Homework I

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Abstract

This is Daniel's homework of "Numerical Algorithms with Case Studies I".

1 The First Problem

For the first problem we need to examine our own code:

```
n = 100;
A = randn(n, n);
x = rand(n, 1);
b = MatVec(A, x);
disp(b)
disp(A*x)
disp(A*x-b)
function b = MatVec(A, x)
    b = zeros(size(x));
    for i = 1:size(A, 1)
        b(i) = A(i, :)*x;
    end
end
```

(a) MatVec

```
n = 100;
x = randn(n, 1);
x2n = NormTwo(x);
disp(x2n)
disp(norm(x, 2))
function x2n = NormTwo(x)
    x2n = sqrt(dot(x, x));
end
```

(b) NormTwo

```
n=100;
[Q, ~]=qr(randn(n, n));
x = randn(n, 1);
y = MatVec(Q, x);
disp(NormTwo(y));
disp(NormTwo(x));
function b = MatVec(A, x)
    b = zeros(size(x));
    for i = 1:size(A, 1)
        b(i) = A(i, :)*x;
    end
end
function x2n = NormTwo(x)
    x2n = sqrt(dot(x, x));
end
```

(c) TestQ

```
>> TestQ
   10.1620

   10.1620

>> TestQ
    9.0591

    9.0591

>> TestQ
    8.9164

    8.9164
```

(d) Results

The correctness of my code is guaranteed, though there could occur slight round-off errors. Surprisingly, $\forall x \in R^n$, $Q \in \{Orthogonal\, Matrices\}$, $s.t. \|Qx\| = \|x\|$. This means orthogonal transformation is a sort of euclid transformation, which doesn't change the two norm of a vector.

2 The Second Problem

Q1 In this problem, the decomposition of A is given by $A = U\Sigma V^T$. Let U $= (u_1, u_2, \dots, u_n)$, and $V^T = (v_1, v_2, \dots, v_n)^T$. As is proved already, U and V^T are both orthogonal matrices, which means either $u_1, u_2, \dots, u_n$ or $v_1, v_2, \dots, v_n$ are linearly independent and $\forall u_i, v_i$, $\|u_i\|_2 = \|v_i\|_2 = 1$. Moreover, u_i and v_i are singular eigenvectors of A associated with singular value σ. Since the SVD is equivalent to

$$AV = U\Sigma$$

Then

$$Av_i = \sigma_i u_i, \quad i = 1, \dots, n \quad and \quad A = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$

$\forall x \in R^n$, $c_1, c_2, \dots, c_n$, $\quad s.t. \|x\|_2 = 1 \, and \, x = \sum_{i=1}^{n} c_i v_i$, so that $\sum_{i=1}^{n} c_i^2 = 1$.

For 2-norm:

$$\begin{aligned}
\|Ax\|_2 &= \|\sum_{i=1}^{n} c_i A v_i\|_2 = \|\sum_{i=1}^{n} c_i \sigma_i u_i\|_2 \\
&= |\sum_{i=1}^{n} c_i \sigma_i| \cdot \|u_i\|_2 = |\sum_{i=1}^{n} c_i \sigma_i| \leq \sigma_1
\end{aligned}$$

$\|Ax\|_2 = \sigma_1$ if and only if $c_1 = 1$, $c_2 = \dots = c_n = 0$

Q2 First, we could easily prove $\|A\|_F^2 = \|\alpha_1\|_2^2 + \|\alpha_2\|_2^2 + \dots + \|\alpha_n\|_2^2$ with definition of the Frobenius norm, as $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ being the column space of A.The singular value decomposition of A is given by $A = U\Sigma V^T$. Obviously,

$$U^T A V = \Sigma = (U^T \alpha_1 V, U^T \alpha_2 V, \dots, U^T \alpha_n V)$$

Let us prove that 2-norm is a unitarily invariant norm:

$$\begin{aligned}
\forall Q \in R^{n \times n}, \quad \|QA\|_2^2 &= \|A^T Q^T Q A\|_2 \\
&= \|A^T A\| = \|A\|_2^2 \\
\|AQ\|_2^2 &= \|AQQ^T A^T\|_2 \\
&= \|AA^T\| = \|A\|_2^2
\end{aligned}$$

As a result,

$$\begin{aligned}
\|A\|_F^2 &= \|\alpha_1\|_2^2 + \|\alpha_2\|_2^2 + \dots + \|\alpha_n\|_2^2 \\
&= \|U^T \alpha_1 V\|_2^2 + \|U^T \alpha_2 V\|_2^2 + \dots + \|U^T \alpha_n V\|_2^2 \\
&= \|U^T A V\|_F^2 \\
&= \|\Sigma\|_F^2 \\
\|A\|_F^2 &= \|\Sigma\|_F^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2
\end{aligned}$$

Q3 The condition number of A is $\kappa(A) = \|A\|_2 \cdot \|A^{-1}\|_2$

$$\begin{aligned}
\|A^{-1}\|_2 &= \|(U\Sigma V^T)^{-1}\|_2 \\
&= \|(V^{-1})^T \Sigma U^{-1}\|_2 \\
&= \|\Sigma^{-1}\|_2 \\
\kappa(A) &= \|A\|_2 \cdot \|A^{-1}\|_2 = \|\Sigma\|_2 \cdot \|\Sigma^{-1}\|_2 \\
&= \frac{\sigma_{max}}{\sigma_{min}} = \frac{\sigma_1}{\sigma_n}
\end{aligned}$$

3 The Third Problem

Given $A = \begin{bmatrix} 5 & 1 & -1 \\ -3 & -1 & -1 \\ 6 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 0 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -3 & -6 \\ -2 & -3 & 2 \end{bmatrix}$. We can easily get the dot products,

$$A \cdot B = \begin{bmatrix} 11 & -17 & 1 \\ -9 & 5 & -3 \\ 16 & -11 & 4 \end{bmatrix}$$

$$A^T \cdot C = \begin{bmatrix} -17 & -4 & 20 \\ -3 & 1 & 6 \\ -3 & -4 & 12 \end{bmatrix}$$

$$C \cdot B^T = \begin{bmatrix} -5 & -5 & 0 \\ 9 & -15 & -15 \\ 5 & -3 & -9 \end{bmatrix}$$

Then we get the inner products,

$$\langle AB,\, C\rangle = -18$$
$$\left\langle B,\, A^T C\right\rangle = -18$$
$$\left\langle A,\, CB^T\right\rangle = -18$$

This could verify that $\langle AB,\, C\rangle = \left\langle B,\, A^T C\right\rangle = \left\langle A,\, CB^T\right\rangle$

4 The Fourth Problem

Let $x = (x_1, x_2, \ldots, x_n)$.

Q1 First prove $\|x\|_\infty \leq \|x\|_2$,

$$\begin{aligned} \|x\|_\infty &= \max\{x_1, x_2, \ldots, x_n\} \\ &= x_{max} = \sqrt{x_{max}^2} \\ &\leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \|x\|_2 \end{aligned}$$

Then prove $\|x\|_2 \leq \|x\|_1$ with Cauchy-Schwarz inequality,

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \leq |x_1| + |x_2| + \cdots + |x_n| = \|x\|_1$$

And we get $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$.

Q2 Consider

$$\begin{aligned} \|x\|_2 &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\ &\leq \sqrt{n \cdot x_{max}^2} = \sqrt{n} \cdot x_{max} \\ &= \sqrt{n} \cdot \|x\|_\infty \end{aligned}$$

And we get $\|x\|_2 \leq \sqrt{n} \cdot \|x\|_\infty$

Q3 Consider the inequality of arithmetic and geometric means,

$$\begin{aligned}
\|x\|_1 &= |x_1| + |x_2| + \cdots + |x_n| \\
&\leq \sqrt{n} \cdot \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\
&= \sqrt{n} \cdot \|x\|_2
\end{aligned}$$