Homework IV

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Abstract

This is Daniel's homework of "Numerical Algorithms with Case Studies II".

1 Problems

Q1 It is clear that the difference quotient $f[x_1, x_2, \cdots, x_k]$ is the leading coefficienct of the $k-1^{th}$ interpolation polynomial $P(x)$ at the points $(x_1, x_2, \cdots, x_k)$. That is to say, the difference function $g(x) = P(x) - f(x)$ has at least k different zero points. Applying Rolle's Theorem, we discover that,

$$g^{(k-1)}(\xi) = (k-1)! \cdot f[x_1, x_2, \cdots, x_k] - f^{(k-1)}(\xi) = 0, \quad \exists \xi \in (min\{x_i\}, max\{x_i\}) \tag{1}$$

$$\therefore f[x_1, x_2, \cdots, x_k] = \frac{f^{(k-1)}(\xi)}{(k-1)!} \tag{2}$$

When $(x_1, x_2, \cdots, x_k) \to (x_*, x_*, \cdots, x_*)$, we have $max\{x_i\} \to x_*$ and $min\{x_i\} \to x_*$. So we have $\xi \to x_*$.

$$\therefore \lim_{(x_1,x_2,\cdots,x_k)\to(x_*,x_*,\cdots,x_*)} f[x_1, x_2, \cdots, x_k] = \frac{f^{(k-1)}(x_*)}{(k-1)!} \tag{3}$$

Q2 We first define the difference of a function.

Definition 1.1. *The 1^{st} difference is denoted as $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$. The 2^{nd} difference is denoted as $\Delta^2 f(x_i) = \Delta f(x_{i+1}) - \Delta f(x_i)$. And the k^{th} difference is denoted as $\Delta^{(k)} f(x_i) = \Delta^{(k-1)} f(x_{i+1}) - \Delta^{(k-1)} f(x_i)$ subsequently.*

After calculating,

$$\Delta^2 f(x_i) = \Delta f(x_{i+1}) - \Delta f(x_i) = f(x_{i+2}) - 2f(x_{i+1}) + f(x_i) \tag{4}$$

We could induce for every $k \in \mathbb{N}$

$$\begin{aligned} \Delta^{(k)} f(x_i) &= \Delta^{(k-1)} f(x_{i+1}) - \Delta^{(k-1)} f(x_i) &(5)\\ &= f(x_{i+k}) - (\binom{k-1}{1} + 1) f(x_{i+k-1}) + \cdots + (-1)^k f(x_i) &(6)\\ &= f(x_{i+k}) - \binom{k}{1} f(x_{i+k-1}) + \binom{k}{2} f(x_{i+k-2}) \cdots + (-1)^k f(x_i) &(7) \end{aligned}$$

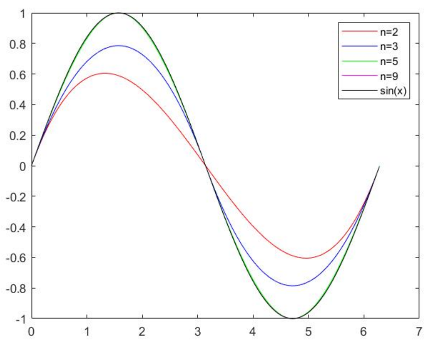
And we can calculate the coefficients of the polynomial with the following table.

$$\begin{array}{cccccc}
x_1 & f(x_1) & \frac{\Delta f(x_1)}{h} & \cdots & \frac{\Delta^{(n-1)} f(x_1)}{(n-1)! h^{(n-1)}} & \frac{\Delta^{(n)} f(x_1)}{n! h^{n}} \\
x_1 + h & f(x_1 + h) & \frac{\Delta f(x_2)}{h} & \cdots & \frac{\Delta^{(n-1)} f(x_2)}{(n-1)! h^{(n-1)}} & \\
\vdots & \vdots & \vdots & \iddots & & \\
x_1 + (n-1)h & f(x_1 + (n-1)h) & & & &
\end{array}$$

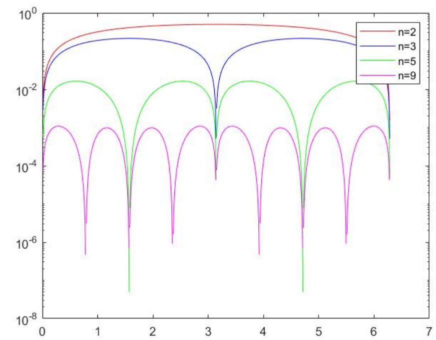
We take out the first row of the table to be the coefficients of our polynomial, that is, $c_0 = f(x_1)$, $c_1 = \frac{\Delta f(x_1)}{h}$, $\cdots$, $c_n = \frac{\Delta^{(n)} f(x_1)}{n! h^n}$. The polynomial can be simplified as follows.

$$P(x) = c_0 + c_1(x - x_1) + \cdots + c_n(x - x_1)(x - x_2)\cdots(x - x_{n-1}) \tag{8}$$

Q3 Apply my algorithm and make the plottings and error analysis (relative),

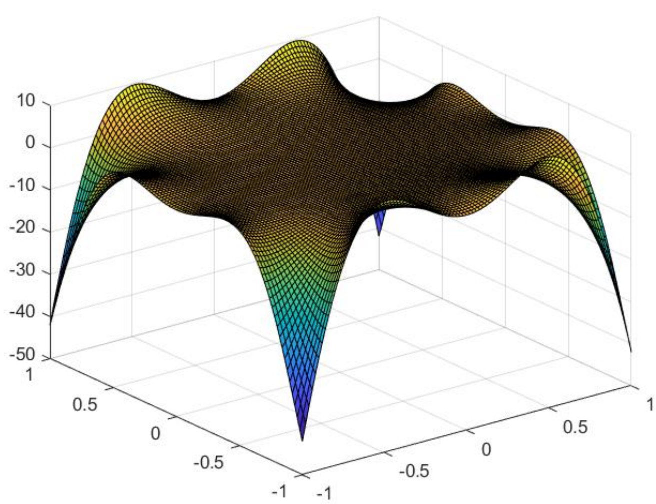


(a) interpolation of sin(x)



(b) error analysis

Q4 We prefer Newton's interpolation over Lagrange's interpolation, because it's easier to implement.



(c) interpolation of my function