Enhancement

Filtering

Additive model

$$\underbrace{f(x,y)}_{\text{observed image}} = \underbrace{u(x,y)}_{\text{unstained image}} + \underbrace{n(x,y)}_{\text{noise}}, \quad (x,y) \in \Omega$$

Method noise ($\mathcal{D}$ is the denoising operator)

$$f(x,y) - \mathcal{D}\{f\}(x,y)$$

Ideal denoising operator $\mathcal{D}$ should have

$$f(x,y) - \mathcal{D}\{f\}(x,y) \sim \mathcal{N}(\cdot|\mu,\sigma)$$

An isotropic Gaussian filter

$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$
$$u(x,y) = G_\sigma(x,y) * f(x,y)$$

Implementation (suppress the high frequency information)

$$u(x,y) = \mathcal{F}^{-1}\left\{e^{-2\pi^2\sigma^2(\xi_1^2+\xi_2^2)} \hat{f}(\xi_1,\xi_2)\right\}$$

Scale Space

Heat Equation

Green's Theorem

$$\underbrace{\oint_{\partial\Omega} \vec{F} \cdot \vec{n} ds}_{\substack{\text{Outward} \\ \text{Flux}}} = \iint_\Omega \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\text{Divergence}} dxdy$$

Heat flow is a vector field

$$V(x,y,t) = -c(x,y)\nabla u(x,y,t)$$

where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, $c(x,y)$ is the thermal conductivity.

Temperature change over Ω_p

$$-\oint_{\partial\Omega_p} V(x,y,t) \cdot n ds = -\iint_{\Omega_p} \operatorname{div}(V(x,y,t)) dxdy$$

Taking the limit of $|\Omega_p| \to 0$, the rate of change becomes

$$\frac{\partial u(x,y,t)}{\partial t} = \operatorname{div}(c(x,y)\nabla u(x,y,t))$$

Isotropic Diffusion

Solve $u(x, y, t)$ from a PDE

$$\begin{gathered}
\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \\
u(x, y, 0)=f(x, y)
\end{gathered}$$

Frequence domain

$$\begin{gathered}
u(x, y, t) \xrightarrow{\mathcal{F}} \hat{u}\left(\xi_1, \xi_2, t\right) \\
\frac{\partial}{\partial t} \hat{u}\left(\xi_1, \xi_2, t\right)=-4 \pi^2\left(\xi_1^2+\xi_2^2\right) \hat{u}\left(\xi_1, \xi_2, t\right) \quad \text{(differentiation property)} \\
\hat{u}\left(\xi_1, \xi_2, 0\right)=\hat{f}\left(\xi_1, \xi_2\right).
\end{gathered}$$

Solution

$$\hat{u}\left(\xi_1, \xi_2, t\right)=\hat{u}\left(\xi_1, \xi_2, 0\right) e^{-4 \pi^2\left(\xi_1^2+\xi_1^2\right) t}$$

Equivalent to Gaussian smoothing

$$\begin{gathered}
g_\tau(x, y)=\frac{1}{4 \pi \tau} e^{-\frac{x^2+y^2}{4 \tau}}, \quad \tau=\frac{\sigma^2}{2} \\
u(x, y, \tau)=g_\tau(x, y) * f(x, y) \\
u(x, y, 0)=f(x, y)
\end{gathered}$$

Anisotropic Diffusion

$$\begin{aligned}
u_t & =\operatorname{div}(c(x, y, t) \nabla u) \\
& =c_x u_x+c u_{x x}+c_y u_y+c u_{y y} \\
& =\nabla c \cdot \nabla u+c \nabla^2 u
\end{aligned}$$

Choice of $c(x, y, t)$

$$\begin{gathered}
c(x, y, t)=e^{-(\|\nabla u\| / k)^2} \\
\text{or } c(x, y, t)=\frac{1}{1+\left(\frac{\|\nabla u\|}{k}\right)^2}.
\end{gathered}$$

Method noise

Taylor expansion trick

$$u(x, y, \tau)=u(x, y, 0)+\tau u_t(x, y, 0)+O\left(\tau^2\right)$$

Hence,

$$f-\mathcal{D}\{f\}=-\tau u_t(x, y, 0)+O\left(\tau^2\right)$$

For different methods

$$\begin{gathered}
f-\mathcal{G F}\{f\}=-\tau \nabla^2 f+O\left(\tau^2\right) \\
f-\mathcal{A D}\{f\}=-\tau \operatorname{div}(c(x, y, 0) \nabla f)+O\left(\tau^2\right)
\end{gathered}$$

Calculus of variations

1. $E(y)=\int_\Omega F(x,y,y')\,dx$, $\min\limits_{y} E(y)$, $\Omega: [a,b]$

$\forall\, \eta(x)$, s.t. $\eta(a)=\eta(b)=0$

$$\Phi(\varepsilon)=E(y+\varepsilon\eta)=\int_\Omega F(x,\, y+\varepsilon\eta,\, y'+\varepsilon\eta')\,dx$$

$$\partial_\eta E=\lim_{\varepsilon\to 0}\frac{\Phi(\varepsilon)-\Phi(0)}{\varepsilon}=\Phi'\big|_{\varepsilon=0}$$

$$=\int_\Omega \frac{\partial F}{\partial \varepsilon}(x,\, y+\varepsilon\eta,\, y'+\varepsilon\eta')\,dx$$

weak $=\underline{\int_\Omega F_y\cdot\eta+F_{y'}\cdot\eta'\,dx}=F_{y'}\cdot\eta\Big|_a^b+\int_\Omega\left(F_y-\frac{d}{dx}(F_{y'})\right)\cdot\eta\,dx$

strong $=\underline{\int_\Omega\left(F_y-\frac{d}{dx}(F_{y'})\right)\cdot\eta\,dx}$

Since η is arbitrary, $\frac{\partial E}{\partial y}=\left(F_y-\frac{d}{dx}(F_{y'})\right)=0$

2. $E(y)=\int_\Omega F(x,y,y',\cdots,y^{(n)})\,dx$

$$\frac{\partial E}{\partial y}=F_y-\frac{d}{dx}F_{y'}+\cdots+(-1)^n\frac{d^n}{dx^n}F_{y^{(n)}}=0$$

3. $E(u)=\iint_\Omega F(x,y,u,u_x,u_y)\,dx\,dy$

$$\frac{\partial E}{\partial u}=F_u-\frac{\partial}{\partial x}F_{u_x}-\frac{\partial}{\partial y}F_y=0$$

4. $E(y,z)=\int_\Omega F(x,y,z,y',z')\,dx$

$$\begin{cases}\frac{\partial E}{\partial y}=F_y-\frac{d}{dx}F_{y'}=0\\[4pt] \frac{\partial E}{\partial z}=F_z-\frac{d}{dx}F_{z'}=0\end{cases}$$

Total Variance Denoising

Energy function

$$\min_u \iint_\Omega \underbrace{\|\nabla u\|}_{\text{total variation}}+\frac{\lambda}{2}(f-u)^2\,dx\,dy.$$

E-L equation

$$
\begin{aligned}
\frac{\delta E}{\delta u} &= \lambda(u-f) - \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2+u_y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2+u_y^2}}\right) \\
&= \lambda(u-f) - \operatorname{div}\left(\frac{\nabla u}{\|\nabla u\|}\right)
\end{aligned}
$$

Method noise

$$
f - TVD\{f\} = -\frac{1}{\lambda}\operatorname{curv}(TVD\{f\})
$$

Solve PDE

[Lecture notes]