April 27, 2021 (Due: 08:00 May 11, 2021)

1. Show that the n-point Gauss–Chebyshev quadrature rule reads

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x \approx \frac{\pi}{n}\sum_{k=1}^{n} f\Big(\cos\frac{(2k-1)\pi}{2n}\Big).$$

2. Develop a quadrature rule for the integral $\int_a^b \cos(mx)f(x)\,\mathrm{d}x$ such that it provides exact results for polynomials of degree up to three.

3. Determine the degrees of exactness of the following 2-D quadrature rules:

$$\int_0^1\int_0^{1-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{6}\Big(f\Big(\frac{1}{2},0\Big)+f\Big(0,\frac{1}{2}\Big)+f\Big(\frac{1}{2},\frac{1}{2}\Big)\Big),$$
$$\int_0^1\int_0^{1-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{6}\Big(f\Big(\frac{2}{3},\frac{1}{6}\Big)+f\Big(\frac{1}{6},\frac{2}{3}\Big)+f\Big(\frac{1}{6},\frac{1}{6}\Big)\Big).$$

Hint: Check whether the quadrature rules provide exact results for bivariate polynomials 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, $\ldots$

4. Let $\mathcal{D}=\{(x,y)\in\mathbb{R}^2\colon x+y\le 1,\ x\ge 0,\ y\ge 0\}$. Estimate

$$\iint_{\mathcal{D}} \mathrm{e}^x \sin y\,\mathrm{d}x\,\mathrm{d}y$$

by partitioning $\mathcal{D}$ with a triangular mesh and applying a composite quadrature rule. Compare your result with the exact one.

5. (optional) Let $\mathcal{D}=\{(x,y)\in\mathbb{R}^2\colon x^2+y^2\le 1\}$ be the closed unit disk. Use numerical integration to estimate

$$\iint_{\mathcal{D}} \mathrm{e}^x \sin y\,\mathrm{d}x\,\mathrm{d}y.$$

Give an *a priori* error estimate if you can.