§5.2 拉格朗日(Lagrange)插值

§5.2.1 Lagrange插值多项式

采用解线性方程组的方法确定系数每次都要重新求解，使用不方便。

设 $\boxed{p_{n+1}(x)=\prod_{i=0}^{n}(x-x_i),}$ 有 $p'_{n+1}(x_j)=\prod_{i=0,i\neq j}^{n}(x_j-x_i)$

记n次多项式：

$$\boxed{l_j(x)=\frac{(x-x_0)(x-x_1)\ldots(x-x_{j-1})(x-x_{j+1})\ldots(x-x_n)}{(x_j-x_0)(x_j-x_1)\ldots(x_j-x_{j-1})(x_j-x_{j+1})\ldots(x_j-x_n)}}$$

$$l_j(x)=\frac{p_{n+1}(x)}{(x-x_j)p'_{n+1}(x_j)},\quad j=0,1,2,\ldots,n$$

显然有：$l_j(x_k)=\delta_{j,k}=\begin{cases}1, & j=k\\ 0, & j\neq k\end{cases}$

满足插值条件$y(x_i)=f(x_i)$的形如

$y(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n$的$Lagrange$插值多项式为：

$$\boxed{y(x)=\sum_{j=0}^{n}f(x_j)l_j(x)}$$

$l_j(x)$称为插值基函数

被插值函数可表示为：

$$\boxed{f(x)=\sum_{j=0}^{n}f(x_j)l_j(x)+E(x)}$$

E(x)为余项

定理5.2 设f(x)在[a,b]上存在n阶连续导数，在(a,b)上存在n+1阶导数，y(x)是满足插值条件的Lagrange插值多项式，对任意a<x<b，**插值余项E(x)**为：

$$\boxed{E(x)=\frac{f^{n+1}(\xi)}{(n+1)!}p_{n+1}(x)}$$

$$设E(x)=f(x)-y(x)=k(x)p_{n+1}(x)$$

$$构造函数F(z)=f(z)-y(z)-k(x)p_{n+1}(z)$$

当$z=x_0,x_1,\ldots,x_n$及z=x时有F(z)=0，即F(z)在[a,b]上存在n+2个零点。递推利用Rolle定理，可知$F^{n+1}(z)$在(a,b)上至少存在一个零点。

$$0 = F^{(n+1)}(\xi) = f^{(n+1)}(\xi) - k(x)(n+1)!$$

$$k(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\boxed{E(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} p_{n+1}(x)}$$

令$\overline{M}_{n+1} = \max_{a \le x \le b} | f^{(n+1)}(x) |$

则$| E(x) | \le \frac{\overline{M}_{n+1}}{(n+1)!} | p_{n+1}(x) |$

结论：

(1)当f(x)为次数不超过n的多项式时，有

$$y(x)=\sum_{j=0}^{n} f(x_j)l_j(x)\equiv f(x)$$

(2)由f(x) ≡1可得

$$\boxed{\sum_{j=0}^{n} l_j(x)\equiv 1}$$

(1)当n=1时

$$l_0(x)=\frac{x-x_1}{x_0-x_1}\qquad l_1(x)=\frac{x-x_0}{x_1-x_0}$$

满足：$l_j(x_k)=\begin{cases}1, & i=k\\ 0, & i\neq k\end{cases}$

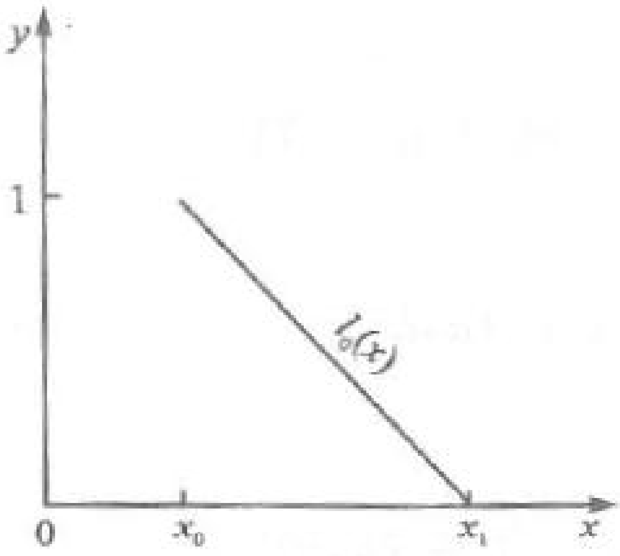


图 5.3 过$(x_0,f(x_0))$的直线

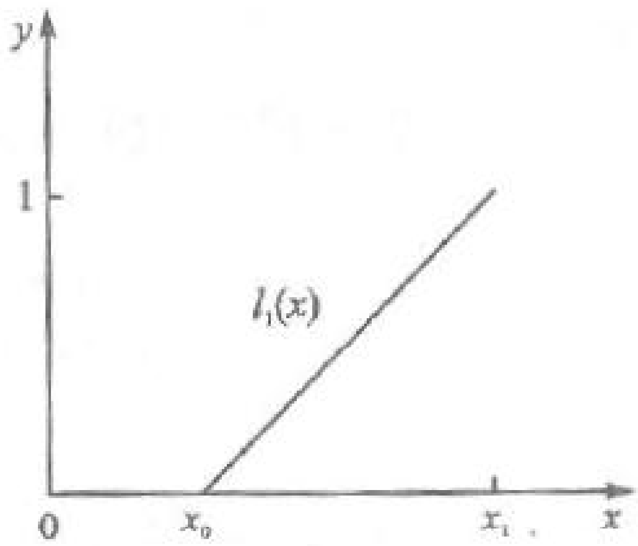


图 5.4 过$(x_1,f(x_1))$的直线

(2)当n=2时，

$$l_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$l_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$l_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

满足： $l_j(x_k)=\begin{cases}1, & i=k\\ 0, & i\neq k\end{cases}$

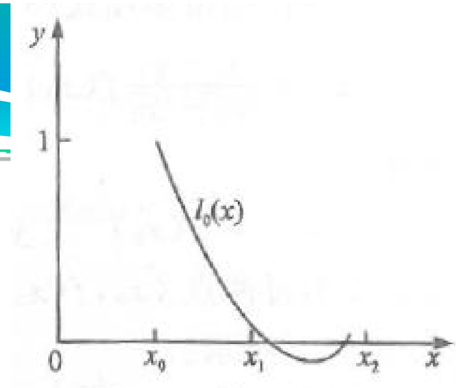


图 5.5　基函数 $l_0(x)$

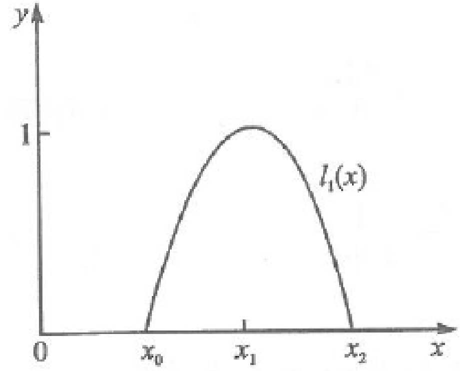


图 5.6　基函数 $l_1(x)$

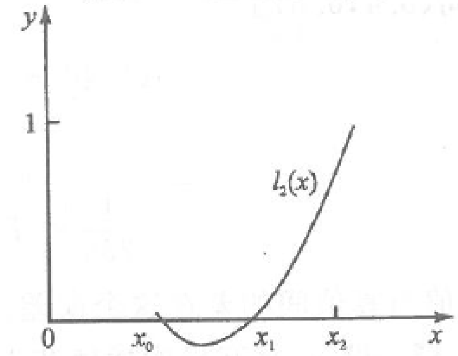


图 5.7　基函数 $l_2(x)$

例**5.1** 已知3个节点的观察数据，求拉格朗日插值多项式。

x	-1	0	1
f(x)	1.0	-2.0	1.0

$$y(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$=3x^2-2$$

例5.2 设f(x)=ln(x)，且已知4个节点值，试估计ln(0.6)的值。

x	0.4	0.5	0.7	0.8
ln(x)	-0.91629	-0.693147	-0.356675	-0.223144

$$l_0(0.6) = -1/6, \quad l_1(0.6) = 2/3,$$

$$l_2(0.6) = 2/3, \quad l_3(0.6) = 0.8$$

$$\ln(0.6) = \sum_{i=0}^{3} f(x_i) l_i(0.6) = -0.509975$$

真值 $\ln(0.6) = -0.510826$

例**5.3** 设x_j是互异的插值节点j=(0,1,2,...,n)，求证：

$$(1)\sum_{j=0}^{n} x_j^k l_j(x) \equiv x^k \qquad (2)\sum_{j=0}^{n} (x_j - x)^k l_j(x) \equiv 0$$

(1)设$f(x) = x^k$,拉格朗日插值多项式为：

$$y(x) = \sum_{j=0}^{n} x_j^k l_j(x) \qquad E(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} p_{n+1}(x)$$

$$\because f^{(n+1)}(x) \equiv 0$$

$$\therefore E(x) = 0, \quad \sum_{j=0}^{n} x_j^k l_j(x) \equiv x^k$$

$$(2)(x_j - x)^k = \sum_{i=0}^{k} (-1)^{k-1} \binom{k}{i} x_j^i x^{k-i}$$

$$\sum_{j=0}^{n} (x_j - x)^k l_j(x) = \sum_{j=0}^{n} \left[\sum_{i=0}^{k} (-1)^{k-1} \binom{k}{i} x_j^i x^{k-i} \right] l_j(x)$$

$$= \sum_{i=0}^{k} (-1)^{k-1} \binom{k}{i} x^{k-i} \sum_{j=0}^{n} x_j^i l_j(x) = \sum_{i=0}^{k} (-1)^{k-1} \binom{k}{i} x^{k-i} x^i$$

$$= (x - x)^k \equiv 0$$

拉格朗日插值公式：

$$\boxed{y(x)=\sum_{j=0}^{n}\frac{p_{n+1}(x)}{(x-x_j)p'_{n+1}(x_j)}f(x_j)}$$

$$\boxed{y(x)=\sum_{j=0}^{n}\left[\prod_{\substack{k=0\\k\neq j}}^{n}\frac{x-x_k}{x_j-x_k}\right]f(x_j)}$$

§5.2.2 高次插值多项式的问题

插值多项式的阶数是否越高越好？

Runge对函数$f(x)=\dfrac{1}{1+x^2}(-5\le x\le 5)$取等距差值节点

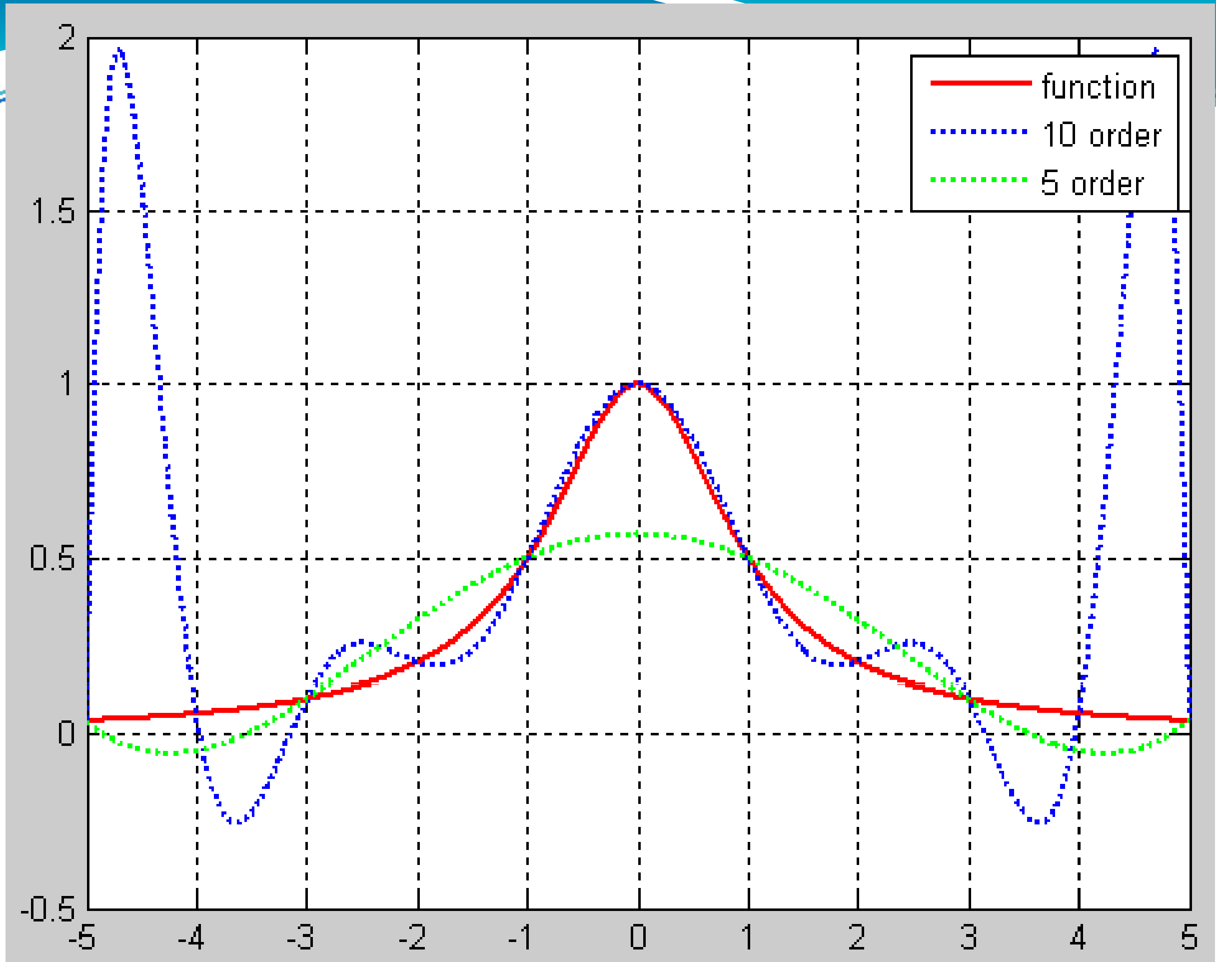
$$x_k = -5 + kh \quad (h=\frac{10}{n}, k=0,1,\dots,n)$$

作Lagrange插值多项式：

$$L_n(x)=\sum_{k=0}^{n}\left(\prod_{i=0,i\ne k}^{n}\frac{x-x_i}{x_k-x_i}\right)\cdot\frac{1}{1+x_k^2}$$

在接近区间端点时，$L_{10}(x)$与f(x)偏离很大！

$L_{10}(\pm\ 4.8)=1.80438$



$$\lim_{n\to\infty} \max_{3.63<|x|\le 5} |f(x)-L_n(x)| = \infty$$

插值多项式不收敛现象称为**Runge现象**

解决方法之一：**分段插值**。

§5.3.1 分段线性Lagrange插值

相邻节点分段线性插值：

$$L_h^{(k)}(x) = y_k \frac{x - x_{k+1}}{x_k - x_{k+1}} + y_{k+1} \frac{x - x_k}{x_{k+1} - x_k}$$

$$R_l(x) = f(x) - L_h(x) = f(x) - L_h^{(k)}(x)$$

$$= f''(\xi)/2 \cdot (x - x_k)(x - x_{k+1})$$

$$M_2 = \max_{a \le x \le b} |f''(x)|, \quad h = \max_{0 \le k \le n-1} (x_{k+1} - x_k)$$

不失一般性，以x_k=0,h=1为例：

$$(x - x_k)(x - x_{k+1}) = x^2 - x < 0$$

$$(x - \frac{1}{2})^2 \ge 0 \Rightarrow x^2 - x \ge -\frac{1}{4} \quad \Rightarrow |x^2 - x| \le \frac{1}{4}$$

$$\boxed{\max_{a \le x \le b} |R_1(x)| \le \frac{M_2}{2} \max_{a \le x \le b} |(x - x_k)(x - x_{k+1})| \le \frac{M_2}{8} h^2}$$

§5.3.2 分段二次Lagrange插值

$$u \in [x_k, x_{k+1}]$$

当$|u - x_k| \le |u - x_{k+1}|$时，取$x_{k-1}$，否则取$x_{k+2}$

$$\boxed{L_h^{(k)}(u) = \sum_{j=k}^{k+2} y_j \left(\prod_{\substack{r=k \\ r \neq j}}^{k+2} \frac{u - x_r}{x_j - x_r} \right)}$$

例5.4 给出y=f(x)的数据如下：

I	0	1	2	3	4	5
x_i	0. 30	0. 40	0. 55	0. 65	0. 80	1. 05
y_i	0. 30163	0. 41075	0. 57815	0. 69675	0. 87335	1. 18885

用分段二次插值多项式计算f(x)在x=0.36, 0.42, 0.75, 0.98处的近似值。

(1) x=0.36, 0.42时选择前三个点：

$$L_h(u)=\sum_{j=0}^{2} y_j\left(\prod_{\substack{r=0\\ r\neq j}}^{2}\frac{u-x_r}{x_j-x_r}\right)$$

$$f(0.36)=0.3669,\quad f(0.42)=0.4328$$

x=0.75, 0.98时选择后三个点：

$$L_h(u)=\sum_{j=3}^{5} y_j\left(\prod_{\substack{r=3\\ r\neq j}}^{5}\frac{u-x_r}{x_j-x_r}\right)$$

$$f(0.75)=0.8134,\qquad f(0.98)=1.0978$$

§ 5.4 牛顿插值法

设已知函数$f(x)$在$n+1$个节点$x_0, x_1, \ldots x_n$的函数值，牛顿插值基函数为：

$$\begin{cases} \varphi_0(x) = 1 \\ \varphi_j(x) = (x-x_0)(x-x_1)\ldots(x-x_{j-1}) = \prod_{i=0}^{j-1}(x-x_i) \end{cases}$$

$$\boxed{p_n(x) = \sum_{j=0}^{n} a_j \varphi_j(x) = a_0 + \sum_{j=1}^{n} a_j \prod_{k=0}^{j-1}(x-x_k)}$$

$$a_0 = f_0, \quad a_1 = \frac{f_1 - f_0}{x_1 - x_0}, \quad a_2 = \left(\frac{f_2 - f_0}{x_2 - x_0} - \frac{f_1 - f_0}{x_1 - x_0}\right) / (x_2 - x_1)$$

§5.4.1 均差

定义5.1 设$f(x)$在互异的节点$x_0, x_1, \ldots x_n$的函数值为$f_0, f_1, \ldots f_n$，则$f(x)$关于x_k, x_i的一阶均差(差商)定义为：

$$\boxed{f[x_i, x_k] = \frac{f_k - f_i}{x_k - x_i} \quad (k \neq i)}$$

二阶均差：

$$\boxed{f[x_i, x_j, x_k] = \frac{f[x_i, x_k] - f[x_i, x_j]}{x_k - x_j}}$$

k阶均差：

$$\boxed{f[x_0, x_1, \ldots, x_{k-1}, x_k] = \frac{f[x_0, x_1, \ldots, x_{k-2}, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_{k-1}}}$$

数学归纳法可证明：

$$f[x_0, x_1, \ldots, x_k] = \sum_{j=0}^{k} f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{1}{x_j - x_i}$$

对称性：

$$f[x_0, \ldots, x_i, \ldots, x_j, \ldots, x_k] = f[x_0, \ldots, x_j, \ldots, x_i, \ldots, x_k]$$

递推计算公式：

$$\boxed{f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}}$$

表 5－1　均差表

x_k	$f(x_k)$	$f[x_k,x_{k+1}]$	$f[x_k,x_{k+1},x_{k+2}]$	$f[x_k,x_{k+1},x_{k+2},x_{k+3}]$	$f[x_k,x_{k+1},x_{k+2},x_{k+3},x_{k+4}]$
x_0	f_0				
		$f[x_0,x_1]$			
x_1	f_1		$f[x_0,x_1,x_2]$		
		$f[x_1,x_2]$		$f[x_0,x_1,x_2,x_3]$	
x_2	f_2		$f[x_1,x_2,x_3]$		$f[x_0,x_1,x_2,x_3,x_4]$
		$f[x_2,x_3]$		$f[x_1,x_2,x_3,x_4]$	
x_3	f_3		$f[x_2,x_3,x_4]$		
		$f[x_3,x_4]$			
x_4	f_4				

§5.4.2 牛顿插值公式及其余项

由一阶均差定义可得：

$$f(x) = f(x_0) + f[x, x_0](x - x_0)$$

由**k+1**阶均差可得：

$$f[x, x_0, \ldots, x_k] = \frac{f[x, x_0, \ldots, x_{k-1}] - f[x_0, x_1, \ldots, x_k]}{x - x_k}$$

$$f[x, x_0, \ldots, x_{k-1}] = f[x_0, x_1, \ldots, x_k] + f[x, x_0, \ldots, x_k](x - x_k)$$

递推可得:

$$f(x)=f(x_0)+\{f[x_0,x_1]+f[x,x_0,x_1](x-x_1)\}(x-x_0)$$

$$=f(x_0)+f[x_0,x_1](x-x_0)+f[x,x_0,x_1](x-x_1)(x-x_0)$$

$$=f(x_0)+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2]\prod_{j=0}^{1}(x-x_j)+\ldots$$

$$+f[x_0,x_1,\ldots,x_n]\prod_{j=0}^{n-1}(x-x_j)+f[x,x_0,\ldots,x_n]\prod_{j=0}^{n}(x-x_j)$$

$$\boxed{f(x)=f(x_0)+\sum_{k=1}^{n}f[x_0,x_1,\ldots,x_k]\prod_{j=0}^{k-1}(x-x_j)}$$

$$\boxed{+f[x,x_0,\ldots,x_n]\prod_{j=0}^{n}(x-x_j)=N_n(x)+R_n(x)}$$

插值多项式：

$$N_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

插值余项：

$$R_n(x) = f[x, x_0, \ldots, x_n] \prod_{j=0}^{n} (x - x_j)$$

由于：

$$R_n(x_i) = f[x_i, x_0, \ldots, x_n] \prod_{j=0}^{n} (x_i - x_j) = 0$$

$i = 0,1,\ldots,n$时有：$N_n(x_i) = f(x_i)$

由于插值多项式的唯一性，因此：

$$\boxed{N_n(x) = L_n(x)} \qquad \boxed{R_n(x) = E(x)}$$

若f(x)在[a, b]存在n+1阶导数，则余项：

$$R_n(x) = f[x, x_0, \dots, x_n]p_{n+1}(x) = E(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}p_{n+1}(x)$$

均差与导数关系：

$$\boxed{f[x, x_0, \dots, x_n] = \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad \xi \in (a,b)}$$

考虑到**x**的任意性，有：

$$\boxed{f[x_0,\ldots,x_k]=\frac{f^{(k)}(\xi)}{k!}\quad \xi\in(a,b)}$$

当**k**阶均差接过于常数时，近似误差估计：

$$R_k(x)=f(x)-N_k(x)\approx f[x_0,x_1,\ldots,x_{k+1}]\prod_{j=0}^{k}(x-x_j)$$

例5.5 给出y=f(x)的数据如下：

x_i	1	2	4	6	7
$f(x_i)$	4	1	0	1	1

i	x_i	$f(x_i)$	$f[x_0, x_i]$	$f[x_0, x_1, x_i]$	$f[x_0, x_1, x_2, xi]$	…
0	1	4				
1	2	1	−3			
2	4	0	−4/3	5/6		
3	6	1	−3/5	3/5	−7/60	
4	7	1	−1/2	1/2	−1/9	1/180

$$f[x_0, x_1, \dots, x_{k-1}, x_k] = \frac{f[x_0, x_1, \dots, x_{k-2}, x_k] - f[x_0, x_1, \dots, x_{k-1}]}{x_k - x_{k-1}}$$

$$N_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

插值多项式：

$$N_4(x) = 4 - 3(x-1) + \frac{5}{6}(x-1)(x-2) - \frac{7}{60}(x-1)(x-2)(x-4) + \frac{1}{180}(x-1)(x-2)(x-4)(x-6)$$

插值余项：

$$f(x) - N_4(x) = \frac{f^{(5)}(\xi)}{5!}(x-1)(x-2)(x-4)(x-6)(x-7)$$

$$\xi \in (\min(x,1), \max(x,7))$$