Holographic Correlation Functions of Fermionic Fields

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1 Introduction

One of the Holographic Principle's significant applications is calculating the correlation function of a non-equilibrium system. Classical mechanics tells us that when a system is perturbed by an external source, we may predict the response that will be employed to solve the non-equilibrium system. According to the linear response theorem, the operator's reaction may be expressed as a retarded Green's function:

$$\delta \langle O(t,x)\rangle = -\int_{-\infty}^{\infty} d^4x' G_R^{\mathcal{O}\mathcal{O}}(t-t', x-x')\phi^{(0)}(t',x') \tag{1}$$

and in Fourier space:

$$\delta \langle O(k)\rangle = -G_R^{\mathcal{O}\mathcal{O}}(k)\phi^0 \tag{2}$$

Practically, we can find some crucial physical quantities here, like conductivity, for example.

$$\delta \langle j^x\rangle \;=\; i\omega\sigma A_x^{(0)} \;=\; -\,G_R^{xx} A_x^{(0)} \tag{3}$$

Thus,

$$\sigma(\omega) = -\frac{G_R^{xx}(\omega)}{i\omega}$$

This shows us that the linear response of a system to an external source can be determined by Green's function, which also gives us the physical quantities. This important statement will give us the way to apply the concept of AdS/CFT. The correspondence result provides a method for calculating the linear response by taking into account the boundary action and source at the boundary.

$$\langle O\rangle = \frac{\delta \underline{S}[\phi^{(0)}]}{\delta\phi^0}$$

After that, we may use this relation to find out the retarded Green function (correlation function) (1)

Set-up: Metric and Gamma Matrices

In this report, we will demonstrate how to calculate the spinor correlation function in $(S)AdS_{3+1}$ using the gamma matrices representation, such that the metric tensor relates to space-time.

$$ds^2 = \frac{L^2}{z^2}\Big(-f(z)dt^2 + dx^2 + dy^2 + \frac{dz^2}{f(z)}\Big) \tag{4}$$

where $f(z) = 1 - \left(\frac{z}{z_h}\right)^3$,where z_h is the horizon, L stands for the AdS radius

For convenient calculation we map horizon $zh = 1$ and $L = 1$:

$$ds^2 = \frac{1}{z^2}\Big(- f(z)dt^2 + dx^2 + dy^2 + \frac{dz^2}{f(z)}\Big) \qquad ; \qquad f(z) = 1 - z^3 \tag{5}$$

In this case, one of the convenient way to decompose Ψ is that $\gamma^t = i\sigma_2$, $\gamma^1 = \sigma_1$, and $\gamma^2 = \sigma_3$

$$\Gamma^z = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \Gamma^t = \begin{pmatrix} 0 & \gamma^t \\ \gamma^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$\Gamma^x = \begin{pmatrix} 0 & \gamma^x \\ \gamma^x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \Gamma^y = \begin{pmatrix} 0 & \gamma^y \\ \gamma^y & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

2 Effective Spinor Action and The Boundary Term

Similarly, in AdS/CFT correspondence, we simply begin by separating boundary action from bulk action.

$$S = \int d^{3+1}x\sqrt{-g}\Big(\frac{i}{2}\bar{\Psi}(\overrightarrow{\not{D}}_M - \overleftarrow{\not{D}}_M)\Psi - m\bar{\Psi}\Psi\Big)w \tag{6}$$

where $\not{D} = \Gamma^M D_M = \Gamma^M\big(\partial_M + \frac{1}{4}\omega_M^{bc}\Gamma_{bc}\big)$.

This looks so complicated and so boring, if we consider every single terms. However, if we consider carefully, Γ_{bc} andω do not contribute boundary term. So, we can consider just the partial derivative term in z direction, which we will extract the pure boundary terms as follows

$$\begin{aligned} S &= \int d^{3+1}x\sqrt{-g}\ \frac{i}{2}(\bar{\Psi}\Gamma^z\overrightarrow{\partial_z}\Psi - \bar{\Psi}\Gamma^z\overleftarrow{\partial_z}\Psi) + \cdots \\ &= \int d^{3+1}x\sqrt{-g}\ \frac{i}{2}(\overrightarrow{\partial_z}(\bar{\Psi}\Gamma^z\Psi) - (\bar{\Psi}\Gamma^z\Psi)\overleftarrow{\partial_z} - \underbrace{\bar{\Psi}\Gamma^z(\Psi\overleftarrow{\partial_z}) + (\overrightarrow{\partial_z}\bar{\Psi})\Gamma^z\Psi) + \cdots}_{\text{EOM}} \end{aligned}$$

The last two terms will contribute equation of motion. Then, when we know that $\bar{\Psi}\gamma^z\Psi$ is a vector. So, we apply divergence theorem which provides

$$S = \int d^3x\sqrt{-h}\ i\big[(\bar{\Psi}\Gamma^z\Psi)_{\rm R} - (\bar{\Psi}\Gamma^z\Psi)_{\rm L}\big] + \int d^{3+1}dx\sqrt{-g} \quad \text{EOM}$$

Then, the variation of subscript R will be operated on Ψ, on the other hand, the variation of subscript L will be operated on $\bar{\Psi}$, So, we may write the bulk-boundary as

$$\delta S_{bulk-bdy} = \int d^3x\sqrt{-h}\ i\big[\bar{\Psi}\Gamma^z\delta\Psi - \delta\bar{\Psi}\Gamma^z\Psi\big] \tag{7}$$

Here we plug-in the spinor field as

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \quad \text{and} \quad \bar{\Psi} = \begin{pmatrix} \bar{\psi}_- \\ \bar{\psi}_+ \end{pmatrix}^T$$

So now we know well how to write equation (7)

$$\delta S_{bulk-bdy} = \frac{i}{2}\int d^3x\sqrt{-h}\big[\bar{\psi}_-\delta\psi_+ - \bar{\psi}_+\delta\psi_- - \delta\bar{\psi}_-\psi_+ + \delta\bar{\psi}_+\psi_-\big] \tag{8}$$

We now meet the trouble, because at the boundary we can fix just only ψ_- or ψ_+ but this action contains variation of both types which mean this boundary action is not well defined (it does not vanish when we fix boundary conditions) So, we add the extra boundary action

$$S_{bdy} = \frac{i}{2}\int d^3x\sqrt{-h}\bar{\Psi}\Psi \tag{9}$$

$$\delta S_{bdy} = \frac{i}{2}\int d^3x\sqrt{-h}\big[\bar{\Psi}\delta\Psi + \delta\bar{\Psi}\Psi\big] \tag{10}$$

$$= \frac{i}{2}\int d^3x\sqrt{-h}\big[\bar{\psi}_-\delta\psi_+ + \bar{\psi}_+\delta\psi_- + \delta\bar{\psi}_-\psi_+ + \delta\bar{\psi}_+\psi_-\big] \tag{11}$$

So, (8)+(11), the boundary action no longer depends on variation of ψ_-, which means we are able to fix just only ψ_+ at boundary and make the action well-defined

$$\delta S_{bulk-bdy} + \delta S_{bdy} = i\int d^dx\sqrt{-h}\Big[\bar{\psi}_-\delta\psi_+ + \delta\bar{\psi}_+\psi_-\Big]$$

We finally can write the well-defined action for fermionic field (also adding the normarization factor) by

$$S_{eff} = \int d^{3+1}x\sqrt{-g}\Big(\bar{\Psi}(\frac{i}{2}(\overrightarrow{\not{D}}_M - \overleftarrow{\not{D}}_M) - m\Big)\Psi + \frac{i}{2}\int d^3x\sqrt{-h}\bar{\Psi}\Psi \tag{12}$$

So, in this sense, if we consider ψ_+ as the source ψ_- is nothing but the condensation (response). Moreover, we can re-define the spinor by

$$\chi = (-gg^{zz})^{1/4}\Psi \tag{13}$$

$$\Psi = (-gg^{zz})^{-1/4}e^{-i\omega t + ik_ix^i}\chi \tag{14}$$

which is the crucial simplification to avoid the spin connection, the derivation is included in Appendix A. From this re-define, we can write the second term of (12) as

$$S_{bdy} = \frac{i}{2}\int_{bdy} d^3x \quad \bar{\chi}\chi \tag{15}$$

$$= \frac{i}{2}\int_{bdy} d^3x \quad \chi^\dagger\Gamma^t\chi \tag{16}$$

3 Solving Dirac Equations

In order to calculate Green's function, we may start by consider the solution of the Dirac equation at the boundary to find a set of boundary conditions. Furthermore, we will try to write down the solution of To begin, consider the structure of the Dirac equation. We derive the equations of motion at the boundary by redefining the dirac spinor as equation (14).

$$(\partial_z \mp \frac{m}{z})\chi_\pm = \mp i k_\mu \gamma^\mu \chi_\mp \tag{17}$$

the detail is included in Appendix A. We see that the equation of motion is coupled. However, we can decouple them by just the substituting, minus spinor equation on the plus one:

$$\begin{aligned}
(\partial_z + \frac{m}{z})(\partial_z - \frac{m}{z})\chi_- &= (k\cdot\gamma)^2\chi_- \\
(\partial_z + \frac{m}{z})(\partial_z\chi_- - \frac{m}{z}\chi_-) &= (k\cdot\gamma)^2\chi_- \\
\partial_z^2\chi_- + \frac{m}{z}\partial_z\chi_- + \frac{m}{z^2}\chi_- - \frac{m}{z}\partial_z\chi_- - \frac{m^2}{z^2}\chi_- &= (k\cdot\gamma)^2\chi_- \\
\partial_z^2\chi_- - \frac{m}{z^2}(-1+m)\chi_- &= (k\cdot\gamma)^2\chi_- \qquad \text{for} \qquad \chi_-
\end{aligned} \tag{18}$$

in the same way,

$$\partial_z^2\chi_+ - \frac{m}{z^2}(1+m)\chi_+ = (k\cdot\gamma)^2\chi_+ \qquad \text{for} \qquad \chi_+ \tag{19}$$

the solution of equations (18),(19) are **Bessel functions**

3.1 Solutions Near Boundary

So far we know from the scalar case that at the boundary the Bessel functions can be reduced as the power of z^a Now we can make sure that both χ_+ and χ_- behave as power law at boundary. So, I supposed that $\chi_- = z^a$ and then substitute in equation (17)

$$\begin{aligned}
(\partial_z + \frac{m}{z})z^a &= -ik\cdot\gamma\chi_+ \\
(a-m)z^{a-1} &= -ik\cdot\gamma\chi_+
\end{aligned}$$

Then, we must substitute the above relation in χ_+

$$\begin{aligned}
(\partial_z + \frac{m}{z})\frac{(a-m)}{-ik\cdot\gamma}z^{a-1} &= ik\cdot\gamma z^a \\
(a-1+m)(a-m)z^{a-2} &= (k\cdot\gamma)^2 z^a \\
(a-1+m)(a-m) &= (k\cdot\gamma)^2 z^2 \\
&\sim 0
\end{aligned}$$

So, we get the solution $a = m, 1-m$, then we can write the solution of χ_- as

$$\chi_- = C(k)z^{1-m} + D(k)z^m$$

we can get the χ_+ by the same calculation

$$\chi_+ = A(k)z^{-m} + B(k)z^{1+m}$$

So, we need to define source for both χ_+ and χ_-, I choose standard quantization for plugging in the m value as $-1/2 < m < 0$. Let's use one fourth for example,

$$\chi_+ = \underbrace{Az^{-1/4}}_{\text{dominate term}} + Bz^{1-1/4} \quad , \quad \chi_- = Cz^{1-1/4} + \underbrace{Dz^{1/4}}_{\text{dominate term}}$$

Since the we can relate B and D as shown in Appendix B, thus we can say that D is nothing but response. So, we will reserve A and D as source and condensation respectively. Moreover, let we consider the spinors $\chi_\pm$ and their solutions

$$\chi_\pm\big|_{bdy} = \lim_{z\to 0} z^{\mp m}\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = \sum_{i=1,2}\begin{pmatrix} c_i y_\pm^{(i)} \\ c_i z_\pm^{(i)} \end{pmatrix}$$

where the index i represents each independent eigenvectors due to coupled differential equations. we may write down solutions on each eigenvectors as

$$y_+^{(i)} = A_1^{(i)} z^{-m} + B_1^{(i)} z^{1+m} \quad , \quad y_-^{(i)} = C_1^{(i)} z^{1-m} + D_1^{(i)} z^m \tag{20}$$

$$z_+^{(i)} = A_2^{(i)} z^{-m} + B_2^{(i)} z^{1+m} \quad , \quad z_-^{(i)} = C_2^{(i)} z^{1-m} + D_2^{(i)} z^m \tag{21}$$

3.1.1 Example: k_y is neglected

Since k_y is neglected, we can write down the equation of motions as

$$\begin{pmatrix} \partial_z \mp \frac{m}{z} & 0 \\ 0 & \partial_z \mp \frac{m}{z} \end{pmatrix}\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = \begin{pmatrix} \mp i(-\omega + k_x) & 0 \\ 0 & \mp i(\omega + k_x) \end{pmatrix}\begin{pmatrix} z_\mp \\ y_\mp \end{pmatrix}$$

From this linear system differential equations, the solution will be spanned on the eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. we then write them separately as the matrix from .

$$\chi_+\big|_{bdy} = \lim_{z\to 0} z^{-m}\begin{pmatrix} y_+ \\ z_+ \end{pmatrix} = y_+^{(0)} c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_+^{(0)} c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} y_+^{(0)} & 0 \\ 0 & z_+^{(0)} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{22}$$

Since χ_+ contain all of the source terms, we may define $\chi_+|_{bdy} = \mathcal{S}\vec{c}$. In the same way, we can perform the same calculation for χ_- and we define $\chi_-|_{bdy} = \mathcal{C}\vec{c}$. We will re-call these kind of representation again when we calculate the green's function. Numerically, this is the first set of boundary conditions, which we need to solve the green's function.

3.2 Solutions Near the Horizon

On one hand, we obtained a set of boundary conditions at the boundary on the other hand, we consider horizon regularity to determine the other set of boundary conditions. After we determine the horizon regularization by choosing in-falling condition, we will be able to relate source and condensation terms. In order to show analytically, let we consider the case that k_y is neglected to avoid coupling problem.

3.2.1 Example: k_y is neglected

$$\partial_z y_+ + \frac{(izkz_- - my_+)}{z\sqrt{f(z)}} - \frac{i\omega z_-}{f(z)} = 0$$
$$\partial_z z_+ + \frac{(izky_- - mz_+)}{z\sqrt{f(z)}} + \frac{i\omega y_-}{f(z)} = 0$$
$$\partial_z y_- + \frac{(-izkz_+ + my_-)}{z\sqrt{f(z)}} + \frac{i\omega z_+}{f(z)} = 0$$
$$\partial_z z_- + \frac{(-izky_+ + mz_+)}{z\sqrt{f(z)}} - \frac{i\omega y_-}{f(z)} = 0$$

These set equations can be decoupled as similar as near boundary by substitution each equation into their partner. However, as we performed in scalar case (Appendix C), we will get the leading order terms as

$$y_+ = A_1(z-1)^a$$
$$z_+ = A_2(z-1)^a$$
$$y_- = D_1(z-1)^a$$
$$z_- = D_2(z-1)^a$$

From here, we put them on the set of Dirac equations at near horizon. Then, expand the solution near the horizon, we will meet a non zero term at $z \to 1$ in each solutions. So, we need to make the terms equal zero. we will get the relation between the coefficients as follows

$$D_1 = A_2 \quad , \quad D_2 = -A_1 \qquad a = \frac{i\omega}{3}$$

and

$$D_1 = -A_2 \quad , \quad D_2 = A_1 \qquad a = \frac{-i\omega}{3} \tag{23}$$

Now, we are able to relate coefficients of source and condensation. Mathematically, we obtain a set of boundary conditions. In numerical point of view, this is one of the main parts of our calculations.

4 Calculation of Correlation Functions

Let we re consider the results of section 3.1 . Since we consider the $\chi_+(\chi_-)$ as the 2×2 matrix, we see that $\chi_+\big|_{bdy} = \mathcal{S}\vec{c}$ is nothing but boundary values J. Additionally, we also write the $\chi_-\big|_{bdy} = \mathcal{C}\vec{c}$, So, we can re-write the boundary action as

$$\begin{aligned}
S_{bdy} &= \frac{i}{2}\int_{bdy} d^3x \;\; \bar{\chi}_+\chi_- - \bar{\chi}_-\chi_+ = \frac{i}{2}\int_{bdy} d^3x \;\; J^\dagger i\sigma_2\mathcal{C}\vec{c} + h.c. \\
&= \frac{i}{2}\int_{bdy} d^3x \;\; (\mathcal{S}\vec{c})^\dagger i\sigma_2\mathcal{C}\vec{c} + h.c. \\
&= \frac{1}{2}\int_{bdy} d^3x \;\; (\mathcal{S}\vec{c})^\dagger - \sigma_2\mathcal{C}\mathcal{S}^{-1}\chi_+ + h.c. \\
&= \frac{1}{2}\int_{bdy} d^3x \;\; J^\dagger(-\sigma_2\mathcal{C}\mathcal{S}^{-1})J + h.c.
\end{aligned}$$

The term between boundary value J is definitely the Green function. Let us test the above relation by using the simplest case.

4.0.1 Example: k_y is neglected

After we define the source and condensation matrices as

$$J = S\vec{c} = \begin{pmatrix} y_+^{(0)} & 0 \\ 0 & z_+^{(0)} \end{pmatrix}\vec{c} \qquad \chi_-\big|_{bdy} = C\vec{c} = \begin{pmatrix} 0 & y_-^{(0)} \\ z_-^{(0)} & 0 \end{pmatrix}\vec{c}$$

So, we can find the Green's functions directly by

$$-\sigma_2\mathcal{C}\mathcal{S}^{-1} = -\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & y_-^{(0)} \\ z_-^{(0)} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{y_+^{(0)}} & 0 \\ 0 & \frac{1}{z_+^{(0)}} \end{pmatrix} = \begin{pmatrix} \frac{iz_-^{(0)}}{y_+^{(0)}} & 0 \\ 0 & \frac{-iy_-^{(0)}}{z_+^{(0)}} \end{pmatrix}$$

From here we see that, if we know the exactly the boundary condition at the horizon,

$$\xi_\pm\Big|_h = i \tag{24}$$

we will numerically get the final result of the Green's function using flow equation.

$$\sqrt{\frac{g_{ii}}{g_{zz}}}\partial_z\xi_\pm = -2m\sqrt{g_{ii}}\xi_\pm \mp (k \mp \omega) \pm (k \pm m)\xi_\pm^2$$

The canonical calculation for this green's function, following Hong Lui publications will be included on the Appendix B, and we will see that the results of green's function is same.

Now, let we see what will happen if we do not neglect k_y. The Dirac equations can be written by

$$\sqrt{\frac{g_{ii}}{g_{zz}}}(\partial_z \mp \sqrt{g_{zz}}m)y_\pm = \mp i(-\omega + k_x)z_\mp \mp k_y y_\mp \tag{25}$$

$$\sqrt{\frac{g_{ii}}{g_{zz}}}(\partial_z \mp \sqrt{g_{zz}}m)z_\pm = \mp i(\omega + k_x)y_\mp \pm k_y z_\mp \tag{26}$$

So, we cannot decoupled them as previous. However, this is nothing but system of differential equations.

$$\begin{pmatrix} \partial_z \mp \frac{m}{z} & 0 \\ 0 & \partial_z \mp \frac{m}{z} \end{pmatrix}\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = \begin{pmatrix} \mp i(-\omega + k_x) & \mp k_y \\ \pm k_y & \mp i(\omega + k_x) \end{pmatrix}\begin{pmatrix} z_\mp \\ y_\mp \end{pmatrix}$$

Let we recall the from of solution at boundary of $\chi_\pm$

$$\chi_\pm\big|_{bdy} = \lim_{z\to 0} z^{-m}\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = \sum_{i=1,2}\begin{pmatrix} c_i y_\pm^{(i)} \\ c_i z_\pm^{(i)} \end{pmatrix}$$

So, in the same way, we can define the 2×2 matrix form of $\chi_\pm$

$$\chi_+\big|_{bdy} = \begin{pmatrix} y_+^{(1)} & y_+^{(2)} \\ z_+^{(1)} & z_+^{(2)} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathcal{S}\vec{c} \tag{27}$$

$$\chi_-\big|_{bdy} = \begin{pmatrix} y_-^{(1)} & y_-^{(2)} \\ z_-^{(1)} & z_-^{(2)} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathcal{C}\vec{c} \tag{28}$$

Which corresponds to the Green function

$$-\sigma_2\mathcal{C}\mathcal{S}^{-1} = \begin{pmatrix} \dfrac{i(z_-^{(1)}z_+^{(2)} - z_-^{(2)}z_+^{(1)})}{y_+^{(1)}z_+^{(2)} + y_+^{(2)}z_+^{(1)}} & -\dfrac{i(z_-^{(1)}y_+^{(2)} + z_-^{(2)}y_+^{(1)})}{y_+^{(2)}z_+^{(1)} + y_+^{(1)}z_+^{(2)}} \\ -\dfrac{i(y_-^{(2)}z_+^{(1)} + y_-^{(1)}z_+^{(2)})}{y_+^{(2)}z_+^{(1)} + y_+^{(1)}z_+^{(2)}} & \dfrac{i(y_-^{(2)}y_+^{(1)} - y_-^{(1)}y_+^{(2)})}{y_+^{(2)}z_+^{(1)} + y_+^{(1)}z_+^{(2)}} \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \tag{29}$$

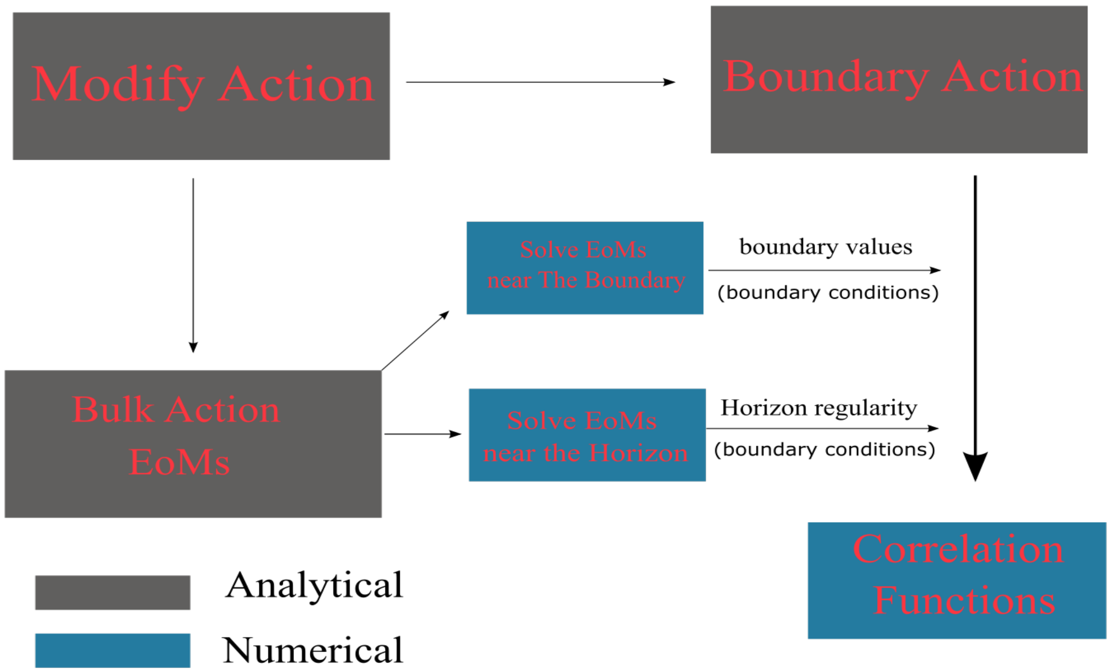
We can find the trace which is given by

$$TrG = i\frac{y_-^{(2)}y_+^{(1)} - y_-^{(1)}y_+^{(2)} + z_-^{(2)}z_+^{(1)} - z_-^{(1)}z_+^{(2)}}{y_+^{(2)}z_+^{(1)} + y_+^{(1)}z_+^{(2)}}$$

We can check this result with the case that $k_y = 0$ by assume that there are now the cross term between each eigenvectors $y_+^{(2)} = z_+^{(1)} = y_-^{(1)} = z_-^{(2)} = 0$ the Green function will reduce to a diagonal matrix

$$G = \begin{pmatrix} \frac{iz_-^{(1)}}{y_+^{(1)}} & 0 \\ 0 & \frac{-iy_-^{(2)}}{z_+^{(2)}} \end{pmatrix}$$

These are all of the set up before pure numerical calculation. We see that if we can solve all of the components with the initial conditions, we will get the complete Green function.


Modify Action
Boundary Action
Solve EoMs
near The Boundary
boundary values
(boundary conditions)
Bulk Action
EoMs
Solve EoMs
near the Horizon
Horizon regularity
(boundary conditions)
Correlation
Functions
Analytical
Numerical

APPENDIX A

Here we show the derivation of re-defined Dirac spinor, we start by considering Dirac equation represented by vierbien formalism:

$$\Big(e_a^M\Gamma^a\big(\partial_M+\frac{1}{4}\omega_M^{bc}\Gamma_{bc}\big)-m\Big)\Psi=0 \tag{30}$$

We using the statement that:

$$\frac{1}{4}\omega_M^{bc}e_a^M\Gamma^a\Gamma_{bc}=\frac{1}{4}\Gamma^z\partial_z\ln(-gg^{zz}) \tag{31}$$

So, if we substitute (31) on (30) and simplify them we get

$$e_a^M\Gamma^a\partial_M\Psi+\frac{1}{4}\omega_M^{bc}e_a^M\Gamma^a\Gamma_{bc}\Psi-m\Psi=0$$
$$e_a^M\Gamma^a\partial_M\Psi+\Gamma^z\partial_z\ln\big((-gg^{zz})^{1/4}\big)\Psi-m\Psi=0$$

where $\partial_z\ln\big((-gg^{zz})^{1/4}\big))=(-gg^{zz})^{-1/4}\partial_z(-gg^{zz})^{-1/4}$, if A $=(-gg^{zz})^{-1/4}$, we have

$$e_a^M\Gamma^a\partial_M\Psi+\frac{\Gamma^z}{A}(\partial_zA)\Psi-m\Psi=0$$
$$e_a^M\Gamma^aA\partial_M\Psi+\Gamma^z(\partial_zA)\Psi-mA\Psi=0$$
$$\Gamma^zA\partial_z\Psi+\Gamma^\mu A\partial_\mu\Psi+\Gamma^z(\partial_zA)\Psi+\underbrace{\Gamma^\mu(\partial_\mu A)\Psi}_{\text{inserted by 0}}-mA\Psi=0$$
$$e_a^M\Gamma^a\partial_M(A\Psi)-mA\Psi=0$$

We see that this kind of simplification is true only A is a function of z. Fortunately, for AdS space-time it depends on only z. So, we can write

$$\chi=(-gg^{zz})^{1/4}\Psi$$

Then, we can consider the Dirac equation without contribution from spin connection which given by

$$(\Gamma^{\underline{z}}\partial_z+\Gamma^{\underline{\mu}}\partial_\mu-m)\chi=0$$
$$(\frac{1}{\sqrt{g_{zz}}}\Gamma^z\partial_z+\frac{1}{\sqrt{g_{\mu\mu}}}\Gamma^\mu\partial_\mu-m)\chi=0$$
$$\frac{1}{\sqrt{g_{zz}}}(\Gamma^z\partial_z-\sqrt{g_{zz}}m)\chi-\frac{1}{\sqrt{-g_{tt}}}\Gamma^t\partial_t\chi+\frac{1}{\sqrt{g_{ii}}}\Gamma^i\partial_i\chi=0$$
$$\sqrt{\frac{g_{ii}}{g_{zz}}}(\Gamma^z\partial_z-\sqrt{g_{zz}}m)\chi-\big(-\sqrt{\frac{g_{ii}}{-g_{tt}}}i\omega\Gamma^t+ik_i\Gamma^i\big)\chi=0$$
$$\sqrt{\frac{g_{ii}}{g_{zz}}}(\Gamma^z\partial_z-\sqrt{g_{zz}}m)\chi-i(-E(z,\omega)\Gamma^{\underline{t}}+k_i\Gamma^i)\chi=0$$
$$\sqrt{\frac{g_{ii}}{g_{zz}}}(\Gamma^z\partial_z-\sqrt{g_{zz}}m)\chi-iK_\mu\Gamma^\mu\chi=0$$

where $K_\mu=\{-E(z,\omega),k_i\}=\{-\sqrt{\frac{g_{ii}}{-g_{tt}}}\omega,k_i\}$

APPENDIX B

According to Houng Lui papers, we follow the green function calculation easily by consider term by term as follows

$$S_{bdy} = \frac{i}{2}\int_{bdy} d^3x\bar{\chi}\chi = \frac{i}{2}\int_{bdy} d^3x\bar{\chi}_-\chi_+ + \bar{\chi}_+\chi_- = \frac{i}{2}\int_{bdy} d^3x - z_-^*y_+ + y_-^*z_+ - z_+^*y_- + y_+^*z_-$$

Here we should to fix either χ_+ or χ_- at the boundary, in this case we fix χ_+, So the variation of the boundary term can be reduced as

$$\delta S_{eff} = i\int_{bdy} d^3x \ - z_-^*\delta y_+ + y_-^*\delta z_+ + c.c. \tag{32}$$

Then, the solution of dirac equations solutions near the boundary are given by

$$y_+ = A_1 z^{-m} + B_1 z^{1+m} \quad , \quad y_- = C_1 z^{1-m} + D_1 z^m \tag{33}$$
$$z_+ = A_2 z^{-m} + B_2 z^{1+m} \quad , \quad z_- = C_2 z^{1-m} + D_2 z^m \tag{34}$$

from here we see that,

$$\begin{aligned}\frac{z_-}{y_+} &= z^{2m}\Big(\frac{-D_2 - C_2 z^{1-2m}}{A_1 + B_1 z^{1+2m}}\Big)\\ \frac{y_-}{z_+} &= z^{2m}\Big(\frac{-D_1 - C_1 z^{1-2m}}{A_2 + B_2 z^{1+2m}}\Big)\end{aligned}$$

As we consider D and A are matrices, and if we define

$$\xi_+ = \frac{iy_-}{z_+} \quad , \quad \xi_- = -\frac{iz_-}{y_+}$$

where we are able to relate $A, B, C, \text{and} D$ by following procedure Let's consider

$$\begin{aligned}(\partial_z - \frac{m}{z})\chi_+ &= -ik_\mu\gamma^\mu\chi_-\\ (\cancel{Bmz^{m-1}} + A(1-m)z^{-m} - \cancel{Bmz^{m-1}} - Amz^{-m}) &= -ik\cdot\gamma(Dz^{1+m} + Cz^{-m})\\ -A(2m-1)z^{-m} &= -ik\cdot\gamma(Dz^{1+m} + Cz^{-m})\\ A &= \frac{ik\cdot\gamma}{2m-1}C\end{aligned}$$

in the same way,

$$\begin{aligned}(\partial_z + \frac{m}{z})\chi_- &= ik_\mu\gamma^\mu\chi_+\\ (D(1+m)z^m - \cancel{Cmz^{-m-1}} + Dmz^m + \cancel{Cmz^{-m-1}}) &= ik\cdot\gamma(Bz^m + Az^{1-m})\\ D(2m+1)z^m &= ik\cdot\gamma(Bz^m + Az^{1-m})\\ D &= \frac{ik\cdot\gamma}{2m+1}B\end{aligned}$$

APPENDIX C

Solution on $SAdS_{d+1}$ background
Let us consider SAdS background metric tensor,

$$ds^2 = \frac{L^2}{z^2}\left(- f(z)dt^2 + d\vec{x} + \frac{dz^2}{f(z)}\right) \tag{35}$$

where $f(z) = 1 - \left(\frac{z}{z_h}\right)^d$, we see that at $z_h \to \infty$ the metric tensor reduces to pure AdS.

Normally, for convenient calculation we may map horizon to 1 and we may simply write $\left(\frac{z}{z_h}\right)^d$ to z. However, z_h contains thermodynamic information about the black hole, and I need to show it in the last results. As a result, I continue to consider the horizon to be z_h. Now, let's consider the equation of motion for scalar field

$$\left(\frac{1}{\sqrt{|g|}}\partial_m\left(\sqrt{|g|}g^{mn}\partial_n\right) - m^2\right)\phi = 0 \tag{36}$$

and then, we assume the solution of the field by using separation of variable,

$$\phi = \phi(z)e^{ik^\mu x_\mu} = \phi(z)e^{-i\omega t + k\cdot\vec{x}}$$

We start from $\vec{x}$ components, it's easily to get $\quad -\left(\frac{z}{L}\right)^2 k^2\phi$
For t-z component, let easily begin with

$$\frac{1}{\sqrt{\left(\left(\frac{L}{z}\right)^2\right)^{d+1}}}\partial_m\left(\sqrt{\left(\left(\frac{L}{z}\right)^2\right)^{d+1}}g^{mn}\partial_n\phi(z)e^{-i\omega t + k\cdot\vec{x}}\right) = 0$$

$$\frac{1}{\left(\frac{L}{z}\right)^{d+1}}\partial_m\left(\left(\frac{L}{z}\right)^{d+1}g^{mn}\partial_n\phi(z)e^{-i\omega t + k\cdot\vec{x}}\right) = 0$$

and then we see that the metric component is time-independent, so, for time component

$$\frac{z^2}{L^2 f(z)}\partial_t^2\phi(z)e^{-i\omega t + k\cdot\vec{x}} = \left(\frac{z}{L}\right)^2\frac{\omega^2}{f(z)}\phi \tag{37}$$

Then, let us consider the non-trivial one, the z component

$$\frac{1}{\left(\frac{L}{z}\right)^{d+1}}\partial_z\left(\left(\frac{L}{z}\right)^{d+1}g^{zz}\partial_z\phi(z)e^{-i\omega t + k\cdot\vec{x}}\right) = \frac{z^{d+1}}{L^2}\partial_z\left(z^{-(d+1)}z^2 f(z)\phi'\right) = 0$$

$$
\begin{aligned}
\frac{z^{d+1}}{L^2}\partial_z\left(z^{-(d-1)}f(z)\phi'\right) &= 0\\
\frac{z^{d+1}}{L^2}\left(z^{-(d-1)}(f(z)\phi')' - (d-1)z^{-d}f(z)\phi'\right) &= 0\\
\frac{z^{d+1}}{L^2}\left(z^{-(d-1)}\big(f(z)\phi'' + f'(z)\phi'\big) - (d-1)z^{-d}f(z)\phi'\right) &= 0\\
\frac{z^2}{L^2}\left(f(z)\phi'' + f'(z)\phi' - \frac{(d-1)}{z}f(z)\phi'\right) &= 0
\end{aligned}
$$

From above result we can write (2) as

$$
f(z)\phi'' + \left(f'(z) - \frac{(d-1)}{z}f(z)\right)\phi' + \left(\frac{\omega^2}{f(z)} - k^2 - \frac{L^2}{z^2}m^2\right)\phi = 0 \tag{38}
$$

It is almost impossible to solve the full solution outlined above. However, we need just the solution near the horizon, so we may simplify more by considering $f(z)$ where $z \to z_h$

$$
\begin{aligned}
f(z) &= 1 - \left(\frac{z}{z_h}\right)^d\\
&\sim -\frac{d}{z_h}(z - zh) \qquad\qquad z \to z_h\\
f'(z) &\sim -\frac{d}{z_h}
\end{aligned}
$$

So, we can re-write (4)

$$
-\frac{d}{z_h}(z - zh)\phi'' + \left(-\frac{d}{z_h} + \frac{d(d-1)}{z_h z}(z - z_h)\right)\phi' + \left(-\frac{z_h\omega^2}{d(z - zh)} - k^2 - \frac{L^2}{z^2}m^2\right)\phi = 0 \tag{39}
$$

And then, ignoring some terms of the above equation, we will get a simple differential equation as follows.

$$
\begin{aligned}
\frac{d}{z_h}(z - z_h)\phi'' + \frac{d}{z_h}\phi' + \frac{z_h\omega^2}{d(z - z_h)}\phi &= 0\\
\phi'' + \frac{\phi'}{z - z_h} + \frac{z_h^2\omega^2}{d^2(z - z_h)^2}\phi &= 0
\end{aligned}
$$

This differential equation gives us the solution as a power-law solution,

$$
\phi \sim c_1(z - zh)^{\frac{z_h^2\omega^2}{d^2}} \tag{40}
$$

So, we know that the leading term of the solution near the horizon will in the form that

$$
\phi \sim (z - z_h)^{\alpha} \tag{41}
$$

We can substitute ϕ to check what is α exactly will be. So, we tern it on equation (5), we will see that $(z-z_h)^{\alpha-1}$ terms will dominate the equation.

$$
\begin{aligned}
-\frac{d}{z_h}(z-zh)\big[\alpha(\alpha-1)(z-z_h)^{\alpha-2}\big]+\Big(-\frac{d}{z_h}+\frac{d(d-1)}{z_h z}(z-z_h)\Big)\alpha(z-z_h)^{\alpha-1}+ \\
...+\Big(-\frac{z_h\omega^2}{d(z-zh)}-k^2-\frac{L^2}{z^2}m^2\Big)(z-z_h)^{\alpha}=0 \\
-\frac{d}{z_h}\big(\alpha(\alpha-1)\big)(z-z_h)^{\alpha-1}-\frac{d}{z_h}\alpha(z-z_h)^{\alpha-1}-\frac{z_h\omega^2}{d}(z-zh)^{\alpha-1}=0 \\
\alpha(\alpha-1)+\alpha+\frac{z_h^2\omega^2}{d^2}=0
\end{aligned}
$$

So, now we have the solution of α by solve above

$$\alpha=\pm i\frac{z_h\omega}{d}$$

re-write equation (7) and we get the solution near the horizon as,

$$\phi\sim(z-z_h)^{\pm\frac{iz_h\omega}{d}} \tag{42}$$

As we introduced that the horizon z_h contains thermodynamics information $z_h=1/\pi T$. So, the information provides

$$
\begin{aligned}
\phi &\sim (z-zh)^{\pm\frac{i\omega}{d\pi T}} && (43)\\
&\sim c_1(z-zh)^{\frac{i\omega}{d\pi T}}+c_2(z-zh)^{\frac{-i\omega}{d\pi T}} && (44)
\end{aligned}
$$

Numerically, we can set the ansatz solutions for the horizon region as above solution.