1. (a) show $B = M^TM$ Symmetric

Let $M = [\, I \;\; P \,]$ $\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad P = \begin{bmatrix} P_x & -P_y \\ P_y & P_x \end{bmatrix}$

$$B = \begin{bmatrix} I \\ P^T \end{bmatrix} [\, I \;\; P \,] = \begin{bmatrix} I & P \\ P^T & P^TP \end{bmatrix} \Rightarrow \text{symmetric}$$

(b) show B is p.s.d.

$$B = \begin{bmatrix} 1 & 0 & P_x & -P_y \\ 0 & 1 & P_y & P_x \\ P_x & P_y & P_x^2+P_y^2 & 0 \\ -P_y & P_x & 0 & P_x^2+P_y^2 \end{bmatrix}$$

if B is p.s.d, all eigenvalues λ_i of B,

$\lambda_i \geq 0$

$\rightarrow$ find λ : $\det(B - \lambda I) = 0$

$$\det \begin{bmatrix} 1-\lambda^{+} & 0^{-} & P_x^{+} & -P_y^{-} \\ 0 & 1-\lambda & P_y & P_x \\ P_x & P_y & P_x^2+P_y^2-\lambda & 0 \\ -P_y & P_x & 0 & P_x^2+P_y^2-\lambda \end{bmatrix}$$

$$(1-\lambda)\begin{bmatrix} 1-\lambda & P_y & P_x \\ P_y & P_x^2+P_y^2-\lambda & 0 \\ P_x & 0 & P_x^2+P_y^2-\lambda \end{bmatrix}$$

① $\Rightarrow (1-\lambda)\left[(1-\lambda)(P_x^2+P_y^2-\lambda)^2 - (P_x^2+P_y^2)(P_x^2+P_y^2-\lambda)\right]$

$$+P_x \begin{bmatrix} 0 & 1-\lambda & P_x \\ P_x & P_y & 0 \\ -P_y & P_x & P_x^2+P_y^2-\lambda \end{bmatrix}$$

② $\Rightarrow P_x\left[P_x^3 + P_xP_y^2 - (1-\lambda)P_x(P_x^2+P_y^2-\lambda)\right]$

$$+P_y \begin{bmatrix} 0 & 1-\lambda & P_y \\ P_x & P_y & P_x^2+P_y^2-\lambda \\ -P_y & P_x & 0 \end{bmatrix}$$

③ $\Rightarrow P_y\left[(1-\lambda)(-P_y)(P_x^2+P_y^2-\lambda) + P_x^2P_y + P_y^3\right]$

① : $(1-\lambda)^2(P_x^2+P_y^2-\lambda)^2 - (1-\lambda)(P_x^2+P_y^2)(P_x^2+P_y^2-\lambda)$

② : $P_x^4 + P_x^2P_y^2 - P_x^2(1-\lambda)(P_x^2+P_y^2-\lambda)$

③ : $P_y^4 + P_x^2+P_y^2 - P_y^2(1-\lambda)(P_x^2+P_y^2-\lambda)$

$\det(B-\lambda I) = ① + ② + ③ \Rightarrow$

$$(1-\lambda)^2(P_x^2+P_y^2-\lambda)^2 - 2(1-\lambda)(P_x^2+P_y^2-\lambda)(P_x^2+P_y^2) + (P_x^2+P_y^2)^2$$

$$= \left[(1-\lambda)(P_x^2+P_y^2-\lambda) - (P_x^2+P_y^2)\right]^2$$

$$= \left[\cancel{(P_x^2+P_y^2)} - \lambda - \lambda(P_x^2+P_y^2) + \lambda^2 - \cancel{(P_x^2+P_y^2)}\right]^2$$

$$= \left[\lambda^2 - \lambda(P_x^2+P_y^2+1)\right]^2 = \lambda^2\left[\lambda - (P_x^2+P_y^2+1)\right]^2 = 0$$

$\lambda_1 = \lambda_2 = 0 \qquad \lambda_3 = \lambda_4 = P_x^2+P_y^2+1 \geq 1$

all $\lambda \geq 0 \Rightarrow B$ is p.s.d.

ⓒ Show $\sum_{i=1}^{n} B_i$ is p.s.d.

$x^T B_i x \geq 0. \quad \forall x \in \mathbb{R}^4, \quad \forall B_i. \ i \in \{1, 2 \cdots n\}$

$$\sum_{i=1}^{n}(x^T B_i x) = x^T \sum_{i=1}^{n} B_i x \geq 0$$

$\Rightarrow \sum_{i=1}^{n} B_i$ is p.s.d.

2. $$x^* = \arg\min_{x\in\mathbb{R}^4} \sum_{i=1}^{n} \| M_i x - \pi_i \|_2^2$$

$$= \arg\min_{x\in\mathbb{R}^4} \sum_{i=1}^{n} (M_i x - \pi_i)^T (M_i x - \pi_i)$$

$$= \arg\min_{x\in\mathbb{R}^4} \sum_{i=1}^{n} (x^T M_i^T M_i x - 2\pi_i^T M_i x + \underbrace{\pi_i^T \pi_i}_{\text{const.}})$$

$$= \arg\min_{x\in\mathbb{R}^4} \left[x^T \sum_{i=1}^{n} (M_i^T M) x + \sum_{i=1}^{n} (-2\pi_i^T M_i) x \right]$$

$$M = \sum_{i=1}^{n} M_i^T M_i \qquad g^T = \sum_{i=1}^{n} (-2\pi_i^T M_i) \Rightarrow x^* = \arg\min_{x\in\mathbb{R}^4} x^T M x + g^T x.$$

$$x = \begin{bmatrix} t_x \\ t_y \\ \cos\theta \\ \sin\theta \end{bmatrix}, \quad x^T W x = 1. \quad \text{Let. } x = \begin{bmatrix} T \\ \theta \end{bmatrix} \Rightarrow x^T = [T^T \theta^T], \text{ where } \theta^T\theta = 1$$

$$\text{Let. } W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$x_3^2 + x_4^2 = 1.$

$$x^T W x = [T^T \theta^T] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T \\ \theta \end{bmatrix} = [T^T A + \theta^T C \quad T^T B + \theta^T D] \begin{bmatrix} T \\ \theta \end{bmatrix}$$

$$= T^T A T + \theta^T C T + T^T B T + \theta^T D \theta = 1, \quad \text{recall } \theta^T\theta = 1$$

$$\Rightarrow A = B = C = 0^{2\times 2}, \ D = I^{2\times 2} \Rightarrow W = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

3. Bi = ~~$M_i \Rightarrow$ p.s.d.~~ $\Rightarrow M_i^T M_i$ p.s.d. $\Rightarrow \sum_{i=1}^{n} M_i^T M_i$ p.s.d.

eigenvalue of W: $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 1 > 0$. $\Rightarrow \lambda_i \geq 0, \forall i \in \{1,2,3,4\}$

W is p.s.d.