Lecture 21

Produced by Dr. Worldwide

Welcome to the 305

Multiple Maxima/Minima

- When solving non-linear problems, it is important to consider the possibility that there may be multiple local solutions (maxima/minima)
- There is no method that guarantees we find all such points
- In problems not guaranteed to have a unique interior optimal solution, it is a good idea to run the GRG algorithm starting at several initial points
- Consider the following nonlinear problem

Maximize	$f(x) = 1 + x + \sqrt{x}\sin(2x)$
Subject to	$0 \leq x \leq 9$

Multiple Maxima/Minima

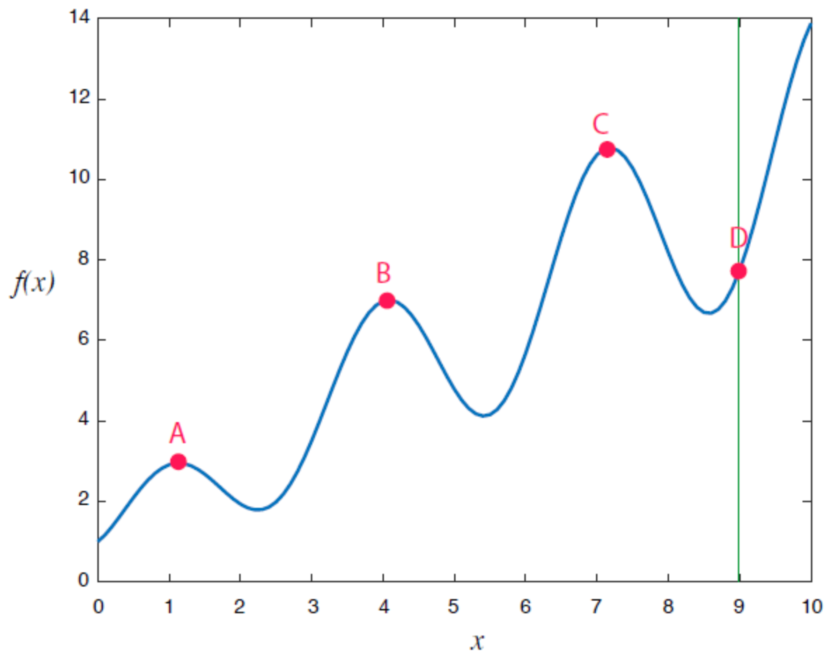
- Consider the graph
  $f(x) = 1 + x + \sqrt{x}\sin(2x)$

- Four different local maxima

- Q: What is the answer to our problem?

Maximize $\quad f(x) = 1 + x + \sqrt{x}\sin(2x)$

Subject to $\quad 0 \leq x \leq 9$



Multiple Maxima/Minima

- Download MultipleMaxima.xlsx from link Sheet 1 on course website
- Consider the following part of the spreadsheet

26	Starting Value	Objective Function
27	0	1

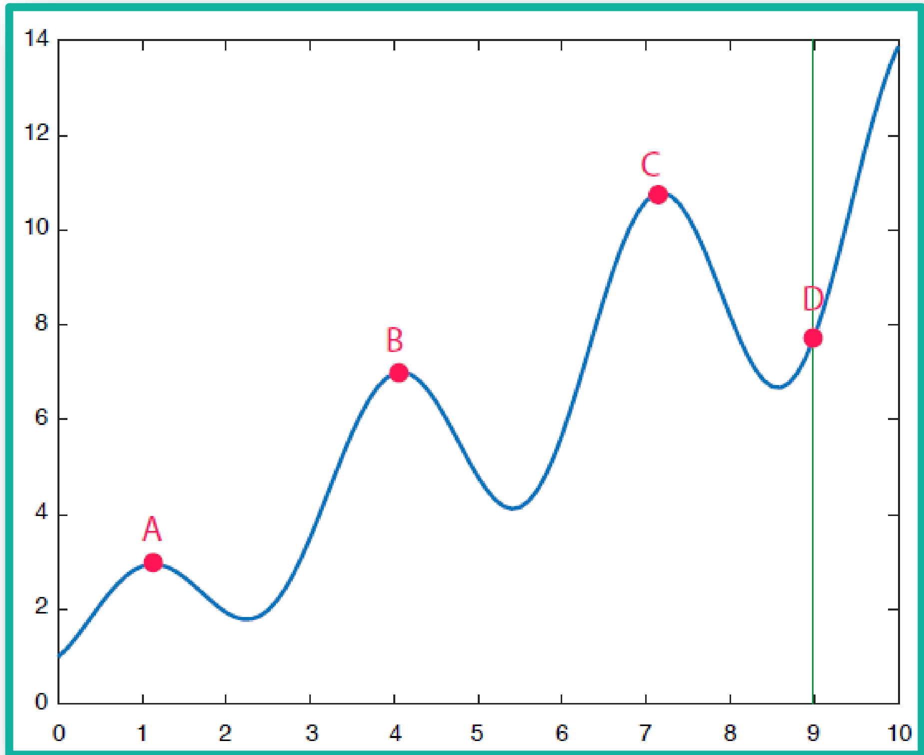
- Run solver with four different starting values

$x = 0, \qquad x = 4, \qquad x = 8, \qquad x = 9$

- Q: Do all four starting values lead to the same solution?

Multiple Maxima/Minima

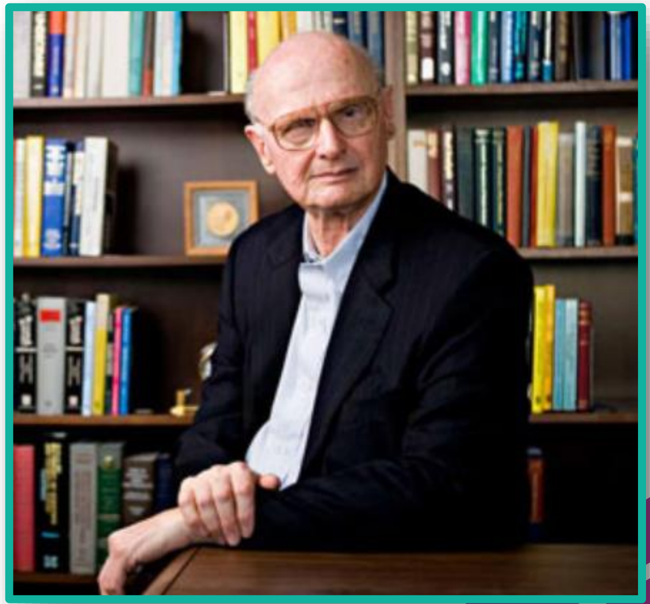
- Optimal solution under all initial values
- Q: Do the answers make sense?



Starting Values	Optimal Solution	Maxima
0	1.13	2.95
4	4.08	7.01
8	7.18	10.79
9	9	7.75

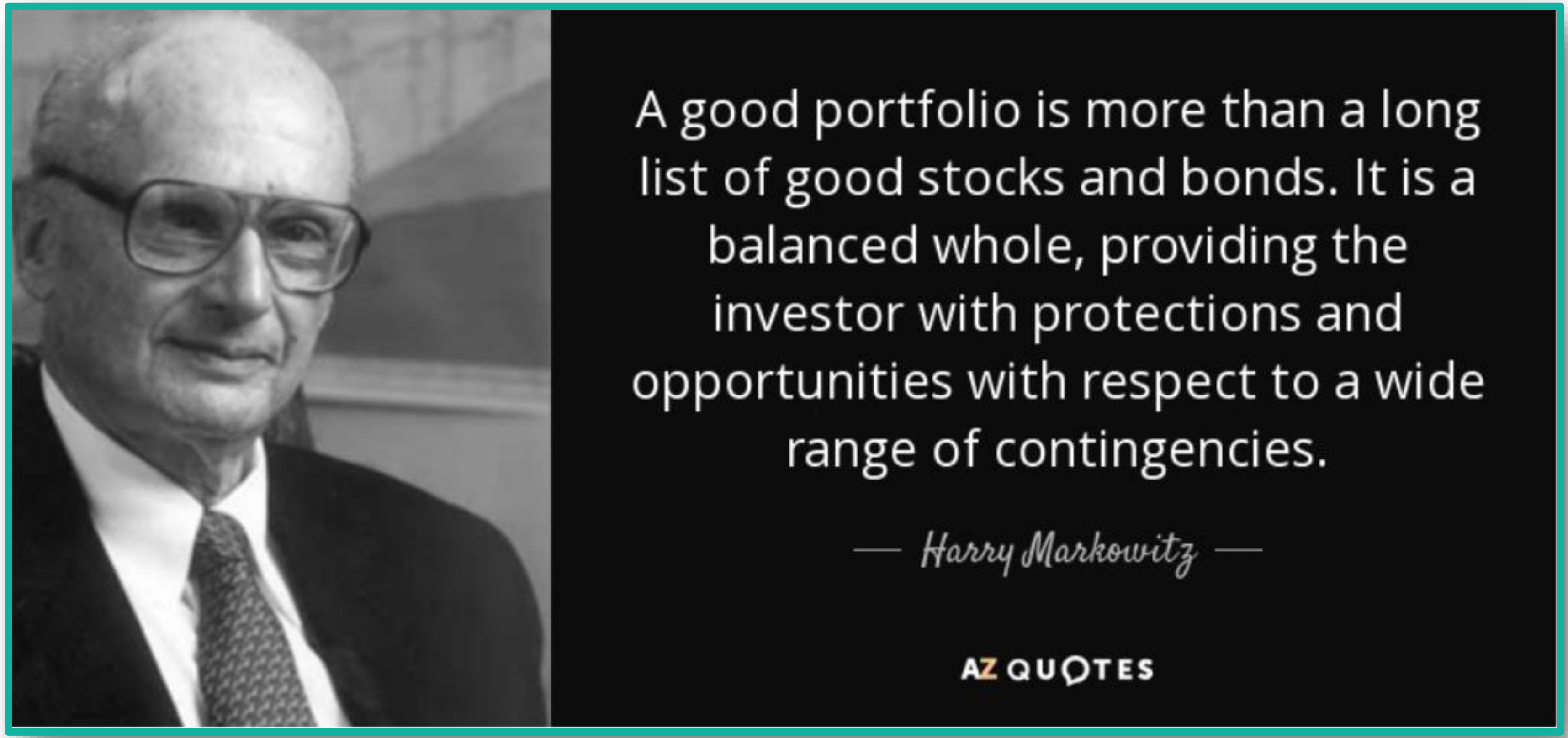
Ex: Investment Portfolio

- An investor can choose among *n* different investment opportunities
- An investment portfolio is a selection of how much to invest in each option
- Popular model for portfolios is the Markowitz model
  - Minimize risk (variance of the portfolio)
  - Maximize return on investment
- Different investments are assumed to be correlated
  - Positively correlated
  - Negatively correlated
- Diversification protects against these correlations



Ex: Investment Portfolio

- Dope quote from Harry Markowitz



Ex: Investment Portfolio

- Let x_i denote the proportion of money invested in option $i \in \{1,2,\cdots,n\}$
- Let σ_i^2 denote the variance of investment option $i \in \{1,2,\cdots,n\}$
- Let ρ_{ij} denote the correlation between investment option $i \in \{1,2,\cdots,n\}$ and investment option j $\in \{1,2,\cdots,n\}$ where i $\neq$ j
- The variance of the portfolio is given by

$$S = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + \cdots + x_n^2\sigma_n^2 + \sum_{i=1}^{n} \sum_{1\leq j\leq n, j\neq i} x_i x_j \rho_{ij}\sigma_i\sigma_j$$

$$= (x_1, x_2, \ldots, x_n) \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \ldots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \ldots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & \vdots & \ldots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \rho_{n2}\sigma_n\sigma_2 & \ldots & \sigma_n^2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Ex: Investment Portfolio

- Let r_i denote the expected return on investment of option $i \in \{1,2,\cdots,n\}$
- Expected return on investment from the portfolio is given by

$$R = r_1x_1 + r_2x_2 + \cdots + r_nx_n = (r_1, r_2, \ldots, r_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- Vector/Matrix notation
  - $\boldsymbol{x}' = [x_1, x_2, \cdots, x_n]$ (Vector of decision variables of portfolio)
  - $\boldsymbol{r}' = [r_1, r_2, \cdots, r_n]$ (Vector of expected returns)
  - $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \ldots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \ldots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & \vdots & \ldots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \rho_{n2}\sigma_n\sigma_2 & \ldots & \sigma_n^2 \end{bmatrix}$ (Variance/covariance matrix)

Ex: Investment Portfolio

- The nonlinear program we desire to solve

Minimize	$\boldsymbol{x}'\boldsymbol{\Sigma}\mathbf{x}$
Subject to	$\boldsymbol{r}'\mathbf{x} \geq r_m$
	$x_1 + x_2 + \cdots + x_n = 1$
	$x_i \geq 0$

- Objective function is nonlinear and quadratic
- Q: What are the units of the different values $x_1, x_2, \cdots, x_n$?
- Q: What does r_m represent in this linear program?

Ex: Investment Portfolio

- Suppose an investor wants to build a portfolio from the following stocks:

Stock (x_i)	Annual return (r_i)	Variance
1. Altacam	.08	.009
2. Bestco	.09	.015
3. Com.com	.16	.040
4. Delphi	.12	.023

- Consider the correlation matrix of the stocks

$$\begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & 1 & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & 1 & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & 1 \end{bmatrix} = \begin{bmatrix} 1 & .4 & .3 & .6 \\ .4 & 1 & .2 & .7 \\ .3 & .2 & 1 & .4 \\ .6 & .7 & .4 & 1 \end{bmatrix}$$

Ex: Investment Portfolio

- The covariance matrix can be computed as follows:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{14}\sigma_1\sigma_4 \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_4 \\ \rho_{31}\sigma_3\sigma_1 & \rho_{32}\sigma_3\sigma_2 & \sigma_3^2 & \rho_{34}\sigma_3\sigma_4 \\ \rho_{41}\sigma_4\sigma_1 & \rho_{42}\sigma_4\sigma_2 & \rho_{43}\sigma_4\sigma_3 & \sigma_4^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.009 & 0.00464758 & 0.0056921 & 0.008632497 \\ 0.00464758 & 0.015 & 0.004898979 & 0.013001923 \\ 0.0056921 & 0.004898979 & 0.04 & 0.012132601 \\ 0.008632497 & 0.013001923 & 0.012132601 & 0.023 \end{bmatrix}$$

- The investor wants a total annual return of at least 0.11 (11%)

- Download Markowitz.xlsx from link Sheet 2 on course website

Ex: Investment Portfolio

- Notice the formula for the objective function

21	Computing the portfolio variance:	
22	x'*Sigma*x =	0
23		
24	Portfolio variance:	0

= SUMPRODUCT(B4:B7,MMULT(A16:D19,B4:B7))

- Try the alternative approach

  = MMULT(TRANSPOSE(B4:B7),MMULT(A16:D19,B4:B7))

- Examine what the constraints look like in Solver

Subject to the Constraints:

B25 >= 0.11
B26 = 1

Ex: Admissions at State

- State has increased its tuition for all students in each of the last 5 years
- University administration always thought the number of applications received was independent of tuition
- Drops in applications and enrollment prove this idea to be wrong
- University admissions officials developed the following relationships between the number of applicants (x_i) and cost of tuition (t_i)

$x_1 = 21000 - 12t_1$ (Relationship for in-state applicants)
$x_2 = 35000 - 6t_2$ (Relationship for out-of-state applicants)

- University desires to develop a planning model to indicate the in-state and out-of-state tuitions, as well as, the number of students that could be expected to enroll in the freshman class

Ex: Admissions at State

- Constraints based on resources
  - Not enough classroom space for more than 1,400 students
  - Needs at least 700 freshmen to meet all its class size objectives
  - At most 800 dorm rooms available for freshmen
- Historical expectations
  - 55% of all in-state freshmen desire to live in dorms
  - 72% of all out-of-state freshmen desire to live in dorms
- Uphold the academic standards of the institution
  - Average SAT is 960 for in-state students
  - Average SAT is 1150 for out-of-state students
  - University wants the entering freshmen to average 1,000

Ex: Admissions at State

- Legislative requirements
  - State is supported by the state LOL ☺
  - The legislature wants to make sure that State doesn't just admit out-of-state students because they pay more money or have better SAT scores
  - Policy that no more than 55% of the entering freshman can be out-of-state students

- Q: How much should State charge, what would the total tuition be, and how many in-state and out-of-state students should they expect?

- Decision variables
  - We have a choice between x_1 and x_2 or t_1 and t_2
  - Related through the following equations

$$x_1 = 21000 - 12t_1$$
$$x_2 = 35000 - 6t_2$$

Ex: Admissions at State

- Objective function
  - Goal is to maximize the revenue in tuition
  - Total tuition based off in-state and out-of-state students

$$x_1 t_1 + x_2 t_2 = x_1 \times \frac{(21000 - x_1)}{12} + \mathrm{x}_2 \times \frac{(35000 - x_2)}{6}$$

- Constraints
  - Maximum number of freshmen

    $x_1 + x_2 \leq 1400$

  - Minimum number of freshmen

    $x_1 + x_2 \geq 700$

  - Maximum number of dormitories

    $0.55x_1 + 0.72x_2 \leq 800$

Ex: Admissions at State

- Constraints
  - Average SAT Scores

$$\frac{960x_1 + 1150x_2}{x_1 + x_2} \geq 1000$$

$$(960 - 1000)x_1 + (1150 - 1000)x_2 = -40x_1 + 150x_2 \geq 0$$

  - Maximum for out-of-state students

$$\frac{x_2}{x_1 + x_2} \leq 0.55$$

$$-0.55x_1 + (1 - 0.55)x_2 = -0.55x_1 + 0.45x_2 \leq 0$$

Ex: Admissions at State

- Nonlinear program

Maximize $\quad x_1 \times \frac{(21000-x_1)}{12} + \mathrm{x}_2 \times \frac{(35000-x_2)}{6}$

Subject to

$$x_1 + x_2 \leq 1400$$
$$-x_1 - x_2 \leq -700$$
$$0.55x_1 + 0.72x_2 \leq 800$$
$$40x_1 - 150x_2 \leq 0$$
$$-0.55x_1 + 0.45x_2 \leq 0$$
$$x_1, x_2 \geq 0$$

- Download Admissions.xlsx from link Sheet 3 on course website

Ex: Admissions at State

- Use Excel to find the optimal solution is $x_1 = 559.44$ and $x_2 = 683.76$

- Consequently, we find the optimal tuition for in-state and out-of-state students

$$t_1 = \frac{21000 - x_1}{12} = \frac{21000 - 559.44}{12} = \$1{,}703.38$$

$$t_2 = \frac{35000 - x_2}{6} = \frac{35000 - 683.76}{6} = \$5{,}719.37$$

- Expected total tuition from freshmen is $x_1 t_1 + x_2 t_2 = \$4{,}863{,}622.37$

Ex: Admissions at State

- Solve again in Excel and save sensitivity report

Constraints

Cell	Name	Final Value	Lagrange Multiplier
D5	Max. students Computed	1243.201243	0
D6	Min. students Computed	-1243.201243	0
D7	Dormitory capacity Computed	800	5949.524432
D8	SAT average Computed	-80186.48019	0
D9	Max. out-of-state Computed	0	2937.233899

- Q: What does it mean that when the Lagrange multiplier is 0?

The End

Dale