§ 5.4.3 差分

定义5.2 设f(x)在等距节点

$$x_k = x_0 + kh \quad (k = 0,1,\dots,n)$$

上的函数值为f_k，则称

$$\Delta f_k = f_{k+1} - f_k, \nabla f_k = f_k - f_{k-1}$$

分别为一阶向前差分和一阶向后差分。

$$\Delta^m f_k = \Delta^{m-1} f_{k+1} - \Delta^{m-1} f_k, \nabla^m f_k = \nabla^{m-1} f_k - \nabla^{m-1} f_{k-1}$$

为**m**阶向前差分和**m**阶向后差分。

数学归纳可知： $\Delta^m f_k = \nabla^m f_{k+m}$

在等距节点时，均差与差分的关系：

$$f[x_0,x_1]=\frac{f_1-f_0}{x_1-x_0}=\frac{\Delta f_0}{h}$$

$$f[x_0,x_1,x_2]=\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}=\frac{\Delta f_1-\Delta f_0}{2h^2}=\frac{\Delta^2 f_0}{2h^2}$$

$$\boxed{f[x_0,x_1,\ldots,x_k]=\frac{\Delta^k f_0}{k!h^k}\quad (k=1,2,\ldots)}$$

$$f[x_0,x_1,\ldots,x_k]=f[x_k,x_{k-1},\ldots,x_0]=\frac{\nabla^k f_k}{k!h^k}\quad (k=1,2,\ldots)$$

差分与导数的关系：

$$f[x_0,\ldots,x_k]=\frac{f^{(k)}(\xi)}{k!}\qquad \boxed{\Delta^k f_0=h^k f^{(k)}(\xi)}$$

§5.4.4 等节距节点的插值公式

设节点 $x_k = x_0 + kh \quad (k = 0,1,\dots,n)$

记 $x = x_0 + th \quad (t > 0)$

则 $x - x_k = (t - k)h$

$$f[x_0, x_1, \dots, x_k]\prod_{j=0}^{k-1}(x - x_j) = \frac{\Delta^k f_0}{k!h^k}t(t-1)\dots(t-k+1)h^k$$

$$= \frac{\Delta^k f_0}{k!}\prod_{j=0}^{k-1}(t - j)$$

牛顿向前插值公式：

$$\boxed{N_n(x_0+th)=f_0+\sum_{k=1}^{n}\frac{\Delta^k f_0}{k!}\prod_{j=0}^{k-1}(t-j)}$$

牛顿向前插值余项：

$$\boxed{R_n(x_0+th)=\frac{f^{(n+1)}(\xi)}{(n+1)!}t(t-1)\dots(t-n)h^{n+1}}$$

差分代替导数的余项： $\boxed{\Delta^k f_0=h^k f^{(k)}(\xi)}$

$$R_n(x_0+th)=\frac{\Delta^{n+1} f_0}{(n+1)!}t(t-1)\dots(t-n)$$

设 $x_{n-k} = x_n - kh \quad (k = 0,1,\dots,n)$

$$x = x_n + th \quad (t < 0)$$

则 $x - x_{n-k} = (t+k)h$

牛顿向后插值公式：

$$\boxed{N_n(x_0 + th) = f_n + \sum_{k=1}^{n} \frac{\nabla^k f_n}{k!} \prod_{j=0}^{k-1} (t+j)}$$

牛顿向后插值余项：

$$\boxed{R_n(x_n + th) = \frac{f^{(n+1)}(\xi)}{(n+1)!} t(t+1)\dots(t+n)h^{n+1}}$$

$$R_n(x_n + th) = \frac{\nabla^{n+1} f_n}{(n+1)!} t(t+1)\dots(t+n)$$

【例 5.6】 给定 $f(x)=\cos x$ 的函数表如下：

k	0	1	2	3	4	5	6
x_k	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x_k)$	1.0000	0.99500	0.98007	0.95534	0.92106	0.87758	0.82534

用 Newton 插值法计算 cos 0.048 及 cos 0.566 的近似值，并估计误差。

$f(x_k)$	Δf_k	$\Delta^2 f_k$	$\Delta^3 f_k$	$\Delta^4 f_k$	$\Delta^5 f_k$	$\Delta^6 f_k$
1.000						
0.99500	−0.00500					
0.98007	−0.01493	−0.00993				
0.95534	−0.02473	−0.00980	0.00013			
0.92106	−0.03428	−0.00955	0.00025	0.00012		
0.87758	−0.04348	−0.00920	0.00035	0.00010	−0.00002	0.00001
0.82534	−0.05224	−0.00876	0.00044	0.00009	−0.00001	

$h=0.1$ 当 $x=0.048$ 时，$t=\dfrac{x-x_0}{h}=0.48$

$$N_4(x_0+th)=f_0+\Delta f_0 t+\frac{\Delta^2 f_0}{2!}t(t-1)+\frac{\Delta^3 f_0}{3!}t(t-1)(t-2)+$$

$$\frac{\Delta^4 f_0}{4!}t(t-1)(t-2)(t-3)=$$

$$f_0+t\left(\Delta f_0+(t-1)\left(\frac{\Delta^2 f_0}{2!}+(t-2)\left(\frac{\Delta^3 f_0}{3!}+(t-3)\left(\frac{\Delta^4 f_0}{4!}\right)\right)\right)\right)$$

$$=1.0000+0.48\left(-0.00500-0.52\left(\frac{-0.00993}{2}-1.52\left(\frac{0.00013}{6}-\right.\right.\right.$$

$$\left.\left.\left.2.53\,\frac{(0.00012)}{24}\right)\right)\right)=0.99884\approx\cos 0.048$$

$$|R_4(0.048)|\leqslant\left|\,\left|\frac{M_5}{5!}t(t-1)(t-2)(t-3)(t-4)\right|\,\right|h^5=1.5845\times 10^{-7}$$

$$M_5=|\sin 0.6|=0.565$$

§5.5 埃尔米特(Hermite)插值

构造插值多项式H(x)，不仅要求在某些节点上函数值相等，还要求一阶导数甚至高阶导数值相等。

$$f(x_1), f'(x_1), \cdots, f^{(m_1)}(x_1);$$
$$f(x_2), f'(x_2), \cdots, f^{(m_2)}(x_2);$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f(x_n), f'(x_n), \cdots, f^{(m_n)}(x_n).$$

$$N = \sum_{i=1}^{n} m_i + n$$

寻找一个次数<=N-1的多项式H(x)，满足:

$$f(x_i) = H(x_i) \quad (i = 0,1,\ldots,2)$$
$$f'(x_i) = H'(x_i)$$
$$f^{(m_i)}(x_i) = H^{(m_i)}(x_i)$$

当只有一个节点时，为m_1次泰勒多项式：

$$H(x)=f(x_1)+f'(x_1)(x-x_1)+\frac{f''(x_1)}{2!}(x-x_1)^2+$$

$$\ldots+\frac{f^{(m_1)}(x_1)}{m_1!}(x-x_1)^{m_1}$$

余项：

$$R(x)=f(x)-H(x)=\frac{f^{(m_1+1)}(\xi)}{(m_1+1)!}(x-x_1)^{m_1+1}$$

求埃尔米特多项式的待定常数步骤：

（1）确定多项式次数为(条件数减1)；

（2）利用k个函数值作k-1次拉格朗日插值多项式，后面添加一个或几个的($x-x_i$)乘积项，项前设一些特定常数，使该假设不影响k-1次拉格朗日插值多项式在节点上的取值，又恰能符合埃尔米特多项式的总次数，并使待定常数个数恰为给定导数个数。

（3）用导数条件确定步骤2中设定的待定常数。

【例 5.8】 设 $f(x)=x^3$，已知 $x=0,1,2$ 的某些函数值和导数值，在以下不同已知条件下，求埃尔米特插值多项式。

(1) 已知 $f(0),f(1),f(2),f'(1)$；

(2) 已知 $f(0),f(1),f'(0),f'(1)$；

(3) 已知 $f(1),f'(1),f''(1)$；

(4) 已知 $f(0),f(1),f'(1),f''(1)$。

x	0	1	2
$f(x)$	0	1	8
$f'(x)$	0	3	12
$f''(x)$	0	6	12

(1) 已知 $f(0),f(1),f(2),f'(1)$，则埃尔米特插值多项式必 3 次的

$$H_3(x)=L_2(x)+Ax(x-1)(x-2)=$$
$$3x^2-2x+Ax(x-1)(x-2)$$

$$H_3'(1)=4-A=3\Rightarrow A=1$$

$$H_3(x)=3x^2-2x+x(x-1)(x-2)=x^3$$

(2) 已知 $f(0), f(1), f'(0), f'(1)$，埃尔米特插值多项式也是 3 次的。

$$H_3(x) = L_1(x) + (Ax+B)x(x-1) = x + (Ax+B)x(x-1)$$

$$H_3'(x) = 1 + Ax(x-1) + (Ax+B)(x-1) + (Ax+B)x$$

$$\begin{cases} H'(0) = -B + 1 = 0 \\ H'(1) = A + B + 1 = 3 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \end{cases}$$

$$H_3(x) = x + (x+1)x(x-1) = x^3$$

(3) 已知 $f(1), f'(1), f''(1)$ ，只有 3 个条件，应该是二次插值。

$$H_2(x) = f(1) + (Ax + B)(x - 1) = 1 + (Ax + B)(x - 1)$$

$$H_2'(x) = A(x - 1) + Ax + B = 2Ax - A + B$$

$$H_2''(x) = 2A$$

$$\begin{cases} H_2'(1) = A + B = 3 \\ H_2''(1) = 2A = 6 \end{cases} \Rightarrow \begin{cases} A = 3 \\ B = 0 \end{cases}$$

$$H_2(x) = 1 + (3x + 0)(x - 1) = 3x^2 - 3x + 1$$

(4) 已知 $f(0), f(1), f'(1), f''(1)$ ，应该是三次插值。

$$H_3(x) = L_1(x) + (Ax+B)x(x-1) = x + (Ax+B)(x-1)$$

$$\begin{cases} H_3'(1) = 1 + A + B = 3 \\ H''_3(1) = 4A + 2B = 6 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \end{cases}$$

$$H_3(x) = x + (x+1)x(x-1) = x^3$$

§ 5.7 数据拟合的最小二乘法

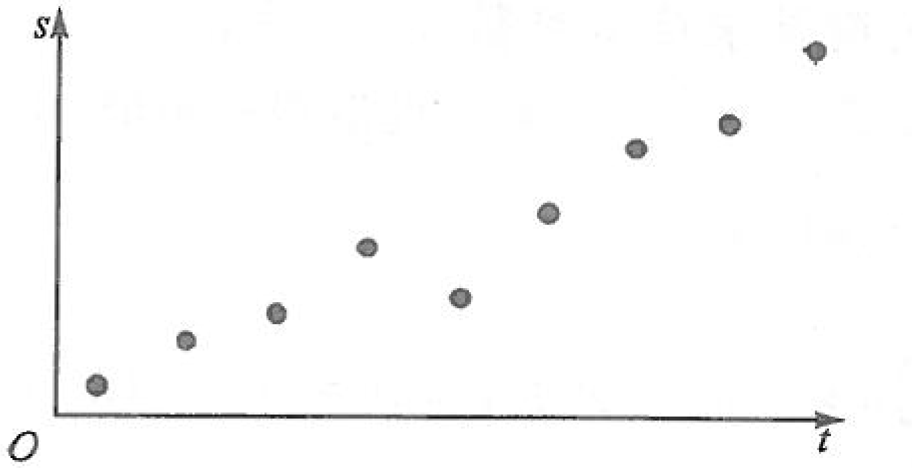

图 5－11　某物体的直线运动数据在坐标平面

$$S(t) = at + b$$

记：$\delta_i = S(t_i) - s_i$

最小化：

$$\boxed{\|\delta\|_2^2 = \sum_{i=0}^{m} \omega_i \delta_i^2 = \sum_{i=0}^{m} \omega_i (S(t_i) - s_i)^2}$$

推广至一般情形：

$$\Phi = span\{\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)\}$$

$$S^*(x) = \sum_{j=0}^{n} a_j^* \varphi_j(x) \quad (n \le m)$$

满足：

$$\boxed{\|\delta\|_2^2 = \sum_{i=0}^{m} \omega_i (S^*(x_i) - y_i) = \min_{S \in \Phi} \sum_{i=0}^{m} \omega_i (S(x_i) - y_i)^2}$$

数据拟合的最小二乘法。

§ 5.7.1 法方程组

记：

$$\Psi(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{m} \omega_i (S(x_i) - y_i)^2 = \sum_{i=0}^{m} \omega_i (\sum_{j=0}^{n} a_j \varphi_j(x_i) - y_i)^2$$

由

$$\frac{\partial \Psi}{\partial a_k} = 0 (k = 0, 1, \ldots, n)$$

得：

$$\sum_{i=0}^{m} \omega_i (\sum_{j=0}^{n} a_j \varphi_j(x_i) - y_i) \varphi_k(x_i) = 0$$

即：

$$\sum_{j=0}^{n} \left(\sum_{i=0}^{m} \omega_i \varphi_j(x_i) \varphi_k(x_i) \right) a_j = \sum_{i=0}^{m} \omega_i y_i \varphi_k(x_i)$$

令：

$$\varphi_r = (\varphi_r(x_0), \varphi_r(x_1), \ldots, \varphi_r(x_m)) \quad (r = 0, 1, \ldots, n)$$

$$f = (y_0, y_1, \ldots, y_m)$$

定义内积：

$$(\varphi_j,\varphi_k)=\sum_{i=0}^{m}\omega_i\varphi_j(x_i)\varphi_k(x_i) \quad (f,\varphi_k)=\sum_{i=0}^{m}\omega_i y_i\varphi_k(x_i)$$

$$\sum_{j=0}^{m}(\varphi_j,\varphi_k)a_j=(f,\varphi_k) \quad (k=0,1,\dots,n)$$

称为函数系在离散点上的法方程

$$\boxed{\begin{pmatrix} (\varphi_0,\varphi_0) & (\varphi_0,\varphi_1) & \cdots & (\varphi_0,\varphi_n) \\ (\varphi_1,\varphi_0) & (\varphi_1,\varphi_1) & \cdots & (\varphi_1,\varphi_n) \\ \vdots & \vdots & \cdots & \vdots \\ (\varphi_n,\varphi_0) & (\varphi_n,\varphi_1) & \cdots & (\varphi_n,\varphi_n) \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}=\begin{pmatrix} (f,\varphi_0) \\ (f,\varphi_1) \\ \vdots \\ (f,\varphi_n) \end{pmatrix}}$$

由基函数组成的**Gram**行列式，存在唯一解**a***。

记： $$S^*(x)=\sum_{j=0}^{n} a_j^* \varphi_j(x)$$

可证明：

$$\sum_{j=0}^{m} \omega_i (S^*(x_i)-y_i)^2 \le \sum_{j=0}^{m} \omega_i (S(x_i)-y_i)^2$$

平方误差可表示为：

$$\|\delta\|_2^2 = |(f,f)-(S^*,f)| = \left| \sum_{i=0}^{m} \omega_i y_i^2 - \sum_{i=0}^{m} a_k^* (\varphi_k, f) \right|$$

【例 5.8】 求拟合下列数据的最小二乘解。

i	0	1	2	3	4	5	6
x_i	0.0	0.2	0.4	0.6	0.8	1.0	1.2
y_i	0.9	1.9	2.8	3.3	4.0	5.7	6.5

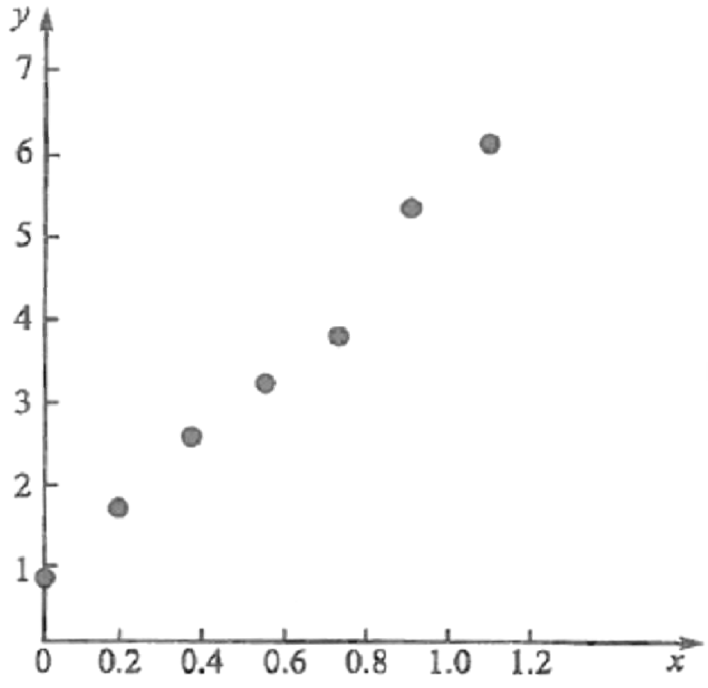

图 5－12 点 (x_i, y_i)

选用线性函数：

$$p_1(x) = a_0 + a_1 x$$

$$n=1, m=6, \omega_i \equiv 1$$

$$(\varphi_0, \varphi_1) = (\varphi_1, \varphi_0) = \sum_{i=0}^{6} \omega_i x_i = 4.2$$

$$(\varphi_0, \varphi_0) = \sum_{i=0}^{6} \omega_i = 7$$

$$(\varphi_1, \varphi_1) = \sum_{i=0}^{6} \omega_i x_i^2 = 3.64$$

$$(f, \varphi_0) = \sum_{i=0}^{6} \omega_i y_i = 25.1$$

$$(f, \varphi_1) = \sum_{i=0}^{6} \omega_i x_i y_i = 20.18$$

$$\begin{pmatrix} 7 & 4.2 \\ 4.2 & 3.64 \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{pmatrix} 25.1 \\ 20.18 \end{pmatrix}$$

$$a_0 = 0.843, \quad a_1 = 4.57$$

$$p_1(x) = 0.843 + 4.57x$$

【例 5.9】 求拟合下列数据的最小二乘解。

x_i	0.24	0.65	0.95	1.24	1.73	2.01	2.23	2.52	2.77	2.99
y_i	0.23	−0.26	−1.10	−0.45	0.27	0.10	−0.29	0.24	0.56	1.00
ω_i	1	1	0.8	0.9	1	1	1	1	0.9	0.9

所求函数可用 $y = \cos x$，$y = \ln x$ 和 $y = e^x$ 的线性组合表示

$$\varphi_0(x) = \ln x，\varphi_1(x) = \cos x，\varphi_2(x) = e^x$$

$$\begin{pmatrix} 6.5651 & -5.1453 & 59.407 \\ -5.1453 & 4.8457 & -45.969 \\ 59.407 & -45.969 & 934.96 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1.4481 \\ -2.0891 \\ 24.619 \end{pmatrix}$$

$$a = -0.99480 \quad b = -1.1957 \quad c = 0.030752$$

$$S(x) = -0.99480 \ln x - 1.1957 \cos x + 0.030752 e^x$$

多项式拟合：$P_n(x)=\sum_{j=0}^{n}a_jx^j$

基底 $\varphi_j(x)=x^j\,(j=0,1,\cdots,n)$ ，从而

$$(\varphi_j,\varphi_k)=\sum_{i=0}^{m}\omega_i x_i^{j+k}\qquad (j,k=0,1,\cdots,n)$$

$$(f,\varphi_k)=\sum_{i=0}^{m}\omega_i x_i^k y_i\qquad (k=0,1,\cdots,n)$$

$$\begin{pmatrix} \sum_{i=0}^{m}\omega_i & \sum_{i=0}^{m}\omega_i x_i & \cdots & \sum_{i=0}^{m}\omega_i x_i^n \\ \sum_{i=0}^{m}\omega_i x_i & \sum_{i=0}^{m}\omega_i x_i^2 & \cdots & \sum_{i=0}^{m}\omega_i x_i^{n+1} \\ \vdots & \vdots & & \vdots \\ \sum_{i=0}^{m}\omega_i x_i^n & \sum_{i=0}^{m}\omega_i x_i^{n+1} & \cdots & \sum_{i=0}^{m}\omega_i x_i^{2n} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}=\begin{pmatrix} \sum_{i=0}^{m}\omega_i y_i \\ \sum_{i=0}^{m}\omega_i x_i y_i \\ \vdots \\ \sum\omega_i x_i^n y_i \end{pmatrix}$$

例**5.10** 给定数据(x_i,f_i), i=0,1,2,4

选择适当模型求最小二乘拟合函数。

i	x_i	f_i	$\ln f_i$	x^2_i	x_iY_i	$\Phi^*(x)$
0	1.00	5.10	1.529	1.000	1.629	5.09
1	1.25	5.79	1.756	1.5625	2.195	5.78
2	1.50	6.53	1.876	2.2500	2.814	6.56
3	1.75	7.45	2.008	3.0625	3.514	7.44
4	2.00	8.46	2.135	4.000	4.270	8.44

选模型： $y=ae^{bx}$,取对数:$\ln y=\ln a+bx$

令$Y=\ln y, A=\ln a$，求$Y=A+bx$,取$\varphi_0(x)=1,\varphi_1(x)=x$

$$(\varphi_0,\varphi_0)=5,(\varphi_0,\varphi_1)=(\varphi_1,\varphi_0)=7.5,(\varphi_1,\varphi_1)=11.875$$

$$(Y,\varphi_0)=\sum_{i=0}^{4}Y_i=9.404,(Y,\varphi_1)=\sum_{i=0}^{4}Y_ix_i=14.422$$

法方程：

$$\begin{cases} 5A + 7.50b = 9.404 \\ 7.50A + 11.875b = 14.422 \end{cases}$$

解得：

$$A = 1.122, b = 0.5056, a = e^{A} = 3.071$$

$$y = 3.071e^{0.5056x} = \varphi^*(x)$$

【例5.11】给定数据：

x	1.0	1.4	1.8	2.2	2.6
y	0.931	0.473	0.297	0.224	0.168

求形如 $y=\dfrac{1}{a+bx}$ 的拟合函数。

令 $Y=\dfrac{1}{y}$ 则拟合函数为：$Y=a+bx$

i	1	2	3	4	5
x_i	1.0	1.4	1.8	2.2	2.6
Y_i	1.074	2.114	3.367	4.464	5.592

$$\begin{bmatrix} s_0 & s_1 \\ s_1 & s_2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} T_0 \\ T_1 \end{bmatrix}$$

$$s_0 = 5, s_1 = \sum_{i=0}^{4} x_i = 9, s_2 = \sum_{i=0}^{4} x_i^2 = 17.8$$

$$T_0 = \sum_{i=0}^{4} Y_i = 16.971, T_1 = \sum_{i=0}^{4} x_i Y_i = 35.3902$$

$$\begin{bmatrix} 5 & 9 \\ 9 & 17.8 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 16.971 \\ 35.3902 \end{bmatrix}$$

$$a = -2.0535, \quad b = 3.0265$$

$$y = \frac{1}{-2.0535 + 3.0265x}$$

本章小结

- 插值法概述
  - 插值多项式存在的唯一性
- 拉格朗日插值法

$$\boxed{y(x)=\sum_{j=0}^{n} f(x_j)l_j(x)}$$

$$\boxed{l_j(x)=\frac{(x-x_0)(x-x_1)\ldots(x-x_{j-1})(x-x_{j+1})\ldots(x-x_n)}{(x_j-x_0)(x_j-x_1)\ldots(x_j-x_{j-1})(x_j-x_{j+1})\ldots(x_j-x_n)}}$$

- 牛顿插值法

$$\boxed{f[x_i,x_k]=\frac{f_k-f_i}{x_k-x_i}(k\neq i)}$$

$$\boxed{f[x_0,x_1,\ldots,x_{k-1},x_k]=\frac{f[x_0,x_1,\ldots,x_{k-2},x_k]-f[x_0,x_1,\ldots,x_{k-1}]}{x_k-x_{k-1}}}$$

$$\boxed{f(x)=f(x_0)+\sum_{k=1}^{n}f[x_0,x_1,\ldots,x_k]\prod_{j=0}^{k-1}(x-x_j)}$$

$$N_n(x_0+th)=f_0+\sum_{k=1}^{n}\frac{\Delta^k f_0}{k!}\prod_{j=0}^{k-1}(t-j)$$

$$N_n(x_0+th)=f_n+\sum_{k=1}^{n}\frac{\nabla^k f_n}{k!}\prod_{j=0}^{k-1}(t+j)$$

- # 数据拟合的最小二乘法

$$(\varphi_j,\varphi_k)=\sum_{i=0}^{m}\omega_i\varphi_j(x_i)\varphi_k(x_i)\quad (f,\varphi_k)=\sum_{i=0}^{m}\omega_i y_i\varphi_k(x_i)$$

$$\begin{pmatrix}(\varphi_0,\varphi_0) & (\varphi_0,\varphi_1) & \cdots & (\varphi_0,\varphi_n)\\ (\varphi_1,\varphi_0) & (\varphi_1,\varphi_1) & \cdots & (\varphi_1,\varphi_n)\\ \vdots & \vdots & \cdots & \vdots\\ (\varphi_n,\varphi_0) & (\varphi_n,\varphi_1) & \cdots & (\varphi_n,\varphi_n)\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ \vdots\\ a_n\end{pmatrix}=\begin{pmatrix}(f,\varphi_0)\\ (f,\varphi_1)\\ \vdots\\ (f,\varphi_n)\end{pmatrix}$$

课 后 作 业

第五章习题的1、2、3、4、8、9、14(1)。