Brownian Motion

Reflection Principle

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Reflection Principle

Suppose that $\{B_t\}_{t \geqslant 0}$ is a standard Brownian motion. Define the running maximum and minimum as

$$M_t = \max_{0 \leqslant s \leqslant t} B_s \text{ and } m_t = \min_{0 \leqslant s \leqslant t} B_s. \tag{1}$$

Obviously, we have $M_t \geqslant 0 \geqslant m_t$ since $B_0 = 0$. Define a stopping time T_a for any $a \in \mathbb{R}$:

$$T_a = \min \{t : B_t = a\}, \tag{2}$$

which is the first time when $\{B_t\}_{t \geqslant 0}$ hits level a. By convention, $T_a = \infty$ when $\{B_t\}_{t \geqslant 0}$ never hits level a. We give two claims without proof.

Theorem

$T_a < \infty$ almost surely.

Theorem

(Strong independent increments) Process $\{X_t\}_{t \geqslant 0}$ given by $X_t = B_{T_a + t} - a$ is a standard Brownian motion and is independent of $\{B_t\}_{0 \leqslant t \leqslant T_a}$.

Reflection Principle

Theorem

(Reflection principle) For each $a \geqslant 0$ *we have*

$$\mathbb{P}\left(M_t \geqslant a\right) = 2\mathbb{P}\left(B_t \geqslant a\right) = \frac{2}{\sqrt{2\pi t}} \int_a^{\infty} \exp\left(-\frac{x^2}{2t}\right) dx. \tag{3}$$

For each $a \leqslant 0$ *we have*

$$\mathbb{P}\left(m_t \leqslant a\right) = 2\mathbb{P}\left(B_t \leqslant a\right) = \frac{2}{\sqrt{2\pi t}} \int_{-\infty}^{a} \exp\left(-\frac{x^2}{2t}\right) dx. \tag{4}$$

Proof:

Reflection Principle

Theorem

For each $a \geqslant 0, y \geqslant 0$ *we have*

$$\mathbb{P}\left(M_t \geqslant a, B_t \leqslant a - y\right) = \mathbb{P}\left(B_t \geqslant a + y\right). \tag{5}$$

For each $a \leqslant 0, y \geqslant 0$ *we have*

$$\mathbb{P}\left(m_t \leqslant a, B_t \geqslant a + y\right) = \mathbb{P}\left(B_t \leqslant a - y\right). \tag{6}$$

Proof: