HW2 — Nitishree Supekar

3.2

(6) $(x^3+2x)/(2x+1)$ is $O(x^2)$

Ans) for this :- when $x>1 \Rightarrow [x^2 > x \Rightarrow x^3 > x^2]$

so $\frac{x^3+2x}{2x+1}$ becomes $\frac{x^3+2x^3}{2x} = \frac{3x^3}{2x} = \frac{3}{2}x^2 = C[x^2]$

here $C = 3/2$, $k=1$ & $O[x^2]$

(12) To show $x \log x$ is $O(x^2)$ but that x^2 is not $O(x \log x)$

→ (a) Since we know that $x > \log x$ for all x.

so for $x>0$; $x \cdot \log x < x \cdot x$

$\Rightarrow x \log < x^2$

$\therefore c=1$ and $k=0$

→ (b) Since for x^2 is not $\log x \rightarrow \frac{x^2}{x \log x} = \frac{x}{\log x}$

and

$\forall x > 1$ $\log x < \sqrt{x}$

so $\frac{x}{\log x} > \frac{x}{\sqrt{x}}$ ~~for all~~ Hence x^2 is not $O(x \log x)$

3.3

(36) By greedy algorithm :

Making change for n cents using quarters, dimes, nickels, pennies is $O(n)$.

Ans) Since for n cents we need to iterate first n times (any 1 pennies denominations used)

and since each time one comparison is made, the next time the n decreases $(n, n-1)$

so the highest comparison is n. &

for Big O we take higher degree of polynomial $\rightarrow O(n)$

Q3) $\boxed{5.1}$

(40) To prove if $A_1, A_2 \ldots A_n$ and B are sets, then
$$(A_1 \cap A_2 \cap \ldots \cap A_n) \cup B$$
$$= (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_n \cup B).$$

Ans) By Induction :-

Basis step : at $n = 1$
$$(A_1 \cap A_2 \cap \ldots \cap A_n) \cup B \Rightarrow \underline{A_1 \cup B}$$

Inductive step :
$$(A_1 \cap A_2 \cap \ldots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \ldots (A_n \cup B)$$

To prove :-
$$(A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1}) \cup B = (A_1 \cup B) \cap \ldots \cap (A_{n+1} \cup B)$$
$$\Rightarrow \cancel{[[(A_1 \cup B) \cup (A_2 \cup B) \cup (A_3 \cup B) \ldots (A_n \cup B)] \cap A_{n+1}] \cup B}$$
$$[(A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}] \cup B$$
$$\Rightarrow [[A_1 \cap A_2 \cap \ldots \cap A_n] \cup B] \cap (A_{n+1} \cup B)$$
$$= \underbrace{(A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_n \cup B)}_{\text{from above inductive step}} \cap (A_{n+1} \cup B)$$

Hence proved.

(54) Given set of $(n+1)$ positive integers, none exceeding $2n$ there is atleast one integer in the set that divides another integer in the set.

Ans)

At the last →

$\boxed{5.2}$

Q8) Given: Store offers gift certificates in denominations of 25 dollars & 40 dollars.
To get possible total amounts formed with these gift certificates.

Ans) Let $P(n)$ = possible amounts formed from 25 \$ & 40 \$ certificates
Since $25 = 5 \times 5$ & $40 = 5 \times 8 = \underline{\underline{5(n)}}$
We can form some numbers using 5 & 8.
After calculating :-
let $n \geq 28$
By strong induction :-
$P(n)$ is true for all $n \geq 28$.

Starting

$$\left.\begin{aligned} n = 28 &= 8+5+5+5+5 \\ n = 29 &= 8+8+8+5 \\ n = 30 &= 5+5+5+5+5+5 \\ n = 31 &= 8+8+5+5+5 \end{aligned}\right\} \text{come true}$$

$n = 32 = 8+8+8+8$

Second Induction Step :-
$P(j)$ comes true if $28 \leq j \leq k$ & $k \geq 32$
$k-4 \geq 32-4 \Rightarrow k-4 \geq 28$
since $k-4$ is true
$P(k-4)$ is true thus we can form
$5(k-4)$ dollars $\rightarrow 5(k-4)+25 = \boxed{5(k+1)}$

$\boxed{6.1}$

34) Given :- set of 10 elements to map with

→ (a) a set of 2 elements.

Since $\{1\} \xrightarrow{f} \{1\} \rightarrow 1^1$ function we get

$\{1,2\} \xrightarrow{f} \{1,2\} \rightarrow 2^1$ functions.

$\{1\} \xrightarrow{f} \{1,2,3\} \rightarrow 3^1$ functions

Thus for a set of 10 elements

we get → $\underline{\underline{2^{10} \text{ functions}}} = \underline{\underline{1024}}$

→ (b) a set of 3 elements.

⇒ $\underline{\underline{3^{10} \text{ elements}}} = \underline{\underline{59049}}$

→ (c) a set of 4 elements

⇒ $\underline{\underline{4^{10} \text{ elements}}} = \underline{\underline{1048576}}$

→ (d) a set of 5 elements

⇒ $\underline{\underline{5^{10} \text{ elements}}} = \underline{\underline{9765625}}$

$\boxed{6.2}$

34) Given :- people have 1,000,000 hair on head

population of NY city = 8,008,278.

To prove atleast 9 people have same number of hair on their head -

Ans) By formula $N = 8,008,278$ and $k = 1,000,001$

$$\left\lceil \frac{N}{k} \right\rceil \approx \lceil 8.008 \rceil$$

$$= 9$$

$\boxed{6.3}$

24) Given :- 10 women, 6 men standing in a line.
→ No 2 men should stand next to eachother

Ans) _ W M W M W M W M W M W M W W W W

This makes $P(10, 10) \times P(11, 6)$
men can have 10 + 1 (1 - start position)

$\Rightarrow 10! \times \frac{11!}{5!} = 3628800 \times 332640$

$= \underline{\underline{1207084032000}}$

$\boxed{6.5}$

(14) Given : $x_1 + x_2 + x_3 + x_4 = 17$
To find all possible solutions for the equation :

Ans) This can be written as -
$C[17 + 4 - 1, 17) = \frac{20!}{17!\,3!} = \underline{\underline{1140 \text{ solutions}}}$

(44) Given :- Dozen books to be stored on 4 distinguishable ways

Ans) (a) → all books indistinguishable -
By this -

$\underbrace{x_1 + x_2 + x_3 + x_4}_{\text{4 indistinguisable shelves}} = 12$ (books)

$= C[12 + 4 - 1, 12] = 455$

(b) Given :- no 2 books are same

Ans) Numbering books $b_1, b_2 \ldots b_{12}$ (positions don't matter)

Since b_1 can have 4 ways to be placed

$b_2 \rightarrow 5$ ways

$b_3 \rightarrow 6$ ways

$\vdots$

$b_{12} \rightarrow 15$ ways to place

total $\rightarrow 4 \cdot 5 \cdot 6 \cdot 7 \cdots 15 = 217945728000$

$\boxed{5.1}$

Q 54) Given set of $n+1$ positive integers, none exceeding $2n$, there ~~are~~ is atleast 1 integer in the set that divides another integer in the set.

~~Eg~~ Ans) By Induction;-

Let $P(n)$ be the proposition "If A is set of n+1 positive Integers none exceeding $2n$ then there exists one integer in A that divides another integer in A"

let $S = \{ n \in N \mid P(n) = T \}$

(i) $1 \in S$ {Base case}

(ii) If $n \in S$ then $n+1 \in S$ (Inductive step)

for (i) $\rightarrow$ when $n = 1$ the set A can only consist of 2 positive integers no larger than 2.

Since there are 2 integers $\rightarrow A = \{1, 2\} \Rightarrow 1 \mid 2$

for inductive hypothesis –

Assume $P(n)$ is true OR If ~~A~~ A is a set consisting of $n+1$ positive integers no larger than $2n$, then there exists elements $a, b \in A$ so that $a | b$.

We need to prove $P(n+1)$ is true :–

$P(n+1)$ is statement → "If B is a set of $(n+1)+1$ positive integers, none exceeding $2(n+1)$ then there is atleast 1 integer in B that divides another integer in B.

Case 1 If $2n+1 \notin B$ & $2n+2 \notin B$ then every element in B is less than or equal to $2n$.

Take out one element x from B.

Thus $B \setminus \{x\}$ which has $n+1$ positive integers none exceed $2n$.

$a, b \in B \setminus \{x\}$ so that $a | b$. since a & b are in B so then 2 elements divides other.

Case 2 If $2n+1 \in B$ or $2n+2 \in B$ but not both so for every element in B is less than or equal to $2n$ with exactly one exception. ($2n+1$ or $2n+2$) after taking it out a set of $n+1$ elements remain and none exceed $2n$. Thus by hypothesis a & b $a | b$ and these elements are in B.

Case 3 If $2n+1 \in B$ & $2n+2 \in B$. If we throw $2n+1$ out ∴ $2n+2$ is in B which is more than $2n$. and same for $2n+1$.

Consider $B / \{2n+2\}$ and add element $n+1$ at $n+1 \notin B$ This gives us set $C = (B \setminus \{2n+2\}) \cup \{n+1\}$

now If there is an element $a \in B$ that divides $n+1$ then it must divide $2(n+1)$ also.
So now we throw $2n+2$ out so there is no element in C that can possibly divide by $n+1$.
Thus we have a set C that contains $n+2$ positive integers & exactly one of them exceeds $2n$. This case is reduced to ~~and~~ previous case and thus true.