1. $\hat{\beta}=(X'X)^{-1}X'Y$

MLE:

$$L(\beta,\sigma)=\frac{1}{(2\pi\sigma)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\varepsilon^T\varepsilon\right)$$

$$=\frac{1}{(2\pi\sigma)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)\right)$$

since $-\frac{1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)\leq 0 \quad \forall\ Y-X\beta\in\mathbb{R}^n$

the exponencial term will be minimized by $Y=X\beta$

$\therefore$ An MLE of β can also be $\hat{\beta}=(X'X)^{-1}X'Y$

2. Let any linear combination of β be $c'\beta=c_0\beta_0+c_1\beta_1+\cdots+c_n\beta_n$

$\therefore$ for OLS estimator, $c'\hat{\beta}=c'(Z'Z)^{-1}Z'Y$ Mark $a=Z(Z'Z)^{-1}c$

And for any unbiased estimator $d'Y$, such that $E(d'Y)=d'Z\beta=c'\beta$

$\therefore Z'd=c$

Calculate Variance, $Var(d'Y)=\sigma^2 d'd=\sigma^2(a+(d-a))'(a+(d-a))$

$$=\sigma^2\ a'a+2(d-a)'a+(d-a)'(d-a)$$

$(d-a)'a=(d-Z(Z'Z)^{-1}c)'(Z(Z'Z)^{-1}c)=c'((Z'Z)^{-1}-(Z'Z)^{-1})c=0$

$\therefore Var(d'Y)\geq\sigma^2 a'a=Var(c'\hat{\beta})$

3. $E((n-r)\hat{\sigma}^2)=E(\hat{\varepsilon}'(y-\hat{y}))=E(\hat{\varepsilon}'(y+\hat{y}))$

$$=E(y'y)-E(\hat{y}'\hat{y})=\sum_{i=1}^{n}E(y_i^2)-E(\hat{y}_i^2)+n(E(y_i))^2-n(E(\hat{y}_i))^2$$

$$=\sum_{i=1}^{n}Var(Y_i)-Var(\hat{Y}_i)$$

$$=n\sigma^2-r\sigma^2=(n-r)\sigma^2$$

$\because \hat{Y}_i=e_i'\hat{Y}=e_i'Z(Z'Z)^{-1}Z'Y\sim N(\beta_i,(H)^{-1}_{ii}\cdot\sigma^2)$, $\sum_{i=1}^{n}(H)^{-1}_{ii}=tr(H^{-1})=r$

$\therefore E(\hat{\sigma}^2)=\sigma^2$

4. (1) $\hat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta+\varepsilon)$ is normal distribution

$E(\hat{\beta}) = \beta$. $Var(\hat{\beta}) = \sigma^2 (X'X)^{-1} X'X (X'X)^{-1} = \sigma^2 (X'X)^{-1}$

$\therefore \hat{\beta} \sim N(\beta, \sigma^2 (X^T X)^{-1})$

(2) $(N-p)\hat{\sigma}^2 = \hat{\varepsilon}'\hat{\varepsilon} = (y-\hat{y})'(y-\hat{y})$

$= y'(I-H)y$ where H is hat matrix $X(X'X)^{-1}X'$

since $(I-H)(I-H) = I-H$ is idempotent

and $tr(I-H) = n-p$

$\therefore \exists Q^T Q = I$, s.t. $Q^T(I-H)Q = Q^T \begin{pmatrix} \overbrace{1 \; \ddots \; 1}^{n-p} & \\ & \underbrace{0 \; \ddots \; 0}_{p} \end{pmatrix} Q$

$\therefore \hat{\varepsilon}'\hat{\varepsilon} = \xi'\xi = \sum_{i=1}^{n-p} \xi_i^2 \sim \chi^2_{n-p} \cdot \sigma^2$

$\therefore (N-p)\hat{\sigma}^2 = \sigma^2 \chi^2_{n-p}$

5. $E(y) = E(e^{x^T\beta+\varepsilon}) = e^{x^T\beta} E(e^{\varepsilon})$

$= e^{x^T\beta} \int_{-\infty}^{\infty} e^{\varepsilon} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\varepsilon^2}{2\sigma^2}} d\varepsilon$

$= e^{x^T\beta} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\varepsilon-\sigma^2)^2}{2\sigma^2} + \frac{\sigma^4}{2\sigma^2}} d\varepsilon$

$= e^{x^T\beta + \frac{\sigma^2}{2}}$

6. define $y = (y_1, \cdots, y_n)^T$, $\bar{y} = (\bar{y}, \cdots, \bar{y})^T$, $\hat{y} = (\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_n)^T$

$TSS = (y-\bar{y})'(y-\bar{y}) = (y-\hat{y}+\hat{y}-\bar{y})(y-\hat{y}+\hat{y}-\bar{y})$

$= (y-\hat{y})'(y-\hat{y}) + 2(y-\hat{y})(\hat{y}-\bar{y}) + (\hat{y}-\bar{y})'(\hat{y}-\bar{y})$

$\because (\hat{y}-\bar{y})'(y-\hat{y}) = (Hy)'(y-\bar{y}) + (\mathbb{1}\mathbb{1}^T y)'(y-\hat{y})$

$= y'H(I-H)y + y\mathbb{1}\mathbb{1}^T(I-H)y$

$\mathbb{1}^T(I-H) = \mathbb{1}^T - \mathbb{1}^T X(X^TX)^{-1}X^T$, $X = (\mathbb{1} \,|\, X_1 \,|\, \cdots \,|\, X_n)$

$\therefore \mathbb{1}^T$ is eigenvector of H. $\mathbb{1}^T H = \mathbb{1}$ $\therefore \mathbb{1}^T(I-H) = 0$

$\therefore TSS = RSS + ESS$