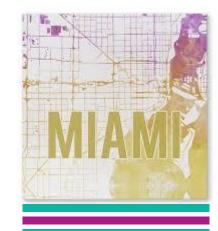


Goal Programming



- All prior linear programming problems have had a single objective function
- Companies may have multiple criteria in consideration for a decision
- Sometimes the multiple objectives conflict
- Company may want to maximize profit and minimize pollution
- Goal programming is linear programming for multiple objectives or criteria





- Trying to choose $x = number\ of\ bowls\$ and $y = number\ of\ mugs\$ to maximize the profit function
- Recall the original linear program

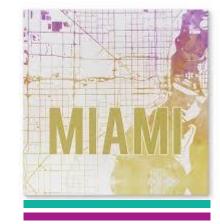
Maximize
$$40x + 50y$$

Subject to:
$$x + 2y \le 40$$
 (Labor)

$$4x + 3y \le 120 \tag{Clay}$$

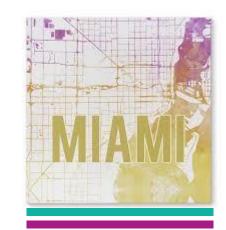
$$x, y \ge 0$$

Objective function reflects a single goal





- Suppose Beaver Creek wanted to achieve other goals while maximizing profit
- In order of importance:
 - To avoid layoffs, they want to use at least 40 hours of labor per day
 - They want to achieve a satisfactory profit level of \$1,600 per day
 - To avoid having clay dry out, they prefer to keep no more than 120 lb of clay on hand each day
 - To avoid overhead costs due to keeping the factory open past normal hours, they want to minimize the amount of overtime
- We reformulate our linear programming model using goal programming
- Transform linear programming model constraints into goals





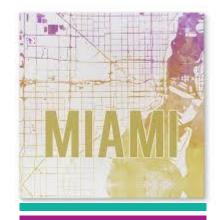
- Goal 1: Avoid underutilization of labor
 - Original constraint $x + 2y \le 40$
 - Reformulation to a goal constraint

$$x + 2y + d_1^- - d_1^+ = 40$$
 (Labor)

- Two new variables d_1^- and d_1^+ are nonnegative and represent the underutilized time and overtime, respectively
- Q: What if the optimal solution had $d_1^- > 0$?
- Q: What if the optimal solution had $d_1^+ > 0$?
- The top priority corresponding to minimization of d_1^-

Minimize
$$P_1d_1^-$$

• The P_1 indicates the priority of this goal (not a coefficient)





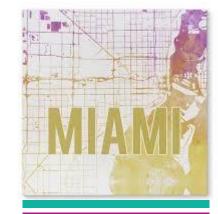
- Goal 2: Achieve daily profit of \$1,600
 - Original objective function Z = 40x + 50y
 - Reformulation to a goal constraint

$$40x + 50y + d_2^- - d_2^+ = 1600$$
 (Profit)

- Two new variables d_2^- and d_2^+ are nonnegative and represent the amount of profit less than \$1,600 and more than \$1,600
- The second priority corresponding to minimization of d_2^- is added

Minimize
$$P_1d_1^-$$
, $P_2d_2^-$

- The comma between the terms indicates that we are minimizing them sequentially, not simultaneously
- Q: Why are we not minimizing d_2^+ ?



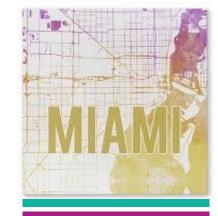


- Goal 3: Avoid waste of material
 - Original constraint $4x + 3y \le 120$
 - Reformulation to a goal constraint

$$4x + 3y + d_3^- - d_3^+ = 120 (Clay)$$

- Two new variables d_3^- and d_3^+ are nonnegative and represent the amount of clay less than 120 lbs and more than 120 lbs
- The company cannot keep more than 120 lbs in storage
- The third priority corresponds to minimization of d_3^+ is added

Minimize
$$P_1 d_1^-, P_2 d_2^-, P_3 d_3^+$$



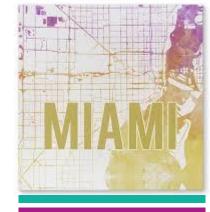


- Goal 4: Avoid overtime costs
 - Recall the modified goal constraint for labor

$$x + 2y + d_1^- - d_1^+ = 40$$
 (Labor)

- Already attempting to minimize d_1^-
- To ensure we don't exceed the maximum labor, we involve d_1^+
- Finalization of objective function

Minimize
$$P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$$





Full goal programming model

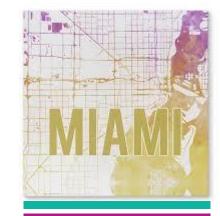
Minimize
$$P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$$
 Subject to
$$x + 2y + d_1^- - d_1^+ = 40 \qquad \text{(Labor)}$$

$$40x + 50y + d_2^- - d_2^+ = 1600 \qquad \text{(Profit)}$$

$$4x + 3y + d_3^- - d_3^+ = 120 \qquad \text{(Clay)}$$

$$x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+ \geq 0$$

- The variables $\{d_1^-,d_1^+,d_2^-,d_2^+,d_3^-,d_3^+,d_4^-,d_4^+\}$ are called deviational variables
- We minimize the four different objective functions individually by priority





- Modification 1: Maximum of 10 hours of overtime
 - Recall the goal constraint for labor

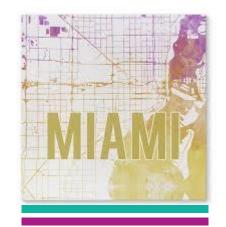
$$x + 2y + d_1^- - d_1^+ = 40$$
 (Under hours)

- Remember that d_1^+ represents overtime
- We want $0 \le d_1^+ \le 10$
- Use same strategy as before by adding a goal constraint

$$d_1^+ + d_4^- - d_4^+ = 10$$
 (Over hours)

- Possible goal constraint of all deviational variables
- Two new variables d_4^- and d_4^+ are nonnegative and represent the amount of overtime hours less than 10 hours and more than 10 hours
- New objective function

Minimize
$$P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+$$

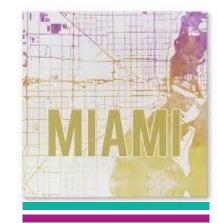




- Modification 2: Maximum number of bowls and mugs made daily
 - Pottery company has limited warehouse space
 - They can only produce at most 30 bowls and 20 mugs each day
 - Profit for bowls (\$40) less than profit for mugs (\$50)
 - Consider the new constraints

$$x + d_5^- = 30$$
 (Bowls)
 $y + d_6^- = 20$ (Mugs)

- We want to minimize d_5^- and d_6^-
- Q: Why not include positive deviational variables d_5^+ and d_6^+ ?
- Q: For which item is it more important to achieve this goal?





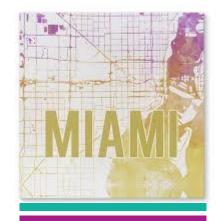
- Modification 2: Maximum number of bowls and mugs made daily
 - Positive deviational variables are unnecessary since it is imperative to not exceed the warehouse space
 - We need to achieve the goal for mugs more than the goal for bowls because the profit is higher for mugs
 - If goals were of equal importance, we would minimize

Minimize
$$P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, P_5d_5^- + P_5d_6^-$$

- We can make the degree of importance in proportion to the profit
- The goal for mugs is more important than the goal for bowls by a ratio of 5 to 4

Minimize
$$P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, 4P_5d_5^- + 5P_5d_6^-$$

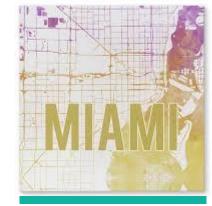
• The coefficients 4 and 5 are referred to as weights





• Full modified goal programming model

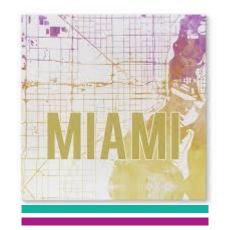
| Minimize | $P_1d_1^-$, $P_2d_2^-$, $P_3d_3^+$, $P_4d_4^+$, $4P_5d_5^- + 5P_5d_6^-$ | |
|------------|--|---|
| Subject to | $x + 2y + d_1^ d_1^+ = 40$ $40x + 50y + d_2^ d_2^+ = 1600$ $4x + 3y + d_3^ d_3^+ = 120$ $d_1^+ + d_4^ d_4^+ = 10$ $x + d_5^- = 30$ $y + d_6^- = 20$ $x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^-$ | (Labor)(Profit)(Clay)(Overtime)(Bowls)(Mugs)≥ 0 |



Excel for Goal Programming



- Builds off linear programming using Excel Solver
- Solve the linear program multiple times with different objective functions
- Go in order of priority
- After finding the optimal solution, we add the optimal value attained in the first objective function as a new constraint and move on to the next objective function
- Possible that while solving for a given priority, we simultaneously optimize other lower ranked priorities

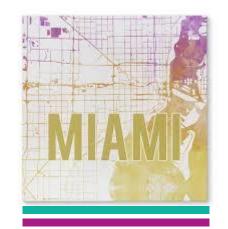




- Download GoalProgramming.xlsx from link Sheet 1 on course website
- See tab Priority 1 for minimization of d_1^-
 - Optimal solution

$$x = 15$$
 $y = 20$ $d_1^+ = 15$ $d_4^+ = 5$ $d_5^- = 15$ $d_1^-, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_6^- = 0$

- It is optimal to set $d_1^- = 0$
- In our system of linear constraints, we have employees working at least 40 hr
- Move on to P2 for minimization of d_2^-
 - Notice from last solution $d_2^- = 0$
 - Optimal solution from P1 minimizes objective function from P2
- Unnecessary to consider P3 since $d_3^+ = 0$ under optimal solution of P1

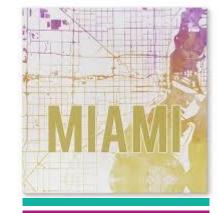




- See tab Priority 4 for minimization of d_4^+
 - To ensure none of the optimal values achieved thus far change when we attempt to minimize d_4^+ , we add the values attained as constraints
 - We add one constraint for each goal we have already attained

Minimize d_4^+

Subject to $x + 2y + d_1^- - d_1^+ = 40 \qquad \text{(Labor)}$ $40x + 50y + d_2^- - d_2^+ = 1600 \qquad \text{(Profit)}$ $4x + 3y + d_3^- - d_3^+ = 120 \qquad \text{(Clay)}$ $d_1^+ + d_4^- - d_4^+ = 10 \qquad \text{(Overtime)}$ $x + d_5^- = 30 \qquad \text{(Bowls)}$ $y + d_6^- = 20 \qquad \text{(Mugs)}$ $d_1^-, d_2^-, d_3^+ = 0 \qquad \text{(Mugs)}$

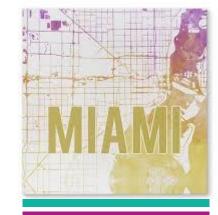




- See tab Priority 4 for minimization of d_4^+
 - Optimal solution

$$x = 15$$
 $y = 20$ $d_1^+ = 15$ $d_4^+ = 5$ $d_5^- = 15$ $d_1^-, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_6^- = 0$

- Solution did not change and $d_4^+ = 5$ in both cases
- Not possible to reduce the value of d_4^+ without violating the optimal solutions for the three goals that have higher priority
- This indicates that the overtime must be exceed by 5 hours to fulfill other constraints from higher priority goals





- See tab Priority 5 for minimization of $4d_5^- + 5d_6^-$
 - Add result from previous priority rank as a constraint

Minimize
$$4d_5^- + 5d_6^-$$

$$x + 2y + d_{1}^{-} - d_{1}^{+} = 40$$

$$40x + 50y + d_{2}^{-} - d_{2}^{+} = 1600$$

$$4x + 3y + d_{3}^{-} - d_{3}^{+} = 120$$

$$d_{1}^{+} + d_{4}^{-} - d_{4}^{+} = 10$$

$$x + d_{5}^{-} = 30$$

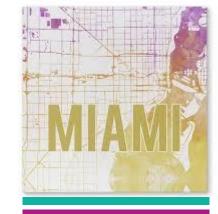
$$y + d_{6}^{-} = 20$$

$$d_{1}^{-}, d_{2}^{-}, d_{3}^{+} = 0$$

$$d_{4}^{+} = 5$$

$$x, y, d_{1}^{+}, d_{2}^{+}, d_{3}^{-}, d_{4}^{-}, d_{5}^{-}, d_{6}^{-} \ge 0$$

(Labor)
(Profit)
(Clay)
(Overtime)
(Bowls)
(Mugs)
(New Constraints)
(New Constraints)

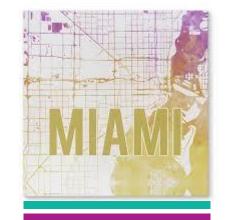




- See tab Priority 5 for minimization of $4d_5^- + 5d_6^-$
 - Optimal solution

$$x = 15$$
 $y = 20$ $d_1^+ = 15$ $d_4^+ = 5$ $d_5^- = 15$ $d_1^-, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_6^- = 0$

- Solution still did not change
- Optimal solution stays optimal
- Final Solution
 - Produce 15 bowls and 20 mugs
 - Hours of work: 15 + 2(20) = 55 (Over by 15 hours)
 - Profit: 40(15) + 50(20) = 1600
 - Pounds of clay: 4(15) + 3(20) = 120
 - Overtime beyond 10 hours: $d_4^+ = 5$
 - Slack for bowls below 30: $d_5^- = 15$
 - Slack for mugs below 20: $d_4^+ = 0$



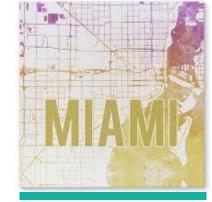


Full modified goal programming model

| Minimize | $P_1d_1^-$, $P_2d_2^-$, $P_3d_3^+$, $P_4d_4^+$, $4P_5d_5^-+$ | $5P_5d_6^-$ |
|----------|--|-------------|
|----------|--|-------------|

$$x + 2y + d_1^- - d_1^+ = 40$$
 (Labor)
 $40x + 50y + d_2^- - d_2^+ = 1600$ (Profit)
 $4x + 3y + d_3^- - d_3^+ = 120$ (Clay)
 $d_1^+ + d_4^- - d_4^+ = 10$ (Overtime)
 $x + d_5^- = 30$ (Bowls)
 $y + d_6^- = 20$ (Mugs)

$$x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \ge 0$$









The End





