Lecture 15

Produced by Dr. Worldwide

Welcome to the 305

Goal Programming

- All prior linear programming problems have had a single objective function
- Companies may have multiple criteria in consideration for a decision
- Sometimes the multiple objectives conflict
- Company may want to maximize profit and minimize pollution
- Goal programming is linear programming for multiple objectives or criteria

Ex: Beaver Creek Pottery

- Trying to choose $x = number\ of\ bowls$ and y $= number\ of\ mugs$ to maximize the profit function

- Recall the original linear program

Maximize $40x + 50y$

Subject to:
$x + 2y \leq 40$ (Labor)
$4x + 3y \leq 120$ (Clay)
$x, y \geq 0$

- Objective function reflects a single goal

Ex: Beaver Creek Pottery

- Suppose Beaver Creek wanted to achieve other goals while maximizing profit
- In order of importance:
  - To avoid layoffs, they want to use at least 40 hours of labor per day
  - They want to achieve a satisfactory profit level of $1,600 per day
  - To avoid having clay dry out, they prefer to keep no more than 120 lb of clay on hand each day
  - To avoid overhead costs due to keeping the factory open past normal hours, they want to minimize the amount of overtime
- We reformulate our linear programming model using goal programming
- Transform linear programming model constraints into goals

Ex: Beaver Creek Pottery

- Goal 1: Avoid underutilization of labor
  - Original constraint $x + 2y \leq 40$
  - Reformulation to a goal constraint

$$x + 2y + d_1^- - d_1^+ = 40 \qquad \text{(Labor)}$$

  - Two new variables d_1^- and d_1^+ are nonnegative and represent the underutilized time and overtime, respectively
  - Q: What if the optimal solution had $d_1^- > 0$?
  - Q: What if the optimal solution had $d_1^+ > 0$?
  - The top priority corresponding to minimization of d_1^-

Minimize $\quad P_1 d_1^-$

  - The P_1 indicates the priority of this goal (not a coefficient)

Ex: Beaver Creek Pottery

- Goal 2: Achieve daily profit of \$1,600
  - Original objective function $Z = 40x + 50y$
  - Reformulation to a goal constraint

$$40x + 50y + d_2^- - d_2^+ = 1600 \qquad \text{(Profit)}$$

  - Two new variables d_2^- and d_2^+ are nonnegative and represent the amount of profit less than \$1,600 and more than \$1,600
  - The second priority corresponding to minimization of d_2^- is added

Minimize $\quad P_1 d_1^-, P_2 d_2^-$

  - The comma between the terms indicates that we are minimizing them sequentially, not simultaneously
  - Q: Why are we not minimizing d_2^+?

Ex: Beaver Creek Pottery

- Goal 3: Avoid waste of material
  - Original constraint $4x + 3y \leq 120$
  - Reformulation to a goal constraint

$$4x + 3y + d_3^- - d_3^+ = 120 \qquad \text{(Clay)}$$

  - Two new variables d_3^- and d_3^+ are nonnegative and represent the amount of clay less than 120 lbs and more than 120 lbs
  - The company cannot keep more than 120 lbs in storage
  - The third priority corresponds to minimization of d_3^+ is added

Minimize $\quad P_1 d_1^-, P_2 d_2^-, P_3 d_3^+$

Ex: Beaver Creek Pottery

- Goal 4: Avoid overtime costs
  - Recall the modified goal constraint for labor

$$x + 2y + d_1^- - d_1^+ = 40 \qquad \text{(Labor)}$$

  - Already attempting to minimize d_1^-
  - To ensure we don't exceed the maximum labor, we involve d_1^+
  - Finalization of objective function

Minimize $\quad P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$

Ex: Beaver Creek Pottery

- Full goal programming model

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$

Subject to

$$x + 2y + d_1^- - d_1^+ = 40 \quad \text{(Labor)}$$
$$40x + 50y + d_2^- - d_2^+ = 1600 \quad \text{(Profit)}$$
$$4x + 3y + d_3^- - d_3^+ = 120 \quad \text{(Clay)}$$
$$x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+ \geq 0$$

- The variables $\{d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+\}$ are called deviational variables
- We minimize the four different objective functions individually by priority

Ex: Beaver Creek Pottery

- Modification 1: Maximum of 10 hours of overtime
  - Recall the goal constraint for labor

$$x + 2y + d_1^- - d_1^+ = 40 \qquad \text{(Under hours)}$$

  - Remember that d_1^+ represents overtime
  - We want $0 \leq d_1^+ \leq 10$
  - Use same strategy as before by adding a goal constraint

$$d_1^+ + d_4^- - d_4^+ = 10 \qquad \text{(Over hours)}$$

  - Possible goal constraint of all deviational variables
  - Two new variables d_4^- and d_4^+ are nonnegative and represent the amount of overtime hours less than 10 hours and more than 10 hours
  - New objective function

Minimize $\quad P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+$

Ex: Beaver Creek Pottery

- Modification 2: Maximum number of bowls and mugs made daily
  - Pottery company has limited warehouse space
  - They can only produce at most 30 bowls and 20 mugs each day
  - Profit for bowls (\$40) less than profit for mugs (\$50)
  - Consider the new constraints

$$x + d_5^- = 30 \qquad \text{(Bowls)}$$
$$y + d_6^- = 20 \qquad \text{(Mugs)}$$

  - We want to minimize d_5^- and d_6^-
  - Q: Why not include positive deviational variables d_5^+ and d_6^+ ?
  - Q: For which item is it more important to achieve this goal?

Ex: Beaver Creek Pottery

- Modification 2: Maximum number of bowls and mugs made daily
  - Positive deviational variables are unnecessary since it is imperative to not exceed the warehouse space
  - We need to achieve the goal for mugs more than the goal for bowls because the profit is higher for mugs
  - If goals were of equal importance, we would minimize

Minimize $\quad P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+, P_5 d_5^- + P_5 d_6^-$

  - We can make the degree of importance in proportion to the profit
  - The goal for mugs is more important than the goal for bowls by a ratio of 5 to 4

Minimize $\quad P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+, 4P_5 d_5^- + 5P_5 d_6^-$

  - The coefficients 4 and 5 are referred to as weights

Ex: Beaver Creek Pottery

- Full modified goal programming model

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+, 4P_5 d_5^- + 5P_5 d_6^-$

Subject to

$$x + 2y + d_1^- - d_1^+ = 40 \quad \text{(Labor)}$$
$$40x + 50y + d_2^- - d_2^+ = 1600 \quad \text{(Profit)}$$
$$4x + 3y + d_3^- - d_3^+ = 120 \quad \text{(Clay)}$$
$$d_1^+ + d_4^- - d_4^+ = 10 \quad \text{(Overtime)}$$
$$x + d_5^- = 30 \quad \text{(Bowls)}$$
$$y + d_6^- = 20 \quad \text{(Mugs)}$$
$$x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0$$

Excel for Goal Programming

- Builds off linear programming using Excel Solver
- Solve the linear program multiple times with different objective functions
- Go in order of priority
- After finding the optimal solution, we add the optimal value attained in the first objective function as a new constraint and move on to the next objective function
- Possible that while solving for a given priority, we simultaneously optimize other lower ranked priorities

Ex: Beaver Creek Pottery

- Download GoalProgramming.xlsx from link Sheet 1 on course website

- See tab Priority 1 for minimization of d_1^-
  - Optimal solution

    $$x = 15 \quad y = 20 \quad d_1^+ = 15 \quad d_4^+ = 5 \quad d_5^- = 15$$
    $$d_1^-, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_6^- = 0$$

  - It is optimal to set $d_1^- = 0$
  - In our system of linear constraints, we have employees working at least 40 hr

- Move on to P2 for minimization of d_2^-
  - Notice from last solution $d_2^- = 0$
  - Optimal solution from P1 minimizes objective function from P2

- Unnecessary to consider P3 since $d_3^+ = 0$ under optimal solution of P1

Ex: Beaver Creek Pottery

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- See tab Priority 4 for minimization of d_4^+
  - To ensure none of the optimal values achieved thus far change when we attempt to minimize d_4^+, we add the values attained as constraints
  - We add one constraint for each goal we have already attained

Minimize $\quad d_4^+$

Subject to

$$
\begin{aligned}
x + 2y + d_1^- - d_1^+ &= 40 && \text{(Labor)}\\
40x + 50y + d_2^- - d_2^+ &= 1600 && \text{(Profit)}\\
4x + 3y + d_3^- - d_3^+ &= 120 && \text{(Clay)}\\
d_1^+ + d_4^- - d_4^+ &= 10 && \text{(Overtime)}\\
x + d_5^- &= 30 && \text{(Bowls)}\\
y + d_6^- &= 20 && \text{(Mugs)}\\
d_1^-, d_2^-, d_3^+ &= 0\\
x, y, d_1^+, d_2^+, d_3^-, d_4^-, d_4^+, d_5^-, d_6^- &\geq 0
\end{aligned}
$$

Ex: Beaver Creek Pottery

- See tab Priority 4 for minimization of d_4^+
  - Optimal solution

    $$x = 15 \quad y = 20 \quad d_1^+ = 15 \quad d_4^+ = 5 \quad d_5^- = 15$$
    $$d_1^-, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_6^- = 0$$

  - Solution did not change and $d_4^+ = 5$ in both cases
  - Not possible to reduce the value of d_4^+ without violating the optimal solutions for the three goals that have higher priority
  - This indicates that the overtime must be exceed by 5 hours to fulfill other constraints from higher priority goals

Ex: Beaver Creek Pottery

- See tab Priority 5 for minimization of $4d_5^- + 5d_6^-$
  - Add result from previous priority rank as a constraint

Minimize	$4d_5^- + 5d_6^-$	
Subject to	$x + 2y + d_1^- - d_1^+ = 40$	(Labor)
	$40x + 50y + d_2^- - d_2^+ = 1600$	(Profit)
	$4x + 3y + d_3^- - d_3^+ = 120$	(Clay)
	$d_1^+ + d_4^- - d_4^+ = 10$	(Overtime)
	$x + d_5^- = 30$	(Bowls)
	$y + d_6^- = 20$	(Mugs)
	$d_1^-, d_2^-, d_3^+ = 0$	(New Constraints)
	$d_4^+ = 5$	(New Constraints)
	$x, y, d_1^+, d_2^+, d_3^-, d_4^-, d_5^-, d_6^- \geq 0$	

Ex: Beaver Creek Pottery

- See tab Priority 5 for minimization of $4d_5^- + 5d_6^-$
  - Optimal solution

    $x = 15 \quad y = 20 \quad d_1^+ = 15 \quad d_4^+ = 5 \quad d_5^- = 15$
    $d_1^-, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_6^- = 0$

  - Solution still did not change
  - Optimal solution stays optimal

- Final Solution
  - Produce 15 bowls and 20 mugs
  - Hours of work: $15 + 2(20) = 55$ (Over by 15 hours)
  - Profit: $40(15) + 50(20) = 1600$
  - Pounds of clay: $4(15) + 3(20) = 120$
  - Overtime beyond 10 hours: $d_4^+ = 5$
  - Slack for bowls below 30: $d_5^- = 15$
  - Slack for mugs below 20: $d_4^+ = 0$

Ex: Beaver Creek Pottery

- Full modified goal programming model

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+, 4P_5 d_5^- + 5P_5 d_6^-$

Subject to

$x + 2y + d_1^- - d_1^+ = 40$ (Labor)

$40x + 50y + d_2^- - d_2^+ = 1600$ (Profit)

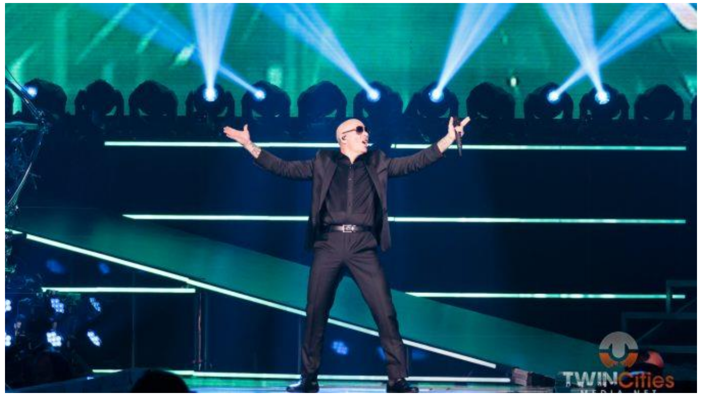
$4x + 3y + d_3^- - d_3^+ = 120$ (Clay)

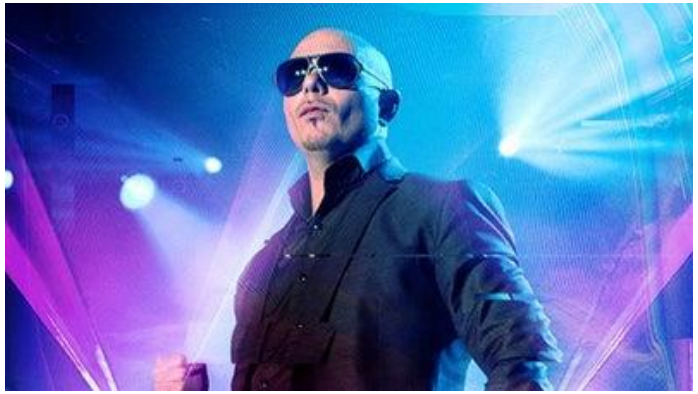
$d_1^+ + d_4^- - d_4^+ = 10$ (Overtime)

$x + d_5^- = 30$ (Bowls)

$y + d_6^- = 20$ (Mugs)

$x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0$





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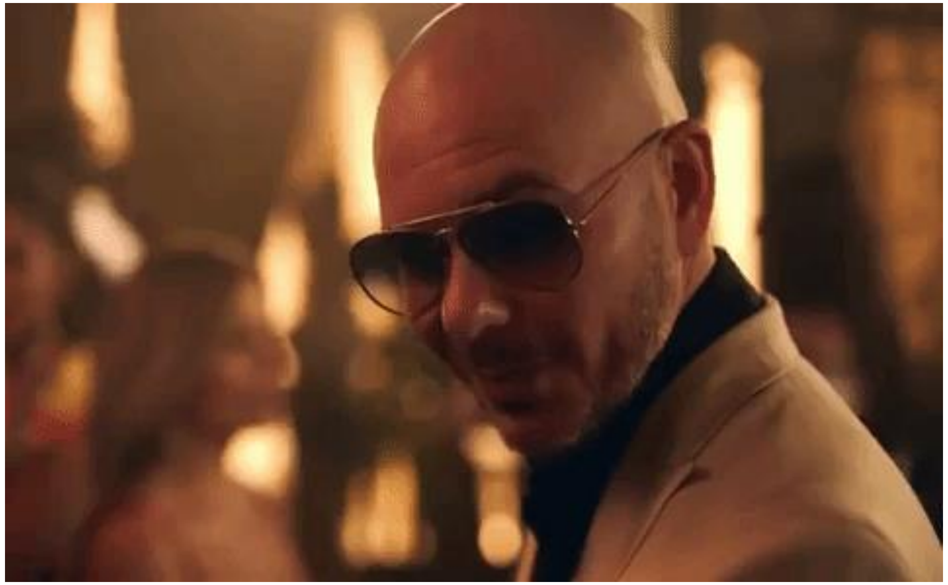

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