June 1, 2021 (Due: 08:00 June 15, 2021)

1. The Dirac delta function $\delta(x)$ can be treated as the derivative of the Heaviside step function

$$H(x) = \frac{1 + \operatorname{sign}(x)}{2} = \begin{cases} 0, & (x < 0) \\ 1/2, & (x = 0) \\ 1, & (x > 0) \end{cases}.$$

Use this fact to compute the derivative of a 1-D linear element.

2. Use finite element method (with linear elements on $n+1$ equispaced nodes) to solve the boundary value problem

$$\begin{cases} -u''(x) + u(x) = x^2, & (0 < x < 1) \\ u(0) = 0, \ u(1) = 1. \end{cases}$$

Try a few different values of n and compare your solutions with the exact one.

3. Solve the partial differential equation

$$\begin{cases} \dfrac{\partial^2 u(x,y)}{\partial x^2} + \dfrac{\partial^2 u(x,y)}{\partial y^2} = 0, & (-1 < x < 1, \ -1 < y < 1) \\ u(x,-1) = u(x,1) = x + 1, & (-1 < x < 1) \\ u(-1,y) = y^2 - 1, \ u(1,y) = y^2 + 1, & (-1 < y < 1) \end{cases}$$

using the finite element method. Visualize your solution.

4. (optional) Solve the partial differential equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2}, & (0 < x < 1) \\ u(x,0) = 1 - x, & (0 < x < 1) \\ u(0,t) = 1, \ u(1,t) = 0, & (t \geq 0) \end{cases}$$

with different finite difference schemes. Observe the convergence and error propagation using a few different step sizes.

5. (optional) Find the solution of the Fredholm integral equation

$$x(t) - \int_0^1 8stx(s)\,\mathrm{d}x = \max\{1 - 4t, -1\}.$$