Lecture 5

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Welcome to the 305

Sensitivity Analysis

- A sensitivity analysis is how we investigate the effect changes in the objective function and constraints have on the optimal solution

- Types of changes
  - Changes in the objective function coefficients
  - Changes in the constraint quantity values
  - Changes in the constraint coefficients
  - Additional constraints
  - Additional decision variables

- Excel’s Solver can handle changes in the first two types

- Other types involve rerunning Excel’s Solver with different information

Changing Objective Function

- Q: How much can parameters change without changing the optimal solution?

- Changes in objective function parameters lead to changes in the direction of level curves in a graph

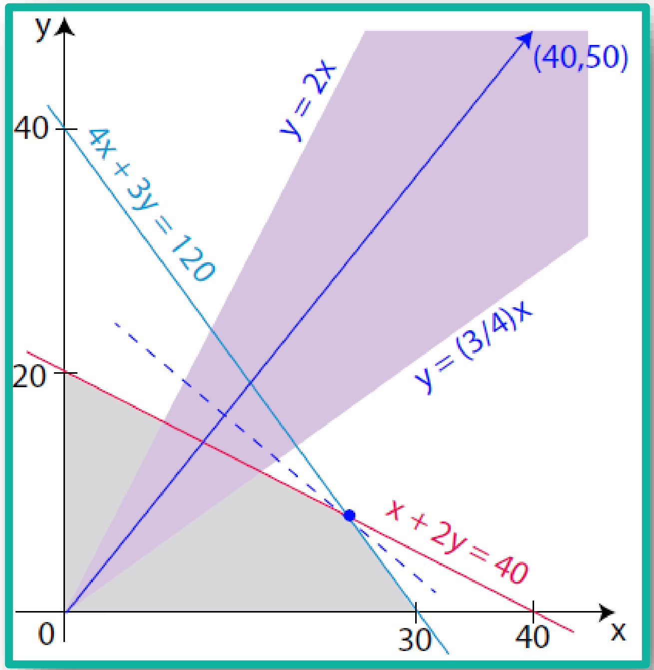
- Consider the Beaver Creek linear program

  Maximize $40x + 50y$
  Subject to $x + 2y \leq 40$
  $4x + 3y \leq 120$
  $x \geq 0$
  $y \geq 0$

- Recall optimal solution was (24,8)

Changing Objective Function

- We will "tilt" the objective function coefficients $(a, b) = (40{,}50)$ until the optimal solution changes



- Any vector in direction of (a, b), where (a, b) is in the purple region will lead to the same optimal solution

Changing Objective Function

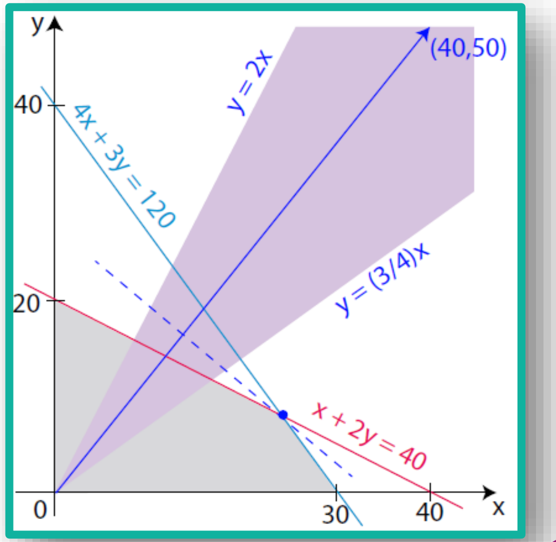
- Edges of purple cone are perpendicular to constraints
  - Line $y = 2x$ is perpendicular to $x + 2y = 40$ $\left(y = 20 - \frac{1}{2}x\right)$
  - Line $y = \frac{3}{4}x$ is perpendicular to $4\text{x} + 3\text{y} = 120$ $\left(y = 40 - \frac{4}{3}x\right)$

- Points in the purple cone are characterized by slopes

- Points (a, b) in purple cone must satisfy $\frac{3}{4} \leq \frac{b}{a} \leq 2$

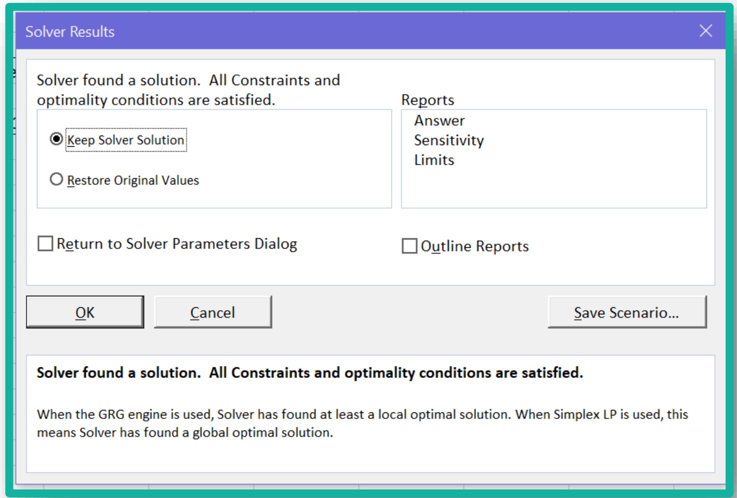
- If we fix $b = 50$, then $25 \leq a \leq \frac{200}{3}$

- If we fix $\text{a} = 40$, then $30 \leq b \leq 80$

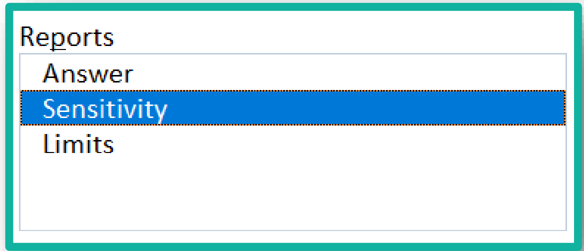


Ex: Beaver Creek Pottery

- Download BeaverCreek.xlsx from website link called Sheet 1
- We start by considering changes to the objective function $40x + 50y$
- Attempt to solve linear program produces menu of options



Select Sensitivity



- Selecting OK creates a new sheet in Excel file

Ex: Beaver Creek Pottery

- Sensitivity analysis for objective function displayed below

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B11	Bowls = Bowl	24	0	40	26.66666667	15
B12	Mugs = Bowl	8	0	50	30	20

- If we fix $b = 50$, then $25 = 40 - 15 \leq a \leq 40 + 26.67 = \frac{200}{3}$
- If we fix a $= 40$, then $30 = 50 - 20 \leq b \leq 50 + 30 = 80$
- Next, we consider changes to the constraint quantit

Changing Constraint Quantity

- Changing constraint quantities (c, d) from $(40{,}120)$

  Subject to
  $$\begin{aligned} x + 2y &\leq c \\ 4x + 3y &\leq d \\ x &\geq 0 \\ \mathrm{y} &\geq 0 \end{aligned}$$

- Changes in these quantities cause vertical shifts of constraints in graph
- Constraint is binding at the optimal solution if the constraint holds with equality
- Constraint is non-binding at optimal solution if the constraint fails with equality
- In this sensitivity analysis, there is not concern about different optimal solutions

Changing Constraint Quantity

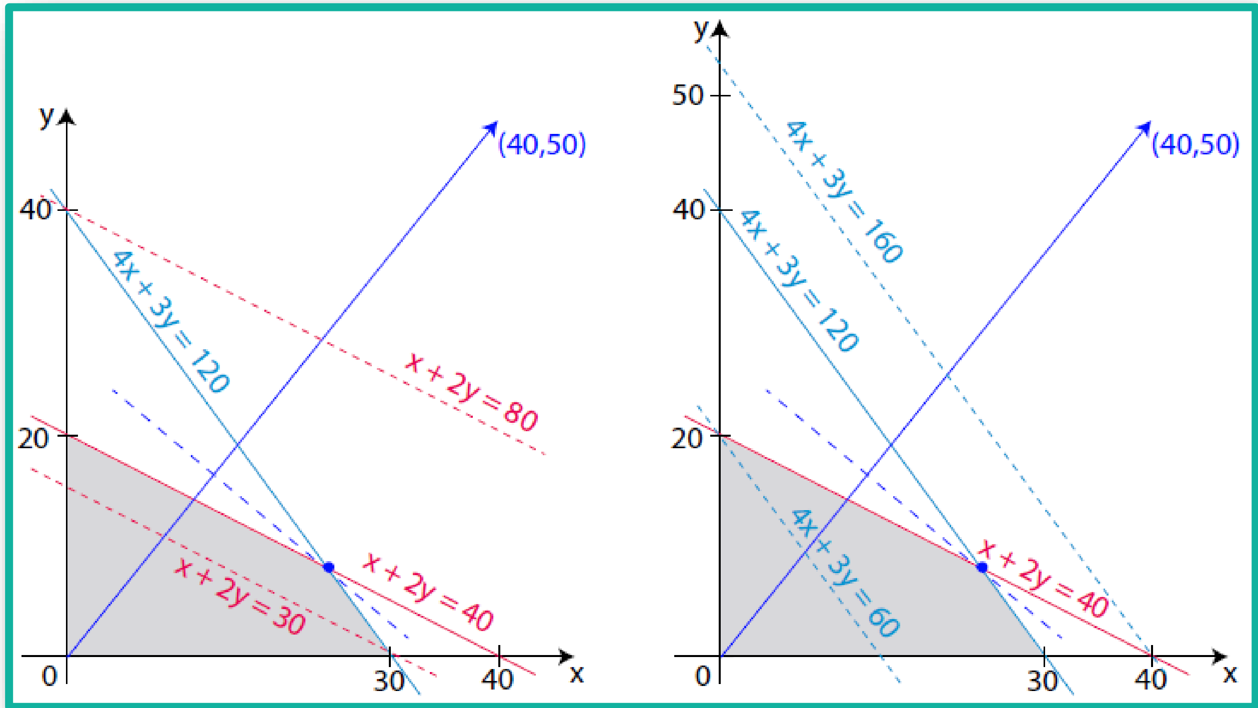
- Binding constraints at the optimal solution (24,8)
  - $x + 2y = 24 + 2(8) = 40 = \mathrm{c}$
  - $4x + 3y = 4(24) + 3(8) = 120 = \mathrm{d}$

- Non-binding constraints at the optimal solution (24,8)
  - $x = 24 \neq 0$
  - $y = 8 \neq 0$

- Q: How can we change (c, d) while keeping the first two constraints binding at optimality? (i.e. we want the optimal solution to occur at the intersection of the lines $x + 2y = \mathrm{c}$ and $4x + 3y = \mathrm{d}$)

Changing Constraint Quantity

- Adjustment of y-intercepts of lines until lines don't intersect and one of the constraints is no longer "necessary"



- We can change $30 \leq c \leq 80$ and $60 \leq d \leq 160$

Ex: Beaver Creek Pottery

- Follow same steps in Excel from previous sensitivity analysis

- Sensitivity analysis for constraint quantities displayed below

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$D\$6	Labor (hr/unit) Usage	40	16	40	40	10
\$D\$7	Clay (lb/unit) Usage	120	6	120	40	60

- If we fix c $= 40\ hours\ of\ labor$, then $30 = 40 - 10 \leq c \leq 40 + 40 = 80$

- If we fix d $= 120\ lbs.of\ clay$, then $60 = 120 - 60 \leq d \leq 120 + 40 = 160$

Shadow Prices

- The shadow price or dual value of a constraint (resource) correspond to the maximum amount that one would be willing to pay for one additional unit of that resource

- Standard sensitivity reports include these shadow prices

- In general, there is one shadow price for each constraint

Constraints

Cell	Name	Final Value	Shadow Price
D6	Labor (hr/unit) Usage	40	16
D7	Clay (lb/unit) Usage	120	6

Ex: Beaver Creek Pottery

- The shadow prices of labor and clay are 16 and 6, respectively

- Implication for labor
  - If we increase labor hours from 40 to $40 + x$, the profit increases by $16x$
  - We shouldn't pay more than $16 per hour of labor

- Implication for clay
  - If we increase pounds of clay from 120 to $120 + y$, the profit increases by $6y$
  - We shouldn't pay more than $6 per pound of clay

- This only can be applied for constraint quantity values in the limits under the specificity analysis

Ex: Quick-Screen Clothing

- Quick-Screen is a clothing manufacturing company specializing in the production of commemorative shirts immediately following major sporting events and they have a contract to produce shirts for winning team of a college football bowl game on New Year's Day between State and Tech

- They will produce two different sweatshirts and two different t-shirts with one of each having writing on front (F) only and the other having writing on both front (F) and back (B)

- All items will be produced by the box where each box contains a dozen items

- Q: How much of each of the items should be produced to maximize profit?

Ex: Quick-Screen Clothing

- Decision Variables
  - $x_1 = Number\ of\ Boxes\ of\ Sweatshirts\ - F$
  - $x_2 = Number\ of\ Boxes\ of\ Sweatshirts - B/F$
  - $x_3 = Number\ of\ Boxes\ of\ T - shirts - F$
  - $x_4 = Number\ of\ Boxes\ of\ T - shirts - B/F$

- Consider the following table showing resource requirements, unit costs, and profit of every dozen (box) of shirts

	Processing time (hr.) per dozen	Cost per dozen	Profit dozen
Sweatshirt - F	0.10	$36	$90
Sweatshirt - B/F	0.25	48	125
T-shirt - F	0.08	25	45
T-shirt - B/F	0.21	35	65

Ex: Quick-Screen Clothing

- Objective Function
  - Goal: Maximize profit on shirts
  - $Z = 90x_1 + 125x_2 + 45x_3 + 65x_4$

- Constraints
  - Only have 72 hours of processing time to produce all items: $0.1x_1 + 0.25x_2 + 0.08x_3 + 0.21x_4 \leq 72$
  - Company has a budget of $25,000: $36x_1 + 48x_2 + 25x_3 + 35x_4 \leq 25{,}000$
  - Trailer truck will pick up shirts and can accommodate 1,200 standard-size boxes where each standard-size box holds 12 T-shirts and a box of 12 sweatshirts is 3 times the size of the standard-size box: $3(\mathrm{x}_1 + \mathrm{x}_2) + x_3 + x_4 \leq 1{,}200$
  - They have 500 dozens of blank sweatshirts: $x_1 + x_2 \leq 500$
  - They have 500 dozens of blank T-shirts: $x_3 + x_4 \leq 500$
  - Nonnegativity: $x_1, x_2, x_3, x_4 \geq 0$

Ex: Quick-Screen Clothing

- Download ProductMix.xlsx from website link called Sheet 2
- Before Excel solver

A product mix								
Products:	Sweatshirt-F (dozen)	Sweatshirt-B/F (dozen)	T-shirt-F (dozen)	T-shirt-B/F (dozen)				
Profit per dozen:	90	125	45	65				
Resources:					Usage	Constraint	Available	Left over
Processing time	0.1	0.25	0.08	0.21	0	<=	72	72
Cost	36	48	25	35	0	<=	25000	25000
Truck capacity	3	3	1	1	0	<=	1200	1200
Blank sweatshirts	1	1	0	0	0	<=	500	500
Blank T-shirts	0	0	1	1	0	<=	500	500
Production:								
Sweatshirts-F =	0							
Sweatshirts-B/F =	0							
T-shirt-F =	0							
T-shirt-B/F =	0							
Profit =	0							

Ex: Quick-Screen Clothing

- After Excel solver

A product mix								
Products:	Sweatshirt-F (dozen)	Sweatshirt-B/F (dozen)	T-shirt-F (dozen)	T-shirt-B/F (dozen)				
Profit per dozen:	90	125	45	65				
Resources:					Usage	Constraint	Available	Left over
Processing time	0.1	0.25	0.08	0.21	72	<=	72	0
Cost	36	48	25	35	21593.333	<=	25000	3406.6667
Truck capacity	3	3	1	1	1200	<=	1200	0
Blank sweatshirts	1	1	0	0	233.33333	<=	500	266.66667
Blank T-shirts	0	0	1	1	500	<=	500	0
Production:								
Sweatshirts-F =	175.55556							
Sweatshirts-B/F =	57.777778							
T-shirt-F =	500							
T-shirt-B/F =	0							
Profit =	45522.222							

Ex: Quick-Screen Clothing

- Recommended optimal solution to maximize profit at $45,522.22
  - $x_1 = 175.56$
  - $x_2 = 57.78$
  - $x_3 = 500$
  - $x_4 = 0$

- Sensitivity report for objective function coefficients

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B15	Sweatshirts-F = (dozen)	175.5555556	0	90	11.92307692	40
B16	Sweatshirts-B/F = (dozen)	57.77777778	0	125	13.21428571	11.92307692
B17	T-shirt-F = (dozen)	500	0	45	1E+30	4.111111111
B18	T-shirt-B/F = (dozen)	0	-10.33333333	65	10.33333333	1E+30

Ex: Quick-Screen Clothing

- Sensitivity report for constraint quantities

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F10	Blank sweatshirts Usage	233.3333333	0	500	1E+30	266.6666667
F11	Blank T-shirts Usage	500	4.111111111	500	185.7142857	500
F7	Processing time Usage	72	233.3333333	72	26.33333333	8.666666667
F8	Cost Usage	21593.33333	0	25000	1E+30	3406.666667
F9	Truck capacity Usage	1200	22.22222222	1200	260	316

The End

Dale