Homework II

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Abstract

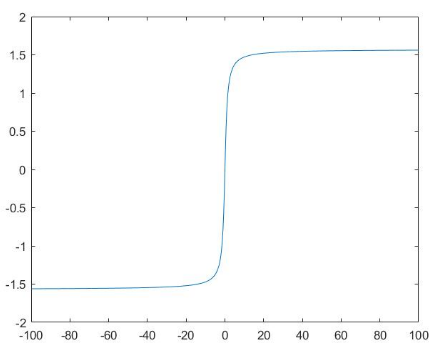
This is Daniel's homework of "Numerical Algorithms with Case Studies II".

1 Problems

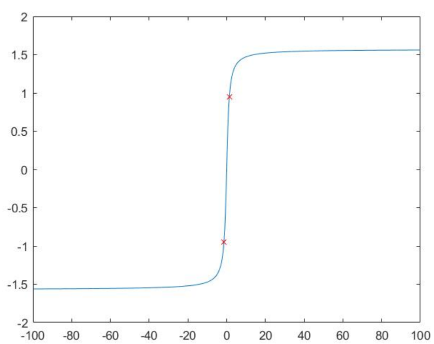
Q1 We first observe the figure of $f(x) = \arctan(x)$, and make a few guesses that the intersection of the tangent at α with the axis coincidentally falls on $-\alpha$. I will give out the direct equation to look for α,

$$f(\alpha) = \arctan(\alpha) - \frac{2\alpha}{1+\alpha^2} = 0 \tag{1}$$

Use Newton method, calculate the zero point of $f(\alpha)$ with Matlab. $\alpha \approx 1.3917452002$



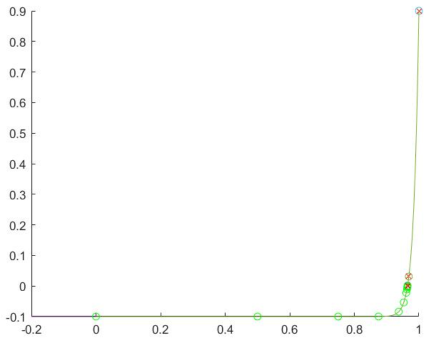
(a) arctan(x)



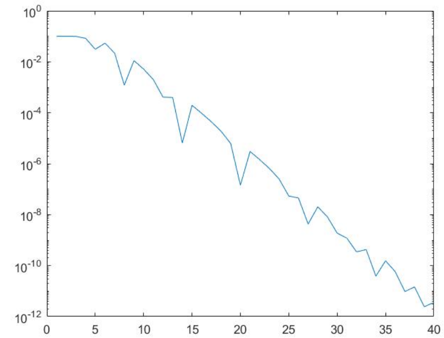
(b) Newton method fails

After this, conduct Newton method for $\arctan(x)$ with the initial guess. The result is upset, since we couldn't get to the zero point. Instead, the x is always skipping from α to $-\alpha$.

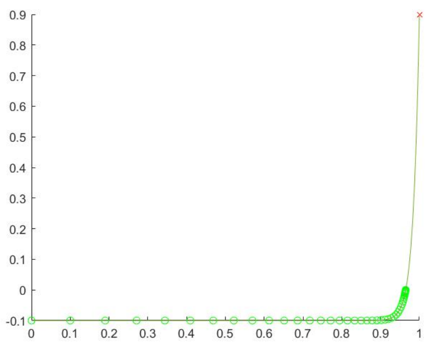
Q2 The solution with method bisection gives $x_1 = 0.964661619910657$, while the other gives $x_2 = 0.964661619911055$. Basically, x_1 and x_2 are identifical with accuracy of 10^{-11}.
Here are the convergence analysis and searching procedure of bisection method and regula falsi method.



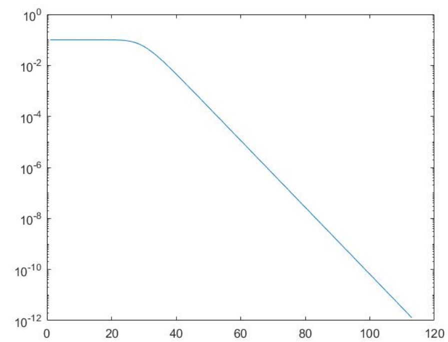
(c) bisection



(d) convergence



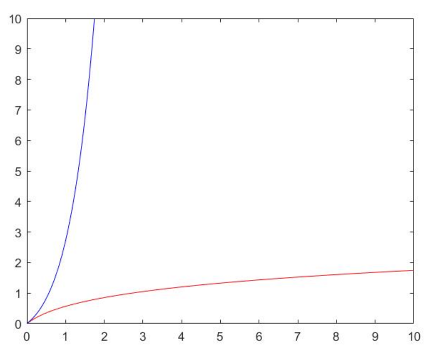
(e) regula falsi



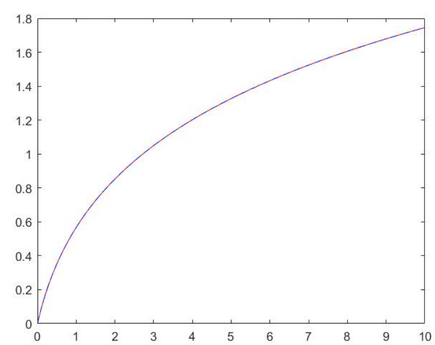
(f) convergence

Note that both loss function are output in semilogy(). And bisection method converges far better than regula falsi in this case.

Q3 Here is the $x = f(y)$ and $y = f^{-1}(x)$ plot in one figure. And they perfectly match with each other if we fold across $y = x$.



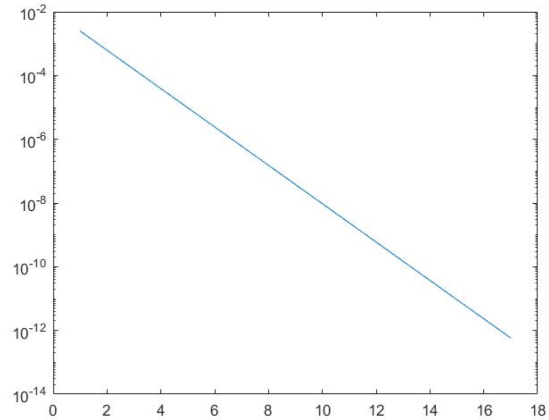
(g) $x = ye^y$ and $y = lambertW(x)$



(h) match after folding

Q4

(a) The solution is $x = 3.141591575732336$, which is not exactly π because of the truncating error.



(i) Newton loss

Through observation, we conclude that Newton method is linearly convergent in this case.

(b) Use the Newton iteration formula $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, and the Taylor expansion $f(x_k) = f(x_*) + f'(x_*)(x_k - x_*) + \frac{1}{2!}f''(\xi)(x_k - x_*)^2, (x_k < \xi < x_*)$. As $f(x_*) = 0$ and $f'(x_*) = 0$, we could get an estimation of convergence,

$$\|x_{k+1} - x_*\| = \|x_k - x_* - \frac{f'(\eta)}{2f'(\xi)}(x_k - x_*)\|, \quad (x_k < \eta < x_* \quad and \quad x_k < \xi < x_*) \tag{2}$$

$$= |1 - \frac{f'(\eta)}{2f'(\xi)}|\|x_k - x_*\| \leq C\|x_k - x_*\| \tag{3}$$

This exhibits linear convergence.

(c) Similarly, we apply a further Taylor expansion $f(x_k) = f(x_*) + f'(x_*)(x_k - x_*) + \cdots + \frac{1}{(m+1)!}f^{(m+1)}(\xi)(x_k - x_*)^{m+1}, (x_k < \xi < x_*)$. With the modified method $x_{k+1} = x_k - \frac{(m+1)f(x_k)}{f'(x_k)}$ and the given property $f'(x_*) = \cdots = f^{(m)}(x) = 0 \neq f^{(m+1)}(x)$, we could get an estimation of convergence as below,

$$\|x_{k+1} - x_*\| = \|x_k - x_* - \frac{f^{(m+1)}(\eta)}{f^{(m+1)}(\xi)}(x_k - x_*)\|, \quad (x_k < \eta < x_* \quad and \quad x_k < \xi < x_*) \tag{4}$$

$$= |1 - \frac{f^{(m+1)}(\eta)}{f^{(m+1)}(\xi)}|\|x_k - x_*\| = \frac{|f^{(m+1)}(\xi) - f^{(m+1)}(\eta)|}{|f^{(m+1)}(\xi)|}\|x_k - x_*\| \tag{5}$$

$$= \frac{|f^{(m+2)}(\tau)|}{|f^{(m+1)}(\xi)|}\|\eta - \xi\|\|x_k - x_*\|, \quad (\xi < \tau < \eta) \tag{6}$$

$$\leq \frac{|f^{(m+2)}(\tau)|}{|f^{(m+1)}(\xi)|}\|x_k - x_*\|^2 \tag{7}$$

This exhibits quadratic convergence.