ASSIGNMENT

Topic: Maximum and minimum, Partial fraction, Integration

Course Title: Mathematics I
Course Code: MAT 101

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Solution to the question no: 1

Finding the maximum and minimum values

a) $f(x) = 4e^{2x} + 9e^{-2x}$

Given that, $f(x) = 4e^{2x} + 9e^{-2x}$ ——— (i)

$\therefore f'(x) = 4.2e^{2x} + 9(-2)e^{-2x}$

$f'(x) = 8e^{2x} - 18e^{-2x}$ ——— (ii)

Again $f''(x) = 16e^{2x} + 36e^{-2x}$ ——— (iii)

For critical point, $f'(x) = 0$

$\therefore 8e^{2x} - 18e^{-2x} = 0$

$\Rightarrow 8e^{2x} = 18e^{-2x}$

$\Rightarrow \dfrac{e^{2x}}{e^{-2x}} = \dfrac{18}{8}$

$\Rightarrow e^{4x} = \dfrac{9}{4}$

$\Rightarrow \ln e^{4x} = \ln \frac{9}{4}$

$\Rightarrow 4x = \ln \frac{9}{4}$ $[\because \ln e = 1]$

$\therefore x = \frac{1}{4} \ln \frac{9}{4}$

$\therefore$ The critical point is $\frac{1}{4} \ln \frac{9}{4}$

From (III), $f''(x) = 16e^{2x} + 36e^{-2x}$

$$f''\left(\frac{1}{4}\ln\frac{9}{4}\right) = 16 \cdot e^{2\cdot\frac{1}{4}\ln\left(\frac{9}{4}\right)} + 36e^{-2\cdot\frac{1}{4}\ln\left(\frac{9}{4}\right)}$$

$$= 16e^{\frac{1}{2}\ln\left(\frac{9}{4}\right)} + 36e^{-\frac{1}{2}\ln\left(\frac{9}{4}\right)}$$

$$= 16e^{\ln\left(\frac{9}{4}\right)^{1/2}} + 36e^{\ln\left(\frac{9}{4}\right)^{-1/2}}$$

$$= 16\left(\frac{9}{4}\right)^{1/2} + 36\left(\frac{9}{4}\right)^{-1/2} \quad [\because e^{\ln x} = x]$$

$$= 16 \cdot \frac{3}{2} + 36 \cdot \left(\frac{4}{9}\right)^{1/2}$$

$$= 24 + 36 \cdot \frac{2}{3}$$

$$= 24 + 24 = 48 > 0$$

$\therefore$ There is a minimum value at point $x = \frac{1}{4}\ln\frac{9}{4}$

The minimum value is

$$f\left(\frac{1}{4}\ln\frac{9}{4}\right) = 4e^{2\cdot\frac{1}{4}\ln\frac{9}{4}} + 9e^{-2\cdot\frac{1}{4}\ln\frac{9}{4}}$$

$$= 4e^{\frac{1}{2}\ln\frac{9}{4}} + 9e^{-\frac{1}{2}\ln\frac{9}{4}}$$

$$= 4e^{\ln\left(\frac{9}{4}\right)^{1/2}} + 9e^{\ln\left(\frac{9}{4}\right)^{-1/2}}$$

$$= 4\left(\frac{9}{4}\right)^{1/2} + 9\left(\frac{9}{4}\right)^{-1/2}$$

$$= 4 \cdot \frac{3}{2} + 9 \cdot \frac{2}{3}$$

$$= 6 + 6 = 12$$

(Ans)

b) $f(x) = x^5 - 5x^4 + 5x^3 - 10$

Given that, $f(x) = x^5 - 5x^4 + 5x^3 - 10$ ——— (i)

$\therefore f'(x) = 5x^4 - 20x^3 + 15x^2$ ——— (ii)

Again, $f''(x) = 20x^3 - 60x^2 + 30x$ ——— (iii)

For critical point, $f'(x) = 0$

$$\Rightarrow 5x^4 - 20x^3 + 15x^2 = 0$$

$$\Rightarrow 5x^2(x^2 - 4x + 3) = 0$$

$$\Rightarrow 5x^2(x^2 - 3x - x + 3) = 0$$

$$\Rightarrow 5x^2\{x(x-3) - 1(x-3)\} = 0$$

$$\Rightarrow 5x^2(x-3)(x-1) = 0$$

$$\Rightarrow x = 0, 1, 3$$

Now, $f''(0) = 0$, Therefore $f(x)$ has no maximum or minimum values at $x = 0$

At, $x = 1$, $f''(1) = 20(1)^3 - 60(1)^2 + 30(1)$
$= 20 - 60 + 30$
$= -10 < 0$

Therefore, $f(x)$ has a maximum value at $x = 1$ which is $f(1) = (1)^5 - 5(1)^4 + 5(1)^3 - 10$
$= 1 - 5 + 5 - 10$
$= -9$

Again, at $x=3$, $f''(3) = 20(3)^3 - 60(3)^2 + 30(3) = 540 - 540 + 90$

$= 90 > 0$

Therefore, $f(x)$ has a minimum value at $x=3$

which is $f(3) = (3)^5 - 5(3)^4 + 5(3)^3 - 10$

$= 243 - 405 + 135 - 10$

$= -37$

$\therefore$ The maximum value is -9 and the minimum value is -37.

Solution to the question no: 2

a) $\frac{x}{(x+1)^2(x+2)}$

Let, $\frac{x}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$ ——(i)

Multiplying both sides by the denominator $(x+1)^2(x+2)$

$x = A(x+1)(x+2) + B(x+2) + C(x+1)^2$ ——(ii)

Set, $x = -1$,

$-1 = B(-1+2)$

$\therefore B = -1$

$x = -2$,

$-2 = C(-2+1)^2$

$\Rightarrow C = -2$

$x = 0$,

$0 = A(0+1)(0+2) + B(0+2) + C(0+1)^2$

$\Rightarrow 2A + 2B + C = 0$

$\Rightarrow 2A + 2(-1) - 2 = 0$

$\Rightarrow 2A - 2 - 2 = 0$

$\Rightarrow 2A = 4$

$A = 2$

$\therefore \frac{x}{(x+1)^2(x+2)} = \frac{2}{x+1} + \frac{-1}{(x+1)^2} + \frac{-2}{x+2}$ is the partial fraction.

b) $\dfrac{x}{(x^2+1)(x-4)}$

Let,

$$\frac{x}{(x^2+1)(x-4)} = \frac{A}{x-4} + \frac{Bx+c}{x^2+1} \quad \text{——— (1)}$$

Multiplying (i) by the denominator $(x^2+1)(x-4)$

$$x = A(x^2+1) + (Bx+c)(x-4) \quad \text{——— (ii)}$$

Put $x=4$

$$4 = A(4^2+1) + 0$$

$$\Rightarrow 17A = 4$$

$$\therefore A = \frac{4}{17}$$

Now, put $x=0$ into (ii)

$$0 = A(0^2+1) + \{B(0) + c\}(0-4)$$

$$\Rightarrow A + c(-4) = 0$$

$$\Rightarrow A = 4c$$

$$c = \frac{A}{4} = \frac{\frac{4}{17}}{4} = \frac{1}{17}$$

From (ii) we get,

$$\frac{x}{(x^2+1)(x-4)} = \frac{4}{17(x-4)} + \frac{Bx+\frac{1}{17}}{x^2+1} \quad \text{——— (iii)}$$

Put $x = 1$ into (iii)

$$\frac{1}{(1^2+1)(1-4)} = \frac{4}{17(1-4)} + \frac{B(1)+\frac{1}{17}}{1^2+1}$$

$$\Rightarrow \frac{1}{-6} = \frac{4}{-51} + \frac{B+\frac{1}{17}}{2}$$

$$\Rightarrow B = -\frac{4}{17}$$

Hence,

$$\frac{x}{(x^2+1)(x-4)} = \frac{4}{17(x-4)} + \frac{-\frac{4}{17}x+\frac{1}{17}}{x^2+1}$$

c) $$\frac{x^4+2x^3-2x^2+4x-1}{x^2+2x-3}$$

$$= \frac{x^2(x^2+2x-3)+x^2+4x-1}{x^2+2x-3}$$

$$= \frac{x^2(x^2+2x-3)}{x^2+2x-3} + \frac{x^2+4x-1}{x^2+2x-3}$$

$$= x^2 + \frac{(x^2+2x-3)+2x+2}{x^2+2x-3}$$

$$= x^2 + \frac{(x^2+2x-3)}{(x^2+2x-3)} + \frac{2x+2}{x^2+2x-3}$$

$$= x^2+1+\frac{2x+2}{x^2+2x-3}$$

$$= x^2+1+\frac{2x+2}{x^2+3x-x-3}$$

$$= x^2+1+\frac{2x+2}{x(x+3)-1(x+3)}$$

$$= x^2+1+\frac{2x+2}{(x+3)(x-1)}$$

$$= x^2+1+\frac{1}{x+3}+\frac{1}{x-1}$$

$$\therefore \frac{x^4+2x^3-2x^2+4x-1}{x^2+2x-3} = x^2+1+\frac{1}{x+3}+\frac{1}{x-1}$$

Solution to the question number 3

a) $\int \frac{\tan^{-1}x}{1+x^2} dx$

Here, $\int \frac{\tan^{-1}x}{1+x^2} dx = \int u \, du$

$= \frac{u^2}{2} + c$

$= \frac{(\tan^{-1}x)^2}{2} + c$

$\therefore \int \frac{\tan^{-1}x}{1+x^2} dx = \frac{(\tan^{-1}x)^2}{2} + c$

Let, $\tan^{-1}x = u$

$\frac{1}{1+x^2} = \frac{du}{dx}$

$du = \frac{dx}{1+x^2}$

b) $\int \sin^2 2x \, dx$

Here, $\int \sin^2 2x \, dx = \int \frac{1-\cos 4x}{2} dx$

$= \frac{1}{2}\int (1-\cos 4x) \, dx$

$= \frac{1}{2}\left[\int 1 \, dx - \int \cos 4x \, dx\right]$

$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right] + c$

$= \frac{1}{2}x - \frac{1}{8}\sin 4x + c$

$\therefore \int \sin^2 2x \, dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + c$

$2\sin^2 2x = 1 - \cos 4x$

$\therefore \sin^2 2x = \frac{1-\cos 4x}{2}$

c) $\int \sin^4 x\,dx$

Here, $\int \sin^4 x\,dx = \int (\sin^2 x)^2 dx$

$$= \frac{1}{4}\int (2\sin^2 x)^2 dx$$

$$= \frac{1}{4}\int (1-\cos 2x)^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\left[\int 1\,dx - \int 2\cos 2x\,dx + \int \cos^2 2x\,dx\right]$$

$$= \frac{1}{4}\left[x - 2\int \cos 2x\,dx + \frac{1}{2}\int 2\cos^2 2x\,dx\right]$$

$$= \frac{1}{4}\left[x - 2\frac{\sin 2x}{2} + \frac{1}{2}\int (1+\cos 4x)\,dx\right]$$

$$= \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \cos 4x\,dx\right]$$

$$= \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}x + \frac{1}{2}\cdot\frac{\sin 4x}{4}\right] + c$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right] + c$$

$$= \frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$$

$\therefore \int \sin^4 x\,dx = \frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$

d) $\int \cos^4 2x\,dx$

Here, $\int \cos^4 2x\,dx = \int (\cos^2 2x)^2 dx$

$\cos^2 2\theta = \dfrac{1+\cos 4\theta}{2}$

$$= \int \left(\frac{1+\cos 4x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int [1+\cos^2 4x + 2\cos 4x]\,dx$$

$$= \frac{1}{4}\int \left[1+\left(\frac{1+\cos 8x}{2}\right)+2\cos 4x\right]dx$$

$$= \frac{1}{4}\int \left[1+\frac{1}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]dx$$

$$= \frac{1}{4}\int \left[\frac{3}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]dx$$

$$= \int \left[\frac{3}{8}+\frac{\cos 8x}{8}+\frac{2\cos 4x}{4}\right]dx$$

$$= \int \left[\frac{3}{8}+\frac{\cos 8x}{8}+\frac{\cos 4x}{2}\right]dx$$

$$= \frac{3}{8}\int 1\,dx + \frac{1}{8}\int \cos 8x\,dx + \frac{1}{2}\int \cos 4x\,dx$$

$$= \frac{3}{8}x + \frac{1}{8}\,\frac{\sin 8x}{8} + \frac{1}{2}\,\frac{\sin 4x}{4} + c$$

$$= \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + c$$

$$\therefore \int \cos^4 2x\,dx = \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + c$$

e) $\int \frac{e^x(1+x)}{\sin^2(xe^x)}\,dx$

Let,

$xe^x = u$

$e^x + xe^x = \frac{du}{dx}$

$(e^x + xe^x)\,dx = du$

$e^x(1+x)\,dx = du$

Here, $\int \frac{e^x(1+x)}{\sin^2 x(e^x)}\,dx = \int \frac{du}{\sin^2 u}$

$= \int \operatorname{cosec}^2 u\,du$

$= -\cot u + c$

$= -\cot(xe^x) + c$

$\therefore \int \frac{e^x(1+x)}{\sin^2(xe^{2x})}\,dx = -\cot(xe^x) + c$