Stochastic Calculus and Itô's Lemma

In this lecture. . .

More on a W.P. X_t

Itô's lemma & calculus for functions of a stoch. var. X_t $F = F(X_t)$

By the end of this lecture you will be able to

- understand more about Brownian motion and diffusions

- manipulate functions of random variables

Introduction

Stochastic calculus is very important in the mathematical modeling of financial processes. This is because of the underlying (assumed) random nature of financial markets.

State var. vs. asset price

S_t r_t

Very Important Notation

We have seen X as the 'end result' of a random walk, up to some time t.

We will often work with the amount by which X changes from moment to moment.

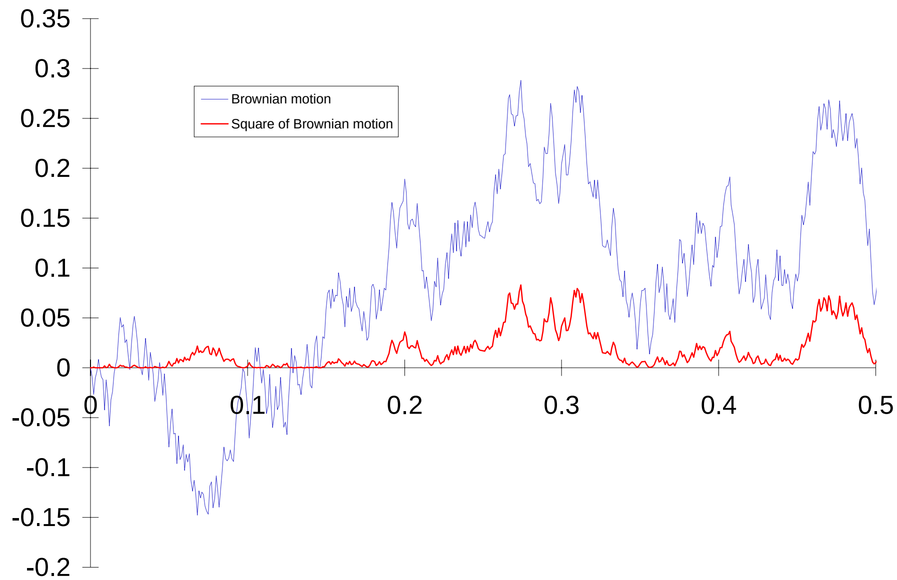
- Think of dX as being an increment in X, i.e. a Normal random variable with mean zero and standard deviation $dt^{1/2}$.

$$t \to t + dt$$

$$X_t \to X_t + \boxed{dX_t}$$

Functions of stochastic variables and Itô's lemma

Now we'll see the idea of a function of a stochastic variable. Below is shown a realization of a Brownian motion $X(t)$ and the function $F(X) = X^2$.



Whenever we have functions of a variable it is natural to want to know how to differentiate and manipulate these functions.

What are the rules of calculus when variables are stochastic?

$x \qquad f = f(x)$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\frac{dW}{dt} = \lim_{\delta t \to 0} \frac{W(t+\delta t) - W(t)}{\delta t} \quad dW$$

$$= \lim_{\delta t \to 0} \frac{O(\sqrt{\delta t})}{\delta t} \sim \lim_{\delta t \to 0} \frac{1}{\sqrt{\delta t}}$$

The first point to note is that in the stochastic world we really have two 'variables.'

These are time t and the Brownian motion X.

X_t $\qquad$ X

$F(X)$

We are used to writing ordinary and partial differential equations in the form

$$\frac{dF}{d\cdot}$$

or

$$\frac{\partial F}{\partial \cdot}$$

where the quantities on the bottom are the independent variables.

So might expect something similar in the stochastic world.

We immediately hit a problem, however.

Because dX is of size $\sqrt{dt}$ it is much bigger than dt.

This means that we have to be careful whenever we think about gradients/slopes/derivatives/sensitivities, since these are limits as dt goes to zero.

For this reason, in the stochastic world we instead work with stochastic differential equations.

These take the form

$$dF = \cdots dt + \cdots dX.$$

$\phi\sqrt{\delta t}$

$$dF = A(X,t)\, dt$$

$$\frac{dF}{dt} = A(X,t)$$

$\phi \sim N(0,1)$

So, what are the rules of calculus?

Since X is stochastic, so is F, and we can ask 'what is the stochastic differential equation for F?'

If $F(X) = X^2$ what is the equation for dF?

$$\underbrace{dX^2}_{(dX)^2} = dt$$

$$\mathbb{E}\left[dX^2\right] = dt$$

$$\lim_{dt \to 0} dX^2 \to dt$$

If $F = X^2$ is it true that $dF = 2X\ dX$?

No.

- The ordinary rules of calculus do not generally hold in a stochastic environment.

Then what are the rules of calculus?

We are going to throw caution to the wind, pretend that there are no problems or subtleties, use Taylor series. . . and see what happens!

Taylor Series . . . and Itô

If we were to do a naive Taylor series expansion of F, completely disregarding the nature of X, and treating dX as a small increment in X, we would get

$$F(X + dX) = F(X) + \frac{dF}{dX}dX + \tfrac{1}{2}\frac{d^2F}{dX^2}dX^2,$$

ignoring higher-order terms.

We could argue that $F(X + dX) - F(X)$ was just the 'change in' F and so

$$\boxed{dF = \frac{dF}{dX}dX + \tfrac{1}{2}\frac{d^2F}{dX^2}\cancel{dX^2}. \quad dt}$$

This is ~~almost~~ correct.

Because of the way that we have defined Brownian motion, and have seen how the quadratic variation behaves, it turns out that the dX^2 term isn't really random at all.

The dX^2 term becomes (as all time steps become smaller and smaller) the same as its average value, dt.

Taylor series and the ‘proper’ Itô are very similar. The only difference being that the correct Itô’s lemma has a dt instead of a dX^2.

- You can, with little risk of error, use Taylor series with the 'rule of thumb'

$$dX^2 = dt.$$

  and in practice you will get the right result.

Let's get some intuition now, and then shortly we will do Itô's lemma properly!

We can now answer the question, "If $F = X^2$ what is dF?" In this example

$$\frac{dF}{dX} = 2X \quad \text{and} \quad \frac{d^2F}{dX^2} = 2.$$

Therefore Itô's lemma tells us that

$$dF = dt + 2X\,dX.$$

This is an example of a **stochastic differential equation**.

Stochastic differential equations

Stochastic differential equations are used to model random quantities, a stock price for example.

They have two parts to them, a **deterministic** and a **random**.

Suppose we want to model a stock price as a random quantity. Let's use S to denote that stock price.

A stochastic differential equation for S would look something like this:

drift × dt   diffusion × dX

$$dS = \text{ Deterministic } + \text{ Random } .$$

In words: "The change in the stock price has a predictable component and a random component."

$$dS = \mu S\, dt + \sigma S\, dX$$

$$\frac{dS}{S} = \mu\, dt + \sigma\, dX$$

$$\phi \sqrt{dt}$$

More precisely

$$dS = \text{ Something } \ dt + \text{ Something else } \ dX.$$

The randomness is captured by the dX term.

But what are these 'somethings'?

In the standard models they would be functions of S and time, t.

$$dS = f(S,t)\; dt + g(S,t)\; dX.$$

The function $f(S,t)$ captures how the predictable bit of the stock model varies with S and t and the $g(S,t)$ function captures the randomness.

$$dS = f(S,t)\,dt + g(S,t)\,dX.$$

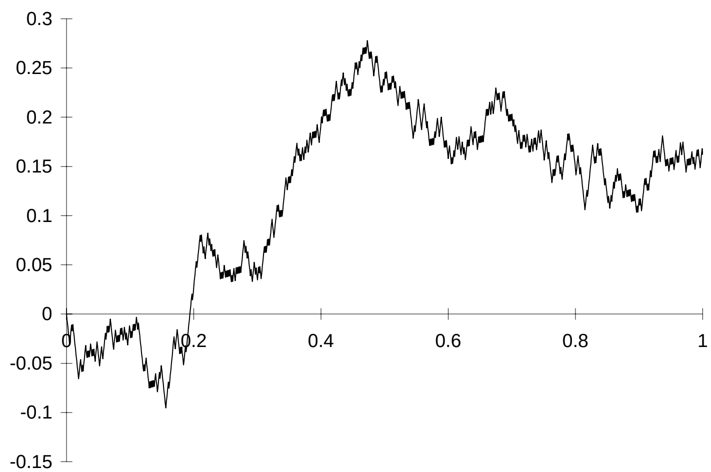
We sometimes call the $f(S,t)$ function the **growth rate** or the **drift**.

The $g(S,t)$ is related to the **volatility** of S.

Some pertinent examples

The first example simple Brownian motion but with a drift:

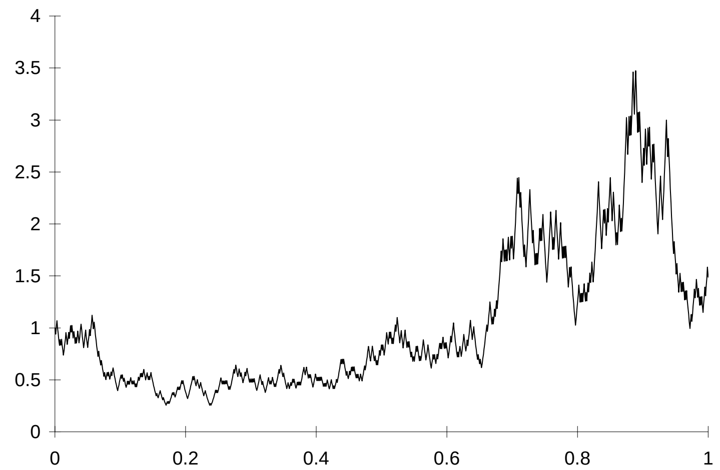
$$dS = \mu \, dt + \sigma \, dX.$$



In this realization S has gone negative.

Our second example is similar to the above but the drift and randomness scale with S:

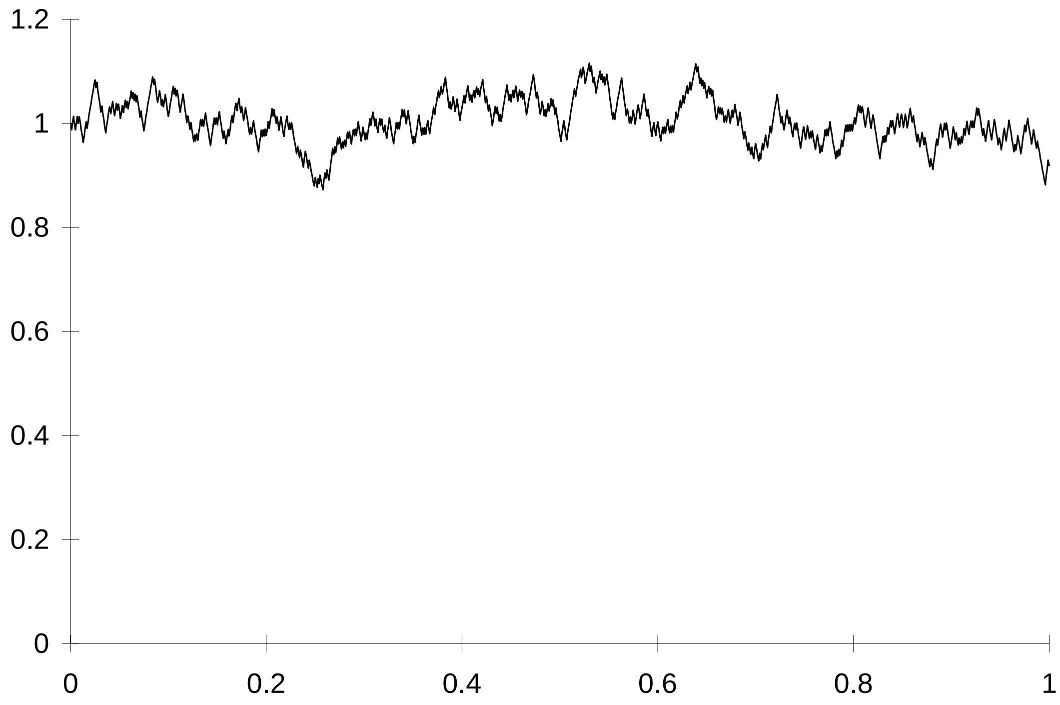
$$dS = \mu S \, dt + \sigma S \, dX.$$



If S starts out positive it can never go negative; the closer that S gets to zero the smaller the increments dS.

The third example is

$$dS = (\nu - \mu S)dt + \sigma\, dX.$$



The random walk

$$dS = (\nu - \mu S)dt + \sigma\, dX$$

is an example of a **mean-reverting** random walk.

If S is large, the negative coefficient in front of dt means that S will move down on average, if S is small it rises on average. There is still no incentive for S to stay positive in this random walk.

With r instead of S this random walk is the Vasicek model for the short-term interest rate.

The final example is similar to the third but we are going to adjust the random term slightly:

$$dS = (\nu - \mu S)dt + \sigma S^{1/2}\,dX.$$

Now if S ever gets close to zero the randomness decreases, perhaps this will stop S from going negative?

This particular stochastic differential equation for S will be important later on, it is the Cox, Ingersoll & Ross model for the short-term interest rate.

$$dr_t = -\gamma(r_t - \bar{r})\,dt + \sigma\sqrt{r}\,dX$$

$$\sqrt{r} \to 0$$

Pursuing this idea further, imagine what might be meant by

$$dW = g(t)\,dt + f(t)\,dX.$$

- Equations like this are called **stochastic differential equations**. Their precise meaning comes, however, from the technically more accurate equivalent stochastic integral.

This equation above is shorthand for

$$W(t) = \int_0^t g(\tau)\,d\tau + \int_0^t f(\tau)\,dX(\tau).$$

The mean square limit $E\left[\left(f(x)-l\right)^2\right] \to 0$

This is useful in the precise definition of stochastic integration.

Examine the quantity

$$E\left[\left(\sum_{j=1}^{n} \underbrace{(X(t_j)-X(t_{j-1}))^2}_{Y(t_j)} - t\right)^2\right]$$

where

$$t_j = \frac{jt}{n}.$$

$$E\left[\left(\sum_{j=1}^{n} Y(t_j) - t\right)^2\right] \qquad Y(t_j) \equiv Y_j$$

This can be expanded as

$$E\left[\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^4\right.$$

$$+2\sum_{i=1}^{n}\sum_{j<i}(X(t_i)-X(t_{i-1}))^2(X(t_j)-X(t_{j-1}))^2$$

$$\left.-2t\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^2+t^2\right].$$

Since $X(t_j)-X(t_{j-1})$ is Normally distributed with mean zero and variance t/n we have

$$E\left[(X(t_j)-X(t_{j-1}))^2\right]=\frac{t}{n}$$

and

$$E\left[(X(t_j)-X(t_{j-1}))^4\right]=\frac{3t^2}{n^2}.$$

Thus the required expectation becomes

$$n\frac{3t^2}{n^2} + n(n-1)\frac{t^2}{n^2} - 2tn\frac{t}{n} + t^2 = O\left(\frac{1}{n}\right).$$

As $n \to \infty$ this tends to zero. We therefore say that

$$\sum_{j=1}^{n} (X(t_j) - X(t_{j-1}))^2 = t$$

in the 'mean square limit.' This is often written, for obvious reasons, as

$$\int_0^t (dX)^2 = t.$$

Whenever we talk about 'equality' in the following 'proof' we mean equality in the mean square sense.

Manipulating stochastic differential equations

An equation of the form

G_t is a stoch process

$$dG = a\,(G, t)\,dt + b\,(G, t)\,dX$$

is called a Stochastic Differential Equation (SDE) for G (or random walk for dG) and consists of two components:

1. $a\,(G, t)\,dt$ is deterministic – coefficient of dt is known as the **drift** or **growth**

2. $b\,(G, t)\,dX$ is random – coefficient of dX is known as the **diffusion** or **volatility**

and we say G evolves according to (or follows) this process.

So if for example we have a random walk

$$dS = \mu S dt + \sigma S dX \tag{1}$$

then the drift is $a(S,t) = \mu S$ and the diffusion is $b(S,t) = \sigma S$.

The process (1) is also called **Geometric Brownian Motion** (GMB) or **Exponential Brownian motion** (EMB) and is a popular model for a wide class of asset prices.

We have previously considered Itô's lemma to obtain the change in a function $f(X)$ when $X \to X + dX$, where X is a standard Brownian motion.

This jump $df = f(X + dX) - f(X)$ is given by

$$df = \frac{df}{dX} dX + \tfrac{1}{2} \frac{d^2 f}{dX^2} dt \tag{2}$$

using the result

$$\lim_{dt \to 0} dX^2 = dt.$$

Suppose we now wish to extend the result (2) to consider the change in an option price $V(S)$ where the underlying variable S follows a geometric Brownian motion.

(Of course, you are not supposed to know anything about options yet. Just think of manipulating functions.)

If we rewrite (1) as

$$\frac{dS}{S} = \mu \, dt + \sigma \, dX \qquad (*)$$

then dS represents the change in asset price S in a small time interval dt.

This expression is the return on the asset.

μ is the average growth rate of the asset and σ the associated volatility (standard deviation) of the returns.

dX is an increment of a Brownian Motion, known as a Wiener process and is a Normally distributed random variable such that $dX \sim N(0, dt)$.

An obvious question we may ask is, what is the jump in $V\left(S+dS\right)$ when $S \rightarrow S + dS$?

We begin (again) by using a Taylor series as in (2), but for $V\left(S+dS\right)$ to get

$$dV = \frac{dV}{dS}\,dS + \tfrac{1}{2}\,\frac{d^2V}{dS^2}\,dS^2.$$

We can proceed further now as we have an expression for dS (and hence dS^2). As dt is very small, any terms in $dt^{\frac{3}{2}}$ or dt^2 are insignificant in comparison and can be ignored. So working to $O\left(dt\right)$

$$dS^2 = \sigma^2 S^2 dt.$$

If we substitute this into the previous expression for dV we get Itô's lemma as applied to $V(S)$:

$$dV = \left(\mu S \frac{dV}{dS} + \frac{1}{2}\,\sigma^2 S^2 \frac{d^2V}{dS^2}\right) dt + \left(\sigma S \frac{dV}{dS}\right) dX. \qquad (3)$$

Note that this is another stochastic differential equation!

It contains a predictable part and a random part.

Suppose that we had a formula for $V(S)$. Let's take a very special case, let's consider

$$V(S) = \log S.$$

Differentiating this once gives

$$\frac{dV}{dS} = \frac{1}{S}.$$

Differentiating this again gives

$$\frac{d^2V}{dS^2} = -\frac{1}{S^2}.$$

Now from (3) we have

$$d(\log S) = \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma dX.$$

Integrating both sides between 0 and t

$$\int_0^t d(\log S) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) d\tau + \int_0^t \sigma \, dX \qquad (t > 0)$$

$$\log \frac{S_t}{S_0} \qquad = \left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma \left(X(t) - X(0)\right).$$

Therefore

$$\log\left(\frac{S(t)}{S(0)}\right) = \left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma\left(X(t) - X(0)\right)$$

Assuming $X(0) = 0$ and $S(0) = S_0$, the exact solution becomes

$$S(t) = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma X(t)\right). \tag{4}$$

$\phi\sqrt{dt}$

$[t, t+dt]$

$S_{t+dt} = S_0 e^{(\mu-\frac{1}{2}\sigma^2)dt + \phi\sqrt{dt}}$

$[0, T]$

$S_T = S_0 e^{(\mu-\frac{1}{2}\sigma^2)T + \phi\sqrt{T}}$

Itô in higher dimensions

In financial problems we often have functions of one stochastic variable S and a deterministic variable t, time: $V(S,t)$. If

$$dS = a(S,t)dt + b(S,t)dX,$$

then the increment dV is given by

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}b^2\frac{\partial^2 V}{\partial S^2}dt.$$

Again, this is shorthand notation for the correct integrated form.

The example we will be seeing a lot of is the sde commonly used to model an equity price

$$dS = \mu S\ dt + \sigma S\ dX,$$

with $V(S,t)$ being the value of an option. The sde for $V(S,t)$ is then

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt.$$

Remember this!

Occasionally, we have a function of two, or more, random variables, and time as well: $V(S_1, S_2, t)$.

Let's write the behaviour of S_1 and S_2 in the general form

$$dS_1 = a_1(S_1, S_2, t)dt + b_1(S_1, S_2, t)dX_1$$

and

$$dS_2 = a_2(S_1, S_2, t)dt + b_2(S_1, S_2, t)dX_2.$$

Note that we have *two* Brownian increments dX_1 and dX_2. We can think of these as being Normally distributed with variance dt, but *they are correlated*.

The correlation between these two random variables we will call ρ. This can also be a function of S_1, S_2 and t but must satisfy

$$-1 \leq \rho \leq 1.$$

- The ‘rules of thumb’ can readily be imagined:

$$dX_1^2 = dt, \quad dX_2^2 = dt \quad \text{and} \quad dX_1 dX_2 = \rho\, dt.$$

Itô’s lemma becomes

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2 + \\ &\quad \tfrac{1}{2} b_1^2 \frac{\partial^2 V}{\partial S_1^2} dt + \rho b_1 b_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} dt + \tfrac{1}{2} b_2^2 \frac{\partial^2 V}{\partial S_2^2} dt. \end{aligned}$$

Summary

Please take away the following important ideas

- Functions of random variables can't be differentiated in quite the same way as functions of deterministic variables.

- Instead of using Taylor series you must use Itô's lemma. However, they are very similar and a simple rule of thumb can usually be used to get from Taylor to Itô.