Lecture 29

Produced by Dr. Worldwide

Welcome to the 305

Stochastic Simulation

- Simulation is used to “simulate” the operations of various kinds of real-world systems or processes with the aid of a computer
- We make assumptions about how a system works
- These assumptions constitute a model
- Models are used to gain some understanding of how a system behaves
- In simple models, we can use analytical (mathematical) methods to obtain exact information on questions of interest
- In complex models, use simulation to evaluate the model numerically

Monte Carlo Process

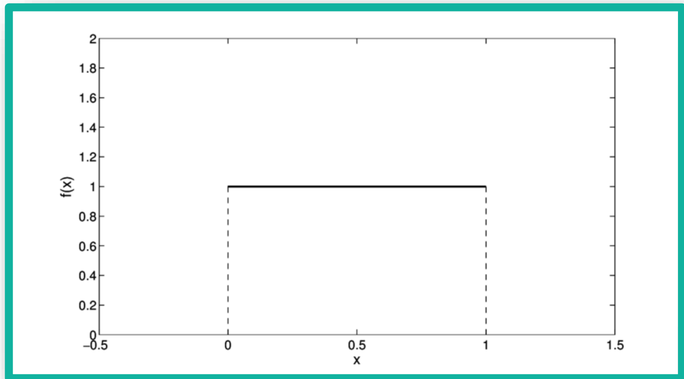
- Large proportion of applications of simulations are for probabilistic models
- Monte Carlo is a technique for selecting numbers randomly from a probability distribution
- Monte Carlo is not a simulation model, but a mathematical process used within a simulation
- Monte Carlo simulation can also be thought of as multiple probability simulation
- Read Link 1 on course website for more description about Monte Carlo

Random Number Generator

- To run a simulation, we may need "observations" from a random process
- If we "know" the distribution of these observations, we might be able to generate them artificially
- Historically, random numbers have been generated physically
    - Throwing dice
    - Dealing cards
    - Drawing balls from urns
- Not until mid 1950's where electronic random number generators were used
- Two kinds of random number generators: physical and numerical
- Computer are equipped with numerical random number generators

Random Number Generator

- Q: Other than my teaching, is anything truly random?
- Numerical random number generators are not really random
- The most basic type of random numbers that we can artificially generate are from the Uniform[0,1] distribution where numbers between 0 and 1 are equally likely
- RAND() function in Excel samples from this distribution
- We can generate random numbers from any other distribution from the RAND() function
- Read Link 2 on course website for more description about RAND()



Ex: Spinning a Wheel

- Suppose we want to simulate the result of spinning a wheel of fortune

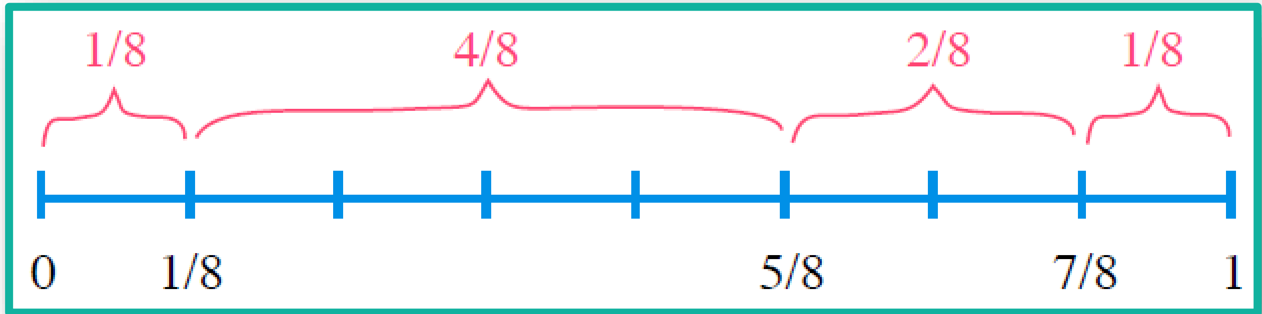
- No physical version of the wheel, but can generate random uniform numbers
- Q: How can we use RAND() to “spin” the wheel?

Ex: Spinning a Wheel

- Compute probabilities of each outcome
- Let X be the result from spinning the wheel once
- Possible outcomes are $X \in \{0,10,20,100\}$
- Each segment of the wheel has the same area
- Probabilities under assumption wheel is "fair"
  - $P(X = 0) = \frac{1}{8}$
  - $P(X = 10) = \frac{4}{8} = \frac{1}{2}$
  - $P(X = 20) = \frac{2}{8} = \frac{1}{4}$
  - $P(X = 100) = \frac{1}{8}$

Ex: Spinning a Wheel

- Represent probabilities within the interval [0,1]



- Use Excel RAND() to generate a U from the Uniform [0,1] distribution and set

$$X = \begin{cases} 0, & \text{if } 0 \le U \le 1/8, \\ 10, & \text{if } 1/8 < U \le 5/8, \\ 20, & \text{if } 5/8 < U \le 7/8, \\ 100, & \text{if } 7/8 < U \le 1. \end{cases}$$

Ex: Spinning a Wheel

- Download SpinWheel.xlsx from link Sheet 1 on course website
- When we use the RAND() function repeatedly, the random numbers generated are independent of each other
- We can use Excel to simulate as many spins of the wheel as we want
- Random sample located in cells E6:E47
- We are simulating 42 spins of this wheel
- Compare theoretical mean and standard deviation to sample statistics

17	True mean	22.5
18	True variance	893.75
19	True standard devi	29.89565186

12	Sample mean	26.9047619
13	Sample standard d	33.67622707

Ex: Spinning a Wheel

- To transform, the Uniform[0,1] numbers generated by the function RAND() we can create a look-up table
  - 1st column contains the values of the function h$(x) = P(X < x)$
  - 2nd column contains the values of x
- Table contain in array B6:C9

Cumulative P(X < x)	x
0.00	0
0.13	10
0.63	20
0.88	100

- Table saved into variable named CDF_Table

Ex: Spinning a Wheel

- The command VLOOKUP(cell, look-up table, 2) will compare the number in the cell to the ranges defined by the first column of the look-up table, and return the value in the second column

- Example of function usage

Cumulative P(X < x)	x
0.00	0
0.13	10
0.63	20
0.88	100

Uniform U	X
0.1185088	0
0.2509473	10
0.1383176	10
0.1524281	10
0.1422752	10

- Value $U = 0.1185$ is between 0 and 0.13 so VLOOKUP returns 0

Simulation for Discrete

- Suppose we want to use simulation to generate samples from a random variable X that takes values

$$\{x_1, x_2, x_3, \cdots, x_k\}$$

  with probabilities $P(X = x_i) = p_i,\ i = 1,2,3, \cdots, k$

- For convenience, we can assume $x_i \leq x_{i+1}$

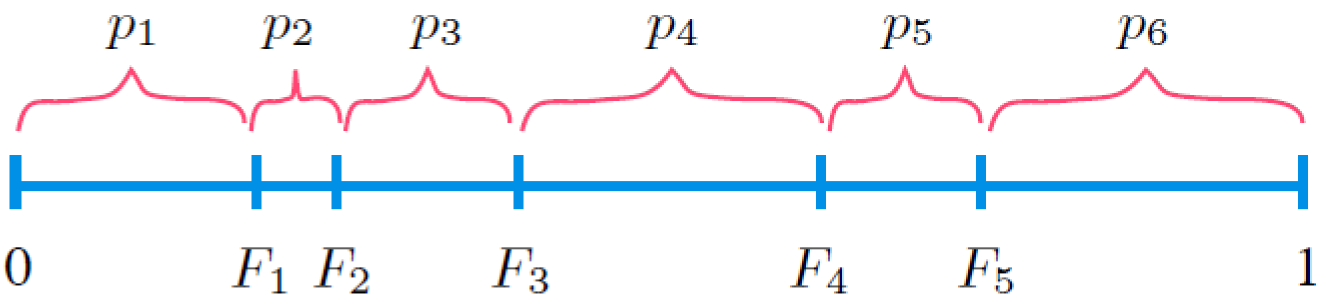
- We want to generate a random number that has the same distribution as X

- Assume we have a computer that can generate random numbers uniformly distributed on the interval $[0,1]$

Simulation for Discrete

- Step 1: Compute the cumulative probabilities

$F_i = P(X \leq x_i) = p_1 + \cdots + p_i, \qquad i = 1,2,3,\cdots,k$



- Note that $F_0 = 0$ and $F_k = 1$

- Step 2: Generate a random number $U \sim Uniform[0,1]$

- Step 3: Return $X = x_i$ if $F_{i-1} < U \leq F_i$

Ex: Cell Phone Production

- Suppose we want to model the production of a cellphone factory
- Factory ships cellphones in boxes containing 100 phones
- Each cellphone has an independent probability $p = 0.01$ of being defective
- We want to simulate a shipment containing 20 boxes
- We are interested in both the number of defective cellphones in a shipment, and the number of boxes containing at least one defective cellphone in a shipment
- If N_i denotes the number of defective cellphones in box i,
  then $N_i \sim Binomial(100{,}0.01)$
- Related to flipping an unfair coin with probability of heads equal to 0.01

Ex: Cell Phone Production

- Instead of simulating the N_i's, we try to simulate coin flips, and evaluate

$$N_i = \sum_{j=1}^{100} X_{ij}$$

- where $X_{ij} = \begin{cases} 1 & \textit{if the jth cellphone in box i is defective} \\ 0 & \textit{otherwise} \end{cases}$
- Download SimBinomial.xlsx from link Sheet 2 on course website
- Q: What do each of the columns represent?
- Q: Where are the simulated N_i's?
- Q: What is the meaning of the Excel syntax =IF(RAND()>0.99,1,0)?

Simulation for Continuous

- Many times we want to sample from a continuous distribution e.g. normal

- Suppose we want to simulate a random variable X having a cumulative distribution function (CDF)

$$F(x) = P(X \leq x)$$

- Then, we compute its inverse function $\mathrm{F}^{-1}(\mathrm{u})$ i.e. the function satisfying

$$F(F^{-1}(x)) = F^{-1}(F(x)) = x$$

- If U is a uniform $Uniform[0,1]$ random variable , then the random variable $F^{-1}(U)$ has the same distribution as X

- This method is called the inverse transform

Ex: Exponential Simulation

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- An exponential random variable has CDF

$$F(x) = 1 - e^{-\lambda x}, \qquad x \geq 0$$

- With $\lambda > 0$ a parameter known as its “rate”

- Exponentials are often used to model the time between random arrivals

- To compute $F^{-1}(U)$, we set $u = F(x)$ and solve for x

$$u = 1 - e^{-\lambda x} \iff e^{-\lambda x} = 1 - u \iff$$

$$-\lambda x = \ln(1 - u) \iff x = -\frac{1}{\lambda}\ln(1 - u)$$

- If $U \sim Uniform[0,1]$, the random variable $X = -\frac{1}{\lambda}\ln(1 - U)$ is an exponentially distributed random variable with rate λ

Ex: Exponential Simulation

- Note that if U is uniformly distributed in $[0,1]$, then $1 - U$ is too
- For the inverse transform method, we can replace U by $1 - U$ when convenient
- In the exponential example, we can set

$$X = -\frac{1}{\lambda} \ln U$$

The End

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