March 9, 2021 (Due: 08:00 March 16, 2021)

1. Using Newton's method to find the root of $\arctan x = 0$ is an overkill, since the unique solution, $x_* = 0$, is trivial. However, this is a good example to see that the convergence of Newton's method relies on the initial guess. The set of real initial guesses such that Newton's method converges to x_* is of the form $(-\alpha, \alpha)$, where $\alpha > 0$. Try to calculate α with at least 10 significant decimal digits. What happens if α is used as the initial guess?

2. Use bisection and *regula falsi*, respectively, over the interval $[0, 1]$ to find the root of $x^{64} - 0.1 = 0$ with absolute accuracy 10^{-12}. Visualize the convergence history of these methods in one figure.

3. The Lambert W-function, $y = W(x)$, is the inverse function of $x = y \exp(y)$ for $y \in [-1, +\infty)$. Make a plot of the Lambert W-function, with at least 100 equally spaced sampling points over x. Verify your plot using the graph of $x = y \exp(y)$.

4. When applying Newton's method to solve the equation $f(x) = 0$, we usually require that $f'(x_*) \neq 0$, i.e., the root x_* is a simple one. Without such a condition, Newton's method is still applicable to find x_* while the convergence is no longer quadratic.

(a) Use Newton's method to solve $1 + \cos x = 0$ around $x_0 = 3$ and plot the convergence history.

Parts (b) and (c) are optional. Let us assume that $f(x)$ is sufficiently smooth to avoid complications in theoretical analysis.

(b) Let x_* be a root of $f(x)$ with multiplicity higher than one, i.e.,

$$h(x_*) = f'(x_*) = 0.$$

Show that Newton's method converges (locally) linearly around x_*.

(c) Let x_* be a root of $f(x)$ with multiplicity $m > 1$, i.e.,

$$f(x_*) = f'(x_*) = \cdots = f^{(m)}(x_*) = 0 \neq f^{(m+1)}(x_*).$$

We can modify Newton's method as

$$x_{k+1} = x_k - \frac{(m+1) f(x_k)}{f'(x_k)}$$

to achieve local quadratic convergence. Try to explain why such a modification improves the convergence.