}{P-Z}, q = 0,1,2, \dots, P-Z-1$$

$$\theta_1 = \frac{180}{2} = 90; \theta_2 = \frac{540}{2} = 270$$

$$\alpha = \frac{\sum Real\ Part\ of\ Poles\ - \sum Real\ Part\ of\ Zeros}{P-Z}$$

$$\alpha = \frac{-1-3}{2} = -2$$

  - By rule 4, we can find the breakaway point:
    - First express real K using $-1/L(s)$ as: $K = -(s+1)(s+3)$

$$\frac{dK}{ds} = \frac{d-(s+1)(s+3)}{ds} = -2s - 4$$

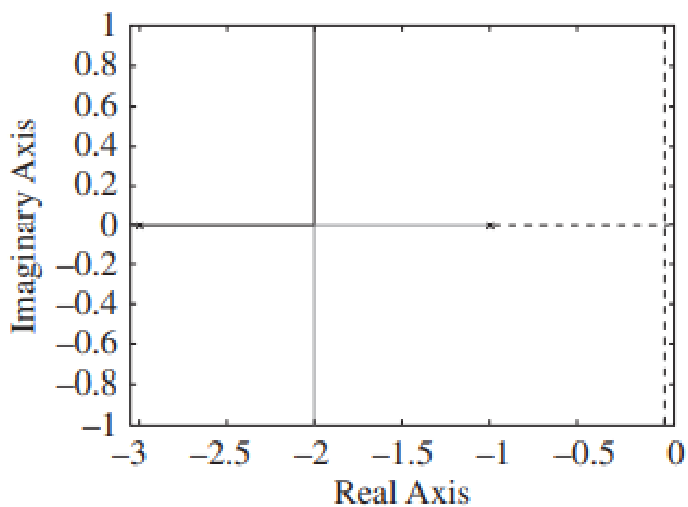
Example Question (Continued)

- Making $\frac{dK}{ds} = \frac{d-(s+1)(s+3)}{ds} = -2s - 4 = 0$, we have the local maximum:

$$-2s - 4 = 0$$

$$s = -2$$

- It can be easily shown that for any system with only two real axis poles, the breakaway point is midway between the two poles.



Practice Question

- Sketch the Root Locus Plots for the Loop Gain:

$$L(s) = \frac{s+5}{(s+1)(s+3)}$$

- **Solution:**
  - From rule 1, the root locus has two branches; $P = 2, Z = 1$
  - One branch ends up at the zero $z = -5$, and one branch ends up at infinite.
  - The branch ends up at infinite has the asymptote angle as:

$$\theta = \frac{(2q+1)180°}{P-Z}, q = 0,1,2,\ldots, P-Z-1$$

$$\theta_1 = \frac{180}{1} = 180$$

  - From Rule 4, expressing K using $-\frac{1}{L(s)}$

$$K = -\frac{(\sigma+1)(\sigma+3)}{\sigma+5}$$

Here $\sigma = s$

$$\frac{dK}{d\sigma} = -\frac{(\sigma+1+\sigma+3)(\sigma+5)-(\sigma+1)(\sigma+3)}{(\sigma+5)^2}$$

$$= -\frac{\sigma^2+10\sigma+17}{(\sigma+5)^2}$$

$$= 0$$

Practice Question (Continued)

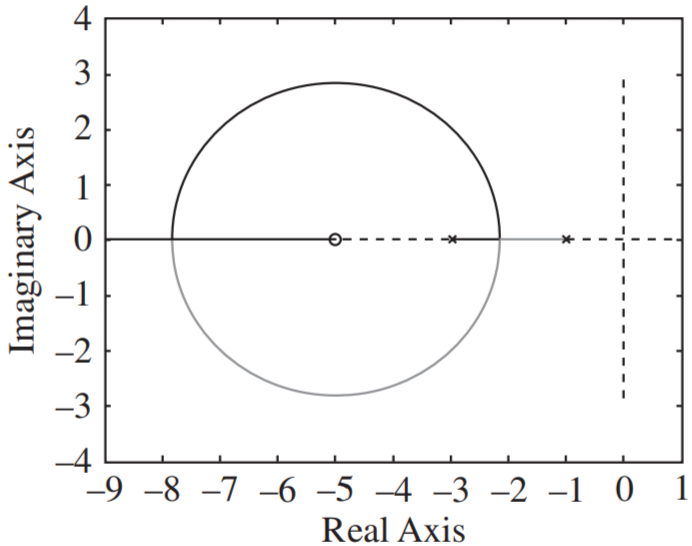
$$\frac{dK}{d\sigma} = -\frac{(\sigma+1+\sigma+3)(\sigma+5)-(\sigma+1)(\sigma+3)}{(\sigma+5)^2}$$

$$= -\frac{\sigma^2+10\sigma+17}{(\sigma+5)^2}$$

$$= 0$$

$$\sigma_b = -2.172 \text{ or } -7.828.$$

- The first value is a breakaway point since it lies between poles;
- The second value is the break-in point.



Root Locus Design

- This figure shows pole locations and the associated time functions.
  - **Real poles** are associated with an **exponential** time response that **decays** for **LHP** poles and **increases** for **RHP** poles.
  - The magnitude of the pole determines the rate of exponential change.
  - A pole at the origin is associated with a unit step.
  - **The real part** of the pole determines the rate of exponential change, and **the imaginary part** determines the frequency of oscillations.

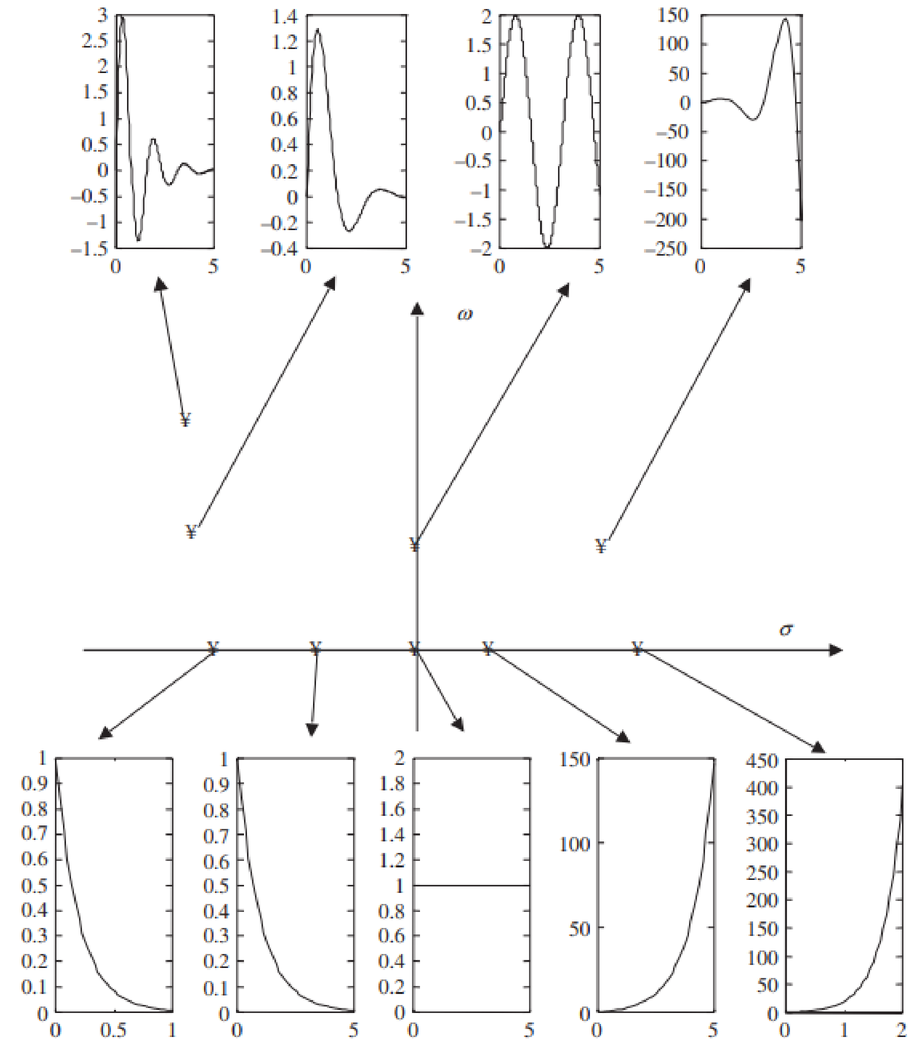


FIGURE 5.3

Pole locations and the associated time responses.

Design specifications and the effect of gain variation

- The objective of control system design is to *construct a system that has a desirable response to standard inputs*.
  - A desirable transient response is one that is sufficiently fast without excessive oscillations.
  - A desirable steady-state response is one that follows the desired output with sufficient accuracy.
- In terms of the response to a unit step input, the transient response is characterized by the following criteria:

1. *Time constant* τ. Time required to reach about 63% of the final value.
2. *Rise time* T_r. Time to go from 10% to 90% of the final value.
3. *Percentage overshoot* (PO).

$$PO = \frac{\text{Peak value} - \text{Final value}}{\text{Final value}} \times 100\%$$

4. *Peak time* T_p. Time to first peak of an oscillatory response.
5. *Settling time* T_s. Time after which the oscillatory response remains within a specified percentage (usually 2 percent) of the final value.

Design specifications and the effect of gain variation (Continued)

- Consider the second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

  - where ζ is the damping ratio and ω_n is the undamped natural frequency.

- Then, the Percentage Overshoot (PO), Peak Time, and Settling Time and be presented as:

$$PO = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$T_s \approx \frac{4}{\zeta\omega_n}$$

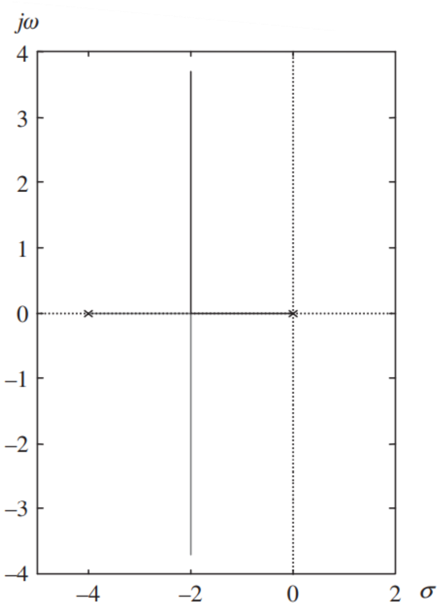
Example

- A position control system with the transfer function of the system is

$$G(s) = \frac{K}{s(s+p)}$$

  - Design a proportional controller for the system to obtain
    - Peak time is less than 5 seconds

- **Solution**
  - The root locus of the system is shown below:



- The root locus remains in the LHP for all K values.
- The closed-loop characteristic equation of the system is given by:

$$s(s+p) + K = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Equating Coefficients $\Downarrow$ $p = 2\zeta\omega_n \quad K = \omega_n^2$

$$\omega_n = \sqrt{K} \quad \zeta = \frac{p}{2\sqrt{K}} \Rightarrow K = \omega_n^2 \text{ And } K = \left(\frac{p}{2\zeta}\right)^2$$

Example Solution

- For a second-order system, the peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

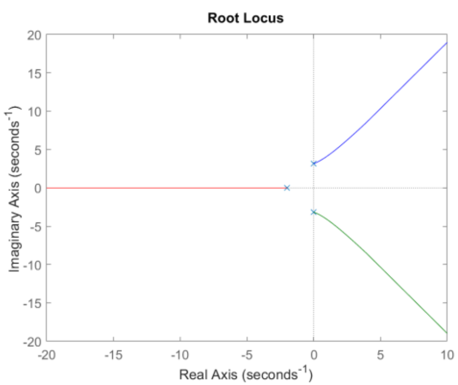
- Let $\omega_n = 1, \zeta = 0.5, T_P = 3.63 < 5$
- Thus, from $K = \omega_n^2$
  - $K = 1$
- From $K = \left(\frac{p}{2\zeta}\right)^2$
  - $K = \left(\frac{p}{1}\right)^2 = p^2$
- Choose the larger K (since we need to satisfy both conditions)

Proportional Control Pro and Con

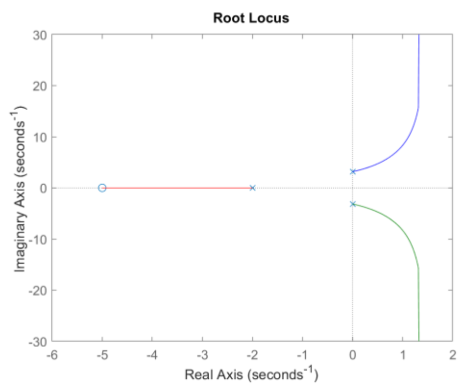
- From the previous example, we can see some advantages and disadvantages of proportional control.
  - **Pro**: The design is simple, and this simplicity carries over to higher-order systems.
  - **Con**: The single free parameter available (i.e., K) limits the designer's choice to one design criterion.
- If more than one aspect of the system time response must be improved, a dynamic controller is needed.
  - We can add another control gain, such as the derivative control.

PD control

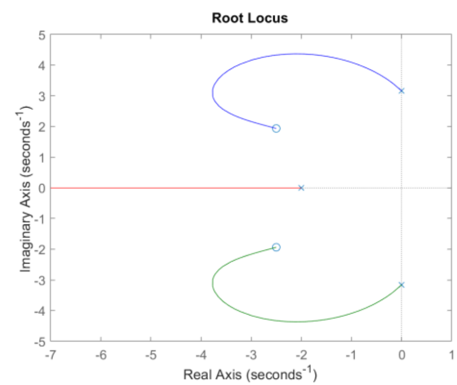
- Adding a zero to the loop gain improves the time response in the system.
  - Adding a zero to the loop gain can pull poles back into the LHD;



0 zero



Add 1 zero



Add 2 zero

  - Thus, adding a zero allows the use of higher gain values (K values) without **destabilizing** the system.
  - Adding a zero is accomplished via adding s in the controller $C(s)$:

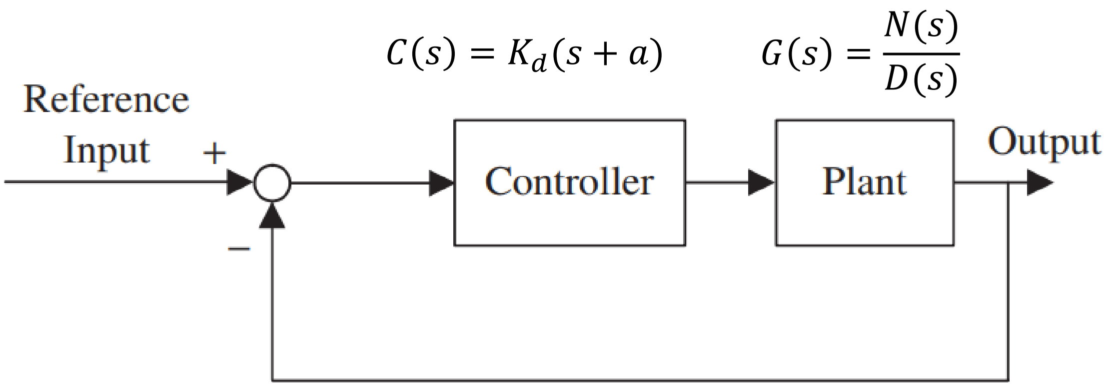
$$C(s) = K_p + K_d s = K_d(s + a)$$

$$a = K_p / K_d$$

This is known as the **Proportional-Derivative** Controller.

PD control TF

- For a **PD controlled closed-loop system**, the transfer function is:



$$G_{cl}(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}$$
$$= \frac{K_d(s+a)N(s)}{D(s) + K_d(s+a)N(s)}$$

Example

- Design a PD controller for the loop gain to meet the following specifications:

$$G(s) = \frac{K}{s(s+p)}$$

  - Specified ζ and ω_n

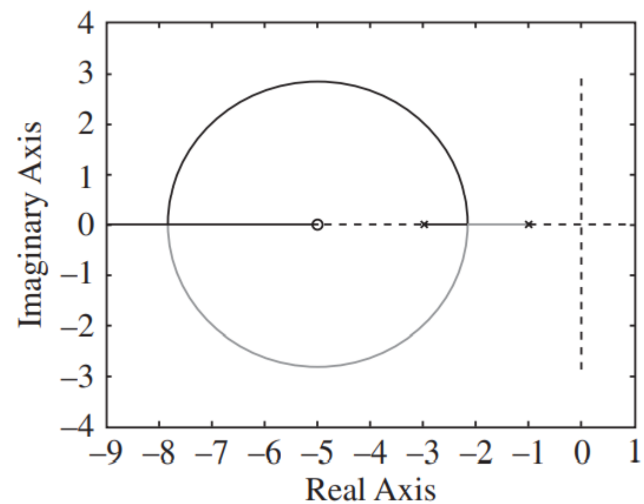
Solution:

- The root locus of the PD-controlled system is shown as:
- This shows that the system gain can be increased with no fear of instability.
- With a PD controller the closed-loop characteristic equation is of the form

$$\begin{aligned} s^2 + ps + K(s+a) &= s^2 + (p+K)s + Ka \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 \end{aligned}$$

- Equating coefficients gives the equations

$$Ka = \omega_n^2 \quad p + K = 2\zeta\omega_n$$



Example Solution

$$Ka = \omega_n^2 \quad p + K = 2\zeta\omega_n$$

- Specified ζ and ω_n (i.e., ζ and ω_n are known)
  - In this case, solve for K and a gives:

$$K = 2\zeta\omega_n - p \quad a = \frac{\omega_n^2}{2\zeta\omega_n - p}$$

  - If $p = 4, \zeta = 0.7, \omega_n = 10\,\frac{rad}{s}$
  - Then, $K = 10$ and $a = 10$.

PID controller

- Adding a zero (PD) may improve the transient response but does not increase the type number of the system.
- Adding **a pole at the origin** increases the type number
  - Increase type number of the system can **decrease the steady-state error**.
- With a **proportional-integral-derivative (PID) controller**, two zeros and a pole are added.
  - This both increases the type number and allows satisfactory reshaping of the root locus.
  - The transfer function of a PID controller is given by

$$C(s) = K_p + \frac{K_i}{s} + K_d s = K_d \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{s}$$

$$2\zeta\omega_n = K_p/K_d, \quad \omega_n^2 = K_i/K_d$$

Example

- Design a PID Controller for an open loop TF:

$$G(s) = \frac{1}{s(s+1)(s+10)}$$

  - To obtain **zero steady-state error** due to ramp input, a damping ratio of 0.7 and an undamped natural frequency of 4 rad/s.

- **Solution:**
  - Observing $G(s)$, it is a type-1 system (one pole at origin).
    - For ramp input, type-1 system has non-zero steady-state error.
    - For type-2 systems, steady-state error for ramp input is 0.
    - How can we change $G(s)$ into type-2 system?
  - Multiple $G(s)$ with a $C(s)$ that can adding another pole at origin:
  - A possible design of $C(s)$ is

$$C(s) = 50\frac{(s+1)(s+0.5)}{s}$$