Lecture 301

Produced by Dr. Worldwide

Welcome to the 305

Simulation for Continuous

- Many times we want to sample from a continuous distribution e.g. normal

- Suppose we want to simulate a random variable X having a cumulative distribution function (CDF)

$$F(x) = P(X \leq x)$$

- Then, we compute its inverse function $\mathrm{F}^{-1}(\mathrm{u})$ i.e. the function satisfying

$$F(F^{-1}(x)) = F^{-1}(F(x)) = x$$

- If U is a uniform $Uniform[0,1]$ random variable , then the random variable $F^{-1}(U)$ has the same distribution as X

- This method is called the inverse transform

Exponential Simulation

- An exponential random variable has cdf

$$F(x) = 1 - e^{-\lambda x}, \qquad x \geq 0$$

- With $\lambda > 0$ a parameter known as its "rate"

- Exponentials are often used to model the time between random arrivals

- To compute $F^{-1}(U)$, we set $u = F(x)$ and solve for x

$$u = 1 - e^{-\lambda x} \iff e^{-\lambda x} = 1 - u \iff$$

$$-\lambda x = \ln(1-u) \iff x = -\frac{1}{\lambda}\ln(1-u)$$

- If $U \sim Uniform[0,1]$, the random variable $X = -\frac{1}{\lambda}\ln(1-U)$ is an exponentially distributed random variable with rate λ

Exponential Simulation

- Note that if U is uniformly distributed in $[0,1]$, then $1 - U$ is too
- For the inverse transform method, we can replace U by $1 - U$ when convenient
- In the exponential example, we can set

$$X = -\frac{1}{\lambda} \ln U$$

Uniform Simulation

- Function RAND() samples $U \sim Uniform[0,1]$
- Q: How can we use RAND() to sample from $Uniform[a, b]$?
- If U $\sim Uniform[0,1]$ and $X = (b-a)U + a$, then X $\sim Uniform[a, b]$
- Q: What happens when U $= 0$ or U $= 1$?
- In Excel the formula is, $(b-a)RAND() + a$

Normal Simulation

- Most popular continuous distribution is the Normal distribution
- If X $\sim Normal(\mu, \sigma^2)$, then we can use the following pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in (-\infty, \infty)$$

- If X $\sim Normal(\mu, \sigma^2)$, then it can be written as X $= \sigma Z + \mu$ where $Z \sim N(0,1)$
- If we can simulate Standard Normal Z, then we can simulate any Normal X
- We will first focus on standard normal random variables

Normal Simulation

- Although there are more efficient methods for simulating Normal random variables, we could use the inverse transform method
- Set $\mathrm{Z} = \Phi^{-1}(U)$ where $U \sim Uniform[0,1]$
- In Excel, the function $NORM.INV(u, \mu, \sigma)$ computes $\mathrm{F}^{-1}(u)$ for the CDF function $\mathrm{F}(x)$ where

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz$$

- Typically, we use NORM.INV to find percentiles (see Link 1 on course website)
- Therefore, the random number $NORM.INV(RAND(), \mu, \sigma)$ is $Normal(\mu, \sigma^2)$

Ex: Arrival Process

- We want to model the # of customers that come to a coffee shop during a day
- Since people walk into the coffee shop at random times, we want to use a model that reflects this fact
- We assume the times between consecutive arrivals are independent and identically distributed (i.i.d.) random variables
- Specifically, if we let τ_i be the time of arrival between the $(\mathrm{i}-1)\mathrm{th}$ and $i\mathrm{th}$, then we can assume that the $\{\tau_i : i \geq 1\}$ are i.i.d.
- The set $\{\tau_i : i \geq 1\}$ are called interarrival times
- This set contains a random sample from a continuous distribution
- An assumption must be made about the distribution having CDF $F(x)$

Ex: Arrival Process

- An assumption must be made about the distribution with cdf
- Suppose that $F(x)$ is invertible (we can algebraically find $F^{-1}(u)$
- Let $N(t)$ denote the number of arrivals in the interval $[0, t]$
- In our example,
  - $N(10) = Number\ of\ customers\ who\ visit\ coffee\ shop\ in\ 10\ minutes$
  - $N(60) = Number\ of\ customers\ who\ visit\ coffee\ shop\ in\ 1\ hour$
  - $N(1440) = Number\ of\ customers\ who\ visit\ coffee\ shop\ in\ 1\ day$
- Q: What is the mean of $N(t)$?
- Q: What is the standard deviation of $N(t)$?

Ex: Arrival Process

- Process for simulation
  - Step 1: Simulate a large enough sample of $\{\tau_i : i \geq 1\}$ based on cdf $\tau_i = F^{-1}(U_i)$ where $U_i \sim Uniform[0,1]$ such that $\sum \tau_i \geq t$
  - Step 2: Count the number of number of τ's that were able to "fit" into the interval $[0, t]$, i.e. find k such that

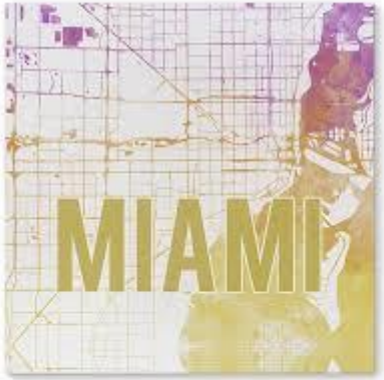
$$\sum_{i=1}^{k} \tau_i \leq t < \sum_{i=1}^{k+1} \tau_i$$

  - Step 3: Return $N(t) = k$
  - Step 4: Repeat steps 1-3

- For right now, we assume $\tau_i \sim Uniform[1,5]$

- Download ArrivalProcess.xlsx from the link Sheet 1 on the course website

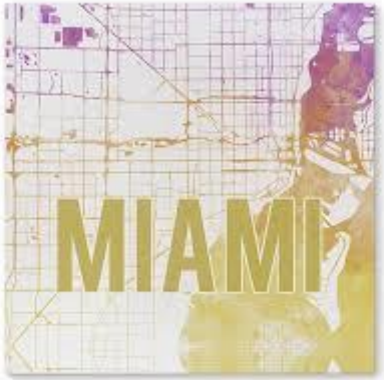
Ex: Arrival Process



- We sample interarrival times according to the formula $(5-1)RAND()+1$
- We get actual arrival time of a customer adding the amount of time elapsed between this customer and the last customer (A8:A39) to the arrival time of the last customer (B8:B39)
- Simulated data

7	Interarrivals	Arrival times			
8	2.21529181	2.21529181			
9	2.37839429	4.5936861		*Number of arrivals*	
10	3.78539481	8.37908091		18	
11	4.18518667	12.5642676			
12	4.86816408	17.4324316			

Ex: Arrival Process



- The cell D10 contains a realization of $N(60)$ which counts the number of customers who arrive within the first 60 minutes

- Notice the Excel formula COUNTIF(B8:B39,"<60")

- Q: What is the problem with the $Uniform[1,5]$ distribution for interarrival times?

- Q: What is needed to estimate the mean and standard deviation of $N(60)$?

- Q: Does it matter if we change our Excel formula from COUNTIF(B8:B39,"<60") to COUNTIF(B8:B39,"<=60")?

Poisson Process

- When the set of interarrival times $\{\tau_i : i \geq 1\}$ follow an exponential distribution with rate λ, we have a Poisson process with rate λ

- Poisson process is a classic way to model random arrivals

- Parameter λ is called the rate, since it corresponds to the average number of arrivals per unit of time

- The cdf for an $EXP(\lambda)$ is $F(x) = 1 - e^{-\lambda x}$

- In simulation, $\tau_i = -\frac{1}{\lambda}\ln(U_i)$ where $\{U_1, U_2, U_3, \cdots\}$ are $i.i.d.\ Uniform[0,1]$

Ex: Coffee Shop Revenue

- A coffee shop is open for 8 hours Monday through Friday
- Customers arrive according to a Poisson process with a rate 5 per hour
- The time between arrivals $\tau_i \sim EXP(5)$
- The amount a customer spends can be approximated using a $Normal(2.5,1)$
- This means the average customer spends $2.50
- Q: What is the problem with using the Normal distribution for the amount spent?
- Any customer who arrives before closing will be served
- We want to simulate the revenue of the coffee shop for one day (8 hours)

Ex: Coffee Shop Revenue

- We must simulate both the arrival times of customers and the amount of money they will spend

- To generate interarrival times we use the inverse transform method

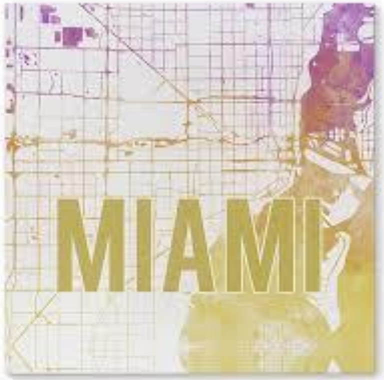
$$\tau_i = -\frac{1}{5}\ln(U_i)$$

where $U_i \sim Uniform[0,1]$

- To generate the amounts spend by different customers, we use the inverse transform method based on the NORM.INV function in Excel with a mean of 2.5 and a standard deviation of 1

- We simulate the times when customers arrive as well as the amounts they spend, then we add the expenses of all the customers who arrived during the time interval $[0,8]$

Ex: Coffee Shop Revenue



- Download CoffeeShop.xlsx from the link Sheet 2 on the course website

- Focus on tab named Revenue

- Descriptions of columns
  - Column A contains interarrival times
  - Column B contains arrival times (notice the calculation)
  - Column C contains the amount spent (notice the use of MAX() function
  - Column D checks to see if the customer made it by closing

- Simulated revenue for a single day

Revenue for the day:		
Sum of column C up to the last arrival in [0,8]		
120.062456		

Ex: Coffee Shop Queue

- Same coffee shop from previous example
- We want to simulate the customer queue
- Assume there is only one person at the cash register, and that this person both takes the order and prepares the coffee
- Moment of silence for this poor worker
- Each customer takes a random amount of time to be served, which we model as an exponential random variable with mean of 5 minutes (1/12 hour)

$$\frac{1}{\lambda} = 5\,min = \frac{1}{12}hr \quad \longrightarrow \quad \lambda = 12$$

- Customers are served in a first-come-first-serve basis

Ex: Coffee Shop Queue

- We want to use simulation to determine the maximum waiting time experienced by a customer during the day, and the number of customers still present at the coffee shop (either in queue or being served) at closing time

- Generate interarrival times $\{\tau_i : i \geq 1\}$ as before, making sure that we have enough to cover the 8-hour interval

- Let χ_i denote the service time of customer i

- To generate service times we use the inverse transform method

$$\chi_i = -\frac{1}{12}\ln(V_i)$$

where $V_i \sim Uniform[0,1]$

- Status of coffee shop changes whenever a customer arrives or leaves

Ex: Coffee Shop Queue

- The number of customers in the coffee shop is the state of the system
- The arrival (A_i) and departure times (D_i) are known as events
- In simulation, we keep track of interarrival times, service times, arrival times, and departure times
- The only random numbers are interarrival times and the service times
- Customer *i* will start his/her service at two potential times
  - At the arrival time if no one is there
  - At the departure time of customer $i-1$ if there is somebody

$$\max\{A_i, D_{i-1}\}, \qquad i = 2, 3, \dots,$$

where the first customer starts always at time A_1

Ex: Coffee Shop Queue

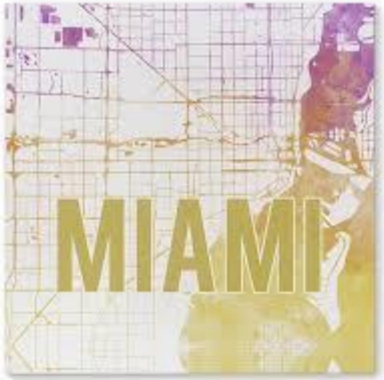
- The number of customers in the coffee shop at the end of the day is

$$Number\ of\ Arrivals\ in\ [0,8] - Number\ of\ Departures[0,8]$$

- Download CoffeeShop.xlsx from the link Sheet 2 on the course website
- Focus on tab named Queue
- Notice that departure time is the sum of arrival time and service time
- Column H shows the waiting times which is the difference between the start of service and arrival time

Maximum waiting time	
0.42893819	hours
25.7362916	minutes

Ex: Coffee Shop Queue



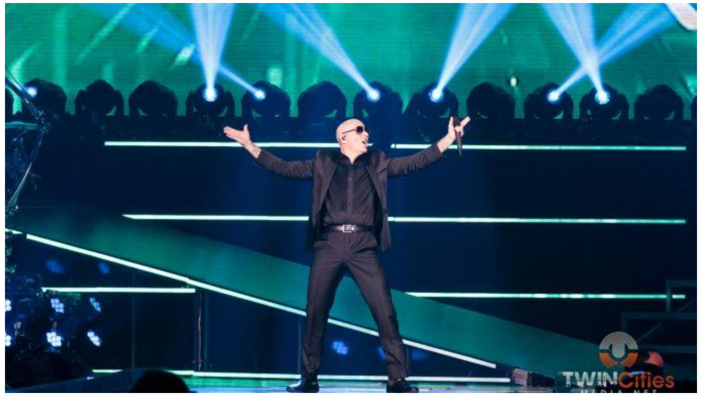
- In the coffee shop it may be unrealistic to assume that customers arrive at the same rate throughout the day

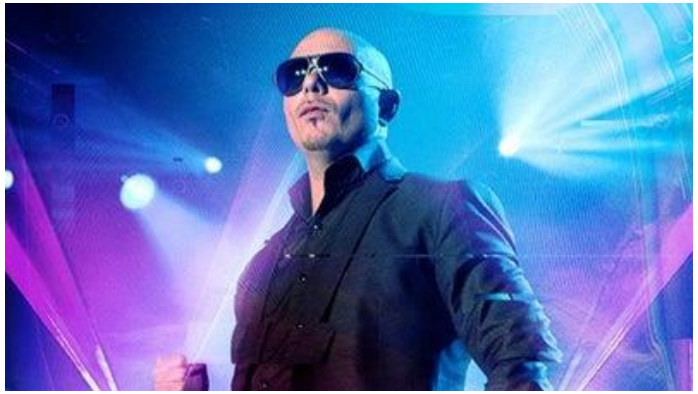
- To make our models more realistic, we can change the rate during different periods of the day

- Assume that we can divide the interval $[0,8]$ into k periods $[0, t_1], (\mathrm{t}_1\mathrm{t}_2], \cdots, (t_{k-1}, 8)]$ such that the arrival rate during each period is constant λ_i

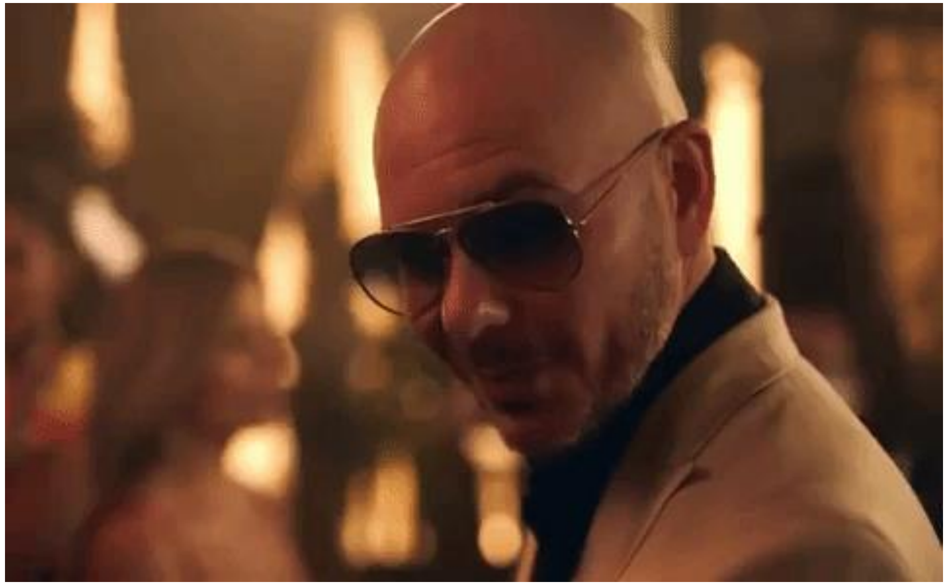
- Simulate the arrivals in each period using the corresponding rate

- In periods with large λ_i , arrivals will be closer to each other, while in periods with small λ_i, they will be more spread out





The End



Dale