Homework XII

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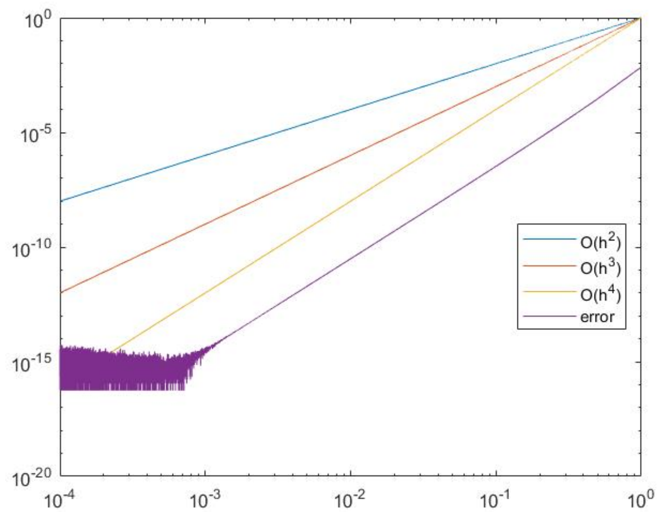
Abstract

This is Daniel's homework of "Numerical Algorithms with Case Studies II".

1 Problems

Q1 Implement the code and examine the truncation error of RK4 on loglog() plots.



(a) truncation error

Most of the truncation error satisfy $O(h^4)$ level, while further accuracy cannot be reached because of rounding error.

Q2 In this implicit method, we will have the following loop invariant in each iteration.

$$\begin{aligned} u_{k+1} &= u_k + h(\theta f(t_k + h, u_{k+1}) + (1-\theta) f(t_k, u_k)) && (1) \\ &= u_k - h\theta\lambda u_{k+1} - h(1-\theta)\lambda u_k && (2) \end{aligned}$$

So after each iteration, we will have,

$$u_{k+1} = \frac{1 - h(1-\theta)\lambda}{1 + h\theta\lambda} u_k \tag{3}$$

And to make $u_k \to 0$ when $k \to \infty$, we have to let,

$$\frac{1 - h(1-\theta)\lambda}{1 + h\theta\lambda} < 1 \tag{4}$$

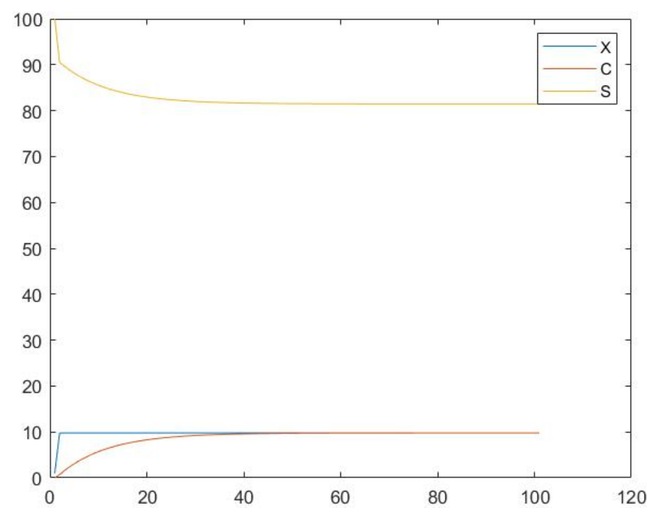
Which gives $h > 0$.
However, this is not sufficient for convergence, because we must take a good step h to make sure every implicit equation converges within each iteration. That gives the condition that $x = (1 - h(1-\theta)\lambda)y - h\theta\lambda x$ must converge.

$$\|x_k + 1 - x_k\| = h\theta\lambda\|x_k - x_{k-1}\| \tag{5}$$

$$h < \frac{1}{\theta\lambda} \tag{6}$$

Therefore, $h \in (0, \frac{1}{\theta\lambda})$. However, numerical studies show that there could be an even tighter upper bound for stepsize.

Q3 The stationary concentrations are approximately $X = 9.7754mg \cdot L^{-1}, C = 9.7749mg \cdot L^{-1}, S = 81.4495mg \cdot L^{-1}$



(b) solution

Q4

(a) We can substitute $x(t)$ with a guess of $x(t) = e^{\lambda t}$ and that gives,

$$(m\lambda^2 + k)(e^{\lambda t}) = 0 \tag{7}$$

Since $e^{\lambda t} \neq 0$, we have $\lambda = i\sqrt{\frac{k}{m}}$. So the solution will be a linear combination of $e^{i\sqrt{\frac{k}{m}}t}$ and $e^{-i\sqrt{\frac{k}{m}}t}$.

$$x(t) = C_1 cos(\sqrt{\frac{k}{m}}t) + C_2 sin(\sqrt{\frac{k}{m}}t) \tag{8}$$

With condition that $x(t) = x_0$, $x'(t) = v_0$, we will have,

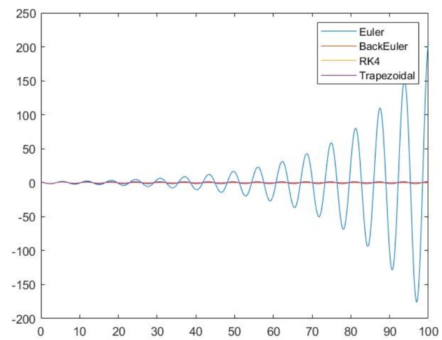
$$x(t) = x_0 cos(\sqrt{\frac{k}{m}}t) + v_0 sin(\sqrt{\frac{k}{m}}t) \tag{9}$$

(b) Let $y(t) = x'(t)$, and the second-order ordinary differential equation is transformed into a two-dimensional system of first-order coupled ordinary differential equations.

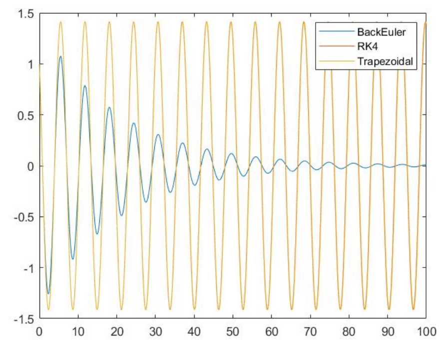
$$\begin{cases} y'(t) = -\dfrac{k}{m}x(t) \\ x'(t) = y \end{cases} \tag{10}$$

This is easy to solve with numerical methods.

(c) Apply four methods, and we can see that Trapezoidal and Runge-Kutta 4 are two stable methods that can be used to solve this system. Forward-Euler method seems to go too big, and Backward-Euler method seems to shrink to small.

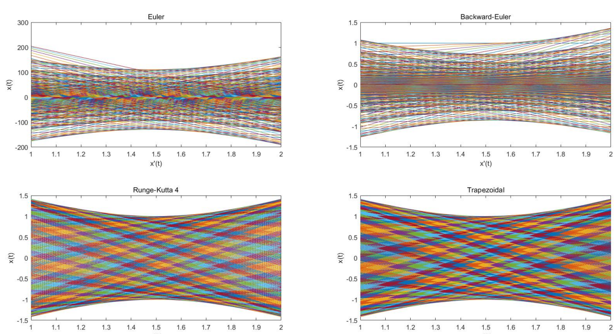


(c) Euler method



(d) other methods

And also shown below is the phase diagram of the solution.



(e) phase diagram

Since both Euler methods are $O(h)$ methods, when h is not so small, the error for $x(t)$ and $x'(t)$ accumulates very fast. What's more, as $x(t)$ and $x'(t)$ are dependent on each other, the final solution slips away from accurate ones quickly.