Homework VIII

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Abstract

This is Daniel's homework of "Numerical Algorithms with Case Studies II".

1 Problems

Q1 Use the short recurrence formula for Chebyshev polynomial $2x\tilde{\omega_n}(x) = \tilde{\omega_{n+1}} + \tilde{\omega_{n-1}}$, we can calculate the eigenvalues of the following matrix to get n Chebyshev points.

$$\lambda_k \begin{pmatrix} \tilde{\omega_0}(\lambda_k) \\ \tilde{\omega_1}(\lambda_k) \\ \tilde{\omega_2}(\lambda_k) \\ \vdots \\ \tilde{\omega_{n-2}}(\lambda_k) \\ \tilde{\omega_{n-1}}(\lambda_k) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\omega_0}(\lambda_k) \\ \tilde{\omega_1}(\lambda_k) \\ \tilde{\omega_2}(\lambda_k) \\ \vdots \\ \tilde{\omega_{n-2}}(\lambda_k) \\ \tilde{\omega_{n-1}}(\lambda_k) \end{pmatrix} \tag{1}$$

Use lemma for symmetric tridiagonal matrices, or rather the known Chebyshev points, we can calculate nodes that $\lambda_i = cos(\dfrac{(2i-1)\pi}{2n})$.

Next, we will calculate the coefficients A_i for the approximation $\int_{-1}^{1} \dfrac{1}{\sqrt{1-x^2}} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} A_i f(cos(\dfrac{(2i-1)\pi}{2n}))$.

Construct a Langrangian basis polynomial, $p_i(x) = \prod_{j\neq i}^{n}(x-\lambda_i)$.

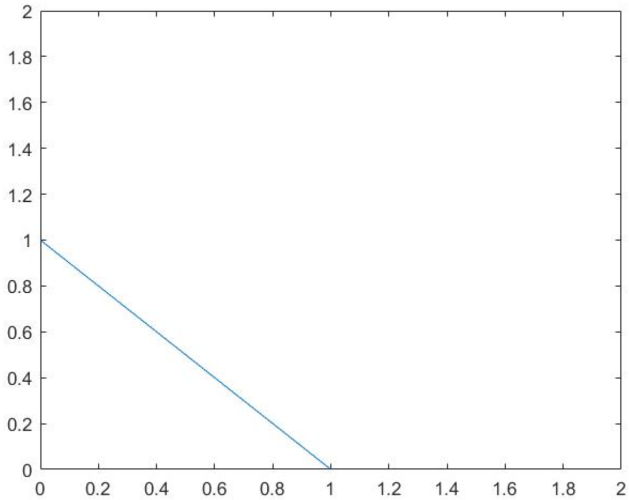
We must have $\int_{-1}^{1} \dfrac{1}{\sqrt{1-x^2}} \tilde{\omega_{n-1}}(x) p_i(x)\mathrm{d}x = A_i \tilde{\omega_{n-1}}(\lambda_i) p_i(\lambda_i) = \int_{-1}^{1} \dfrac{1}{\sqrt{1-x^2}} \tilde{\omega_{n-1}^2}(x)\mathrm{d}x$, because $\mathbf{deg} p(x) = 2n-2$ is in the exactness bound, and $\langle \tilde{\omega_{n-1}}(x), p_i(x)\rangle_{\rho(x)} = \langle \tilde{\omega_{n-1}}(x), \tilde{\omega_{n-1}}(x)\rangle_{\rho(x)}$. Therefore,

$$\begin{aligned} A_i &= \frac{\langle \tilde{\omega_{n-1}}(x), \tilde{\omega_{n-1}}(x)\rangle_{\rho(x)}}{\tilde{\omega_{n-1}}(\lambda_k) \cdot \tilde{\omega_n'}(\lambda_k)} && (2) \\ &= \frac{\frac{\pi}{2}}{cos((n-1)arccos(cos(\frac{(2i-1)\pi}{2n})))(-\frac{n}{sin(\frac{(2i-1)\pi}{2n})} sin(narccos(cos(\frac{(2i-1)\pi}{2n}))))} && (3) \\ &= \frac{\pi}{n} && (4) \end{aligned}$$

So, the quadrature rule reads $\int_{-1}^{1} \dfrac{1}{\sqrt{1-x^2}} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \dfrac{\pi}{n} f(cos(\dfrac{(2i-1)\pi}{2n}))$

Q2

Q3 The integral area is symmetric on x, y,



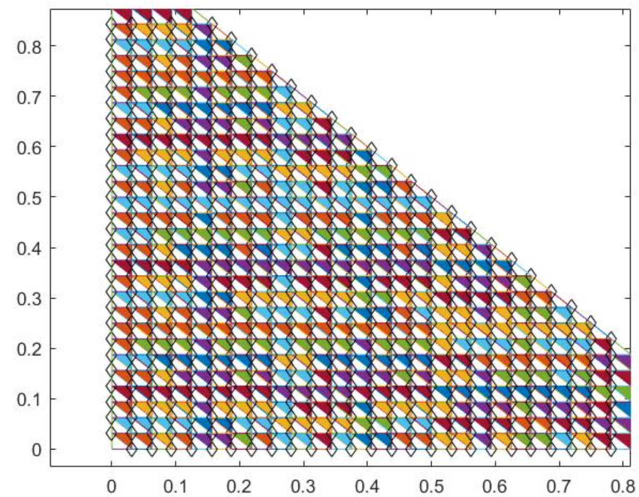
(a) integral area

Evaluate $\int_0^1 \int_0^{1-y} f(x,y)\mathrm{d}x\mathrm{d}y$,

$$\int_0^1 \int_0^{1-y} \mathrm{d}x\mathrm{d}y = \int_0^1 1 - y\mathrm{d}y = \frac{1}{2} = \frac{1}{6}(f(\frac{1}{2},0) + f(0,\frac{1}{2}) + f(\frac{1}{2},\frac{1}{2})) = \frac{1}{6}(f(\frac{2}{3},\frac{1}{6}) + f(\frac{1}{6},\frac{2}{3}) + f(\frac{1}{6},\frac{1}{6}))$$
$$\int_0^1 \int_0^{1-y} x\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-y} y\mathrm{d}x\mathrm{d}y = \frac{1}{6} = \frac{1}{6}(f(\frac{1}{2},0) + f(0,\frac{1}{2}) + f(\frac{1}{2},\frac{1}{2})) = \frac{1}{6}(f(\frac{2}{3},\frac{1}{6}) + f(\frac{1}{6},\frac{2}{3}) + f(\frac{1}{6},\frac{1}{6}))$$
$$\int_0^1 \int_0^{1-y} x^2\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-y} y^2\mathrm{d}x\mathrm{d}y = \frac{1}{12} = \frac{1}{6}(f(\frac{1}{2},0) + f(0,\frac{1}{2}) + f(\frac{1}{2},\frac{1}{2})) = \frac{1}{6}(f(\frac{2}{3},\frac{1}{6}) + f(\frac{1}{6},\frac{2}{3}) + f(\frac{1}{6},\frac{1}{6}))$$
$$\int_0^1 \int_0^{1-y} xy\mathrm{d}x\mathrm{d}y = \frac{1}{24} = \frac{1}{6}(f(\frac{1}{2},0) + f(0,\frac{1}{2}) + f(\frac{1}{2},\frac{1}{2})) = \frac{1}{6}(f(\frac{2}{3},\frac{1}{6}) + f(\frac{1}{6},\frac{2}{3}) + f(\frac{1}{6},\frac{1}{6}))$$
$$\int_0^1 \int_0^{1-y} x^3\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-y} y^3\mathrm{d}x\mathrm{d}y = \frac{1}{20} \neq \frac{1}{6}(f(\frac{1}{2},0) + f(0,\frac{1}{2}) + f(\frac{1}{2},\frac{1}{2})) \neq \frac{1}{6}(f(\frac{2}{3},\frac{1}{6}) + f(\frac{1}{6},\frac{2}{3}) + f(\frac{1}{6},\frac{1}{6}))$$
$$\int_0^1 \int_0^{1-y} x^2y\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-y} xy^2\mathrm{d}x\mathrm{d}y = \frac{1}{60} \neq \frac{1}{6}(f(\frac{1}{2},0) + f(0,\frac{1}{2}) + f(\frac{1}{2},\frac{1}{2})) \neq \frac{1}{6}(f(\frac{2}{3},\frac{1}{6}) + f(\frac{1}{6},\frac{2}{3}) + f(\frac{1}{6},\frac{1}{6}))$$

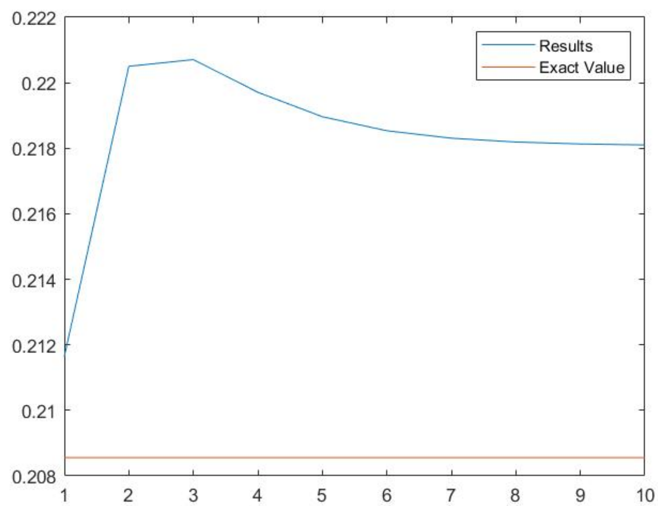
Therefore, the maximum exactness is quadratic.

Q4 The integral area should be partitioned as follows recursively,



(b) integral area

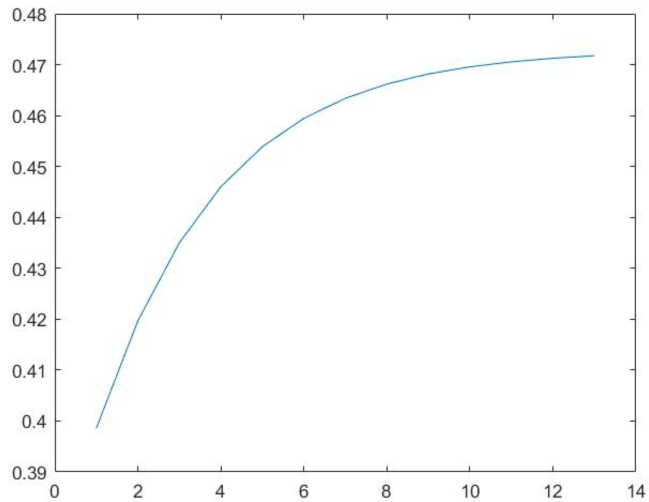
Where the colored triangles will use reversed formula, while uncolored triangles will directly apply the 2^{nd} formula given in Q3.



(c) Result

Curiously, the result doesn't seem to converge to the exact value $\frac{1}{2}(cos(1) + e - sin(1)) - 1$.

Q5 It seems we can do a substitution on variables that $u = x^2, v = y^2$, so that the integration becomes $\int\int_{x^2+y^2\leq 1} e^x sin(y)\mathrm{d}x\mathrm{d}y = \int_0^1\int_0^{1-v} e^{\sqrt{u}} sin(\sqrt{v})\frac{1}{4\sqrt{uv}}\mathrm{d}x\mathrm{d}y$. Use the same code now, and the result is 0.47125.



(d) Result