Homework II

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October 3, 2021

Abstract

This is Daniel's homework of "Statistical Learning".

(1) The OLS is $\hat{\beta} = (\mathrm{X}'\mathrm{X})^{-1}\,\mathrm{X}'\mathrm{Y}$. And the likelihood function of β is,

$$L(\beta, \sigma) = \frac{1}{(2\pi\sigma)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\varepsilon^\top \varepsilon\right)$$

Since $-(\mathrm{Y} - \mathrm{X}\beta)(\mathrm{Y} - \mathrm{X}\beta) \leqslant 0, \quad \forall \mathrm{Y} - \mathrm{X}\beta \in \mathbb{R}^n$, the exponential term will be minimized by $\mathrm{Y} = \mathrm{X}\beta$.

Therefore, MLE of β can also be $\hat{\beta} = (\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathrm{Y}$

(2) Let any linear combination of β be $c'\beta = c_0\beta_0 + c_1\beta_1 + \cdots + \mathrm{c_n}\,\beta_n$. For OLS estimator. $c'\hat{\beta} = c'(\mathrm{Z}'\mathrm{Z})^{-1}\mathrm{Z}'\mathrm{Y}$. Mark $a = \mathrm{Z}\,(\mathrm{Z}'\mathrm{Z})^{-1}\,c$. And for any unbiased estimator $d'\mathrm{Y}$, such that $E(d'\mathrm{Y}) = d'\mathrm{Z}\beta = c'\beta$, we must have $\mathrm{Z}'d = c$.

Calculate Variance,

$$\begin{aligned} \operatorname{Var}\left(d'\mathrm{Y}\right) &= \sigma^2 d'd = \sigma^2(a + (d - \alpha))'(a + (d - a)) \\ &= \sigma^2\left(a'a + 2(d - a)'a + (d - a)'(d - a)\right) \end{aligned}$$

And notice that,

$$\begin{aligned} (d - a)'a &= \left(d - \mathrm{Z}\,(\mathrm{Z}')^{-1}\,c\right)'\left(\mathrm{Z}\,(\mathrm{Z}'\mathrm{Z})^{-1}\,c\right) \\ &= c'\left(\left(\mathrm{Z}'\mathrm{Z}\right)^{-1} - (\mathrm{Z}'\mathrm{Z})^{-1}\right)c = 0 \end{aligned}$$

Therefore, $\operatorname{Var}(d'\mathrm{Y}) \geqslant \sigma^2 a'a = \operatorname{Var}\left(c'\hat{\beta}\right)$

(3) $E\left(n - \hat{r}^{\hat{2}}\right) = E(\hat{z}(y - \hat{1}) = E(\varepsilon(y + y)) = E\left(y'y\right) - E\left(\hat{y}'\hat{y}\right) = \sum_{i=1}^n E\left(y_i^2\right) - E\left(\hat{y}_i^2\right) + n\left(E_c y_i\right)\big)^2 - n\left(E\left(\hat{y}_i\right)^2 = \sum_{i=1}^n \operatorname{Var}\left(Y_i\right) - \operatorname{Var}\left(\hat{Y}_i\right) = n\sigma^2 - r\sigma^2 = (n - r)\sigma^2 \because \hat{Y}_r = e_i'\hat{Y} = e'Z\left(z'z^{-1}\right)'Y \sim N\left(\beta_i, (H)^{-1} \cdot \sigma^2\right), \sum_{i=1}^n (H)_{ii}^{-1} = \operatorname{tr}\left(H^{-1}\right) = r \therefore E\left(\hat{\sigma}^2\right) = \sigma^2$