A Brief Proof for the Monotonic of the Neural Network

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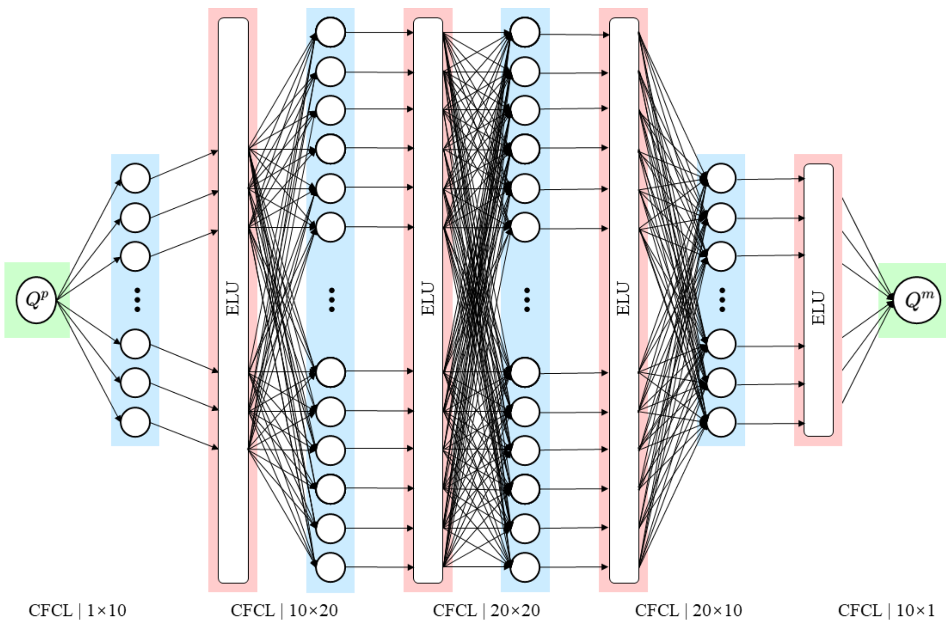

Fig. 1: The architecture of our monotonic neural network. We denote the constrained fully connected layer as CFCL and denote the parameterization of CFCL as "input features × output features".

As shown in Fig. 1, we denoted the perceptual quality as Q^p, the output of the k-th constrained fully connected layer (CFCL) as $\mathbf{M}^{(k)} \in \mathbb{R}^{o_k \times 1}$. The weight of each CFCL is denoted as $\mathbf{W}^{(k)} \in \mathbb{R}^{o_k \times o_{k-1}}$ with the bias $\mathbf{b}^{(k)} \in \mathbb{R}^{o_k \times 1}$. Herein, each element of $\mathbf{W}^{(k)}$ is constrained to be strictly positive. $f(\cdot)$ is the activation function. Given Q^p, the predicted quality Q^m can be obtained as follows,

$$
\begin{aligned}
\mathbf{M}^{(1)} &= \mathbf{W}^{(1)} Q^p + \mathbf{b}^{(1)} \\
\mathbf{M}^{(2)} &= \mathbf{W}^{(2)} f(\mathbf{M}^{(1)}) + \mathbf{b}^{(2)} \\
\mathbf{M}^{(3)} &= \mathbf{W}^{(3)} f(\mathbf{M}^{(2)}) + \mathbf{b}^{(3)} \\
\mathbf{M}^{(4)} &= \mathbf{W}^{(4)} f(\mathbf{M}^{(3)}) + \mathbf{b}^{(4)} \\
\mathbf{M}^{(5)} &= \mathbf{W}^{(5)} f(\mathbf{M}^{(4)}) + \mathbf{b}^{(5)} \\
Q^m &= \mathbf{M}^{(5)}
\end{aligned}
\tag{1}
$$

According to the chain rule, we can derive that,

$$
\begin{aligned}
\frac{\partial Q^m}{\partial Q^p} &= \frac{\partial Q^m}{\partial \mathbf{M}^{(5)}} \cdot \frac{\partial \mathbf{M}^{(5)}}{\partial \mathbf{M}^{(4)}} \cdot \frac{\partial \mathbf{M}^{(4)}}{\partial \mathbf{M}^{(3)}} \cdot \frac{\partial \mathbf{M}^{(3)}}{\partial \mathbf{M}^{(2)}} \cdot \frac{\partial \mathbf{M}^{(2)}}{\partial \mathbf{M}^{(1)}} \cdot \frac{\partial \mathbf{M}^{(1)}}{\partial Q^p} \\
&= 1 \cdot (\mathbf{W}^{(5)} \frac{\partial f(\mathbf{M}^{(4)})}{\partial \mathbf{M}^{(4)}}) \cdot (\mathbf{W}^{(4)} \frac{\partial f(\mathbf{M}^{(3)})}{\partial \mathbf{M}^{(3)}}) \cdot (\mathbf{W}^{(3)} \frac{\partial f(\mathbf{M}^{(2)})}{\partial \mathbf{M}^{(2)}}) \cdot (\mathbf{W}^{(2)} \frac{\partial f(\mathbf{M}^{(1)})}{\partial \mathbf{M}^{(1)}}) \cdot \mathbf{W}^{(1)}
\end{aligned}
\tag{2}
$$

and,

$$
\begin{aligned}
\frac{\partial f(\mathbf{M}^{(k)})}{\partial \mathbf{M}^{(k)}} &= \begin{bmatrix}
\frac{\partial f(m_1^{(k)})}{\partial m_1^{(k)}} & \frac{\partial f(m_1^{(k)})}{\partial m_2^{(k)}} & \cdots & \frac{\partial f(m_1^{(k)})}{\partial m_{o_k}^{(k)}} \\
\frac{\partial f(m_2^{(k)})}{\partial m_1^{(k)}} & \frac{\partial f(m_2^{(k)})}{\partial m_2^{(k)}} & \cdots & \frac{\partial f(m_2^{(k)})}{\partial m_{o_k}^{(k)}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f(m_{o_k}^{(k)})}{\partial m_1^{(k)}} & \frac{\partial f(m_{o_k}^{(k)})}{\partial m_2^{(k)}} & \cdots & \frac{\partial f(m_{o_k}^{(k)})}{\partial m_{o_k}^{(k)}}
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{\partial f(m_1^{(k)})}{\partial m_1^{(k)}} & 0 & \cdots & 0 \\
0 & \frac{\partial f(m_2^{(k)})}{\partial m_2^{(k)}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{\partial f(m_{o_k}^{(k)})}{\partial m_{o_k}^{(k)}}
\end{bmatrix}.
\end{aligned}
\tag{3}
$$

In the CFCL, the ELU [1] is adopted as the activation function $f(\cdot)$, which is continuous and differentiable at all points. In detail, the ELU is defined by

$$
f(x) = \begin{cases} x & \text{if } x > 0, \\ \exp(x) - 1 & \text{if } x \leq 0. \end{cases}
\tag{4}
$$

Its deviation is,

$$f'(x) = \begin{cases} 1 & \text{if } x > 0, \\ \exp(x) & \text{if } x \leq 0. \end{cases} \tag{5}$$

Thus, the ELU activation function is monotonically increasing and all elements on the diagonal in Equation(3) are positive. Since each element of $\mathbf{W}^{(k)}$ is also strictly positive, we can derive that $\frac{\partial Q^m}{\partial Q^p} > 0$ and the monotonicity of the quality transformer is proved.

References

[1] Djork-Arné Clevert, Thomas Unterthiner, and Sepp Hochreiter, "Fast and accurate deep network learning by exponential linear units (elus)," *arXiv preprint arXiv:1511.07289*, 2015.