Lecture 17

Produced by Dr. Worldwide

Welcome to the 305

Ex: Oakdale School Busing

- Goals listed in order based on priority
  - Achieve a 60%/40% ratio of white to black students at each of the schools
  - Minimize the amount of traveling that students will have to do, ideally no more than 30,000 miles per day
  - Keep all schools close to capacity and minimize overcrowding proportionally allocating the excess among the schools

- Q: How can we formulate and solve a goal programming model to help the board with its dilemma?

- Decision variables
  - x_{ij} = Number of white students from district i assigned to district j
  - y_{ij} = Number of black students from district i assigned to district j
  - $i, j \in \{N, S, E, W\}$

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- Goal 1: Achieve racial balance in all 4 schools
  - Consider perfect balance for North High School

$$\frac{Percent\ White}{Percent\ Black} = \frac{0.6}{0.4}$$

$$\frac{\frac{Total\ White}{Total\ Students}}{\frac{Total\ Black}{Total\ Students}} = \frac{0.6}{0.4}$$

$$\frac{Total\ White}{Total\ Black} = \frac{0.6}{0.4}$$

$$0.4(Total\ White) - 0.6(Total\ Black) = 0$$
$$0.4(x_{NN} + x_{SN} + x_{EN} + x_{WN}) - 0.6(y_{NN} + y_{SN} + y_{EN} + y_{WN}) = 0$$

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- Goal 1: Achieve racial balance in all 4 schools
  - Adding deviational variables for North High School

$$0.4(x_{NN} + x_{SN} + x_{EN} + x_{WN}) - 0.6(y_{NN} + y_{SN} + y_{EN} + y_{WN}) + d_1^- - d_1^+ = 0$$

  - Consider constraints for all schools in the county

$$0.4(x_{NN} + x_{SN} + x_{EN} + x_{WN}) - 0.6(y_{NN} + y_{SN} + y_{EN} + y_{WN}) + d_1^- - d_1^+ = 0$$
$$0.4(x_{NS} + x_{SS} + x_{ES} + x_{WS}) - 0.6(y_{NS} + y_{SS} + y_{ES} + y_{WS}) + d_2^- - d_2^+ = 0$$
$$0.4(x_{NE} + x_{SE} + x_{EE} + x_{WE}) - 0.6(y_{NE} + y_{SE} + y_{EE} + y_{WE}) + d_3^- - d_3^+ = 0$$
$$0.4(x_{NW} + x_{SW} + x_{EW} + x_{WW}) - 0.6(y_{NW} + y_{SW} + y_{EW} + y_{WW}) + d_4^- - d_4^+ = 0$$

  - To accomplish our goal we want all of these deviational variables to be as small as possible
  - First priority objective

Minimize $\quad P_1(d_1^- + d_1^+ + d_2^- + d_2^+ + d_3^- + d_3^+ + d_4^- + d_4^+)$

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- Goal 2: Minimize total travel to not much more than 30,000 miles
  - Recall the following table

	Distance (mi.)			
District/School	*North*	*South*	*East*	*West*
North	–	30	12	20
South	30	–	18	26
East	12	18	–	24
West	20	26	24	–

  - Formulation for constraint based on total miles

$$30(x_{NS} + y_{NS} + x_{SN} + x_{SN}) + 12(x_{NE} + y_{NE} + x_{EN} + y_{EN})$$
$$+20(x_{NW} + y_{NW} + x_{WN} + y_{WN}) + 18(x_{SE} + y_{SE} + x_{ES} + y_{ES})$$
$$+26(x_{SW} + y_{SW} + x_{WS} + y_{WS}) + 24(x_{EW} + y_{EW} + x_{WE} + y_{WE}) + d_5^- - d_5^+ = 30{,}000$$

  - Updated objective function for second priority

Minimize $\quad P_1(d_1^- + d_1^+ + d_2^- + d_2^+ + d_3^- + d_3^+ + d_4^- + d_4^+), P_2(d_5^+)$

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- Goal 3: Minimize overcrowding at each of the 4 schools, proportionally allocating the excess among the schools
  - Recall the following table

District/School	# White students	# Black students	Capacity
North	1000	300	1200
South	450	800	1000
East	1050	400	1000
West	500	500	1200

  - Recall that there are 5,000 total students for capacity of 4,400
  - The excess of 600 students needs to be split between the schools
  - Q: How can we handle this proportionally?

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- Goal 3: Minimize overcrowding at each of the 4 schools, proportionally allocating the excess among the schools
  - We want to manage the excess according to the capacities of the schools
  - Schools that are bigger should take larger portions of the overflow
  - We prefer if North and West take $1200/4400 = 3/11$ of the excess
  - We prefer if South and East take $1000/4400 = 5/22$ of the excess
  - Capacities are expanded to handle the overflow (rounded up)

School	North	South	East	West
Ideal # students	1364	1136	1136	1364

$$1200 + \frac{3}{11}(600) = 1363.636363 \approx 1364$$

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- Goal 3: Minimize overcrowding at each of the 4 schools, proportionally allocating the excess among the schools
  - Constraints with deviational variables

$$x_{NN} + y_{NN} + x_{SN} + y_{SN} + x_{EN} + y_{EN} + x_{WN} + y_{WN} + d_6^- - d_6^+ = 1364$$
$$x_{NS} + y_{NS} + x_{SS} + y_{SS} + x_{ES} + y_{ES} + x_{WS} + y_{WS} + d_7^- - d_7^+ = 1136$$
$$x_{NE} + y_{NE} + x_{SE} + y_{SE} + x_{EE} + y_{EE} + x_{WE} + y_{WE} + d_8^- - d_8^+ = 1136$$
$$x_{NW} + y_{NW} + x_{SW} + y_{SW} + x_{EW} + y_{EW} + x_{WW} + y_{WW} + d_9^- - d_9^+ = 1364$$

  - Updated objective function for third priority

$$\begin{aligned}\text{Minimize}\quad & P_1(d_1^- + d_1^+ + d_2^- + d_2^+ + d_3^- + d_3^+ + d_4^- + d_4^+),\\ & P_2(d_5^+),\\ & P_3(d_6^- + d_6^+ + d_7^- + d_7^+ + d_8^- + d_8^+ + d_9^- + d_9^+)\end{aligned}$$

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- Additional constraints
  - We cannot bus more students than what is currently available at each school

District/School	# White students	# Black students	Capacity
North	1000	300	1200
South	450	800	1000
East	1050	400	1000
West	500	500	1200

  - List of constraints

$$x_{NN} + x_{NS} + x_{NE} + x_{NW} = 1000$$
$$y_{NN} + y_{NS} + y_{NE} + y_{NW} = 300$$
$$x_{SN} + x_{SS} + x_{SE} + x_{SW} = 450$$
$$y_{SN} + y_{SS} + y_{SE} + y_{SW} = 800$$
$$x_{EN} + x_{ES} + x_{EE} + x_{EW} = 1050$$
$$y_{EN} + y_{ES} + y_{EE} + y_{EW} = 400$$
$$x_{WN} + x_{WS} + x_{WE} + x_{WW} = 500$$
$$y_{WN} + y_{WS} + y_{WE} + y_{WW} = 500$$

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- Additional integer constraints

  $x_{ij} \in \{0,1,\cdots\}$
  $y_{ij} \in \{0,1,\cdots\}$

- Download Oakdale-1.xlsx from link Sheet 1 on course website

- Tab called Priority 1
  - Matrices of decision variables

White student	North	South	East	West	Total
North	0	0	0	0	0
South	0	0	0	0	0
East	0	0	0	0	0
West	0	0	0	0	0
Total	0	0	0	0	

Black students	North	South	East	West	Total
North	0	0	0	0	0
South	0	0	0	0	0
East	0	0	0	0	0
West	0	0	0	0	0
Total	0	0	0	0	

Deficit	Surplus
0	0
0	0
0	0
0	0
0	0
0	0
0	0
0	0
0	0

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- Tab called Priority 1
  - Notice all the different constraints and inspect formulas

Constraints:						
	Deficit	Surplus	Computed	Constraint	Value	
0	0	0	0	=	0	*Racial balance at North*
0	0	0	0	=	0	*Racial balance at South*
0	0	0	0	=	0	*Racial balance at East*
0	0	0	0	=	0	*Racial balance at West*
0	0	0	0	=	30000	*Total distance travelled*
0	0	0	0	=	1364	*Overcrowding at North*
0	0	0	0	=	1136	*Overcrowding at South*
0	0	0	0	=	1136	*Overcrowding at East*
0	0	0	0	=	1364	*Overcrowding at Wast*
			0	=	1000	*"Supply" of white students*
			0	=	300	*"Supply" of black students*
			0	=	450	*"Supply" of white students*
			0	=	800	*"Supply" of black students*
			0	=	1050	*"Supply" of white students*
			0	=	400	*"Supply" of black students*
			0	=	500	*"Supply" of white students*
			0	=	500	*"Supply" of black students*

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- Tab called Priority 1
  - First objective function

Minimize $\quad d_1^- + d_1^+ + d_2^- + d_2^+ + d_3^- + d_3^+ + d_4^- + d_4^+$

  - Observe formula for objective function

39	**Objective function:**		
40	d1^- + d1^+ + d2^- + d2^+ + d3^- + d3^+ + d4^- + d4^+		
41	0	=SUM(B20:C23)	

  - Q: What is “B20:C23” referring to and what is “SUM” doing?

  - Use Excel solver to find the optimal solution

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- Tab called Priority 1
  - Optimal solution

White students	North	South	East	West
North	461	538	0	0
South	0	140	65	245
East	0	0	1050	0
West	0	0	0	499

Black students	North	South	East	West
North	300	0	0	0
South	1	452	344	3
East	0	0	400	0
West	7	0	0	493

Deviational	1	2	3	4	5	6	7	8	9
Deficit −	0	0	0	0	0	594	6	0	124
Surplus +	0	0	0	0	152	0	0	724	0

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- Tab called Priority 2
  - Notice the additional constraint and inspect formula

0	=		0	*First goal optimal*

  - Second objective function

Minimize d_5^+

  - Observe formula for objective function

40	**Objective function:**	
41	d5^+	
42	0	=SUM(C24)

  - Use Excel solver to find optimal solution

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- Tab called Priority 2
  - Optimal solution

White students	North	South	East	West
North	306	442	1	251
South	0	223	227	0
East	0	6	1044	0
West	0	1	0	499

Black students	North	South	East	West
North	204	0	96	0
South	0	448	352	0
East	0	0	400	0
West	0	0	0	500

Deviational	1	2	3	4	5	6	7	8	9
Deficit −	0	0	0	0	0	854	16	0	114
Surplus +	0	0	0	0	$\boxed{0}$	0	0	984	0

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- Tab called Priority 3
  - Notice the additional constraint

0	=		0	*First goal optimal*
0	=		0	*Second goal optimal*

  - Second objective function

Minimize $\quad d_6^- + d_6^+ + d_7^- + d_7^+ + d_8^- + d_8^+ + d_9^- + d_9^+$

  - Formula for this objective similar to first objective

41	**Objective function:**			
42	d6^- + d6^+ + d7^- + d7^+ + d8^- + d8^+ + d9^- + d9^+			
43	0			

  - Use Excel solver to find optimal solution

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- Tab called Priority 3
  - Optimal solution

White students	North	South	East	West
North	643.77	37.82	0	318.4
South	174.62	275.37	0	0
East	0	368.4	681.6	0
West	0	0	0	500

Black students	North	South	East	West
North	254.4	0	0	45.6
South	0	454.4	345.6	0
East	291.2	0	108.8	0
West	0	0	0	500

Deviational	1	2	3	4	5	6	7	8	9
Deficit −	0	0	0	0	0	0	0	0	0
Surplus +	0	0	0	0	0	0	0	0	0

- Q: What is the problem with the optimal solution?

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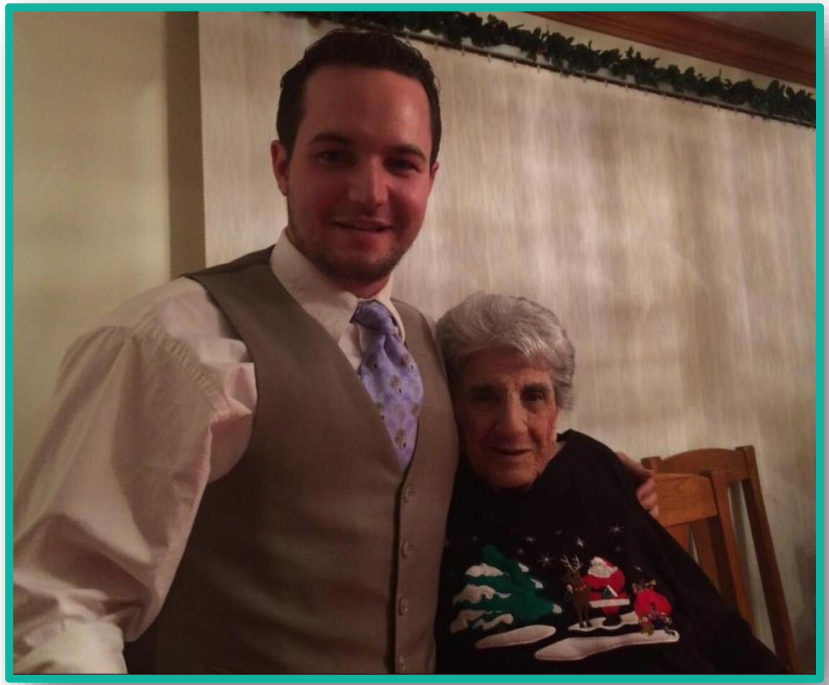
- Tab called Priority 3
  - Q: What do you mean you cannot bus half a person?



  - Try to add integer constraints and see what happens

The End

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