Exam 1: Review

ELEN 472: Introduction to Digital Control

Lingxiao Wang, Ph.D.
Assistant Professor of Electrical Engineering
Louisiana Tech University

Question Cover Range

- Exam 1 has 10 questions (L: Lecture)
  - Q1: Linear Difference Equations (L2)
  - Q2: z-transform and z-transform inversion (L2 & L3)
  - Q3: Final value theorem (L3)
  - Q4: Solve Difference Equations (L4)
  - Q5: Frequency Response (L4)
  - Q6: Modeling of Digital Control Systems (L5)
  - Q7: Systems with Transport Lag (L6)
  - Q8: Steady-state Errors (L7)
  - Q9: Stability of Digital Control Systems (L8)
  - Q10: Nyquist Criterion, Phase Margin and Gain Margin (L9)

Q1: Linear Difference Equations

- **Linear Difference Equations:**

$$
\begin{aligned}
&y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k) \\
&\quad = b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k)
\end{aligned}
$$

  - where $a_0, a_1, \ldots a_{n-1}$ and $b_0, b_1, \ldots, b_n$ are coefficients of $y(.)$ and $u(.)$.

- **Properties:**
  - **System Order**: the difference between the highest and lowest arguments of $y(.)$ and $u(.)$.
  - **Time Invariant**: If these coefficients ($a_0, \ldots, a_{n-1}, b_0, \ldots, b_n$) are constants, then this difference equation is **T**ime **I**nvariant.
  - **Homogeneous**: If $u(.) = 0$, then this difference equation is **Homogeneous**.

Examples

- For each of the following difference equations, determine the order of the equation. Is the equation (a) linear, (b) time invariant, or (c) homogeneous?

$$y(k+2) + 0.8y(k+1) + 0.07y(k) = u(k)$$
$$y(k+4) + \sin(0.4k)y(k+1) + 0.3y(k) = 0$$
$$y(k+1) = -0.1y^2(k)$$

- **Solution:**
  - System 1:
    - The order is 2.
    - The system is ***linear*** and ***time invariant*** since all coefficients are constant.
    - The system is ***not homogeneous*** due to $u(k)$.
  - The order is 4. The system is ***linear*** but ***time varying*** due to the second coefficient. The system is ***homogeneous***.
  - The order is 1. The right-hand side is a nonlinear function of $y(k)$, but does not include $u(k)$ and coefficients that depend on time explicitly. The system is ***nonlinear***, ***time invariant***, and ***homogeneous***.

Q2: Z-transform and Z-transform Inversion

- The following is the definition of the z-transform:

DEFINITION 2.1

Given the causal sequence $\{u_0, u_1, u_2, \ldots, u_k, \ldots\}$, its z-transform is defined as

$$\begin{aligned} U(z) &= u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_k z^{-k} \\ &= \sum_{k=0}^{\infty} u_k z^{-k} \end{aligned}$$

  - The z^{-1} in the above equation can be regarded as a time delay operator.
  - The relationship between z-transform and Laplace transform:

$$z = e^{sT}$$

- **Example:**
  - Obtain the z-transform of the sequence $\{u_k\}_{k=0}^{\infty} = \{1, 3, 2, 0, 4, 0, 0, 0, 0..\}$
- **Solution:**
  - Using the z-transform's definition equation, we have:

$$U(z) = 1 + 3z^{-1} + 2z^{-2} + 4z^{-4}$$

HW1-Q3: Long Division

- Obtain the inverse z-transform of the function:

$$F(z) = \frac{z - 0.1}{z^2 + 0.04z + 0.25}$$

- **Solution:**
  - Using **long division**

$$\begin{array}{r l}
 & \underline{z^{-1} - 0.14z^{-2} - 0.244z^{-3} + \ldots\ldots} \\
z^2 + 0.04z + 0.25 \, \big) & z - 0.1 \\
 & \underline{z + 0.04 + 0.25z^{-1}} \\
 & -0.14 - 0.25z^{-1} \\
 & \underline{-0.14 - 0.0056z^{-1} - 0.035z^{-2}} \\
 & \quad -0.244z^{-1} + 0.035z^{-2}
\end{array}$$

$$F(z) = \frac{z}{z^2 + 0.04z + 0.25} = z^{-1} - 0.14z^{-2} - 0.244z^{-3} + \ldots.$$

$$\{\mathrm{f}(k)\} = \{0, 1, -0.14, -0.244, \ldots.\}$$

Partial Fraction Expansion

- The most convenient method to obtain the partial fraction expansion of a function with simple real roots is the **method of residues**.
- **Step – 1**: For a z-Transform expression $F(z)$, get $\frac{F(z)}{z}$
- **Step – 2**: Express $F(z)/z$ into the sum of individual terms

$$\frac{F(z)}{z} = \sum_{i=0}^{n} \frac{A_i}{z - z_i}$$

  - Where A_i is the partial fraction coefficient of the i-th term of the expansion:

$$A_i = (z - z_i)\frac{F(z)}{z}\Big|_{z \to z_i}$$

- **Step – 3**: Restore $F(z)$ via $\frac{F(z)}{Z} \times Z$
- **Step – 4**: Get inverse z-Transform of individual terms using z-Transform table (Lecture 2)

HW1 – Q4: Partial Fraction Expansion

- Find the inverse transform of the functions using partial fraction expansion and table look-up.

$$F(z) = \frac{z}{z^2 + 0.3z + 0.02}$$

- **Solution:**

$$\frac{F(z)}{z} = \frac{1}{z^2 + 0.3z + 0.02} = \frac{1}{(z+0.1)(z+0.2)} = 10\left\{\frac{1}{z+0.1} - \frac{1}{z+0.2}\right\}$$

$$F(z) = 10\left\{\frac{z}{z+0.1} - \frac{z}{z+0.2}\right\} \quad \{\mathrm{f}(k)\} = 10\left[(-0.1)^k - (-0.2)^k\right]$$

Repeated Roots

- For a function $F(z)$ with a repeated root of multiplicity r, r partial fraction coefficients are associated with the repeated root. The partial fraction expansion is of the form:

$$F(z) = \frac{N(z)}{(z-z_1)^r \prod_{j=r+1}^{n} z - z_j} = \sum_{i=1}^{r} \frac{A_{1i}}{(z-z_1)^{r+1-i}} + \sum_{j=r+1}^{n} \frac{A_j}{z-z_j}$$

  - The coefficients for repeated roots are:

$$A_{1,i} = \frac{1}{(i-1)!} \frac{d^{i-1}}{dz^{i-1}} (z-z_1)^r F(z) \Big]_{z \to z_1}, \qquad i = 1,\ 2, \ldots, r$$

Check Lecture 3 Page 13 – Page 16

Q3: Final Value Theorem

- The final value theorem allows us to calculate the limit of a sequence as k tends to infinity from the z-transform of the sequence.
  - If one is only interested in the final value of the sequence, this constitutes a significant shortcut.
- **Final Value Theorem**:
  - If a sequence approaches a constant limit as k tends to infinity, then the limit is given by

$$\boxed{f(\infty) = \lim_{z \to 1}[(z-1)F(z)]}$$

Final Value Theorem

HW1 – Q5: Final Value Theorem

- Find the final value for the functions if it exists

$$F(z) = \frac{z}{z^2 - 7z + 6}$$

- **Solution:**

$$\begin{aligned} f(\infty) &= \frac{z}{z^2 - 7z + 6} * (z-1) \Big|_{z \to 1} \\ &= \frac{z}{(z-1)(z-6)} * (z-1) \Big|_{z \to 1} \\ &= \frac{z}{z-6} \Big|_{z \to 1} \\ &= -\frac{1}{5} \end{aligned}$$

Q4: Solve Difference Equations

- Recall **linear difference equation expression**:

$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k)$$
$$= b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k)$$

  - Our goal is to solve this equation to **get the expression of $\boldsymbol{y(k)}$**.
  - $u(k)$ will be provided.
- The key to solve this equation is to use **z-Transform** and its **time delay/advance properties.**

Time Delay Property: $\mathscr{Z}\{f(k-n)\} = z^{-n}F(z)$

Time Advance Property: $\mathscr{Z}\{f(k+1)\} = zF(z) - zf(0)$

$$\mathscr{Z}\{f(k+n)\} = z^n F(z) - z^n f(0) - z^{n-1} f(1) - \cdots - zf(n-1)$$

HW1 – Q6: Solve Difference Equations

- Solve the linear difference equation

$$y(k+1) - 0.8y(k) = 1(k), y(0) = 1$$

with the initial conditions $y(0) = 1$.

- **Solution:**
  - **z-transform**: We begin by z-transforming the difference equation using time advance property to obtain

$$zY(z) - zy(0) - 0.8Y(z) = \frac{z}{z-1}$$

$$zY(z) - z - 0.8Y(z) = \frac{z}{z-1}$$

$$(z-0.8)Y(z) = \frac{z}{z-1} + \frac{z(z-1)}{z-1}$$

$$(z-0.8)Y(z) = \frac{z^2}{z-1}$$

$$Y(z) = \frac{z^2}{(z-1)(z-0.8)}$$

HW1 – Q6: Solve Difference Equations

- **Solution:**
  - Inverse z-transform of $Y(z)$:

$$Y(z) = \frac{z^2}{(z-1)(z-0.8)}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-0.8)}$$

$$\frac{Y(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.8)}$$

$$A = (z-1)\frac{Y(z)}{z}\bigg|_{z=1} = \frac{1}{0.2} = 5$$

$$B = (z-0.8)\frac{Y(z)}{z}\bigg|_{z=0.8} = \frac{0.8}{-0.2} = -4$$

$$\frac{Y(z)}{z} = \frac{5}{z-1} + \frac{-4}{z-0.8}$$

$$Y(z) = \frac{5z}{z-1} + \frac{-4z}{z-0.8}$$

$$y(k) = 5 * 1(k) - 4(0.8)^k$$

Q5: Frequency Response of Discrete-time Systems

- Frequency Response of discrete-time systems gives the **magnitude** and **phase response** of the system to the **sinusoids at all frequencies**.

$u(k) = e^{j\omega k}$

Input Signal (Sinusoids)

Discrete-Time System, $h(k)$

$y(k) = H(e^{j\omega}) \cdot e^{j\omega k}$

Output Signal (Sinusoids)

$$y(k) = \boxed{\left|H\left(e^{j\omega}\right)\right| \cdot \cos\left(\omega k + \angle H\left(e^{j\omega}\right)\right)} + j \cdot \boxed{\left|H\left(e^{j\omega}\right)\right| \sin\left(\omega k + \angle H\left(e^{j\omega}\right)\right)}$$

If input $u(t) = \cos(\omega k)$

If input $u(t) = \sin(\omega k)$

HW1-Q9: Frequency Response

- Find the steady-state response of the systems due to the sinusoidal input $u(k) = 0.5\, sin(0.4\, k)$

$$H(z) = \frac{z}{z^2 + 0.4z + 0.03}$$

- **Solution:**
  - Since the input signal is a sin signal, the output signal should only contain sin part, i.e.,

$$\begin{aligned} y(k) &= j \cdot \left|H\left(e^{j\omega}\right)\right| \sin\left(\omega k + \angle H\left(e^{j\omega}\right)\right) \\ &= j|H(e^{j0.4})|\sin(0.4k + \angle H\left(e^{j0.4}\right)) \end{aligned}$$

$$H\left(e^{j0.4}\right) = \frac{1}{e^{j0.4} + 0.4 + 0.03e^{-j0.4}} = \underbrace{\boxed{0.714}}_{|H(e^{j0.4})|}\angle \underbrace{\boxed{-0.273}}_{\angle H\left(e^{j0.4}\right)}$$

Q6: Modeling of Digital Control Systems

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}$$

- Procedures to get $G_{ZAS}(z)$:
  - **Step 1**: get $\frac{G(s)}{s}$
  - **Step 2**: check z-transform pairs (in **Lecture 2 Page 19**) to get $\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}$
    - **Note:** here, we can omit the $\mathcal{L}^{-1}$ notation to make the equation more concise. $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$
  - **Step 3**: multiple $(1 - z^{-1})$ with $\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$ to get $G_{ZAS}(z)$

$$\boxed{\begin{aligned} G_{ZAS}(z) &= (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\ &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \end{aligned}}$$

HW2-Q2

- Known that the dynamic equation of a point mass (m) with force (f) as input and displacement (x) as output is

$$m\ddot{x}(t) = f(t)$$

  - Find $G_{ZAS}(z)$ for the system.

- **Solution:**
  - First get the transfer function of the system:

$$ms^2X(s) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2}$$

  - Then, follow the equation in the previous page to get $G_{ZAS}(z)$

$$G_{ZAS}(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z}\mathcal{Z}\left\{\frac{1}{ms^3}\right\}$$

$$= \frac{z-1}{z}\mathcal{Z}\left\{\frac{1}{2m}\frac{2}{s^3}\right\} = \frac{z-1}{z}\frac{1}{2m}\mathcal{Z}\left\{\frac{2}{s^3}\right\}$$

$$= \frac{z-1}{z}\frac{1}{2m}\frac{z(z+1)T^2}{(z-1)^3} = \frac{1}{2m}\frac{(z+1)T^2}{(z-1)^2}$$

Q7: System with Transport Lag

- **Transport Lag:**
  - Time delay in system's response to the input signals
- **First** model the delay time using

$$T_d = lT - mT, \quad 0 \leq m < 1$$

  - Find l and m
- **Second**, use the equation to model the system with transport lag:

$$\boxed{G_{ZAS}(z) = \left(\frac{z-1}{z^l}\right)\mathscr{Z}_m\{g_s(kT)\}}$$

  - The above equation involves **modified z-transform** $\mathcal{Z}_m$
  - Two commonly-used modified z-transform:

$$\mathscr{Z}_m\{1(kT)\} = \frac{1}{z-1} \qquad \mathscr{Z}_m\{e^{-pkT}\} = \frac{e^{-mpT}}{z - e^{-pT}}$$

HW2-Q3

- For a system

$$G(s) = \frac{\varepsilon \tau s + 1}{\tau s + 1}$$

  - Where τ is the time constant and ε is a known parameter. If a delay is 25 ms and the sampling period is 10 ms, find the $G_{ZAS}(z)$ for the system.

- **Solution:**

For a delay of 25 ms with $T = 10$ ms we have

$$25 = 3 \times 10 - 0.5 \times 10$$

  - $l = 3$ and $m = 0.5$

$$G_{ZAS}(z) = \left(\frac{z-1}{z^l}\right)\mathcal{Z}_m\{g_s(kT)\} \Rightarrow$$

$$G_{ZAS}(z) = \left(1 - z^{-1}\right)\mathrm{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= \left(1 - z^{-1}\right)\left\{\frac{1}{1 - z^{-1}} + \frac{(\varepsilon - 1)e^{-5/\tau}}{1 - e^{-T/\tau}z^{-1}}\right\}z^{-3}$$

$$= \left[1 + \frac{(\varepsilon - 1)(z - 1)e^{-5/\tau}}{z - e^{-T/\tau}}\right]z^{-3}$$

Q8: Steady State Errors

Signal	Type-0	Type-1	Type-2
Sampled step input	$\frac{1}{1+L(1)}$ or $\frac{1}{1+K_p}$	0	0
Sampled ramp input	∞	$\frac{T}{(z-1)L(z)\|_{z=1}}$ or $\frac{1}{K_v}$	0
Sampled parabolic input	∞	∞	$\frac{T^2}{(z-1)^2L(z)\|_{z=1}}$ or $\frac{1}{K_a}$

$L(z) = G_{ZAS}(z)C(z)$

- $G_{ZAS}(z)$ is the system's transfer function
- $C(z)$ is the controller's transfer function

T is the sampling period

Example

- For the following systems with unity feedback, find
  - The position error constant.
  - The velocity error constants.
  - The steady state error due to a unit step input.
  - The steady-state error due to a unit ramp input.

$$G(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$$

- Solution:
  - The position error constant:
    - The system is Type 1 and has an infinite position error constant.
  - The velocity error constants:

$$K_v = \frac{1}{T}(z-1)G(z)\Big|_{z=1} = \frac{0.4(1+0.2)}{T(1-0.1)} = \frac{0.5333}{T}$$

  - The steady state error due to a unit step input:
    - The system is Type 1 and has zero steady-state error due to step.
  - The steady-state error due to a unit ramp input:
    - $e(\infty) = \frac{1}{K_v} = \frac{T}{0.5333}$

Question 9: Stability Test

- Find Poles for low order systems
- Jury test for high order systems

Practice Question

- Determine the stability of the following system:

$$y(k+2) - 0.8y(k+1) + 0.07y(k) = 2u(k+1) + 0.2u(k) \quad k = 0, 1, 2, \ldots$$

- Solution:
  - First, find the transfer function $G(z) = \frac{Y(z)}{U(z)}$

$$G(z) = \frac{2z + 0.2}{z^2 - 0.8z + 0.07} = \frac{2(z + 0.1)}{(z - 0.7)(z - 0.1)}$$

  - Since |0.7|<1 and |0.1|<1, the system is stable.

Question 10: Nyquist Criterion

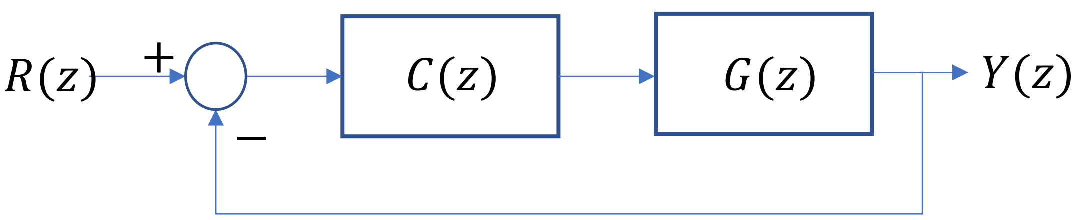
Nyquist Criterion

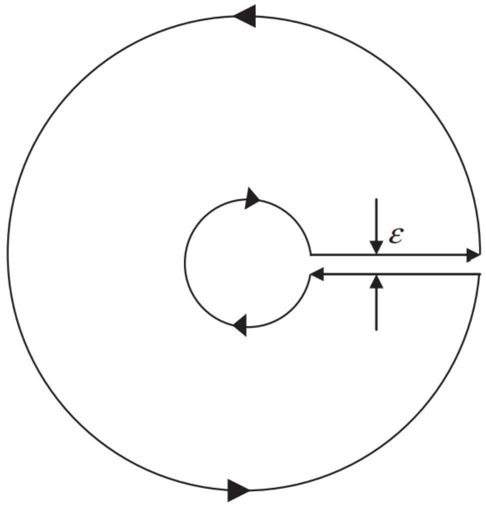
A Closed-Loop system has Z poles **inside** the Nyquist Contour with

$$Z = P - N$$

P is the number of Poles of $L(z)$ **inside** the Nyquist Contour;
N is the number of **anticlockwise contour encirclement** of (-1, 0) given the **Nyquist Curve** of $L(z)$.

$$L(z) = C(z)G(z)$$

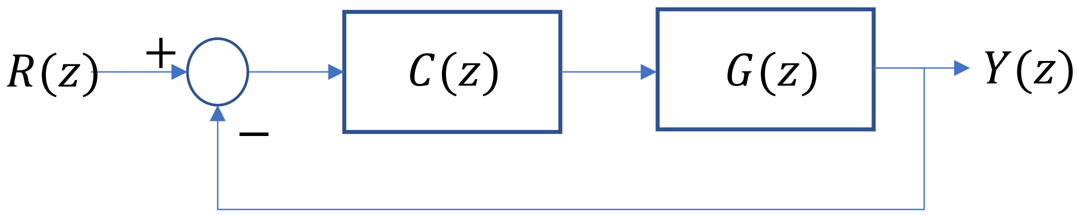




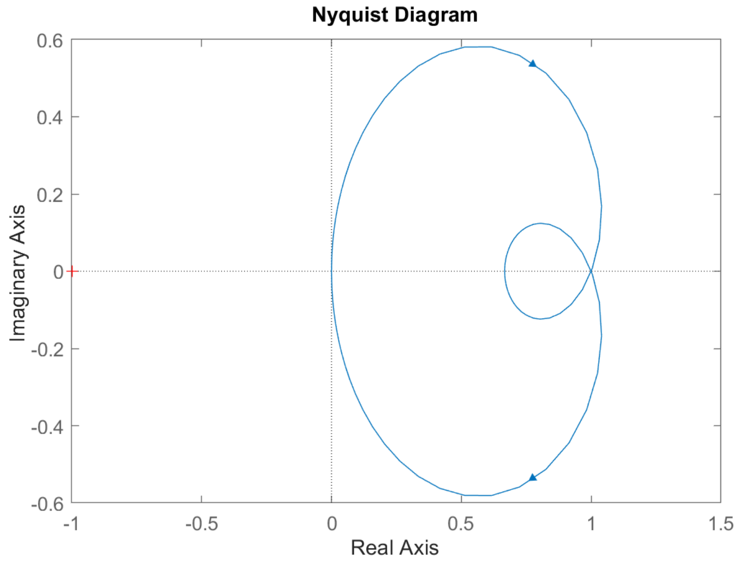
Nyquist Contour

Example

- Consider the following closed-loop discrete-time



  - Where $C(z) = 1$ and $G(z) = \frac{5z+4}{(z-2)(z-3)}$.
  - Given the Nyquist Curve of $G(z)$, determine the stability of this closed-loop system.



Example 2 Solution

- Solution:
  - We first need to determine the value of P.
    - P is the number of poles of loop gain outside the unit circle.
    - In this work, loop gain is

$$G(z) = \frac{5z + 4}{(z - 2)(z - 3)}$$

    - Two poles are at 2 and 3, which are both outside the unit circle.
      - Thus, $P = 2$
  - For a stable system, $Z = P - N$ should be 0
    - Here $P = 2$, thus, $Z = 2 - N = 0$
    - N is the encirclement of (-1, 0) for the Nyquist curve. However, in this diagram, the Nyquist curve does not encircle (-1, 0).
    - Thus, the system is **not stable**.