From the last time, we have already shown the solution of the scalar at the AdS boundary. The solution, however, is not well solved due to the remaining constants. So, we need more boundary conditions by considering the solution at the horizon.

1 Solution on $SAdS_{d+1}$ background

Let us consider SAdS background metric tensor,

$$ds^2 = \frac{L^2}{z^2}\Bigg(- f(z)dt^2 + d\vec{x} + \frac{dz^2}{f(z)}\Bigg) \tag{1}$$

where $f(z) = 1 - \left(\frac{z}{z_h}\right)^d$, we see that at $z_h \to \infty$ the metric tensor reduces to pure AdS.

Normally, for convenient calculation we may map horizon to 1 and we may simply write $\left(\frac{z}{z_h}\right)^d$ to z. However, z_h contains thermodynamic information about the black hole, and I need to show it in the last results. As a result, I continue to consider the horizon to be z_h. Now, let's consider the equation of motion for scalar field

$$\Bigg(\frac{1}{\sqrt{|g|}}\partial_m\Big(\sqrt{|g|}g^{mn}\partial_n\Big) - m^2\Bigg)\phi = 0 \tag{2}$$

and then, we assume the solution of the field by using separation of variable,

$$\phi = \phi(z)e^{ik^\mu x_\mu} = \phi(z)e^{-i\omega t + k\cdot\vec{x}}$$

We start from $\vec{x}$ components, it's easily to get $\quad -\left(\frac{z}{L}\right)^2 k^2\phi$

For t-z component, let easily begin with

$$\frac{1}{\sqrt{\left(\left(\frac{L}{z}\right)^2\right)^{d+1}}}\partial_m\Bigg(\sqrt{\left(\left(\frac{L}{z}\right)^2\right)^{d+1}}g^{mn}\partial_n\phi(z)e^{-i\omega t + k\cdot\vec{x}}\Bigg) = 0$$

$$\frac{1}{\left(\frac{L}{z}\right)^{d+1}}\partial_m\Bigg(\left(\frac{L}{z}\right)^{d+1}g^{mn}\partial_n\phi(z)e^{-i\omega t + k\cdot\vec{x}}\Bigg) = 0$$

and then we see that the metric component is time-independent, so, for time component

$$\frac{z^2}{L^2 f(z)}\partial_t^2\phi(z)e^{-i\omega t + k\cdot\vec{x}} = \left(\frac{z}{L}\right)^2\frac{\omega^2}{f(z)}\phi \tag{3}$$

Then, let us consider the non-trivial one, the z component

$$\frac{1}{\left(\frac{L}{z}\right)^{d+1}}\partial_z\Bigg(\left(\frac{L}{z}\right)^{d+1}g^{zz}\partial_z\phi(z)e^{-i\omega t + k\cdot\vec{x}}\Bigg) = \frac{z^{d+1}}{L^2}\partial_z\Big(z^{-(d+1)}z^2 f(z)\phi'\Big) = 0$$

$$
\begin{aligned}
\frac{z^{d+1}}{L^2}\partial_z\Big(z^{-(d-1)}f(z)\phi'\Big) &= 0 \\
\frac{z^{d+1}}{L^2}\Big(z^{-(d-1)}(f(z)\phi')' - (d-1)z^{-d}f(z)\phi'\Big) &= 0 \\
\frac{z^{d+1}}{L^2}\Big(z^{-(d-1)}\big(f(z)\phi'' + f'(z)\phi'\big) - (d-1)z^{-d}f(z)\phi'\Big) &= 0 \\
\frac{z^2}{L^2}\Big(f(z)\phi'' + f'(z)\phi' - \frac{(d-1)}{z}f(z)\phi'\Big) &= 0
\end{aligned}
$$

From above result we can write (2) as

$$
f(z)\phi'' + \Big(f'(z) - \frac{(d-1)}{z}f(z)\Big)\phi' + \Big(\frac{\omega^2}{f(z)} - k^2 - \frac{L^2}{z^2}m^2\Big)\phi = 0 \tag{4}
$$

It is almost impossible to solve the full solution outlined above. However, we need just the solution near the horizon, so we may simplify more by considering $f(z)$ where $z \to z_h$

$$
\begin{aligned}
f(z) &= 1 - \Big(\frac{z}{z_h}\Big)^d \\
&\sim -\frac{d}{z_h}(z - zh) \qquad\qquad z \to z_h \\
f'(z) &\sim -\frac{d}{z_h}
\end{aligned}
$$

So, we can re-write (4)

$$
-\frac{d}{z_h}(z - zh)\phi'' + \Big(-\frac{d}{z_h} + \frac{d(d-1)}{z_h z}(z - z_h)\Big)\phi' + \Big(-\frac{z_h\omega^2}{d(z - zh)} - k^2 - \frac{L^2}{z^2}m^2\Big)\phi = 0 \tag{5}
$$

And then, ignoring some terms of the above equation, we will get a simple differential equation as follows.

$$
\begin{aligned}
\frac{d}{z_h}\phi' + \frac{z_h\omega^2}{d(z - z_h)}\phi &= 0 \\
\phi' + \frac{z_h^2\omega^2}{d^2(z - z_h)}\phi &= 0
\end{aligned}
$$

This differential equation gives us the solution as a power-law solution,

$$
\phi \sim c_1(z - zh)^{\frac{z_h^2\omega^2}{d^2}} \tag{6}
$$

So, we know that the leading term of the solution near the horizon will in the form that

$$
\phi \sim (z - z_h)^{\alpha} \tag{7}
$$

We can substitute ϕ to check what is α exactly will be. So, we tern it on equation (5), we will see that $(z-z_h)^{\alpha-1}$ terms will dominate the equation.

$$
\begin{aligned}
-\frac{d}{z_h}(z-zh)\big[\alpha(\alpha-1)(z-z_h)^{\alpha-2}\big]+\Big(-\frac{d}{z_h}+\frac{d(d-1)}{z_h z}(z-z_h)\Big)\alpha(z-z_h)^{\alpha-1}+ \\
...+\Big(-\frac{z_h\omega^2}{d(z-zh)}-k^2-\frac{L^2}{z^2}m^2\Big)(z-z_h)^{\alpha}=0 \\
-\frac{d}{z_h}\big(\alpha(\alpha-1)\big)(z-z_h)^{\alpha-1}-\frac{d}{z_h}\alpha(z-z_h)^{\alpha-1}-\frac{z_h\omega^2}{d}(z-zh)^{\alpha-1}=0 \\
\alpha(\alpha-1)+\alpha+\frac{z_h^2\omega^2}{d^2}=0
\end{aligned}
$$

So, now we have the solution of α by solve above

$$\alpha=\pm i\frac{z_h\omega}{d}$$

re-write equation (7) and we get the solution near the horizon as,

$$\phi\sim(z-z_h)^{\pm\frac{iz_h\omega}{d}} \tag{8}$$

As we introduced that the horizon z_h contains thermodynamics information $z_h=1/\pi T$. So, the information provides

$$
\begin{aligned}
\phi &\sim (z-zh)^{\pm\frac{i\omega}{d\pi T}} && (9)\\
&\sim c_1(z-zh)^{\frac{i\omega}{d\pi T}}+c_2(z-zh)^{\frac{-i\omega}{d\pi T}} && (10)
\end{aligned}
$$

In Loreantzian signature, as relevant for real time formalism, the solution with a $-$ sign corresponds to an in-falling boundary condition, which is the condition that will be associated in a natural way with a retared Green's function. So, we may say that $c_1=0$ in this case. This condition will help us to solve the Green's function by considering the solution near the AdS boundary.

2 Holographic Renormalization

From last time, we skipped the details of holographic renormalization. So, let us move to correct them. To begin, recall the scalar action boundary term with a cut-off of ϵ.Let's change the definition of d to stand for spatial to stand for space-time dimensions, which makes AdS_{d+2} to AdS_{d+1}

$$S=\frac{1}{2}\int_{bdy}d^dx\frac{\phi\phi'}{z^{d-1}}\Big|_{z=\epsilon} \tag{11}$$

Let us recall the solution of *phi* and its derivative

$$
\begin{aligned}
\phi(z,k_\mu)&\sim\phi_0 z^{\Delta_-}+\phi_{(+)}z^{\Delta_+}+\cdots\\
\partial_z\phi(z,k_\mu)&\sim\Delta_-\phi_0 z^{\Delta_--1}+\Delta_+\phi_{(+)}z^{\Delta_+-1}+\cdots
\end{aligned}
$$

So, we rewrite (11) as below, and I do not expand it immediately just for notice myself.

$$S_{\text{on-shell}} = \int_{bdy} d^d x \Big[\frac{\Delta_-}{2} \phi_0 \phi_0 \frac{z^{-1+2\Delta_-}}{z^{(d+1)-2}} + \frac{(\Delta_+ + \Delta_-)}{2} \phi_0 \phi_{(+)} \frac{z^{-1+\Delta_- + \Delta_+}}{z^{(d+1)-2}} + \frac{\Delta_+}{2} \phi_{(+)} \phi_{(+)} \frac{z^{-1+2\Delta_+}}{z^{(d+1)-2}} \Big]_{z=\epsilon}$$

Note that power $(d+1)-2$ came from a product of determinant of pure AdS metric tensor and zz component of the metric. So, we can write

$$S_{\text{on-shell}} = \int_{bdy} d^d x \Big[\underbrace{\frac{\Delta_-}{2} \phi_0 \phi_0 \epsilon^{-d+2\Delta_-}}_{\text{Diverge}} + \underbrace{\frac{(\Delta_+ + \Delta_-)}{2} \phi_0 \phi_{(+)} \epsilon^{-d+\Delta_- + \Delta_+} + \frac{\Delta_+}{2} \phi_{(+)} \phi_{(+)} \epsilon^{-d+2\Delta_+}}_{\text{Converge}} \Big]$$

We see that the first term should be cancel by counter term. So, let we consider the first term carefully, especially, $\epsilon^{-d+2\Delta_-}$. This term can be consider as $\epsilon^{-d}\epsilon^{2\Delta_-}$. The first one from squared root of determinant for reduced metric tensor ,called $\sqrt{-\gamma}$. and the second one form the product of the field solution

$$\phi(\epsilon, k_\mu)^2 = \phi_0 \phi_0 \epsilon^{2\Delta_-} + \phi_0 \phi_{(+)} \epsilon^{\Delta_+ + \Delta_-} + \phi_{(+)} \phi_{(+)} \epsilon^{2\Delta_+}$$

if we consider its at boundary the last term vanish and the middle term will contribute the on-shell action. and the first term, will cancel the divergence in on-shell action. So, the counter term is now well defined.

$$S_{CT} = -\frac{\Delta_-}{2} \int_{bdy} d^d x \sqrt{-\gamma} \phi \phi \tag{12}$$

So, let us consider the total action $S_{tot} = S_{\text{on-shell}} + S_{CT}$

$$S_{tot} = \int_{bdy} d^d x \Big[\frac{\Delta_-}{2} \phi_0 \phi_0 \epsilon^{-d+2\Delta_-} + \frac{(\Delta_+ + \Delta_-)}{2} \phi_0 \phi_{(+)} \epsilon^{-d+\Delta_- + \Delta_+} + \frac{\Delta_+}{2} \phi_{(+)} \phi_{(+)} \epsilon^{-d+2\Delta_+} \Big] - \cdots$$
$$\cdots - \Big[\frac{\Delta_-}{2} \phi_0 \phi_0 \epsilon^{-d+2\Delta_-} + \Delta_- \phi_0 \phi_{(+)} \epsilon^{-d+\Delta_- + \Delta_+} + \frac{\Delta_-}{2} \phi_{(+)} \epsilon^{-d+2\Delta_+} \Big]$$

For now the boundary term is finite

$$S_{ren} = \frac{1}{2} \int_{bdy} d^d x \Big[d - 2\Delta_- \Big] \phi_0 \phi_{(+)} = \frac{1}{2} \int_{bdy} d^d x \Big[2\Delta_+ - d \Big] \phi_0 \phi_{(+)}$$

From the in-falling condition, we know from causality that the scalar was generated by source at AdS boundary and fall in the horizon. So, it's reasonable to say that $\phi_{(+)}$ is a function of ϕ_0, which I will write that $\phi_{(+)} = \phi_0 \phi_1$ where ϕ_1 is nothing other than arbitrary function. Therefore, we can rewrite the boundary action as

$$S_{sub} = \frac{1}{2} \int_{bdy} d^d x \Big[\frac{\phi \phi'}{\epsilon^{d-1}} - \frac{\Delta_-}{2} \sqrt{-\gamma} \phi \phi \Big] = \frac{1}{2} \int_{bdy} d^d x \sqrt{-\gamma} \Big[\epsilon \phi \phi' - \frac{\Delta_-}{2} \phi \phi \Big] \tag{13}$$

Finally, the result give use the contribution for Green's function

$$G_R(k_\mu) = (2\Delta_+ - d)\frac{\phi_1(k_\mu)}{\phi_0(k_\mu)} = (d - 2\Delta_-)\frac{\phi_1(k_\mu)}{\phi_0(k_\mu)} \tag{14}$$

3 The work in progress

3.1 Study vielbein for spinor field case

As we know, the transformation of a p-rank tensor has a specific transformation. If we consider the derivative of a vector, we find that the partial derivative of the vector is not a component of the tensor.

$$\partial_{\mu'} V^{\nu'} = \Lambda^\mu_{\mu'}\partial_\mu(\Lambda^{\nu'}_\nu V^\nu) = \Lambda^\mu_{\mu'}\left(\partial_\mu \Lambda^{\nu'}_\nu\right)V^\nu + \underbrace{\Lambda^\mu_{\mu'}\Lambda^{\nu'}_\nu \partial_\mu V^\nu}_{\text{Tensor transformation}}$$

So, under this kind of transformation, we need to write the covariant derivative, which is transformed as a tensor, and we get

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\mu_{\nu\sigma} V^\sigma$$
$$\nabla_{\mu'} V^{\nu'} = \Lambda^\mu_{\mu'}\Lambda^{\nu'}_\nu \nabla_\mu V^\nu$$

Unfortunately, spinor field do not have the same properties as any of the tensors. However, we know how to deal with spinor in inertial coordinates. So, this inspire us to consider mapping between curved space and local inertial frames. Let us consider the map.

$$e^\mu_a = \frac{\partial x^\mu}{\partial y^a} \qquad \text{and} \qquad e^a_\mu = \frac{\partial y^a}{\partial x^\mu} \tag{15}$$

I denote the local coordinates by y^a. for 4-dimensional coordinate we get a set of four vector fields, called *vierbein*. And *vielbein* for n-dimensional coordinates.

Let us consider the properties of vielbein. Firstly,

$$e^\mu_a e^a_\nu = \frac{\partial x^\mu}{\partial y^a}\frac{\partial y^a}{\partial x^\nu} = \delta^\mu_\nu \tag{16}$$

$$e^a_\mu e^\mu_b = \frac{\partial y^a}{\partial x^\mu}\frac{\partial x^\mu}{\partial y^b} = \delta^a_b \tag{17}$$

Then, if we consider metric tensor transformation on arbitrary curved spacetime, the metric tensors have transformation properties as

$$g_{\mu\nu} = \frac{\partial x^\sigma}{\partial x^\mu}\frac{\partial x^\rho}{x^\nu} g_{\sigma\rho}$$

In the same way, we can replace the transformation to transformation between general and local coordinates, which means

$$g_{\mu\nu} = \frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{x^\nu} g_{ab} = \frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{x^\nu}\eta_{ab} \tag{18}$$

Vice versa

$$\eta_{\mu\nu} = \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b} g_{\mu\nu} \tag{19}$$

Now, we can consider transport the vector field using language of local inertial coordinates. We expect that the transformation should to give us a new kind of connection, we call *spin connection* , which corresponds to $V^a(x \to x + dx) = V^a(x) - \omega^a_{b\nu} V^b dx^\nu$

$$\begin{aligned}
V^\mu(x) &= e^\mu_a(x) V^a \\
V^\mu(x \to x + dx) &= e^\mu_a(x + dx) V^a(x \to x + dx) \\
&= \left[e^\mu_a(x) + \partial_\nu e^\mu_a(x) dx^\nu\right]\left[V^a(x) - \omega^a_{b\nu} V^b dx^\nu\right] \\
&= e^\mu_a(x) V^a(x) + \partial_\nu e^\mu_a V^a dx^\nu - e^\mu_a \omega^a_{b\nu} V^b dx^\nu - \mathcal{O}(dx^2) \\
&= V^\mu(x) + \partial_\nu e^\mu_a V^a dx^\nu - e^\mu_a \omega^a_{b\nu} V^b dx^\nu - \mathcal{O}(dx^2)
\end{aligned}$$

Now, let compare with language of general coordinate,

$$\begin{aligned}
V^\mu(x) - \Gamma^\mu_{\sigma\nu} V^\sigma dx^\nu &= V^\mu(x) + \partial_\nu e^\mu_a V^a dx^\nu - e^\mu_a \omega^a_{b\nu} V^b dx^\nu \\
\Gamma^\mu_{\sigma\nu} V^\sigma &= -\partial_\nu e^\mu_a V^a + e^\mu_a \omega^a_{b\nu} V^b \\
\Gamma^\mu_{\sigma\nu} V^\sigma &= -e^a_\sigma \partial_\nu e^\mu_a V^\sigma + e^\mu_a e^b_\sigma \omega^a_{b\nu} V^\sigma
\end{aligned}$$

For now we can erase V^σ and can find out spin connection form

$$\begin{aligned}
\omega^a_{bv} &= e^a_\mu e^\sigma_b \Gamma^\mu_{\sigma\nu} + e^a_\mu e^\sigma_b e^a_\sigma \partial_\nu e^\mu_a \\
&= e^a_\mu \delta^a_b \partial_\nu e^\mu_a + e^a_\mu e^\sigma_b \Gamma^\mu_{\sigma\nu} \\
&= e^a_\mu \partial_\nu e^\mu_b + e^a_\mu e^\sigma_b \Gamma^\mu_{\sigma\nu}
\end{aligned}$$

So, for vector field we can write covariant derivative as

$$D_\nu V^\mu = \partial_\nu V^\mu + \partial_\nu e^\mu_a V^a - e^\mu_a \omega^a_{b\nu} V^b$$

And, of course, substituting $\omega^a_{b\nu}$ on the above yields a covariant derivative on general representation.