Homework II

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Abstract

This is an optional latex source file for homework of "Numerical Algorithms with Case Studies I".

1 Problems

Q1 Conduct the Gaussian elimination on A:

$$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 1 & 1 & -3 & -1 \\ 2 & 3 & -1 & 1 \\ -2 & 3 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -3 & -6 \\ 0 & 3 & -1 & -9 \\ 0 & 3 & -2 & 10 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 7 & 28 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -2 & 3 & 0.875 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 20.125 \end{pmatrix}$$

Therefore, for $A = L \cdot U$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -2 & 3 & 0.875 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 20.125 \end{pmatrix}$$

Q2 Conduct Cholesky decomposition on A:

$$\begin{pmatrix} 3 & -1 & -3 & 1 \\ -1 & 7 & -3 & 7 \\ -3 & -3 & 10 & -4 \\ 1 & 7 & -4 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 1 & 0 \\ \frac{\sqrt{3}}{3} & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{20}{3} & -4 & \frac{22}{3} \\ 0 & -4 & 7 & -4 \\ 0 & \frac{22}{3} & -4 & \frac{26}{3} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & -\frac{\sqrt{3}}{3} & -\sqrt{3} & \frac{\sqrt{3}}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{60}}{3} & 0 & 0 \\ -\sqrt{3} & -\frac{\sqrt{60}}{5} & 1 & 0 \\ \frac{\sqrt{3}}{3} & \frac{11\sqrt{15}}{15} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{23}{5} & \frac{7}{5} \\ 0 & 0 & \frac{7}{5} & \frac{3}{5} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & -\frac{\sqrt{3}}{3} & -\sqrt{3} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{60}}{3} & -\frac{\sqrt{60}}{5} & \frac{11\sqrt{15}}{15} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{60}}{3} & 0 & 0 \\ -\sqrt{3} & -\frac{\sqrt{60}}{5} & \frac{\sqrt{115}}{5} & 0 \\ \frac{\sqrt{3}}{3} & \frac{11\sqrt{15}}{15} & \frac{7\sqrt{115}}{115} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{4}{23} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & -\frac{\sqrt{3}}{3} & -\sqrt{3} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{60}}{3} & -\frac{\sqrt{60}}{5} & \frac{11\sqrt{15}}{15} \\ 0 & 0 & \frac{\sqrt{115}}{5} & \frac{7\sqrt{115}}{115} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{60}}{3} & 0 & 0 \\ -\sqrt{3} & -\frac{\sqrt{60}}{5} & \frac{\sqrt{115}}{5} & 0 \\ \frac{\sqrt{3}}{3} & \frac{11\sqrt{15}}{15} & \frac{7\sqrt{115}}{115} & \frac{2\sqrt{23}}{23} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & -\frac{\sqrt{3}}{3} & -\sqrt{3} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{60}}{3} & -\frac{\sqrt{60}}{5} & \frac{11\sqrt{15}}{15} \\ 0 & 0 & \frac{\sqrt{115}}{5} & \frac{7\sqrt{115}}{115} \\ 0 & 0 & 0 & \frac{2\sqrt{23}}{23} \end{pmatrix}$$

Therefore, for $A = L \cdot L^T$

$$L = \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{60}}{3} & 0 & 0 \\ -\sqrt{3} & -\frac{\sqrt{60}}{5} & \frac{\sqrt{115}}{5} & 0 \\ \frac{\sqrt{3}}{3} & \frac{11\sqrt{15}}{15} & \frac{7\sqrt{115}}{115} & \frac{2\sqrt{23}}{23} \end{pmatrix}$$

Q3 Conduct the Gaussian elimination on A:

$$\begin{pmatrix}1 & 2 & 0 & 0 & 0\\ 1 & -2 & -1 & 0 & 0\\ 0 & 2 & 4 & 1 & 0\\ 0 & 0 & -1 & 3 & 2\\ 0 & 0 & 0 & 2 & -4\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 2 & 0 & 0 & 0\\ 0 & -4 & -1 & 0 & 0\\ 0 & 2 & 4 & 1 & 0\\ 0 & 0 & -1 & 3 & 2\\ 0 & 0 & 0 & 2 & -4\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & -0.5 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 2 & 0 & 0 & 0\\ 0 & -4 & -1 & 0 & 0\\ 0 & 0 & \frac{7}{2} & 1 & 0\\ 0 & 0 & -1 & 3 & 2\\ 0 & 0 & 0 & 2 & -4\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & -0.5 & 1 & 0 & 0\\ 0 & 0 & -\frac{2}{7} & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 2 & 0 & 0 & 0\\ 0 & -4 & -1 & 0 & 0\\ 0 & 0 & -\frac{7}{2} & 1 & 0\\ 0 & 0 & 0 & \frac{23}{7} & 2\\ 0 & 0 & 0 & 2 & -4\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & -0.5 & 1 & 0 & 0\\ 0 & 0 & -\frac{2}{7} & 1 & 0\\ 0 & 0 & 0 & \frac{14}{23} & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 2 & 0 & 0 & 0\\ 0 & -4 & -1 & 0 & 0\\ 0 & 0 & -\frac{7}{2} & 1 & 0\\ 0 & 0 & 0 & \frac{23}{7} & 2\\ 0 & 0 & 0 & 0 & -\frac{120}{23}\end{pmatrix}$$

Therefore, for $A = L \cdot U$

$$L = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & -0.5 & 1 & 0 & 0\\ 0 & 0 & -\frac{2}{7} & 1 & 0\\ 0 & 0 & 0 & \frac{14}{23} & 1\end{pmatrix}$$

$$U = \begin{pmatrix}1 & 2 & 0 & 0 & 0\\ 0 & -4 & -1 & 0 & 0\\ 0 & 0 & -\frac{7}{2} & 1 & 0\\ 0 & 0 & 0 & \frac{23}{7} & 2\\ 0 & 0 & 0 & 0 & -\frac{120}{23}\end{pmatrix}$$

We surprisingly discover that L and U are both bidiagonal matrices.

Q4 Conduct the partial pivoting LU decomposition on A:

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix} \xrightarrow{pivotL.3} \begin{pmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & -0.5 & -1.5 & -1.5 \\ 0 & -0.75 & -1.25 & -1.25 \\ 0 & 1.75 & 2.25 & 4.25 \end{pmatrix}$$

$$\xrightarrow{pivotL.3} \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 1.75 & 2.25 & 4.25 \\ 0 & -0.75 & -1.25 & -1.25 \\ 0 & -0.5 & -1.5 & -1.5 \end{pmatrix} \longrightarrow \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 1.75 & 2.25 & 4.25 \\ 0 & 0 & -\frac{2}{7} & \frac{5}{7} \\ 0 & 0 & -\frac{6}{7} & -\frac{2}{7} \end{pmatrix}$$

$$\xrightarrow{pivotL.4} \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 1.75 & 2.25 & 4.25 \\ 0 & 0 & -\frac{6}{7} & -\frac{2}{7} \\ 0 & 0 & -\frac{2}{7} & \frac{5}{7} \end{pmatrix} \longrightarrow \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 1.75 & 2.25 & 4.25 \\ 0 & 0 & -\frac{6}{7} & -\frac{2}{7} \\ 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

Q5 Let $A = (a_{ij})_{n\times n}$, $b = (b_i)_n^T$,

$$Ax - b = \begin{pmatrix} \sum_i^n a_{1i}x_i - b_1 \\ \sum_i^n a_{2i}x_i - b_2 \\ \vdots \\ \sum_i^n a_{ni}x_i - b_n \end{pmatrix}$$

$$\begin{aligned} f(x) &= \|Ax - b\|_2^2 = \langle Ax - b, Ax - b \rangle \\ &= \sum_j^n (\sum_i^n a_{ji}x_i - b_j)^2 \\ \frac{\partial f}{\partial x_i} &= 2(\sum_i^n a_{ij}) \cdot (\sum_i^n a_{ji}x_j - b_j) \\ \nabla f(x) &= (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})^T \\ &= 2((\sum_i^n a_{i1}) \cdot (\sum_i^n a_{1i}x_1 - b_1), \cdots, 2(\sum_i^n a_{in}) \cdot (\sum_i^n a_{ni}x_n - b_n))^T \end{aligned}$$

And since

$$\begin{aligned} 2A^T(Ax - b) &= 2A^T \begin{pmatrix} \sum_i^n a_{1i}x_i - b_1 \\ \sum_i^n a_{2i}x_i - b_2 \\ \vdots \\ \sum_i^n a_{ni}x_i - b_n \end{pmatrix} \\ &= 2((\sum_i^n a_{i1}) \cdot (\sum_i^n a_{1i}x_1 - b_1), \cdots, 2(\sum_i^n a_{in}) \cdot (\sum_i^n a_{ni}x_n - b_n))^T \end{aligned}$$

The equation $\nabla f(x) = 2A^T(Ax - b)$ is proved.

Q6 The code of LU:

```
function [L,U] = mylu(A)
    n = size(A);
    L = eye(n);
    U = A;

    for i = 1:n-1
        L(i+1:n,i) = U(i+1:n,i)/U(i,i);
        U(i+1:n,i:n) = U(i+1:n,i:n) - L(i+1:n,i) * U(i,i:n);
    end
end
```

Checking the correctness with Frobenius norm:

```
for i = 1:3
    A = randn(10,10);
    [L,U] = mylu(A);
    delta = A - L*U;
    disp(norm(delta,"fro"));
end
```

```
result:
        2.6634e-15

        4.4278e-14

        5.7086e-15
```