Two-Way ANOVA

READING:	6.3 – 6.5 7.3-7.6
EXERCISES:	CH 6. 38 CH 7. 34
ASSIGNED:	NONE
PRODUCER:	DR. MARIO

Example: Grades on Different Exams

- Data

	Barb	Betsy	Bill	Bob	Bud	Mean
Exam #1:	62	94	68	86	50	72
Exam #2:	87	95	93	97	63	87
Exam #3:	74	86	82	70	28	68
Exam #4:	77	89	73	79	47	73
Mean	75	91	79	83	47	75

- Using One-Way ANOVA:
  - No Significant Differences Between Exams
  - Significant Differences Between Students

- Question: *Can we use **both** factors to explain variability in scores?*

Simple Block Design

- **Simple Block Design** has two factors with exactly one observation in each combination of factors

- Example: Examine Effect of Different Treatments at Different Severities
  - Factor A (Treatments) has *I* Levels
  - Factor B (Severity) has *J* Levels
  - Sample Size is $n = I * J$ in a Simple Block Design

- Question: *What is the problem of this design?*

Two-Way ANOVA

- Means Version

Treatment: $i \in \{1,2,\cdots,I\}$
Block: $j \in \{1,2,\cdots,J\}$

$$Y = \mu_{ij} + \epsilon$$

Mean of *Y* for Treatment *i* and Block *j*

- Effects Version (Additive)

$$Y = \mu + \alpha_i + \beta_j + \epsilon$$

Same Assumptions About Error Term

Grand Mean

Effect of Block *j*

Effect of Treatment *i*

Two-Way ANOVA

- Model

$$Y = \mu + \alpha_i + \beta_j + \epsilon$$

Factor A: $i \in \{1,2,\cdots,I\}$
Factor B: $j \in \{1,2,\cdots,J\}$

- Estimation of Model

Parameter	Estimate
μ	$\bar{y}$
α_i	$\bar{y}_i - \bar{y}$
β_j	$\bar{y}_j - \bar{y}$
σ_ϵ	$\sqrt{MSE}$

Group Means for A: $\bar{y}_i$
Group Means for B: $\bar{y}_j$

Two-Way ANOVA

- Partition of Variation

$$SST = \sum_i \sum_j \left(y_{ij} - \bar{y}\right)^2 = (n-1)s^2$$

$$SSA = \sum_i \sum_j (\bar{y}_i - \bar{y})^2 = \sum_i J(\bar{y}_i - \bar{y})^2$$

$$SSB = \sum_i \sum_j \left(\bar{y}_j - \bar{y}\right)^2 = \sum_j I\left(\bar{y}_j - \bar{y}\right)^2$$

$$SSE = SST - SSA - SSB$$

Two-Way ANOVA

- ANOVA Table (Assume One Observation Per Combination)

Source	d.f.	Sum of Squares	Mean Square	F	P-value
Factor A	$I-1$	SSA	$\frac{SSA}{I-1}$	$\frac{MSA}{MSE}$	$F_{I-1,(I-1)(J-1)}$
Factor B	$J-1$	SSB	$\frac{SSB}{J-1}$	$\frac{MSB}{MSE}$	$F_{J-1,(I-1)(J-1)}$
Residual	$(I-1)(J-1)$	SSE	$\frac{SSE}{(I-1)(J-1)}$		
Total	$IJ-1$	$SSTotal$			

Two-Way ANOVA

- Hypothesis Test for Factor A
  - $H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_I = 0$
  - $H_a: \alpha_i \neq 0$ for some i

- Hypothesis Test for Factor B
  - $H_0: \beta_1 = \beta_2 = \cdots = \beta_J = 0$
  - $H_a: \beta_j \neq 0$ for some j

Supplement for Lecture 26

- Fit One-Way ANOVA Models
- Two-Way ANOVA Model

$$Y = \mu + \alpha_i + \beta_j + \epsilon$$

Factor A (Exam) $i \in \{1,2,3,4\}$
Factor B (Student) j $\in \{Barb, Betsy, Bill, Bob, Bud\}$

- Compare to Results from One-Way ANOVA Models

- Confidence Intervals for All Pairwise Comparisons (Tukey HSD)

Interaction Effect

- An **Interaction Effect** Occurs When a Significant Difference is Present at a Specific **Combination** of Factors

$$Y = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon$$

- Example: Readability of Colored Text on Colored Background
  - Yellow Text Hurts Readability ($\alpha_{\mathrm{y}} < 0$)
  - Black Background Hurts Readability ($\beta_{\mathrm{b}} < 0$)
  - Yellow Text on Black Background Helps Readability ($\gamma_{\mathrm{yb}} > 0$)

Interaction Effect

- Book: ***Interaction*** *is a* ***difference*** *of* ***differences.***

- Example:

	White Background	Black Background	Diff
Yellow Text	10	70	**-60**
Gray Text	40	50	**-10**
Diff	**-30**	**20**	

Different Differences

Different Differences

Interaction Effect

- Estimation of Interaction Requires Multiple Observations Per Combo

$$n_{ij} = \#\ of\ Observations\ in\ (i,j)^{th}\ cell$$

- Balanced: $n_{ij} = c$ (Constant for all cells)
  - Randomized Complete Block Design: $c = 1$
  - Factorial Design: $c > 1$

- Use Interaction Plot to Inspect Presence of an Interaction Effect
  - Plot Means of Each Cell Versus One of the Factors With Different Lines for the Other Factor
  - Called Cell Means Plot

Interaction Effect

- ANOVA Table

Source	d.f.	Sum of Squares	Mean Square	F	P-value
Factor A	$I-1$	SSA	$\frac{SSA}{I-1}$	$\frac{MSA}{MSE}$	$F_{I-1,IJ(c-1)}$
Factor B	$J-1$	SSB	$\frac{SSB}{J-1}$	$\frac{MSB}{MSE}$	$F_{J-1,IJ(c-1)}$
A * B	$(I-1)(J-1)$	$SSAB$	$\frac{SSAB}{(I-1)(J-1)}$	$\frac{MSAB}{MSE}$	$F_{(I-1)(J-1),IJ(c-1)}$
Residual	$IJ(c-1)$	SSE	$\frac{SSE}{IJ(c-1)}$		
Total	$n-1$	$SSTotal$			

Interaction Effect

- New Hypothesis Test for Interaction
  - $H_0: \gamma_{11} = \gamma_{12} = \gamma_{21} = \cdots = \gamma_{IJ} = 0$
  - $H_a: \gamma_{ij} \neq 0$ for some i, j

- Similar to the Addition of Interaction Effect in Linear Regression

Example: Glue

- Data Organized in Grid
  - A: Thickness (*thin, moderate, heavy)*
  - B: Glue Type (*plastic, wood)*
  - Y: Force Required to Separate (in Newtons)

DATA	Plastic	Wood
Thin	**52 64**	**72 60**
Moderate	**67 55**	**78 68**
Heavy	**86 72**	**43 51**

$$I = 3$$
$$J = 2$$
$$c = 2$$
$$n = I * J * c = 12$$

Supplement for Lecture 26

- Inspect Interaction Plot
- Fit One-Way ANOVA Models
- Fit Two-Way ANOVA Model Without Interaction
- Fit Two-Way ANOVA Model With Interaction
- Compare the Models and Inspect CI’s for Interaction Effects

Thank You

Make Reasonable Decisions