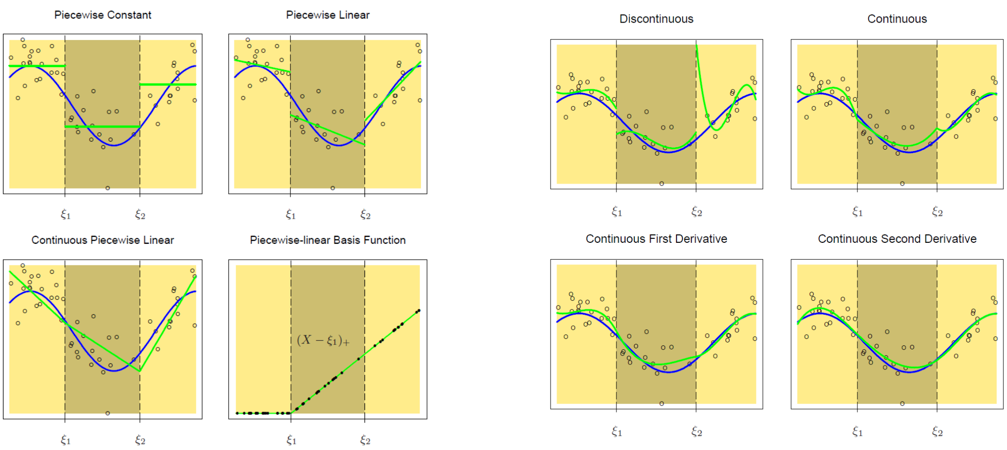
Spline

- $\forall\ \tau_0 < \tau_1 < \cdots < \tau_{K+1}$
  - spline function: a spline function of degree d on $[\tau_0, \tau_{K+1}]$ is a piecewise polynomial of degree d on $[\tau_j, \tau_{j+1})$
  - Knots: points where the spline may not have continuous d^{th} derivative.
- $\forall\ \xi_1 < \xi_2$ (special case with 2 knots)
  - Piecewise Constant:

    $$B_1(X) = I(X < \xi_1) \quad B_2(X) = I(\xi_1 \le X < \xi_2), \quad B_3(X) = I(\xi_2 \le X)$$

  - Piecewise Linear:

    $$B_1(X) = 1 \quad B_2(X) = X \quad B_3(X) = (X - \xi_1)_+ \quad B_4(X) = (X - \xi_2)_+$$



  - Piecewise Cubic:

    $$B_1(X) = 1 \quad B_2(X) = X \quad B_3(X) = X^2 \quad B_4(X) = X^3$$

    $$B_5(X) = (X - \xi_1)_+^3 \quad B_6(X) = (X - \xi_2)_+^3$$

  - Generalized Spline (if knots $\{\xi_l\}_{l=1}^{K}$)

    $$B_j(X) = X^{j-1},\ j = 1, \cdots, M \qquad B_{M+l}(X) = (X - \xi_l)_+^{M-1} \quad l = 1, 2, \cdots, K$$

Why they are spline???

1. We know that $B_j(X)$ $j = 1, \cdots, M+K$ are continuous
2. We know that $B_j^{(1)}(X)$ $j = 1, \cdots, M+K$ are continuous

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M-1. We know that $B_j^{(M-1)}(X)$ $j = 1, \cdots, M+K$ are continuous

$\Rightarrow$ By Algebraic Continuity Property, $\sum_{j=1}^{M+K} \alpha_j B_j(X)$ has continuous $(M-1)^{th}$ derivative and continuous M-th derivative on points other than knots

Q: In 数值II we learnt Cubic Spline. Is spline for regression the same?

A: Yes, but in 数值II spline is used for interpolation, in statistics spline is used for fitting.

Cubic spline: $S_i(x) = ax^3 + bx^2 + cx + d \quad x \in [x_i, x_{i+1}]$

s.t. $S_i(x_i) = f(x_i) \quad S_i(x_{i+1}) = f(x_{i+1}) \quad S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) \quad S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$

Extra constraint: Complete: $S_0'(x_0) = k_0,\ S_{n-1}'(x_n) = k_n$

Natural: $S_0''(x_0) = S_{n-1}''(x_n) = 0$

Periodic: $S_0'(x_0) = S_{n-1}'(x_n) \quad S_0''(x_0) = S_{n-1}''(x_n)$

Not a knot: $S_0'''(x_1) = S_1'''(x_1) \quad S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$

Construct 4 basis polynomials:

$\phi(x) = 3x^2 - 2x^3 \rightsquigarrow \varphi(0) = 0 \quad \varphi(1) = 1 \quad \varphi'(0) = 0 \quad \varphi'(1) = 0$

$\psi(x) = x^3 - x^2 \rightsquigarrow \psi(0) = \psi(1) = 0 \quad \psi'(0) = 0,\ \psi'(1) = 1$

Find $s_i(x) = \alpha_i \varphi\left(\frac{x_{i+1}-x}{h}\right) + \beta_i \varphi\left(\frac{x-x_i}{h}\right) + c_i \psi\left(\frac{x_{i+1}-x}{h}\right) + d_i \psi\left(\frac{x-x_i}{h}\right)$

- $S_i(x_i) = y_i \Rightarrow \alpha_i = y_i$ — $S_i'(x_i) = k_i \Rightarrow c_i = -hk_i$
- $S_i(x_{i+1}) = y_{i+1} \Rightarrow \beta_i = y_{i+1}$ — $S_i'(x_{i+1}) = k_{i+1} \Rightarrow d_i = hk_{i+1}$

And $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$

$$\Rightarrow 6\frac{y_i}{h_i^2} - 6\frac{y_{i+1}}{h_i^2} + 2\frac{k_i}{h_i} + 4\frac{k_{i+1}}{h_i} = -6\frac{y_{i+1}}{h_{i+1}^2} + 6\frac{y_{i+2}}{h_{i+1}^2} - 4\frac{k_{i+1}}{h_{i+1}} - 2\frac{k_{i+2}}{h_{i+1}}$$

Let $\beta_i = \frac{1}{h_i},\ \alpha_i = 2(\beta_i + \beta_{i+1}),\ \eta_i = 3\frac{y_{i+1}-y_i}{h_i^2}$

Then, in matrix form

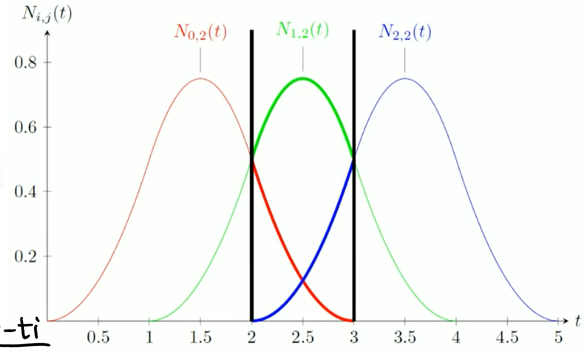
Complete:
$$\begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-3} & \alpha_{n-2} & \beta_{n-2} \\ & & & \beta_{n-2} & \alpha_{n-1} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-1} \end{pmatrix} = \begin{pmatrix} \eta_0 + \eta_1 - \beta_0 k_0 \\ \eta_1 + \eta_2 \\ \vdots \\ \eta_{n-2} + \eta_{n-1} - \beta_n k_n \end{pmatrix}$$

Natural
$$\begin{pmatrix} 2\beta_0 & \beta_0 & & & \\ \beta_0 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-3} & \alpha_{n-2} & \beta_{n-2} \\ & & & \beta_{n-2} & \alpha_{n-1} \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} \eta_0 \\ \eta_0 + \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

B-spline: A B-spline curve of degree k is defined by $S(t)=\sum_{i=0}^{n} N_{i,k}(t)\,\alpha_i$, where $N_{i,k}(t)$ are the basis functions defined using de Boor-Cox recursion formula.

$$N_{i,0}(t)= I(t_i \le t < t_{i+1}), \quad N_{i,j}(t)=\frac{t-t_i}{t_{i+j}-t_i} N_{i,j-1}(t) + \frac{t_{i+j+1}-t}{t_{i+j+1}-t_{i+1}} \cdot N_{i+1,j-1}(t)$$

$$\begin{array}{ccccccc}
N_{0,0}(t) & \rightarrow & N_{0,1}(t) & \cdots & N_{0,k-1}(t) & \rightarrow & N_{0,k}(t) \\
 & \nearrow & & & & \nearrow & \\
N_{1,0}(t) & \rightarrow & N_{1,1}(t) & \cdots & N_{1,k-1}(t) & & \\
\vdots & & \vdots & & & & \\
N_{n-1,0}(t) & \rightarrow & N_{n-1,1}(t) & & & & \\
 & \nearrow & & & & & \\
N_{n,0}(t) & & & & & &
\end{array}$$



· **Matrix form:** $N_{i,j}(t) = \begin{vmatrix} w_{i,k}(t)\ N_{i,j-1}(t) \\ (1-w_{i+1,k}(t))\ N_{i+1,j-1}(t) \end{vmatrix}$, where $w_{i,k}(t)=\frac{t-t_i}{t_{i+k}-t_i}$

· **Properties:**

1. B-spline is a piecewise polynomial which only breaks at knots $\{t_i\}_{i=1}^{n}$
2. $N_{i,j}(t)=0$ for $t\notin(t_i,t_{i+j})$
3. $N_{i,j}(t)>0$ for $t\in(t_i,t_{i+j})$
4. $\sum_{i=1}^{n+j} N_{i,j}(t)=1 \quad \forall\, x\in[t_0,t_n]$