The idea of linear response theory is consideration small space(time) dependent perturbation of equilibrium state.

1 Interaction Picture and Linear Response

So far we known well the two pictures to solve quantum mechanics problems, Schrodinger's and Heisenberg's pictures. on one hand, Schrodinger's picture look at the frame that the operators are fixed and states evolve. On the other hand, the Heisenberg's picture fixes the states and let the operators evolve. So, these two pictures are related by

$$\begin{aligned} \langle\Psi(t)|A_S|\Psi(t)\rangle &= \langle\Psi(0)|U^\dagger A_S U|\Psi(0)\rangle \\ A_H &= U^-1 A_S U \end{aligned}$$

So now what happen if we perturb the state with time dependent perturbation

$$\hat{H}\left|\psi(t)\right\rangle = H^{(0)} + \delta H(t)\left|\psi(t)\right\rangle$$

The above equation shows us that in this type of Hamiltonian there is no eigenstate in Schrodinger's picture and even we use Heisenberg's picture to solve the perturbation it is still too difficult. To simplify this problem we start by consideration inverse time-evolution operator on a state.

$$\left|\Psi(t)\right\rangle = e^{\frac{i}{\hbar}H^{(0)}t}\left|\Psi(t)_I\right\rangle$$

Now, let see what happen when we find time derivative on $\left|\Psi(t)_I\right\rangle$

$$\begin{aligned} \frac{i}{\hbar}\frac{\partial}{\partial t}\left|\Psi(t)_I\right\rangle &= -H^{(0)}\left|\Psi(t)_I\right\rangle + e^{-\frac{i}{\hbar}H^{(0)}}\frac{i}{\hbar}\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle \\ &= -H^{(0)}\left|\Psi(t)_I\right\rangle + U^{(0)\dagger}(H^{(0)}+\delta H)\left|\Psi(t)\right\rangle \\ &= -H^{(0)}\left|\Psi(t)_I\right\rangle + U^{(0)\dagger}(H^{(0)}+\delta H)U\left|\Psi(t)_I\right\rangle \\ &= \quad U^{(0)\dagger}\delta H U^{(0)}\left|\Psi(t)_I\right\rangle \\ &= \quad \delta H_I\left|\Psi(t)_I\right\rangle \end{aligned}$$

So, now we will see that when we consider the state evolve by free Hamiltonian and re-define perturbation Hamiltonian, the Eigenstate will be well defined. So, this is analogous to simple quantum mechanics problems. The picture which state evolve by interaction time evolution operator and the time dependent operators are evolved by free Hamiltonian. This picture is called interaction picture.

So, in order to study linear response we consider ensemble average of an operator which is in presence of external source.

$$\langle\mathcal{O}(x,t)\rangle_s = \mathrm{tr}\Big[\rho(t_0)U_I^\dagger(t,t_0)\mathcal{O}_I(t,x)U_I(t,t_0)\Big] \tag{1}$$

where the operator is defined by

$$\delta H = -\int d^3x J(t,x)\mathcal{O}(x)$$

or

$$\delta S = \int d^4x J(t,x)\mathcal{O}(x)$$

Let's consider interaction picture time evolution operator

$$\begin{aligned} U(t,t_0) &= \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} dt'\delta H(t')\right\} \\ &= 1 - \frac{i}{\hbar}\int_{t_0}^{t}\delta H(t')dt' + \left(\frac{i}{\hbar}\right)^2\int_{t_0}^{t}dt'\int_{t_0}^{t}{}'dt''\delta H(t')\delta H(t'')\cdots \end{aligned}$$

For first perturbation which is analogous to linear response we can write (1) as

$$\begin{aligned} \langle\mathcal{O}(x,t)\rangle_s &= \mathrm{tr}\left[\rho(t_0)\Big(1+\frac{i}{\hbar}\int_{t_0}^{t}\delta H(t')dt'+\cdots\Big)\mathcal{O}_I(t,x)\Big(1-\frac{i}{\hbar}\int_{t_0}^{t}\delta H(t')dt'\cdots\Big)\right] && (2) \\ &= \mathrm{tr}[\rho(t_0)\mathcal{O}_I(t,x)] - \frac{i}{\hbar}\mathrm{tr}\left[\rho(t_0)\int_{t_0}^{t}dt'\Big[\mathcal{O}_I(t,x),\delta H(t')\Big]\right]+\cdots && (3) \end{aligned}$$

So we can see the linear response by consider difference of ensemble average of external source perturbation and equilibrium system.

$$\begin{aligned} \delta\langle\mathcal{O}(x,t)\rangle &= \langle\mathcal{O}(x,t)\rangle_s - \langle\mathcal{O}\rangle \\ &= -\frac{i}{\hbar}\mathrm{tr}\left[\rho(t_0)\int_{t_0}^{t}dt'\Big[\mathcal{O}_I(t,x),\delta H(t')\Big]\right] \\ &= \frac{i}{\hbar}\int_{-\infty}^{t}dt'\int d^3x\left\langle\Big[\mathcal{O}_I(t,x),\mathcal{O}_I(t',x')\Big]\right\rangle J(t',x') \\ &= \frac{i}{\hbar}\int d^4x'\theta(t-t')\left\langle\Big[\mathcal{O}_I(t,x),\mathcal{O}_I(t',x')\Big]\right\rangle J(t',x') \end{aligned}$$

This is the way we calculate retarded Green's function in non homologous differential equation. This easily seen that

$$\delta\langle\mathcal{O}(x,t)\rangle = -\int d^4x G_R^{\mathcal{O}\mathcal{O}}(t-t',x-x')J(t',x')$$

And also in momentum space we get

$$\delta\langle\mathcal{O}(k)\rangle = -G_R^{\mathcal{O}\mathcal{O}}(k)J(k)$$

This shows us that the linear response of a system by external source can be determined by Green's function. This important statement will give us the way to applied the concept of AdS/CFT correspondence.

2 Wave Solution on AdS_{d+2}

In QFT, we perform fields and their interaction on Minkowski space-time background. However, in this work, we include the geometry of the space-time. For scalar field it is the simplest case. Because covariant derivative does not affect on scalar field.

$$
\begin{aligned}
S &= \int d^{d+2}x\sqrt{|g|}\mathcal{L} \\
\delta S &= \int d^{d+2}x\delta(\sqrt{|g|}\mathcal{L}) \\
&= \int d^{d+2}x\left[\frac{\partial}{\partial\phi}\sqrt{|g|}\delta\phi + \frac{\partial}{\partial\partial_m\phi}\delta(\partial_m\phi)\right] = \int d^{d+2}x\left[\frac{\partial}{\partial\phi}\sqrt{|g|}\delta\phi + \frac{\partial}{\partial\partial_m\phi}\partial_\mu\delta\phi\right] \\
&= \int d^{d+2}x\left[\frac{\partial}{\partial\phi}\sqrt{|g|}\mathcal{L} - \partial_m\frac{\partial}{\partial(\partial_\mu\phi)}\sqrt{|g|}\mathcal{L}\right]\delta\phi + \int d^{d+2}x\partial_m\left[\frac{\partial}{\partial(\partial_m\phi)}\sqrt{|g|}\mathcal{L}\delta\phi\right]
\end{aligned} \tag{4}
$$

We see that partial derivative of field does not affect on metric determinant. Then, we apply least-action principle we get

$$
\sqrt{|g|}\frac{\partial\mathcal{L}}{\partial\phi} - \partial_m\sqrt{|g|}\frac{\partial\mathcal{L}}{\partial(\partial_m\phi)} = 0 = \frac{\partial\mathcal{L}}{\partial\phi} - \frac{1}{\sqrt{|g|}}\partial_m\sqrt{|g|}\frac{\partial\mathcal{L}}{\partial(\partial_\phi)}
$$

Let's the scalar field Lagrangian $\mathcal{L} = \frac{1}{2}(\nabla_m\phi)^2 + \frac{1}{2}m\phi^2 = \frac{1}{2}(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2)$

$$
\frac{\partial\mathcal{L}}{\partial\phi} - \frac{1}{\sqrt{|g|}}\partial_m\sqrt{|g|}\frac{\partial\mathcal{L}}{\partial(\partial_m\phi)} = m^2\phi - \frac{1}{\sqrt{|g|}}\partial_m\left(\sqrt{|g|}g^{mn}\partial_n\phi\right)
$$

So, equation of motion is given by

$$
\left(\frac{1}{\sqrt{|g|}}\partial_m\left(\sqrt{|g|}g^{mn}\partial_n\right) - m^2\right)\phi = 0 \tag{5}
$$

In our framework, we interest in AdS space-time. So, let's consider solution of AdS_{d+2}

$$
ds^2 = L^2\left(\frac{-dt^2 + d\vec{x_d}^2 + dr^2}{z^2}\right)
$$

from above we can re-write scalar field as

$$
\phi = \phi(z)e^{ik^\mu x_\mu} = \phi(z)e^{-i\omega t + k\cdot\vec{x}}
$$

So, we substitute above on (5) we found that

$$\frac{1}{\sqrt{\left(\left(\frac{L}{r}\right)^2\right)^{d+2}}}\partial_m\left(\sqrt{\left(\left(\frac{L}{z}\right)^2\right)^{d+2}}g^{mn}\partial_n\phi(z)e^{-i\omega t+k\cdot\vec{x}}\right)-m^2\phi(z)e^{-i\omega t+k\cdot\vec{x}}=0$$

$$\frac{1}{\left(\frac{L}{z}\right)^{d+2}}\partial_m\left(\left(\frac{L}{r}\right)^{d+2}g^{mn}\partial_n\phi(z)e^{-i\omega t+k\cdot\vec{x}}\right)-m^2\phi(z)e^{-i\omega t+k\cdot\vec{x}}=0$$

for t and x_d components we get $\left(\frac{z}{L}\right)^2\omega^2\phi$ and $-\left(\frac{z}{L}\right)^2k^2\phi$
for z component we get

$$\left(\frac{L}{z}\right)^{-(d+2)}\partial_z\left(\left(\frac{L}{z}\right)^d\partial_z\phi\right)=\left(\frac{L}{z}\right)^{-(d+2)}\left(\left(\frac{L}{z}\right)^d\phi''+\frac{d}{z}\left(\frac{L}{z}\right)^d\phi'\right)$$

So, we can write equation of motion as

$$\phi''-\frac{d}{z}\phi'+\left(\omega^2-k^2-\frac{(mL)^2}{z^2}\right)\phi=0 \tag{6}$$

In the case that we simplify nothing we get the solution

$$\phi=z^{\frac{d+1}{2}}J_\alpha[-iz\sqrt{k^2-\omega^2}]A+z^{\frac{d+1}{2}}K_\alpha[-iz\sqrt{k^2-\omega^2}]B \tag{7}$$

where $\alpha=\frac{1}{2}\sqrt{(d+1)^2+4m^2}$

for boundary solution near $z\to 0$ we known so far that the solution will be as the power law solution. So, we suppose that the solution is z^Δ. In this case, we can find Δ by substitute the solution in (6),

$$\Delta(\Delta-d-1)=m^2L^2$$

and then we get

$$\Delta_\pm=\frac{d+1}{2}\pm\sqrt{\frac{(d+1)^2}{4}+m^2L^2}$$

it's easily to seen that $\Delta_+=d+1-\Delta_-$. Then the solution of the scalar can be written as

$$\phi(z,k)=\phi_0z^{\Delta_-}+\phi_+z^{\Delta_+}+\cdots \tag{8}$$

3 GKPW Formula and Green's Function

The ensemble average of equilibrium system gauge field theory can be determine as AdS gravitational theory on boundary.

$$\left\langle\exp\left(i\int\phi_0\mathcal{O}\right)\right\rangle=e^{[iS[\phi]_{bdy}|_{z=0}]} \tag{9}$$

GKP-Witten relation claims that the ensemble average of an operator and be calculated by generating function.

$$\langle \mathcal{O} \rangle_{\phi_0} = \frac{\delta S[\phi_0]_{bdy}}{\delta \phi_0} \tag{10}$$

So, we can easily calculate the Green's functions by considering AdS boundary action of the field. In order to calculate Green's function.

Back to equation (4) in boundary term. If we consider the on-shell action we found that at we can re-write equation (10) in term of canonical momentum.

$$\delta S_{bdy} = \int d^{d+1}x du \partial_m \left[\frac{\partial}{\partial(\partial_m \phi)} \sqrt{|g|} \mathcal{L} \delta\phi \right] = \frac{\partial}{\partial(\partial_m \phi)} \sqrt{-g} \mathcal{L} \delta\phi \Big|_{bdy} = \sqrt{-g} \frac{\partial \mathcal{L}}{\partial(\partial_m \phi)} \delta\phi \Big|_{bdy} = \Pi(z, x_\mu) \delta\phi \Big|_{bdy}$$

where $\Pi(z, x_\mu) = \dfrac{\partial}{\partial(\partial_\mu \phi)} \sqrt{-g} \mathcal{L} = \sqrt{-g} g^{zz} \partial_z \phi$ at the boundary. So, compare with equation (10) we get

$$\langle \mathcal{O} \rangle_{\phi_0} = \frac{\delta S[\phi_0]_{bdy}}{\delta \phi_0} = \lim_{z \to 0} \Pi(z, x_\mu) \tag{11}$$

From linear response we can write the Green's function which determine at the AdS boundary in momentum space as

$$G_R(k_\mu) = \lim_{z \to 0} \left(\frac{\Pi(z, k_\mu)}{\phi(z, k_\mu)} \right) \Big|_{\phi_0 = 0}$$

However, form the reason form holographic renormalization the Green's function will be written by

$$G_R(k_\mu) = \lim_{z \to 0} \left(z^{1+2\Delta_-} \frac{\Pi(z, k_\mu)}{\phi(z, k_\mu)} \right) \Big|_{\phi} \tag{12}$$

The for ϕ has the asymptotic behavior

$$\begin{aligned} \phi(z, k_\mu) &\sim \phi_0 z^{\Delta_-} + \phi_{(+)} z^{\Delta_+} + \cdots \\ \partial_z \phi(z, k_\mu) &\sim \Delta_- \phi_0 z^{\Delta_- - 1} + \Delta_+ \phi_{(+)} z^{\Delta_+ - 1} + \cdots \end{aligned}$$

We see that canonical momentum has the asymptotic behavior

$$\begin{aligned} \Pi(z, k_\mu) &\sim \left(\frac{L}{z} \right)^{d+1} \left(\Delta_- \phi_0 z^{\Delta_- - 1} + \Delta_+ \phi_{(+)} z^{\Delta_+ - 1} + \cdots \right) \qquad (13) \\ &= L^{d+1} \left(\Delta_- \phi_0 z^{-1-\Delta_+} + \Delta_+ \phi_{(+)} z^{-1-\Delta_-} + \cdots \right) \qquad (14) \end{aligned}$$

So, if we substitute asymptotic solution and find the Green's function we found that the Green's function is proportional to ratio function.

$$G_R(k_\mu) \propto \frac{\phi_{(+)}}{\phi_0} \tag{15}$$

Since the first step until here we have already shows that the boundary action give us the way to find the green's function. However, we can perform it directly form asymptotic form of boundary action **or** just consider the ratio function which can determine from differential equation.

3.1 Massive Scalar Field

For massive scalar field action $S = \frac{1}{2}\int d^{d+2}x\sqrt{|g|}\left[(\nabla_m\phi)^2 + m^2\phi^2\right]$ we have

$$\Delta(\Delta - d - 1) = m^2L^2$$

for specific case that $L = 1$, $d = 2$, and $m^2 = -2$ we get $\Delta = (1, 2)$

$$\begin{aligned}\phi(z, x_\mu) &= c_1 z e^{-z\sqrt{k^2-\omega^2}} + \frac{c_2 z e^{z\sqrt{k^2-\omega^2}}}{2\sqrt{k^2-\omega^2}}\\ &\sim z\left(\frac{c_2}{2\sqrt{k^2-\omega^2}} + c_1\right) + z^2\left(\frac{c_2}{2} - c_1\sqrt{k^2-\omega^2}\right) + O\left(z^3\right)\end{aligned}$$

We can calculate the Green's function by ratio function of z^{Δ_+} and z^{Δ_-}. We get the source Green's function

$$G_R(\omega, k) \propto \frac{1}{\sqrt{k^2-\omega^2}}$$

Let's consider analytic form of massive scalar by considering asymptotic behavior of the action on AdS_4

$$\begin{aligned}S &= -\frac{1}{2}\int d^{2+2}x\sqrt{|g|}\left[(\nabla_m\phi)^2 + m^2\phi^2\right]\\ &\sim -\int d^{2+1}xdz\frac{1}{2z^2}\left[\phi'^2 + m^2\phi^2\right]\\ &= -\int d^3xdz\Big[\partial_z\Big(\frac{\phi\phi'}{2z^2}\Big) - \Big(\frac{\phi'}{2z^2}\Big)'\phi + \frac{m^2}{2z^2}\phi^2\Big]\\ &= -\int_{bdy} d^3x\frac{\phi\phi'}{2z^2}\Big|_{z=0} + \int d^3xdz\frac{1}{2}\underbrace{\left[\frac{\phi''}{z^2} - \frac{2\phi'}{z^3} - \frac{m^2}{2z^2}\phi\right]}_{\text{EOM}}\phi\end{aligned}$$

Then, If we substitute asymptotic solution of ϕ and ϕ' the integrand of boundary integral can be written as

$$\begin{aligned}\text{integrand} &= \frac{\Delta_-}{2}\phi_0\phi_0 z^{-3+2\Delta_1} + \frac{(\Delta_+ + \Delta_-)}{2}\phi_0\phi_{(+)}z^{-3+\Delta_-+\Delta_+} + \frac{\Delta_+}{2}\phi_{(+)}\phi_{(+)}z^{-3+\Delta_+}\\ &= \frac{\Delta_-}{2}\phi_0\phi_0 z^{-3+2\Delta_1} + \frac{(d+1)}{2}\phi_0\phi_{(+)}z^{-2+d} + \frac{\Delta_+}{2}\phi_{(+)}\phi_{(+)}z^{-3+\Delta_+}\end{aligned}$$

We found that for d= 2 the integrand with boundary value become

$$\frac{\phi\phi'}{2z}\Big|_{z=0} = \frac{3}{2}\phi_0\phi_{(+)} + \text{Divergent terms} \tag{16}$$

We see that (15) satisfy the linear response theory

$$\langle \mathcal{O} \rangle_{\phi_0} = \frac{\delta S[\phi_0]_{bdy}}{\delta \phi_0} \propto \phi_{(+)}$$

3.2 Massless Scalar Field

For massless case, it is analogues to massive case just only the equation of motion is different.

$$\begin{aligned} S &\sim \int d^{d+1}xdz \frac{\phi'^2}{2z^d} \\ &= -\int_{bdy} d^{d+1}x \frac{\phi\phi'}{2z^d}\Big|_{z=0} + \int d^{d+1}xdz \underbrace{\left[\frac{\phi'}{2z^d}\right]'}_{\text{EOM}} \phi \end{aligned}$$

So, the integrand of boundary term can be written as

$$\begin{aligned} \text{integrand} &= \frac{\Delta_-}{2}\phi_0\phi_0 z^{-d+2\Delta_1} + \frac{(\Delta_+ + \Delta_-)}{2}\phi_0\phi_{(+)}z^{-d+\Delta_- + \Delta_+} + \frac{\Delta_+}{2}\phi_{(+)}\phi_{(+)}z^{-d+\Delta_+} \\ &= \frac{\Delta_-}{2}\phi_0\phi_0 z^{-d+2\Delta_1} + \frac{(d+1)}{2}\phi_0\phi_{(+)}z^{-d+d} + \frac{\Delta_+}{2}\phi_{(+)}\phi_{(+)}z^{-d+\Delta_+} \end{aligned}$$

We also get

$$\langle \mathcal{O} \rangle_{\phi_0} = \frac{\delta S[\phi_0]_{bdy}}{\delta \phi_0} \propto \phi_{(+)}$$

So, for solving the Green's function we will focus on the differential equation with confined by some suitable boundary conditions.