Homework V

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Abstract

This is Daniel's homework of "Numerical Algorithms with Case Studies I".

1 Problems

Q1 Rewrite the matrix $Ax = b$ into iteration form: $x_{k+1} = Bx_k + f$

1. Easy to find that for Jacobi iteration,

$$B = D^{-1}(L+U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} \tag{2}$$

This method clearly converges as $\rho(B) = 0$.
And for G-S iteration,

$$B = (D-L)^{-1}U = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} 0 & 2 & -2 \\ 0 & -2 & 3 \\ 0 & 0 & -2 \end{pmatrix} \tag{4}$$

This method can't converge as $\rho(B) = 2 > 1$.

2. Easy to find that for Jacobi iteration,

$$B = D^{-1}(L+U) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{6}$$

This method can't converge as $\rho(B) = \sqrt{\frac{5}{4}} > 1$.
And for G-S iteration,

$$B = (D - L)^{-1}U = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{8}$$

This method converges as $\rho(B) = \frac{1}{2}$.

Q2

1. A is positive definite only if all submatrix of A is positive definite, which gives,

$$det(A) = 1 - a^2 > 0 \tag{9}$$

$$a \in (-1, 1) \tag{10}$$

2. We could compute B and it's spectral radius,

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{pmatrix} \tag{12}$$

$$\rho(B) = a \tag{13}$$

And to make Jacobi iteration converge, we must have $a \in (-1, 1)$

3. We could compute B and it's spectral radius,

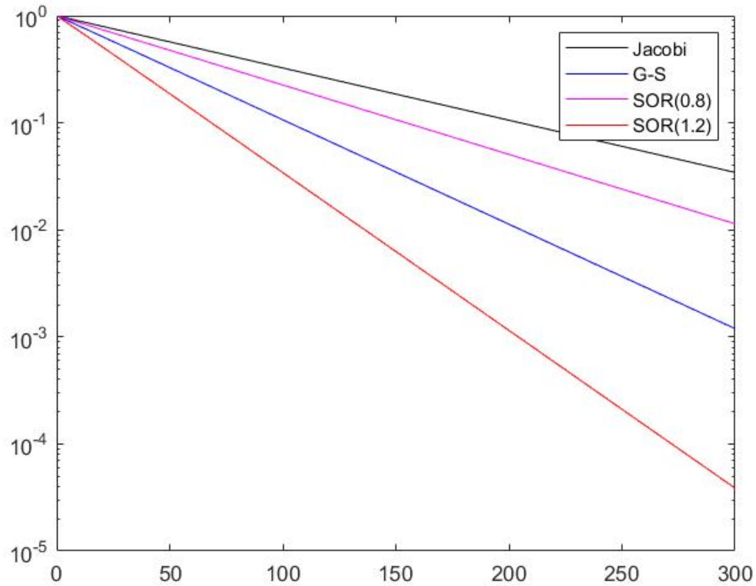
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \tag{15}$$

$$\rho(B) = a \tag{16}$$

And to make G-S iteration converge, we must have $a \in (-1, 1)$

Q3 Through careful examination, we can find that SOR iteration with $\omega = 1.2$ converges far more rapidly than other method. However, my convergence seems to be not so fast as shown in the lecture notes, as my algorithm of forward method to solve a linear system can't guarantee such high accuracy. So the initial case starts at a relatively high residue.



(a) Convergence Analysis

Q4 Check one equivalent statement of strongly convex function,

$$
\begin{aligned}
(\nabla f(y) - \nabla f(x))^T (y - x) &= (Ay - b - (Ax - b))^T (y - x) && (17)\\
&= (y - x)^T A(y - x) > 0 && (18)\\
& && (19)
\end{aligned}
$$

Therefore, $\exists \mu = \frac{(y-x)^T A(y-x)}{2\|y-x\|_2^2}, s.t.$

$$(\nabla f(y) - \nabla f(x))^T (y - x) \geq \mu \|y - x\|_2^2 \tag{20}$$

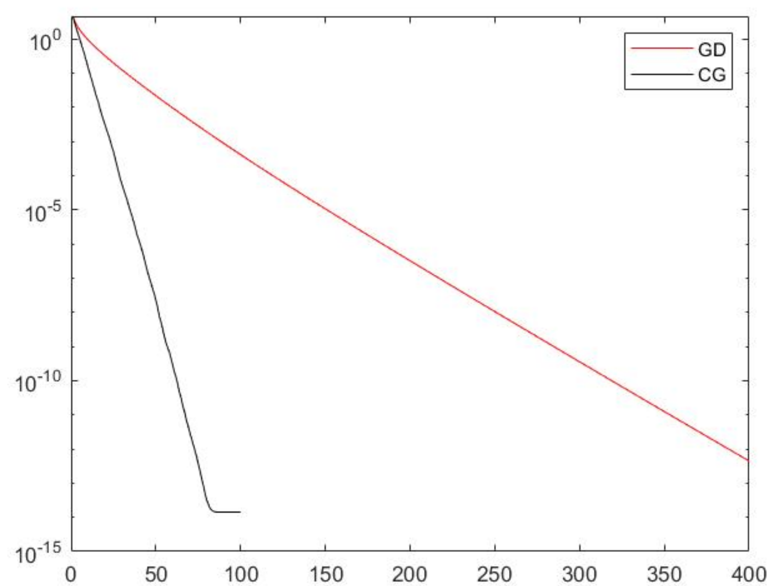
With lemma, we could claim that $f(x)$ is a convex function.

Q5 As A is positive definite, we can find the spectral decomposition of it, that is, $A = Q^T \Lambda Q$, and $p(A) = Q^T p(\Lambda) Q$. Suppose $Qv = (v_1, v_2, \ldots, v_n)^T$

$$\begin{aligned}
\|p(A)v\|_A^2 &= v^T p(A) A p(A) v = v^T Q^T p(\Lambda) \Lambda p(\Lambda) v && (21)\\
&= v^T Q^T \begin{pmatrix} p^2(\lambda_1)\lambda_1 & & & \\ & p^2(\lambda_2)\lambda_2 & & \\ & & \ddots & \\ & & & p^2(\lambda_n)\lambda_n \end{pmatrix} Qv && (22)\\
&= \sum_{i=1}^{n} a_i^2 p^2(\lambda_i)\lambda_i && (23)\\
&\leq p^2(\lambda)_{max} \sum_{i=1}^{n} a_i^2 \lambda_i && (24)\\
&= \|p(A)\|_2^2 v^T A v && (25)\\
&= \|p(A)\|_2^2 \|v\|_A^2 && (26)
\end{aligned}$$

Do square root on both side, and we can get $\|p(A)v\|_A \leq \|p(A)\|_2 \|v\|_A$

Q6 By implementing the two gradient descent method, we could clearly see the difference between the convergence speed of them, that CG converge in approximately 80 steps, while after more than 400 steps GD hasn't given us a satisfying result.



(b) Convergence Analysis