Lecture 19

Produced by Dr. Worldwide

Welcome to the 305

Nonlinear Programming

- Consider problems where the goal is to maximize (minimize) an objective function by changing the values of a set of decision variables $\{x_1, x_2, \cdots, x_k\}$ taking values inside a feasible region

- We have only considered linear objective functions of the following form

$$c_1x_1 + c_2x_2 + \cdots + c_kx_k$$

  and feasible regions defined by linear constraints

- A nonlinear programming problem follows the same format as a linear programming mode with at least one of the following changes
  - Nonlinear objective function
  - Nonlinear constraint

- Nonlinear programs are considerably harder to solve

Nonlinear Programming

- Classic break-even point problem
  - Consider the profit function

    $$Z = vp - c_f - vc_v$$

    where $v = sales\ volume\ (demand)$
    $p = price$
    $c_f = fixed\ cost$
    $c_v = variable\ cost$

  - Break-even point is about identifying what choice of v makes $Z = 0$
  - Unrealistic assumption that volume is independent of price

- Q: How does demand depend on price?

Nonlinear Programming

- Optimizing profit
  - Suppose volume decreases as price increases by the linear function $v = 1500 - 24.6p$
  - This relationship between v and p is visualized below

Nonlinear Programming

- Optimizing profit
  - A company may want to know what p maximizes Z
  - Substituting this relation into the profit function

$$\begin{aligned} Z &= vp - c_f - vc_v \quad = (1500 - 24.6p)p \ - c_f - (1500 - 24.6p)c_v \\ &= 1500p - 24.6p^2 - c_f - 1500c_v + 24.6pc_v \\ &= -24.6p^2 + (1500 + 24.6c_v)p - \left(c_f + 1500c_v\right) \end{aligned}$$

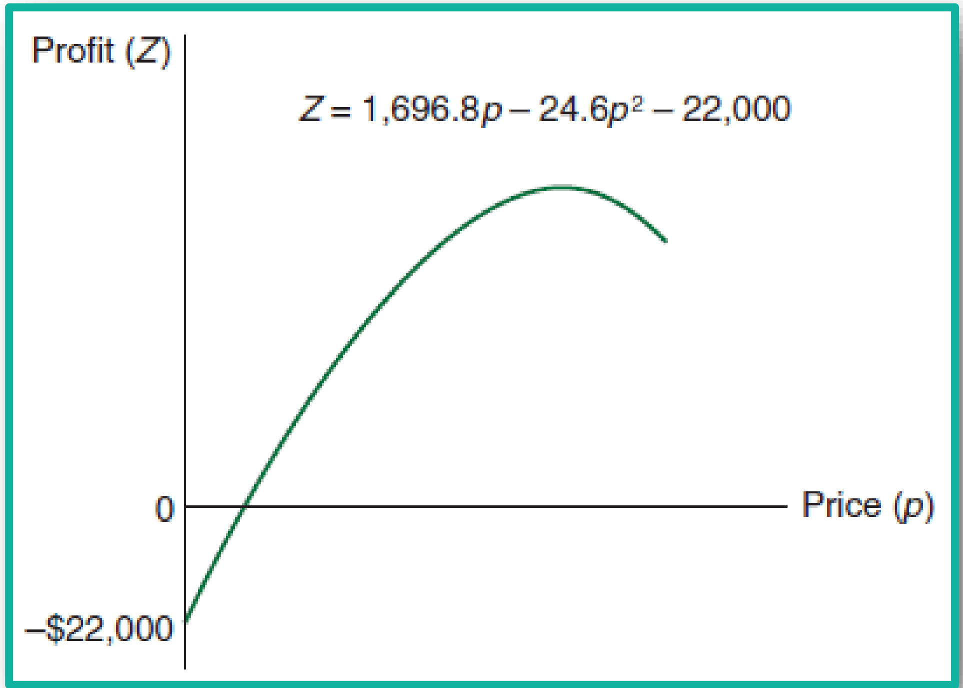
  - Suppose we know that $c_f = \$10{,}000$ and $c_v = \$8$

$$\begin{aligned} Z &= -24.6p^2 + (1500 + 24.6 * 8)p - (10000 + 1500 * 8) \\ &= -24.6p^2 + 1696.8p - 22000 \end{aligned}$$

  - Q: As price increases, does the profit increase or decrease?

Nonlinear Programming

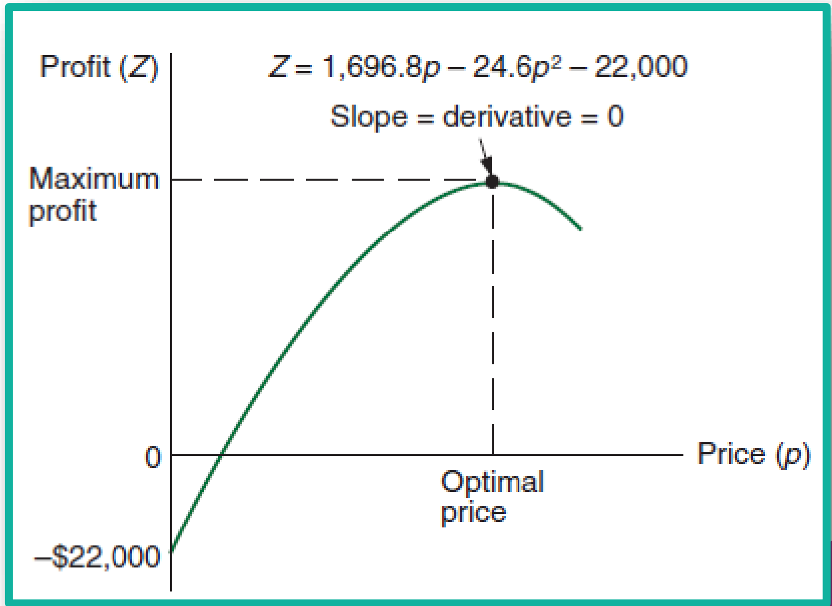
- Optimizing profit
  - Consider the new nonlinear/quadratic profit curve



  - Q: How can we find which price maximizes profit?

Nonlinear Programming

- Optimizing profit
  - Follow steps from calculus to find the maximum (minimum) of a function
    - Take the first derivative
    - Set it equal to zero
    - Solve for the independent variable
    - Check second derivative at the point to see if it is a max or min
      - Negative implies max
      - Positive implies min



Nonlinear Programming

- Optimizing profit
  - Define function

    $Z = -24.6p^2 + 1696.8p - 22000$
  - Derivative of the function based on power rule from Calculus I

    $$Z' = (-24.6)2p + 1696.8 = -49.2p + 1696.8$$
  - Set derivative to zero and solve for the price

    $$0 = -49.2p^* + 1696.8 \quad \rightarrow \quad p^* = \frac{-1696.8}{-49.2} = 34.49$$
  - Second derivative of the function evaluated at $p^* = 34.49$

    $Z'' = -49.2 < 0 \quad \rightarrow \quad concave\ down \quad \rightarrow \quad minimum$

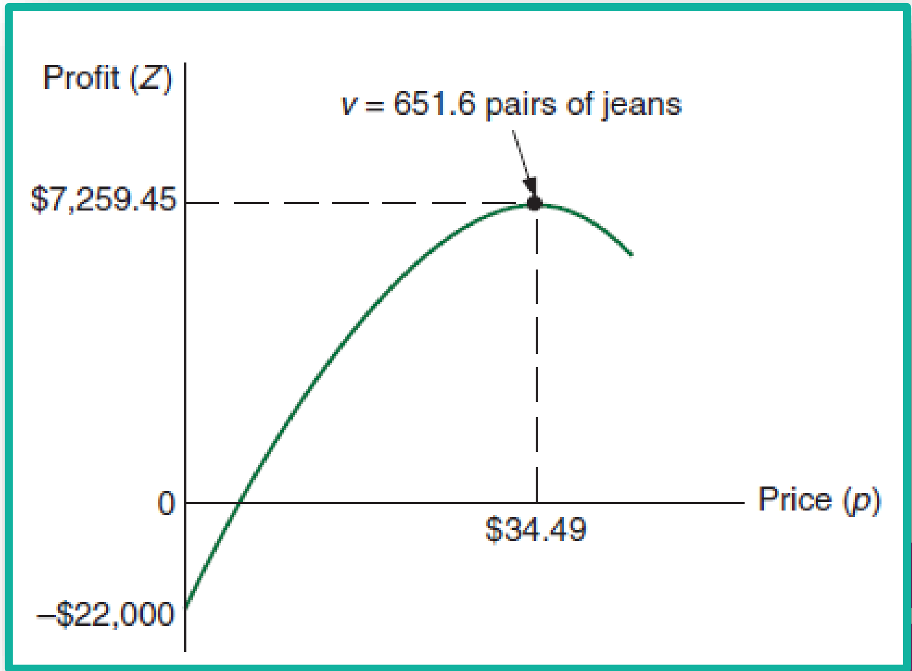
Nonlinear Programming

- Optimizing profit
  - Maximum profit

    $$Z = -24.6(34.49)^2 + 1696.8(34.49) - 22000 = \$7{,}259$$

  - Expected volume or demand

    $$v = 1500 - 24.6(34.49) = 651.6$$



Nonlinear Programming

- An unconstrained optimization model consists of a single nonlinear objective function without any constraints
- When constraints are added, this becomes a constrained optimization model or a nonlinear programming model
- Nonlinear programming models are considerably harder to solve since there are no methods guaranteed to find a solution
- Q: What about the optimal solution of a nonlinear programming model makes it more difficult to find?

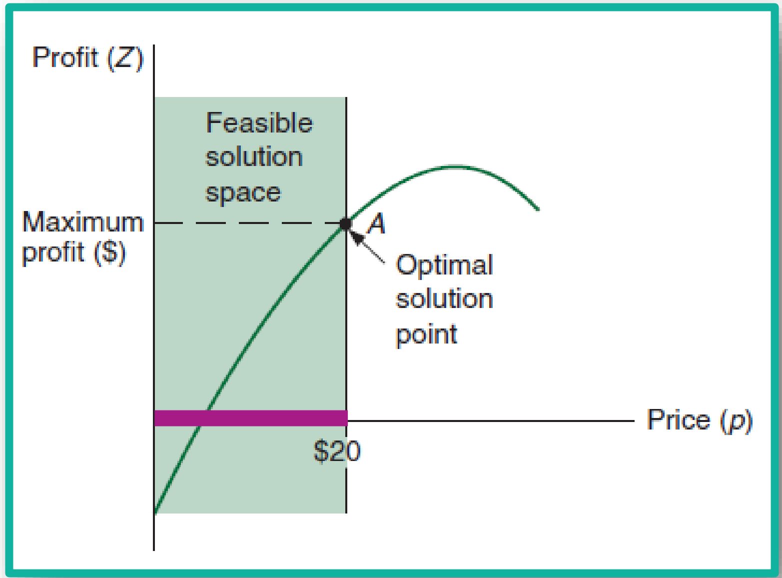
Nonlinear Programming

- A price ceiling is a price control, usually determined by the government, designed to protect consumers from conditions that could make commodities ridiculously expensive

- Optimizing profit with a price ceiling of $20

Maximize $Z = -24.6p^2 + 1696.8p - 22000$

Subject to $0 \le p \le 20$

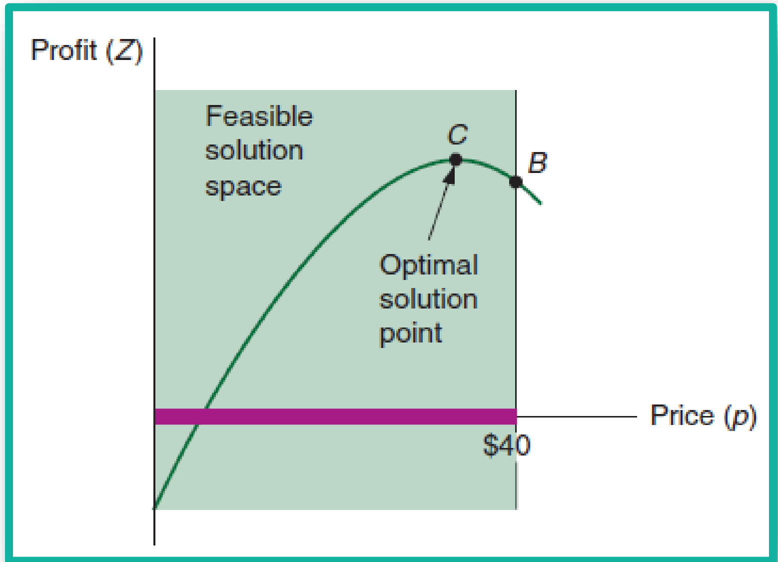


Nonlinear Programming

- Optimizing profit with a price ceiling of $40

  Maximize $\quad Z = -24.6p^2 + 1696.8p - 22000$

  Subject to $\quad 0 \le p \le 40$



- In a constrained optimization model, it is not guaranteed that optimal solutions lie on the boundary of the feasible region

Solving in Excel

- Algorithms for solving nonlinear programming models can be very complex
- Most algorithms can only guarantee that they find a local optimizer rather than a global one
- Excel Solver uses an algorithm called Generalized Reduced Gradient (GRG) to solve nonlinear problems
- This algorithm is designed to find a local optimizer within a certain "tolerance" level, and it can sometimes get "stuck"
- In problems not guaranteed to have a unique interior optimal solution, it is a good idea to run the GRG algorithm starting at several initial points

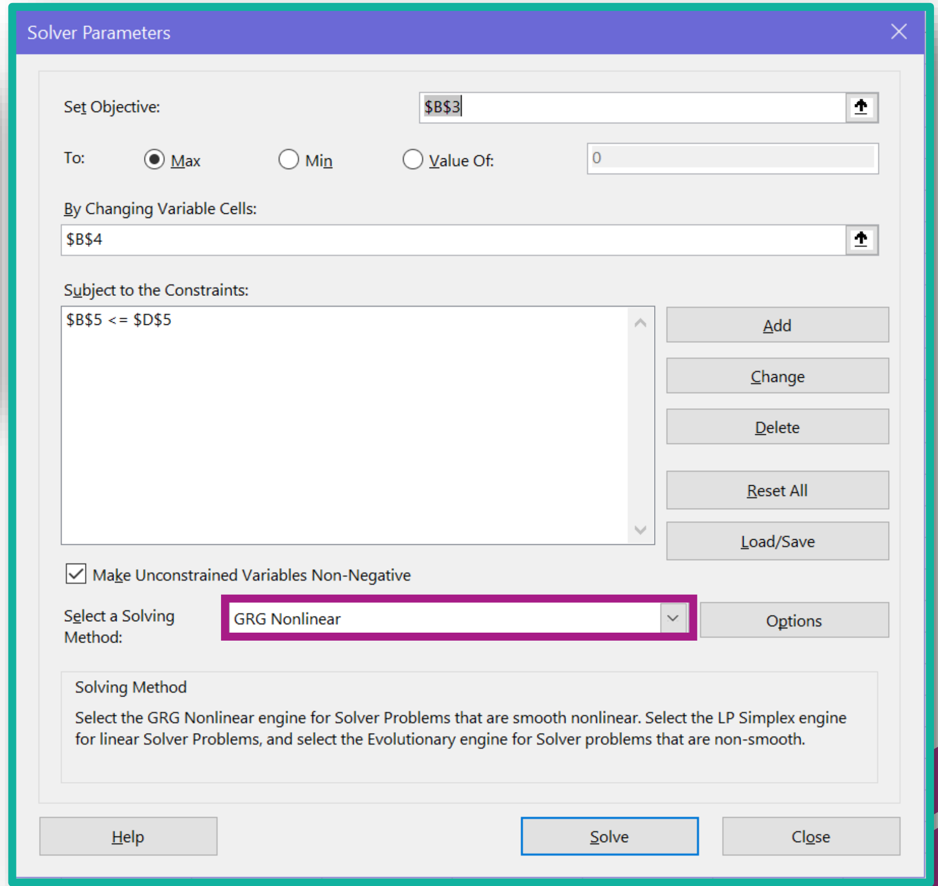
Solving in Excel

- Download NonlinearProfit.xlsx from link Sheet 1 on course website

- Inspect the spreadsheet and Solver

	A	B	C	D
1	**Maximizing nonlinear profit**			
2				
3	Profit:	7259.45366		
4	Variable (p):	34.4878049		
5	*Constraint*	34.4878049	<=	40

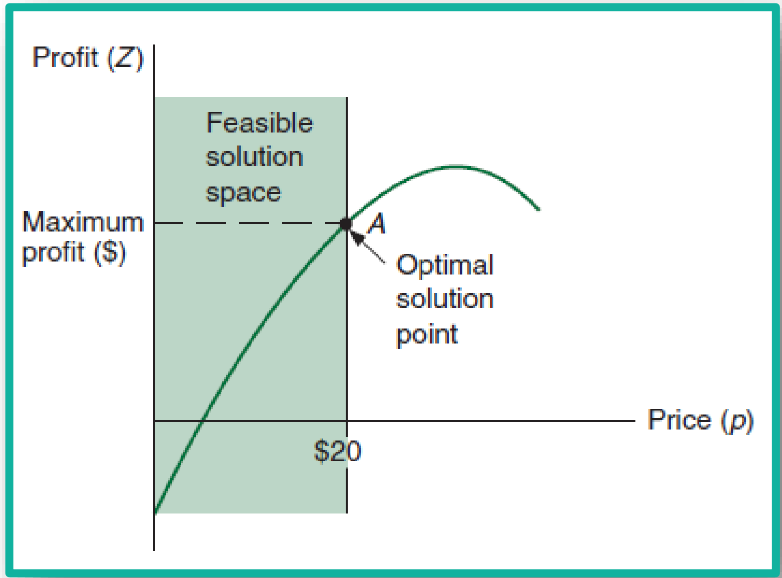
=1696.8*B4-24.6*B4^2-22000

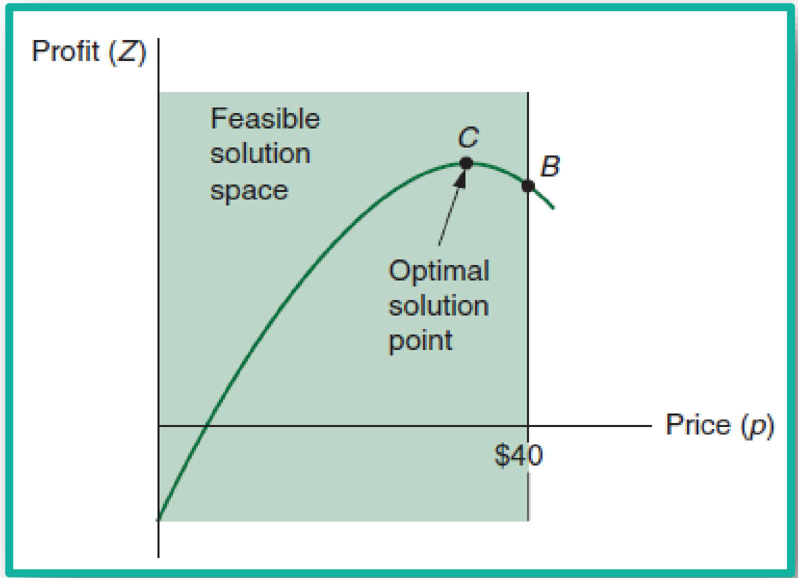


- Solution is for price ceiling of $40

Solving in Excel

- Q: What happens if you adjust the price ceiling to $20?
- Q: Is your answer consistent with what we have previously seen?





- Q: What happens if you completely drop the constraint?

The End

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