March 2, 2021 (Due: 08:00 March 9, 2021)

1. In the lecture we discussed the convergence of the truncated series

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

(a) Show that

$$\left|\frac{\pi^2}{6} - a_n\right| = \Theta\Bigl(\frac{1}{n}\Bigr).$$

(b) Estimate

$$\left|\frac{\pi^2}{6} - \Bigl(a_n + \frac{2}{2n+1}\Bigr)\right|.$$

(c) The convergence can be improved through

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)},$$

in the sense that the remainder of the truncated series is reduced to $\Theta(n^{-2})$. Can you further improve it?

2. In principle, the sine function can be evaluated through Taylor series

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \bigl(x \in (-\infty, +\infty)\bigr).$$

Let us consider two computational schemes to compute the sine function.
(a) Directly truncate the Taylor series. Make sure that the truncation error is less than the rounding error bound for any input x.
(b) First shift x to the interval $(-\pi/2, \pi/2]$, and then apply scheme (a).
Scheme (b) is in general more accurate. Why?

Sample at least 1000 points in $[-10, 10]$ (e.g., using the MATLAB/Octave statement `linspace(-10, 10, 1000)`) and plot the error of scheme (a) relative to scheme (b) (e.g., using the MATLAB/Octave function `semilogy`). Can you explain the result?

3. A backward error can sometimes be interpreted as a forward error in a certain sense. Let us consider the following example.

You are given $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, with A nonsingular. For an approximate solution $\hat{x} \in \mathbb{R}^n$ to the linear system $Ax = b$, try to find two vector norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ such that

$$\|b - A\hat{x}\|_\alpha = \|\hat{x} - x_*\|_\beta,$$

where $x_* = A^{-1}b$ is the exact solution.