Homework III

Name: Shao Yanjun, Number: 19307110036

November 15, 2020

Abstract

This is Daniel's homework of "Numerical Algorithms with Case Studies I".

1 Problems

Q1 With definition, we can find $\omega = \frac{x-\tilde{x}}{\|x-\tilde{x}\|_2}$. And since H is orthogonal, it won't change the two's norm(length) of x. We could compute that $\alpha = 5$ with $\|\tilde{x}\|_2 = \|x\|_2$. Then we compute:

$$\omega = \frac{1}{5\sqrt{2}}[0, -5, 0, 0, 3, 4]$$

$$H = I - 2\omega\omega^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0.64 & -0.48 \\ 0 & 0.8 & 0 & 0 & -0.48 & 0.36 \end{pmatrix}$$

Q2 Conduct CGS on A:

$$
\begin{aligned}
\begin{pmatrix} -1 & -1 & -1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{pmatrix}
&\longrightarrow \begin{pmatrix} -\frac{1}{2} & -1 & -1 \\ \frac{1}{2} & 3 & 3 \\ -\frac{1}{2} & -1 & 5 \\ \frac{1}{2} & 3 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & 3 \\ -\frac{1}{2} & \frac{1}{2} & 5 \\ \frac{1}{2} & \frac{1}{2} & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{3}{\sqrt{26}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{2}{\sqrt{26}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{\sqrt{26}} \\ \frac{1}{2} & \frac{1}{2} & \frac{2}{\sqrt{26}} \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 & 3 \\ 0 & 2 & 7 \\ 0 & 0 & \sqrt{26} \end{pmatrix}
\end{aligned}
$$

Conduct MGS on A:

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -1 & -1 \\ \frac{1}{2} & 3 & 3 \\ -\frac{1}{2} & -1 & 5 \\ \frac{1}{2} & 3 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} -\frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 1 & 3 \\ -\frac{1}{2} & 1 & 5 \\ \frac{1}{2} & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 1 & \frac{13}{2} \\ \frac{1}{2} & 1 & \frac{11}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{13}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{11}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -3 \\ \frac{1}{2} & \frac{1}{2} & -2 \\ -\frac{1}{2} & \frac{1}{2} & 3 \\ \frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 & 3 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{3}{\sqrt{26}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{2}{\sqrt{26}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{\sqrt{26}} \\ \frac{1}{2} & \frac{1}{2} & \frac{2}{\sqrt{26}} \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 & 3 \\ 0 & 2 & 7 \\ 0 & 0 & \sqrt{26} \end{pmatrix}$$

The result perfectly coincides.

Q3 Do HouseholderQR with matlab code:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.1104 & 0.3646 & 0.5812 & -0.7191 \\ 0.4417 & 0.3085 & 0.5313 & 0.6537 \\ 0.7730 & 0.2524 & -0.5479 & -0.1961 \\ 0.4417 & -0.8415 & 0.2823 & -0.1307 \end{pmatrix} \cdot \begin{pmatrix} 9.0554 & 9.4971 & 9.7180 \\ 0 & 2.6086 & 2.1879 \\ 0 & 0 & 1.9427 \\ 0 & 0 & 0 \end{pmatrix}$$

Q4 The code and the figure are handed in together in the file. My observation is that the orthogonality of MGS is way better than CGS, as is shown in the graph.

Q5 Do QR decomposition on A^T that $A^T = Q\begin{pmatrix} R \\ 0 \end{pmatrix}$.Take $R^T = L, Q = \begin{pmatrix} Q_1 \vdots Q_2 \end{pmatrix}$, so $A = \begin{pmatrix} L \vdots 0 \end{pmatrix}\begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix}$. Let $Q_1^T x = y, Q_2^T x = z$, so that $\begin{pmatrix} L \vdots 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = b$ and $y = L^{-1}b$.

$$\min \|x\|_2 = \min \|QQ^T x\|_2 = \min \|Q_1 L^{-1} d + Q_2 z\|_2$$

Do gradient on $\|x\|_2$ and we get the optimized condition $Q_2^T Q_2 z = Q_2^T Q_1 L^{-1} d$. Use QR decomposition to reduce the condition number of this formula and retrieve the answer z of the least square problem. Use y and z to compute x:

$$x = Q\begin{pmatrix} y \\ z \end{pmatrix}$$

The condition number of this least square problem is immensely reduced.