



Lecture 17

Produced by Dr. Worldwide

Welcome to the 305

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- Goals listed in order based on priority
 - Achieve a 60%/40% ratio of white to black students at each of the schools
 - Minimize the amount of traveling that students will have to do, ideally no more than 30,000 miles per day
 - Keep all schools close to capacity and minimize overcrowding proportionally allocating the excess among the schools
- Q: How can we formulate and solve a goal programming model to help the board with its dilemma?
- Decision variables
 - x_{ij} = Number of white students from district i assigned to district j
 - y_{ij} = Number of black students from district i assigned to district j
 - $i, j \in \{N, S, E, W\}$

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- Goal 1: Achieve racial balance in all 4 schools
 - Consider perfect balance for North High School

$$\frac{\text{Percent White}}{\text{Percent Black}} = \frac{0.6}{0.4}$$

$$\frac{\frac{\text{Total White}}{\text{Total Students}}}{\frac{\text{Total Black}}{\text{Total Students}}} = \frac{0.6}{0.4}$$

$$\frac{\text{Total White}}{\text{Total Black}} = \frac{0.6}{0.4}$$

$$0.4(\text{Total White}) - 0.6(\text{Total Black}) = 0$$

$$0.4(x_{NN} + x_{SN} + x_{EN} + x_{WN}) - 0.6(y_{NN} + y_{SN} + y_{EN} + y_{WN}) = 0$$

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- Goal 1: Achieve **racial balance** in all 4 schools
 - Adding deviational variables for North High School

$$0.4(x_{NN} + x_{SN} + x_{EN} + x_{WN}) - 0.6(y_{NN} + y_{SN} + y_{EN} + y_{WN}) + d_1^- - d_1^+ = 0$$

- Consider constraints for all schools in the county

$$0.4(x_{NN} + x_{SN} + x_{EN} + x_{WN}) - 0.6(y_{NN} + y_{SN} + y_{EN} + y_{WN}) + d_1^- - d_1^+ = 0$$

$$0.4(x_{NS} + x_{SS} + x_{ES} + x_{WS}) - 0.6(y_{NS} + y_{SS} + y_{ES} + y_{WS}) + d_2^- - d_2^+ = 0$$

$$0.4(x_{NE} + x_{SE} + x_{EE} + x_{WE}) - 0.6(y_{NE} + y_{SE} + y_{EE} + y_{WE}) + d_3^- - d_3^+ = 0$$

$$0.4(x_{NW} + x_{SW} + x_{EW} + x_{WW}) - 0.6(y_{NW} + y_{SW} + y_{EW} + y_{WW}) + d_4^- - d_4^+ = 0$$

- To accomplish our goal we want **all** of these deviational variables to be as small as possible
- First priority objective

$$\text{Minimize} \quad P_1(d_1^- + d_1^+ + d_2^- + d_2^+ + d_3^- + d_3^+ + d_4^- + d_4^+)$$

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- Goal 2: Minimize **total travel** to not much more than 30,000 miles
 - Recall the following table

District/School	Distance (mi.)			
	North	South	East	West
North	–	30	12	20
South	30	–	18	26
East	12	18	–	24
West	20	26	24	–

- Formulation for constraint based on total miles

$$30(x_{NS} + y_{NS} + x_{SN} + y_{SN}) + 12(x_{NE} + y_{NE} + x_{EN} + y_{EN}) \\ + 20(x_{NW} + y_{NW} + x_{WN} + y_{WN}) + 18(x_{SE} + y_{SE} + x_{ES} + y_{ES}) \\ + 26(x_{SW} + y_{SW} + x_{WS} + y_{WS}) + 24(x_{EW} + y_{EW} + x_{WE} + y_{WE}) + d_5^- - d_5^+ = 0$$

- Updated objective function for second priority

Minimize $P_1(d_1^- + d_1^+ + d_2^- + d_2^+ + d_3^- + d_3^+ + d_4^- + d_4^+), P_2(d_5^+)$

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- Goal 3: Minimize **overcrowding** at each of the 4 schools, **proportionally** allocating the excess among the schools
 - Recall the following table

District/School	<i># White students</i>	<i># Black students</i>	<i>Capacity</i>
North	1000	300	1200
South	450	800	1000
East	1050	400	1000
West	500	500	1200

- Recall that there are 5,000 total students for capacity of 4,400
- The excess of 600 students needs to be split between the schools
- Q: How can we handle this **proportionally**?

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- Goal 3: Minimize **overcrowding** at each of the 4 schools, **proportionally** allocating the excess among the schools
 - We want to manage the excess according to the capacities of the schools
 - Schools that are bigger should take larger portions of the overflow
 - We prefer if North and West take $1200/4400 = 3/11$ of the excess
 - We prefer if South and East take $1000/4400 = 5/22$ of the excess
 - Capacities are expanded to handle the overflow (rounded up)

School	<i>North</i>	<i>South</i>	<i>East</i>	<i>West</i>
Ideal # students	1364	1136	1136	1364

↓

$$1200 + \frac{3}{11}(600) = 1363.636363 \approx 1364$$

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- Goal 3: Minimize **overcrowding** at each of the 4 schools, **proportionally** allocating the excess among the schools

- Constraints with deviational variables

$$x_{NN} + y_{NN} + x_{SN} + y_{SN} + x_{EN} + y_{EN} + x_{WN} + y_{WN} + d_6^- - d_6^+ = 1364$$

$$x_{NS} + y_{NS} + x_{SS} + y_{SS} + x_{ES} + y_{ES} + x_{WS} + y_{WS} + d_7^- - d_7^+ = 1136$$

$$x_{NE} + y_{NE} + x_{SE} + y_{SE} + x_{EE} + y_{EE} + x_{WE} + y_{WE} + d_8^- - d_8^+ = 1136$$

$$x_{NW} + y_{NW} + x_{SW} + y_{SW} + x_{EW} + y_{EW} + x_{WW} + y_{WW} + d_9^- - d_9^+ = 1364$$

- Updated objective function for third priority

$$\begin{aligned} \text{Minimize } & P_1(d_1^- + d_1^+ + d_2^- + d_2^+ + d_3^- + d_3^+ + d_4^- + d_4^+), \\ & P_2(d_5^+), \\ & P_3(d_6^- + d_6^+ + d_7^- + d_7^+ + d_8^- + d_8^+ + d_9^- + d_9^+) \end{aligned}$$

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- Additional constraints
 - We cannot bus more students than what is currently available at each school

District/School	# White students	# Black students	Capacity
North	1000	300	1200
South	450	800	1000
East	1050	400	1000
West	500	500	1200

- List of constraints

$$x_{NN} + x_{NS} + x_{NE} + x_{NW} = 1000$$

$$y_{NN} + y_{NS} + y_{NE} + y_{NW} = 300$$

$$x_{SN} + x_{SS} + x_{SE} + x_{SW} = 450$$

$$y_{SN} + y_{SS} + y_{SE} + y_{SW} = 800$$

$$x_{EN} + x_{ES} + x_{EE} + x_{EW} = 1050$$

$$y_{EN} + y_{ES} + y_{EE} + y_{EW} = 400$$

$$x_{WN} + x_{WS} + x_{WE} + x_{WW} = 500$$

$$y_{WN} + y_{WS} + y_{WE} + y_{WW} = 500$$

- $$y_{ij} \in \{0, 1, \dots\}$$

- Tab called **Priority 1**

- | White student | North | South | East | West | Total |
|----------------|-------|-------|------|------|-------|
| North | 0 | 0 | 0 | 0 | 0 |
| South | 0 | 0 | 0 | 0 | 0 |
| East | 0 | 0 | 0 | 0 | 0 |
| West | 0 | 0 | 0 | 0 | 0 |
| Total | 0 | 0 | 0 | 0 | |
| | | | | | |
| | | | | | |
| Black students | North | South | East | West | Total |
| North | 0 | 0 | 0 | 0 | 0 |
| South | 0 | 0 | 0 | 0 | 0 |
| East | 0 | 0 | 0 | 0 | 0 |
| West | 0 | 0 | 0 | 0 | 0 |
| Total | 0 | 0 | 0 | 0 | |

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- Tab called **Priority 1**
 - Notice all the different constraints and inspect formulas

Constraints:								
	Deficit		Surplus		Computed	Constraint	Value	
0		0		0	0 =		0	Racial balance at North
0		0		0	0 =		0	Racial balance at South
0		0		0	0 =		0	Racial balance at East
0		0		0	0 =		0	Racial balance at West
0		0		0	0 =		30000	Total distance travelled
0		0		0	0 =		1364	Overcrowding at North
0		0		0	0 =		1136	Overcrowding at South
0		0		0	0 =		1136	Overcrowding at East
0		0		0	0 =		1364	Overcrowding at West
					0 =		1000	"Supply" of white students
					0 =		300	"Supply" of black students
					0 =		450	"Supply" of white students
					0 =		800	"Supply" of black students
					0 =		1050	"Supply" of white students
					0 =		400	"Supply" of black students
					0 =		500	"Supply" of white students
					0 =		500	"Supply" of black students

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- Tab called **Priority 1**
 - First objective function

Minimize $d_1^- + d_1^+ + d_2^- + d_2^+ + d_3^- + d_3^+ + d_4^- + d_4^+$

- Observe formula for objective function

39	Objective function:			
40	$d1^- + d1^+ + d2^- + d2^+ + d3^- + d3^+ + d4^- + d4^+$			
41	0	=SUM(B20:C23)		

- Q: What is "B20:C23" referring to and what is "SUM" doing?
- Use Excel solver to find the optimal solution

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- Tab called **Priority 1**
 - Optimal solution

White students	<i>North</i>	<i>South</i>	<i>East</i>	<i>West</i>
<i>North</i>	461	538	0	0
<i>South</i>	0	140	65	245
<i>East</i>	0	0	1050	0
<i>West</i>	0	0	0	499

Black students	<i>North</i>	<i>South</i>	<i>East</i>	<i>West</i>
<i>North</i>	300	0	0	0
<i>South</i>	1	452	344	3
<i>East</i>	0	0	400	0
<i>West</i>	7	0	0	493

Deviation	1	2	3	4	5	6	7	8	9
<i>Deficit -</i>	0	0	0	0	0	594	6	0	124
<i>Surplus +</i>	0	0	0	0	152	0	0	724	0

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- Tab called **Priority 2**
 - Notice the additional constraint and inspect formula

0	=	0	First goal optimal
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- Second objective function
Minimize d_5^+
- Observe formula for objective function

40	Objective function:	
41	d5^+	
42	0	=SUM(C24)

- Use Excel solver to find optimal solution



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Deviational	1	2	3	4	5	6	7	8	9
<i>Deficit –</i>	0	0	0	0	0	854	16	0	114
<i>Surplus +</i>	0	0	0	0	0	0	0	984	0

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- Tab called **Priority 3**
 - Notice the additional constraint

0 =	0	<i>First goal optimal</i>
0 =	0	<i>Second goal optimal</i>

- Second objective function

Minimize $d_6^- + d_6^+ + d_7^- + d_7^+ + d_8^- + d_8^+ + d_9^- + d_9^+$

- Formula for this objective similar to first objective

41	Objective function:				
42	$d_6^- + d_6^+ + d_7^- + d_7^+ + d_8^- + d_8^+ + d_9^- + d_9^+$				
43	0				

- Use Excel solver to find optimal solution

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- Tab called **Priority 3**
 - Optimal solution

White students	<i>North</i>	<i>South</i>	<i>East</i>	<i>West</i>
<i>North</i>	643.77	37.82	0	318.4
<i>South</i>	174.62	275.37	0	0
<i>East</i>	0	368.4	681.6	0
<i>West</i>	0	0	0	500

Black students	<i>North</i>	<i>South</i>	<i>East</i>	<i>West</i>
<i>North</i>	254.4	0	0	45.6
<i>South</i>	0	454.4	345.6	0
<i>East</i>	291.2	0	108.8	0
<i>West</i>	0	0	0	500

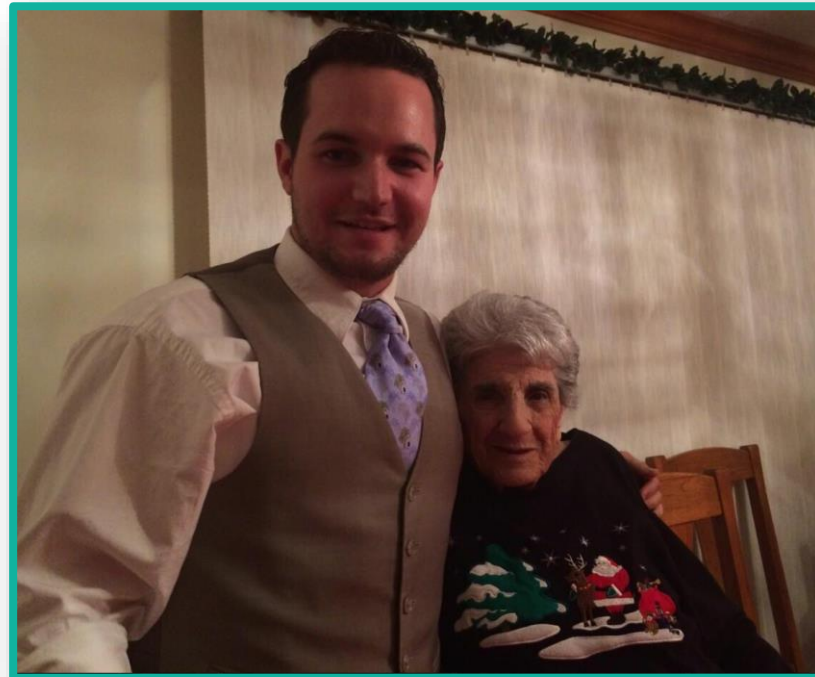
Deviation	1	2	3	4	5	6	7	8	9
<i>Deficit -</i>	0	0	0	0	0	0	0	0	0
<i>Surplus +</i>	0	0	0	0	0	0	0	0	0

- Q: What is the problem with the optimal solution?

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- Tab called **Priority 3**
 - Q: What do you mean you cannot bus half a person?



- Try to add integer constraints and see what happens





The End



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