Homework I

Name: Shao Yanjun, Number: 19307110036

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Abstract

This is Daniel's homework of "Numerical Algorithms with Case Studies II".

1 Problems

Q1

(a) Notice that $\frac{\pi^2}{6} = \sum_{\mathbf{k}=1}^{\infty} \frac{1}{\mathbf{k}^2}$, we could claim that

$$|\frac{\pi^2}{6} - \mathbf{a}_n| = |\sum_{k=n+1}^{\infty} \frac{1}{\mathbf{k}^2}| < |\sum_{k=n}^{\infty} \int_k^{k+1} \frac{1}{\mathbf{t}^2} \mathrm{d}\mathbf{t}| \tag{1}$$

$$= |\int_n^{\infty} \frac{1}{\mathbf{t}^2} \mathrm{d}\mathbf{t}| = \Theta(\frac{1}{\mathbf{n}}) \tag{2}$$

(b) The target value could be intepreted as $|\sum_{k=n+1}^{\infty} \frac{1}{\mathbf{k}^2} - \frac{2}{2\mathbf{n}+1}|$, which suits the following inequality

$$|\sum_{k=n+1}^{\infty} \frac{1}{\mathbf{k}^2} - \frac{2}{2\mathbf{n}+1}| \le \min\{|\sum_{k=n}^{\infty} \int_k^{k+1} \frac{1}{\mathbf{t}^2} \mathrm{d}\mathbf{t} - \frac{2}{2n+1}|, |\sum_{k=n+1}^{\infty} \int_k^{k+1} \frac{1}{\mathbf{t}^2} \mathrm{d}\mathbf{t} - \frac{2}{2n+1}|\} \tag{3}$$

$$= \min\{|\int_n^{\infty} \frac{1}{\mathbf{t}^2} \mathrm{d}\mathbf{t} - \frac{2}{2n+1}|, |\int_{n+1}^{\infty} \frac{1}{\mathbf{t}^2} \mathrm{d}\mathbf{t} - \frac{2}{2n+1}|\} \tag{4}$$

$$= \frac{1}{(n+1)(2n+1)} = \Theta(\frac{1}{\mathbf{n}^2}) \tag{5}$$

(c) Having recognize the decomposition of this series,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{\infty} (\frac{1}{2k(k+1)} - \frac{1}{2(k+1)(k+2)}) \tag{6}$$

$$= \frac{1}{2}(\sum_{k=1}^{\infty}(\frac{1}{k} - \frac{1}{k+1}) - \sum_{k=2}^{\infty}(\frac{1}{k} - \frac{1}{k+1})) = \frac{1}{4} \tag{7}$$

Therefore, the improvement can be furthered by the following formula, whose truncate remainder is reduced to $\Theta(n^{-3})$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} \tag{8}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{2}{k^2(k+1)(k+2)} + \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} \tag{9}$$

$$= \sum_{k=1}^{\infty} \frac{2}{k^2(k+1)(k+2)} + \frac{5}{4} \tag{10}$$

Q2 The packs are uploaded with this pdf along with the semilogy plot. And for the difference in accuracy, I personally believe that with the larger scale of x in each iteration, the fluctuation from the actual value to the calculated value was enlarged at the same time. This was later reflected in the final converged result as a huge disturbence. Meanwhile, shifting all the value of x into the range of $(-\frac{\pi}{2}, \frac{\pi}{2}]$ is a wise improvement, because the impact of the machine error is reduced.

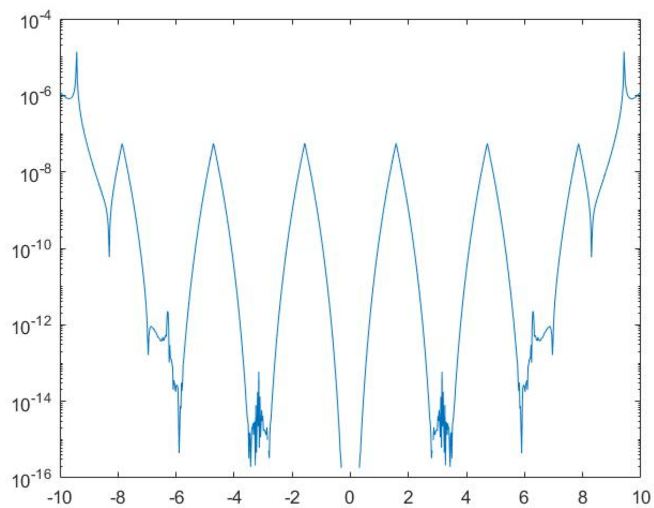

Figure 1: error

In this semilogy plot, we automatically regard the result of function sin_with_shift(x) as the true value of sin(x). It's not hard to observe that as the x starts to stay far away from zero, the calculated value also starts to stay inaccurate.

Q3 Easy to notice that $\mathbf{b} = \mathbf{A}\mathbf{x}_*$, which suggests we look for two vector norms α and β that satisfies $\|\mathbf{A}(\mathbf{x}_* - \hat{\mathbf{x}})\|_\alpha = \|\mathbf{x}_* - \hat{\mathbf{x}}\|_\beta$. Therefore, we take α as norm(2) and define norm(A) for β as follows,

$$\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}} \tag{11}$$

In this way, $\|\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}\|_2 = \|\mathbf{x}_* - \hat{\mathbf{x}}\|_A$.