



Lecture 15

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Welcome to the 305

Goal Programming



- All prior linear programming problems have had a **single** objective function
- Companies may have **multiple criteria** in consideration for a decision
- Sometimes the multiple objectives conflict
- Company may want to maximize profit and minimize pollution
- **Goal programming** is linear programming for multiple objectives or criteria



Ex: Beaver Creek Pottery



- Trying to choose $x = \text{number of bowls}$ and $y = \text{number of mugs}$ to maximize the profit function

- Recall the original linear program

Maximize $40x + 50y$

Subject to: $x + 2y \leq 40$ (Labor)

$4x + 3y \leq 120$ (Clay)

$x, y \geq 0$

- Objective function reflects a single goal

Ex: Beaver Creek Pottery



- Suppose Beaver Creek wanted to achieve other goals while maximizing profit
- In **order of importance**:
 - To avoid layoffs, they want to use at least 40 hours of labor per day
 - They want to achieve a satisfactory profit level of \$1,600 per day
 - To avoid having clay dry out, they prefer to keep no more than 120 lb of clay on hand each day
 - To avoid overhead costs due to keeping the factory open past normal hours, they want to minimize the amount of overtime
- We reformulate our linear programming model using goal programming
- Transform linear programming model constraints into **goals**

Ex: Beaver Creek Pottery



- Goal 1: Avoid **underutilization** of labor
 - Original constraint $x + 2y \leq 40$
 - Reformulation to a **goal constraint**

$$x + 2y + d_1^- - d_1^+ = 40 \quad (\text{Labor})$$

- Two new variables d_1^- and d_1^+ are nonnegative and represent the **underutilized time** and **overtime**, respectively
- Q: What if the optimal solution had $d_1^- > 0$?
- Q: What if the optimal solution had $d_1^+ > 0$?
- The top priority corresponding to minimization of d_1^-

Minimize $P_1 d_1^-$

- The P_1 indicates the priority of this goal (not a coefficient)

Ex: Beaver Creek Pottery



- Goal 2: Achieve daily profit of \$1,600
 - Original objective function $Z = 40x + 50y$
 - Reformulation to a **goal constraint**

$$40x + 50y + d_2^- - d_2^+ = 1600 \quad (\text{Profit})$$

- Two new variables d_2^- and d_2^+ are nonnegative and represent the amount of profit **less than** \$1,600 and **more than** \$1,600
- The second priority corresponding to minimization of d_2^- is added

Minimize $P_1 d_1^-, P_2 d_2^-$

- The comma between the terms indicates that we are minimizing them **sequentially**, not simultaneously
- Q: Why are we not minimizing d_2^+ ?

Ex: Beaver Creek Pottery



- Goal 3: Avoid **waste** of material
 - Original constraint $4x + 3y \leq 120$
 - Reformulation to a **goal constraint**

$$4x + 3y + d_3^- - d_3^+ = 120 \quad (\text{Clay})$$

- Two new variables d_3^- and d_3^+ are nonnegative and represent the amount of clay **less than** 120 lbs and **more than** 120 lbs
- The company cannot keep **more than** 120 lbs in storage
- The third priority corresponds to minimization of d_3^+ is added

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+$

Ex: Beaver Creek Pottery



- Goal 4: Avoid overtime costs
 - Recall the modified goal constraint for labor

$$x + 2y + d_1^- - d_1^+ = 40 \quad (\text{Labor})$$

- Already attempting to minimize d_1^-
- To ensure we don't exceed the maximum labor, we involve d_1^+
- Finalization of objective function

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$

Ex: Beaver Creek Pottery



- Full goal programming model

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$

Subject to

$$\begin{aligned} x + 2y + d_1^- - d_1^+ &= 40 && \text{(Labor)} \\ 40x + 50y + d_2^- - d_2^+ &= 1600 && \text{(Profit)} \\ 4x + 3y + d_3^- - d_3^+ &= 120 && \text{(Clay)} \\ x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+ &\geq 0 \end{aligned}$$

- The variables $\{d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+\}$ are called **deviation** variables
- We minimize the four different objective functions **individually** by **priority**

Ex: Beaver Creek Pottery



- Modification 1: Maximum of 10 hours of overtime

- Recall the goal constraint for labor

$$x + 2y + d_1^- - d_1^+ = 40 \quad (\text{Under hours})$$

- Remember that d_1^+ represents overtime
- We want $0 \leq d_1^+ \leq 10$
- Use same strategy as before by adding a goal constraint

$$d_1^+ + d_4^- - d_4^+ = 10 \quad (\text{Over hours})$$

- Possible goal constraint of all **deviation** variables
- Two new variables d_4^- and d_4^+ are nonnegative and represent the amount of overtime hours **less than** 10 hours and **more than** 10 hours
- New objective function

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+$

Ex: Beaver Creek Pottery



- Modification 2: Maximum number of bowls and mugs made daily
 - Pottery company has **limited warehouse** space
 - They can only produce at most 30 bowls and 20 mugs each day
 - Profit for bowls (\$40) less than profit for mugs (\$50)
 - Consider the new constraints

$$x + d_5^- = 30 \quad (\text{Bowls})$$

$$y + d_6^- = 20 \quad (\text{Mugs})$$

- We want to minimize d_5^- and d_6^-
- Q: Why not include positive deviational variables d_5^+ and d_6^+ ?
- Q: For which item is it more important to achieve this goal?

Ex: Beaver Creek Pottery



- Modification 2: Maximum number of bowls and mugs made daily
 - Positive deviational variables are unnecessary since it is imperative to not exceed the warehouse space
 - We need to achieve the goal for mugs more than the goal for bowls because the profit is higher for mugs
 - If goals were of equal importance, we would minimize

Minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, P_5d_5^- + P_5d_6^-$

- We can make the **degree of importance** in proportion to the profit
- The goal for mugs is more important than the goal for bowls by a **ratio** of 5 to 4

Minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, 4P_5d_5^- + 5P_5d_6^-$

- The coefficients 4 and 5 are referred to as **weights**

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- Full modified goal programming model

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+, 4P_5 d_5^- + 5P_5 d_6^-$

Subject to

$x + 2y + d_1^- - d_1^+ = 40$	(Labor)
$40x + 50y + d_2^- - d_2^+ = 1600$	(Profit)
$4x + 3y + d_3^- - d_3^+ = 120$	(Clay)
$d_1^+ + d_4^- - d_4^+ = 10$	(Overtime)
$x + d_5^- = 30$	(Bowls)
$y + d_6^- = 20$	(Mugs)
$x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0$	

Excel for Goal Programming



- Builds off linear programming using Excel Solver
- Solve the linear program multiple times with different objective functions
- Go in order of priority
- After finding the optimal solution, we add the optimal value attained in the first objective function as a new constraint and move on to the next objective function
- Possible that while solving for a given priority, we simultaneously optimize other lower ranked priorities



Ex: Beaver Creek Pottery



- Download [GoalProgramming.xlsx](#) from link [Sheet 1](#) on course website
- See tab [Priority 1](#) for minimization of d_1^-
 - Optimal solution
$$x = 15 \quad y = 20 \quad d_1^+ = 15 \quad d_4^+ = 5 \quad d_5^- = 15$$
$$d_1^-, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_6^- = 0$$
 - It is optimal to set $d_1^- = 0$
 - In our system of linear constraints, we have employees working at least 40 hr
- Move on to P2 for minimization of d_2^-
 - Notice from last solution $d_2^- = 0$
 - Optimal solution from P1 minimizes objective function from P2
- Unnecessary to consider P3 since $d_3^+ = 0$ under optimal solution of P1

Ex: Beaver Creek Pottery



- See tab **Priority 4** for minimization of d_4^+
 - To ensure none of the optimal values achieved thus far change when we attempt to minimize d_4^+ , we add the values attained as constraints
 - We add one constraint for each goal we have already attained

Minimize d_4^+

Subject to

$$\begin{aligned}x + 2y + d_1^- - d_1^+ &= 40 && \text{(Labor)} \\40x + 50y + d_2^- - d_2^+ &= 1600 && \text{(Profit)} \\4x + 3y + d_3^- - d_3^+ &= 120 && \text{(Clay)} \\d_1^+ + d_4^- - d_4^+ &= 10 && \text{(Overtime)} \\x + d_5^- &= 30 && \text{(Bowls)} \\y + d_6^- &= 20 && \text{(Mugs)} \\d_1^-, d_2^-, d_3^+ &= 0 \\x, y, d_1^+, d_2^+, d_3^-, d_4^-, d_4^+, d_5^-, d_6^- &\geq 0\end{aligned}$$

Ex: Beaver Creek Pottery



- See tab **Priority 4** for minimization of d_4^+

- Optimal solution

$$x = 15 \quad y = 20 \quad d_1^+ = 15 \quad d_4^+ = 5 \quad d_5^- = 15 \\ d_1^-, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_6^- = 0$$

- Solution did not change and $d_4^+ = 5$ in both cases
- Not possible to reduce the value of d_4^+ without violating the optimal solutions for the three goals that have higher priority
- This indicates that the overtime must be exceed by 5 hours to fulfill other constraints from higher priority goals

Ex: Beaver Creek Pottery



- See tab **Priority 5** for minimization of $4d_5^- + 5d_6^-$
 - Add result from previous priority rank as a constraint

Minimize $4d_5^- + 5d_6^-$

Subject to

$$\begin{aligned}x + 2y + d_1^- - d_1^+ &= 40 && \text{(Labor)} \\40x + 50y + d_2^- - d_2^+ &= 1600 && \text{(Profit)} \\4x + 3y + d_3^- - d_3^+ &= 120 && \text{(Clay)} \\d_1^+ + d_4^- - d_4^+ &= 10 && \text{(Overtime)} \\x + d_5^- &= 30 && \text{(Bowls)} \\y + d_6^- &= 20 && \text{(Mugs)} \\d_1^-, d_2^-, d_3^+ &= 0 && \text{(New Constraints)} \\d_4^+ &= 5 && \text{(New Constraints)} \\x, y, d_1^+, d_2^+, d_3^-, d_4^-, d_5^-, d_6^- &\geq 0\end{aligned}$$

Ex: Beaver Creek Pottery



- See tab **Priority 5** for minimization of $4d_5^- + 5d_6^-$

- Optimal solution

$$x = 15 \quad y = 20 \quad d_1^+ = 15 \quad d_4^+ = 5 \quad d_5^- = 15 \\ d_1^-, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_6^- = 0$$

- Solution still did not change
 - Optimal solution stays optimal
- Final Solution
 - Produce 15 bowls and 20 mugs
 - Hours of work: $15 + 2(20) = 55$ (**Over by 15 hours**)
 - Profit: $40(15) + 50(20) = 1600$
 - Pounds of clay: $4(15) + 3(20) = 120$
 - Overtime beyond 10 hours: $d_4^+ = 5$
 - Slack for bowls below 30: $d_5^- = 15$
 - Slack for mugs below 20: $d_4^+ = 0$

Ex: Beaver Creek Pottery



- Full modified goal programming model

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+, 4P_5 d_5^- + 5P_5 d_6^-$

Subject to

$x + 2y + d_1^- - d_1^+ = 40$	(Labor)
$40x + 50y + d_2^- - d_2^+ = 1600$	(Profit)
$4x + 3y + d_3^- - d_3^+ = 120$	(Clay)
$d_1^+ + d_4^- - d_4^+ = 10$	(Overtime)
$x + d_5^- = 30$	(Bowls)
$y + d_6^- = 20$	(Mugs)

$$x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0$$



The End



Dale

