PROBLEM SET - Lucas Stokey (1983)

PhD course in Fiscal Policy - Axelle Ferriere

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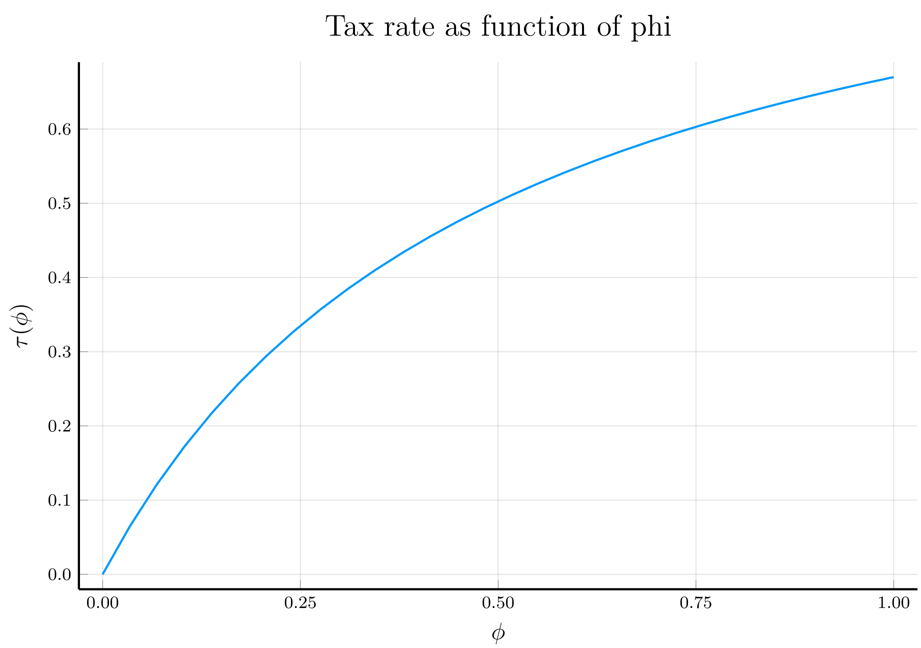
Part One

Question 1 - $t \geq 1$

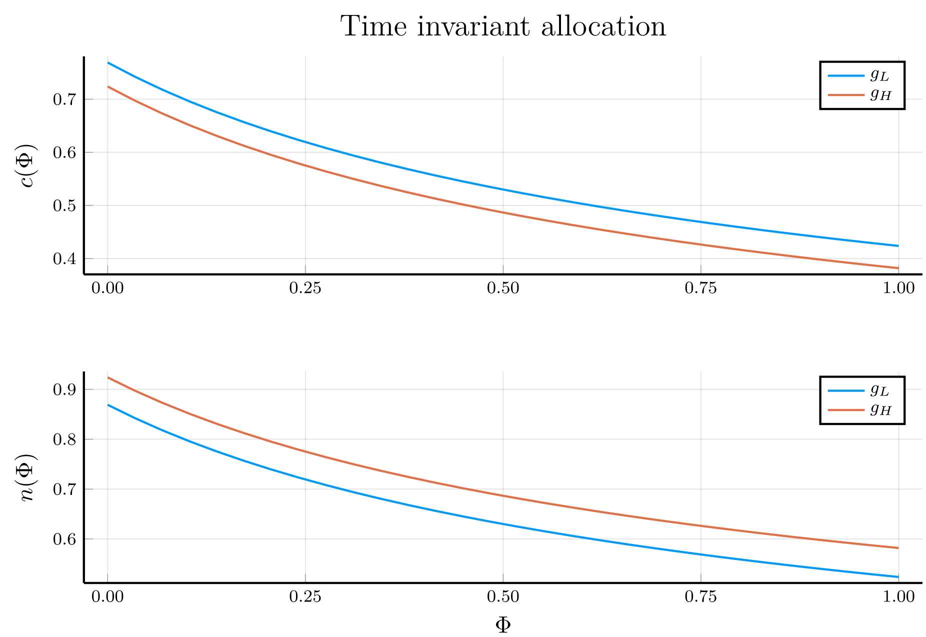
(a) Plot the history-independent and time-invariant tax $\tau(\phi)$



(b) Plot the history-independent ant time invariant allocations $\{c(\phi, g), n(\phi, g)\}$

We compute the allocations from the star equation and the ressource constraint using a non-linear solver.

$$\begin{cases} (1+\phi)(u_c(g) + u_n(g)) + \phi(u_{cc}(g)c(g) + n(g)u_{nn}(g)) & = 0 \\ c(g) + g & = n(g) \end{cases}$$



(c) Write the implementability constraint recursively to find $\{b(\phi, g)\}$ and plot debt policies as a function of ϕ

From the IC we have

$$u_c(g)c(g) + u_n(g)n(g) + \beta \sum_{g'} \Pi(g'|g)u_c(g')b(g') = u_c(g)b(g)$$

We rewrite the problem in matrix form to solve the linear system and because it makes it easy to compute b for larger vector of states. We have

$$b = Ab + k$$

Where

$$b = [b_{g_1} \dots b_{g_n}]' \tag{1}$$

$$A = \beta((U_c)^{\odot -1} U_c') \odot \Pi \tag{2}$$

$$U_c = [u_{c,g_1} \dots u_{c,g_n}]' \tag{3}$$

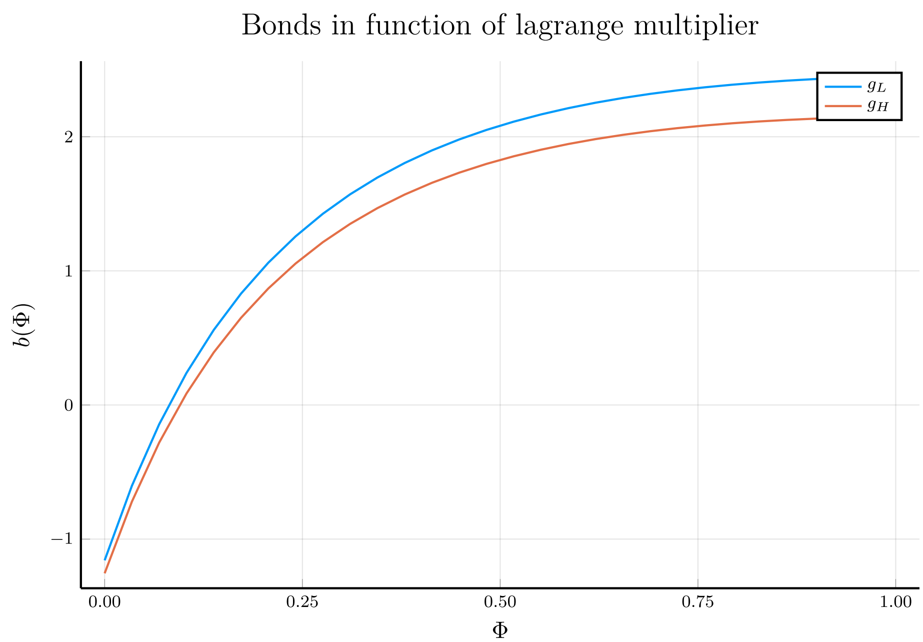
$$\Pi = \begin{bmatrix} \pi(g_1|g_1) & \dots & \pi(g_n|g_1) \\ \vdots & \ddots & \vdots \\ \pi(g_1|g_n) & \dots & \pi(g_n|g_n) \end{bmatrix} \tag{4}$$

$$k = \begin{bmatrix} c_{g_1} + \frac{u_{n,g_1}}{u_{n,g_1}} n_{g_1} \\ \vdots \\ c_{g_n} + \frac{u_{n,g_n}}{u_{n,g_n}} n_{g_n} \end{bmatrix} \tag{5}$$

Where $\odot$ is the Hadamard product (or *elementwise* operator)

The solution to compute debt for each states at a given value of ϕ writes

$$b = (I - A)^{-1} k$$



(d) What can you say about the relationship between ϕ and τ? Between ϕ and c?

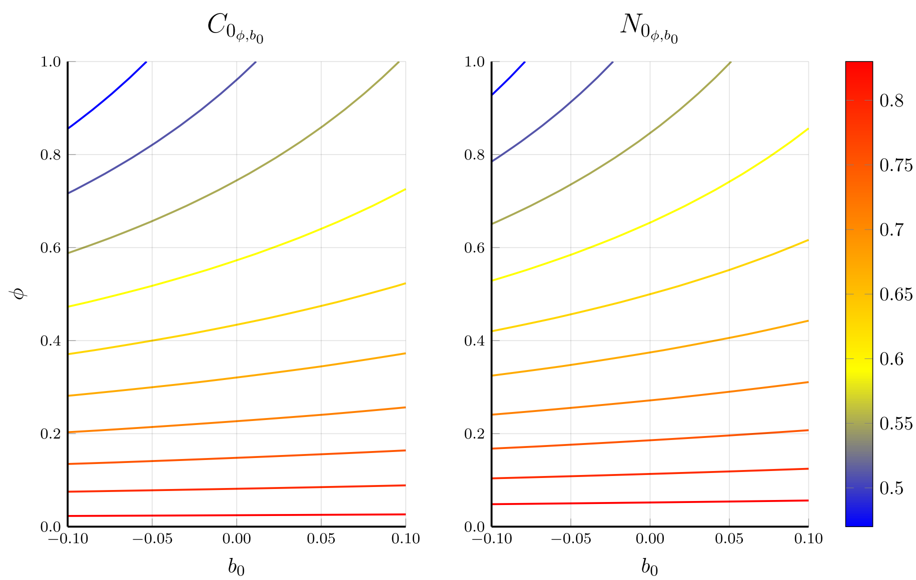
Φ is the Lagrange multiplier of the implementability constraint in the Ramsey plan. It is a measure of the distortion implied by the tax. Hence, it is straightforward to have a positive relationship between the two.

We observe a negative relationship between ϕ and c. This is explained by the substitution effect implied by the tax, which lowers the labor supply and hence the revenue and the consumption.

Question two - $t = 0$

(a) Compute allocations $\{c_0(b_0, \phi), n_0(b_0, \phi)\}$

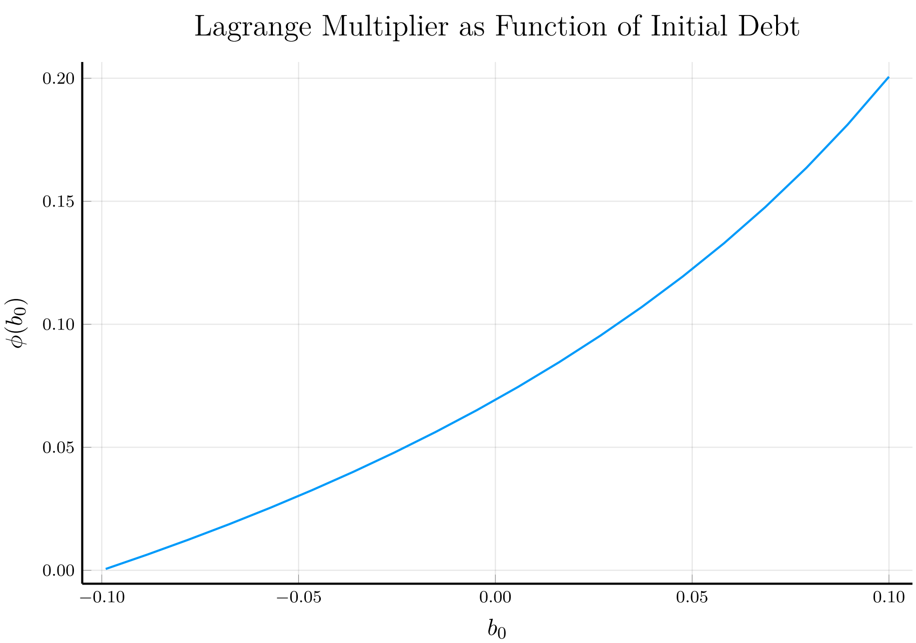
As before, we use the ressource constraint and the star equation, but in time zero. We get c_0 and n_0 as a function of the initial debt b_0 and the lagrange multiplier ϕ.



(b) Use the bisection method to find $\phi(b_0)$

In order to compute $\phi(b_0)$ we use a bisection algorithm on the implementability constraint. We have $IC = F(\phi, b_0, c_0(\phi, b_0), n_0(\phi, b_0)) = F(\phi, b_0)$. For a given value of b_0, written $\bar{b}_0$, we find ϕ with the following algorithm:

- Initialization
  - Define a lower bound ϕ_L and upper bound ϕ_U on ϕ
  - Define a tolerance level ϵ
- `While` $|IC(\phi_U, \bar{b}_0) - IC(\phi_L, \bar{b}_0)| > \epsilon$
  - $\phi_M \leftarrow \frac{\phi_L + \phi_U}{2}$
  - `If` $IC(\phi_M, \bar{b}_0) = 0$
    - `BREAK`
  - `Else if` $IC(\phi_M, \bar{b}_0) \times IC(\phi_L, \bar{b}_0) > 0$
    - $\phi_L \leftarrow \phi_M$
  - `Else`
    - $\phi_U \leftarrow \phi_M$



(c) What can you say about the relation between b_0 and ϕ

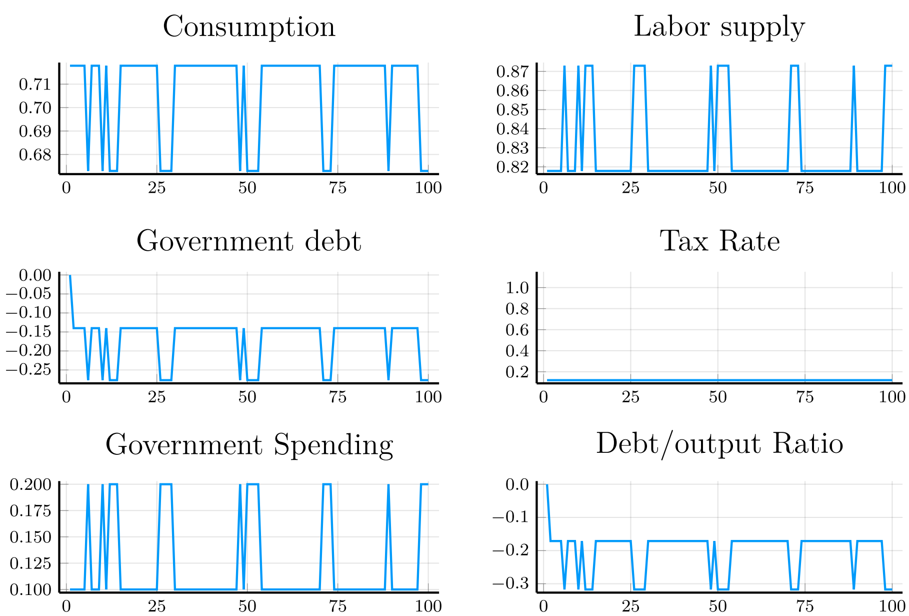
The initial debt b_0 and the distortion measure ϕ are positively related. This is because the debt need to be repaid with distortionary taxes. We see that ϕ is positive even with no initial debt, because the government needs to finance public expenditures in $t = 0$. The distortion disappears for $-b_0 = g_0 = 0.1$, that is when the government can fully finance public expenditures using its initial wealth.

Part Three

Simulating the economy is now straightforward. Fixing b_0 allows the computation of $\phi(b_0)$ which itself allows for the computation of allocations at time zero and for $t \geq 1$

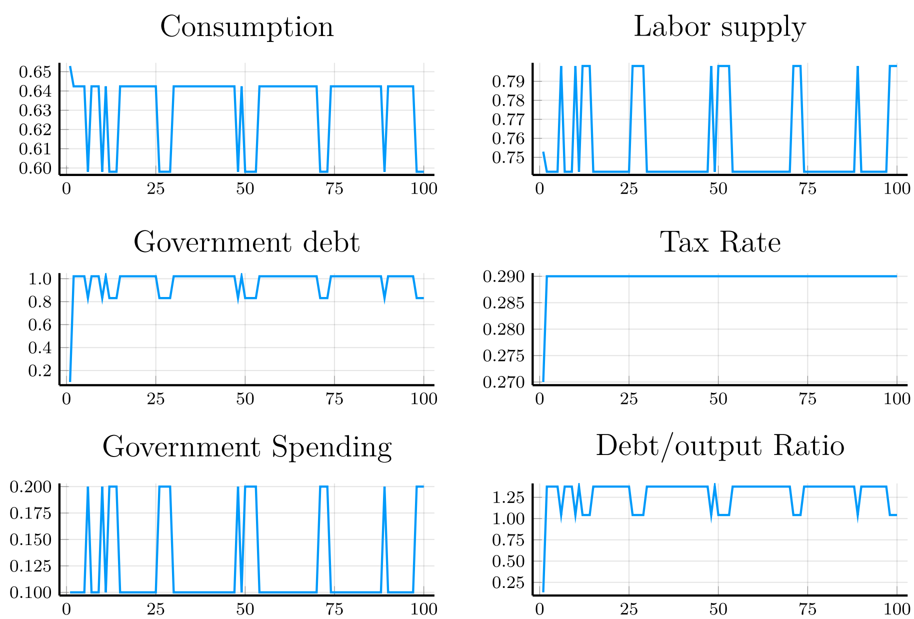
(a) Assume $b_0 = 0$. Simulate the economy for 100 periods.

The tax rate is constant in every periods. If $b_0 = 0$, then $t = 0$ is not different from any other period because $(\star) = (\star_0)$



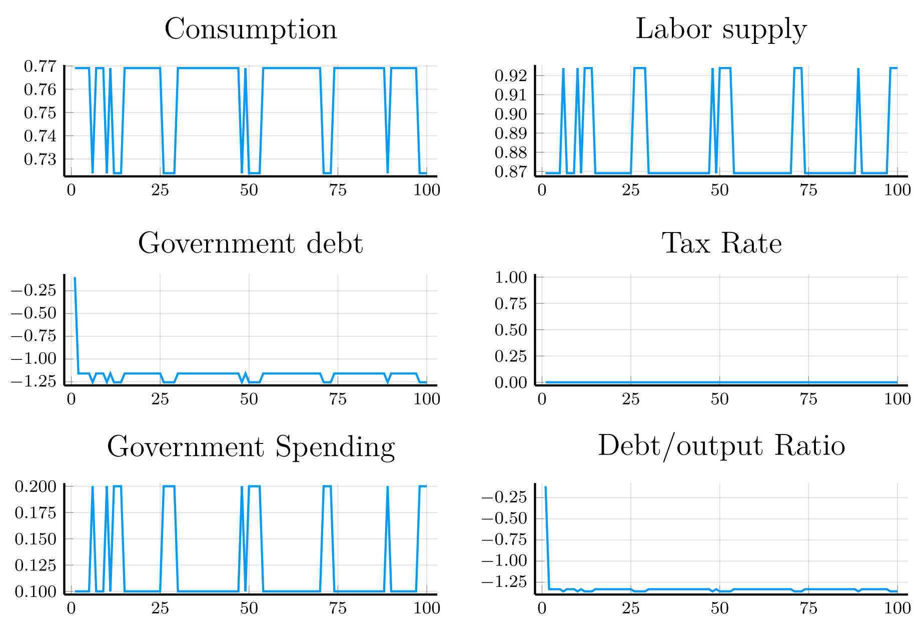
(b) Assume $b_0 = 0.1$. Simulate the economy for 100 periods.

In $t = 0$ the tax rate is below its value at $t \geq 1$, it is also greater than the case where $b_0 = 0$, this is due to the fact that the government needs to finance debt. The government sets τ_0 lower than $\{\tau_t\}_{t \geq 1}$ to incite households to supply labor. This creates a time inconsistency issue because one should expect the government to repeat this behavior at each period.



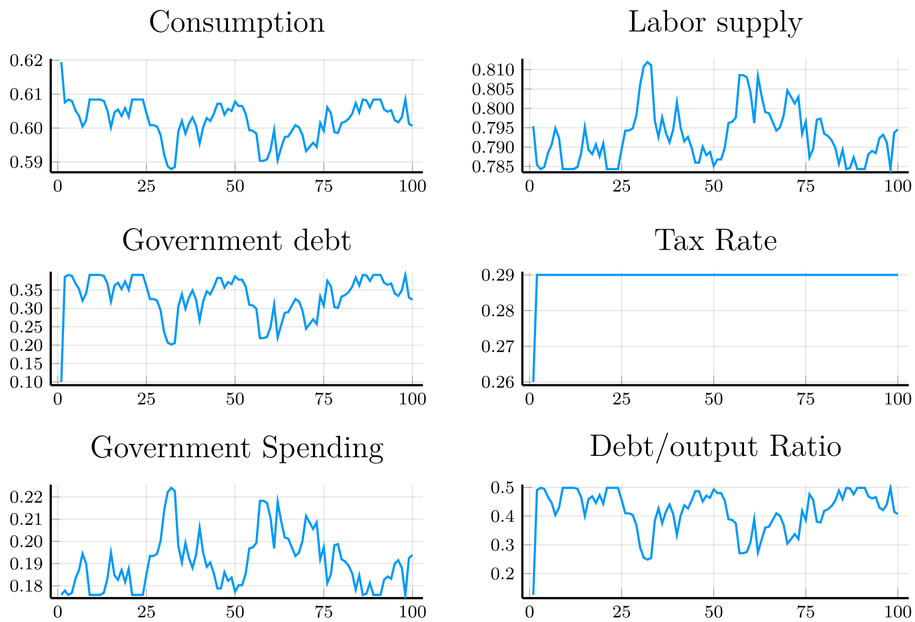
(c) Assume $b_0 = -0.1$. Simulate the economy for 100 periods.

The government has enough assets to finance its expenses so it doesn't need to tax labor, thus $\tau_t = 0 \ \ \forall \ t$



(d) Extension, AR(1) process for g using Tauchen method

In this extension, we use the Tauchen method to simulate an AR(1) process of g_t.



Part two - Theoretical exercise

In the model with complete market and capital, prove that if the government can use lump-sum taxes: (1) the optimal Ramsey plan reaches the first best; (2) the Lagrange multiplier of the iplementability constraint Φ is equal to zero.

Solution

First best

The planner objective is

$$\max_{\{c_t(g^t),n_t(g^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta t \sum_{g^t} \pi_t(g^t|g_0)u(c_t(g^t), 1-n_t(g^t)) \tag{6}$$

$$\text{s.t.} \quad c_t(g^t) + g_t(g^t) + k_{t+1}(g^t) = A_t(g^t)F(k_t(g^{t-1}), n_t(g^t)) + (1-\delta)k_t(g^t) \tag{7}$$

The Lagrangian of this problem is given as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta t \sum_{g^t} \pi_t(g^t|g_0)\{u(c_t(g^t), 1-n_t(g^t)) \tag{8}$$

$$- \lambda_t(g^t)[A_t(g^t)F(k_t(g^{t-1}), n_t(g^t)) \tag{9}$$

$$+ (1-\delta)k_t(g^t) - c_t(g^t) - g_t(g^t) - k_{t+1}(g^t)]\} \tag{10}$$

Competitive equilibrium with lump sum tax

Government

The government budget constraint has to hold:

$$g_t + b_t = T_t + \tau_t^k r_t k_t + \tau_t^n \omega_t n_t + \sum_{g_{t+1}} p_{t+1}(g_{t+1}|g^t) b_{t+1}(g_{t+1}|g^t)$$

Household

With capital, the household's budget constraint becomes:

$$c_t + k_{t+1} + \sum_{g_{t+1}} p_{t+1}(g_{t+1}|g^t) b_{t+1}(g_{t+1}|g^t) = b_t(g^t) \tag{11}$$

$$+ (1-\tau_t^n)\omega_t n_t + (1-\tau_t^k) r_t k_t + (1-\delta)k_t - T_t \tag{12}$$

And the maximization problem problem leads to to the FOCs:

$$\frac{u_x(g^t)}{u_c(g^t)} = (1-\tau_t^n(gt))\omega_t(g^t) \tag{13}$$

$$p_{t+1}(g_{t+1}|g_t) = \beta \pi_{t+1}(g_{t+1}|g_t) \frac{u_c(g^{t+1})}{u_c(g^t)} \tag{14}$$

$$u_c(g^t) = \beta E_t u_c(g^{t+1})[1-\delta + r_{t+1}(g^{t+1})(1-\tau_{t+1}^k(g^{t+1}))] \tag{15}$$

$$\tag{16}$$

We also write the budget constraint in time zero formulation

$$\sum_{t=0}^{\infty} \sum_{g_t} q_t^0(g^t)[c_t + T_t - (1-\tau_t^n)\omega_t \eta_t] = b_0 + [(1-\tau_0^k) r_0 + 1 - \delta] k_0 \tag{17}$$

Firms

Firms optimality conditions yeld:

$$r_t = Fk, t \tag{18}$$

$$\omega_t = F_{n,t} \tag{19}$$

Ramsey plan

We rewrite the household budget constraint without prices using the FOCs to get the implementability constraint:

$$\sum_{t=0}^{\infty}\beta^t\sum_{s_t}\pi_t[u_{c,t}c_t+u_{c,t}T_t+u_{n,t}n_t]=u_{c,0}b_0+u_{c,0}k_0[(1-\tau_0^k)F_{k0}+1-\delta] \quad (20)$$

The program writes

$$\max_{\{c_t,n_t\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta t\sum_{g^t}\pi_t(g^t|g_0)u(c_t,1-n_t) \quad (21)$$

$$s.t \quad \begin{cases} c_t+g_t+k_{t+1}=A_tF(k_t,n_t)+(1-\delta)k_t \\ \sum_{t=0}^{\infty}\beta^t\sum_{s_t}\pi_t[u_{c,t}c_t+u_{c,t}T_t+u_{n,t}n_t]=u_{c,0}b_0 \\ \quad +u_{c,0}k_0[(1-\tau_0^k)F_{k0}+1-\delta] \end{cases} \quad (22)$$

The Lagrangian writes:

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^t\sum_{g^t}\pi_t(g^t)\Big\{u(c_t,x_t) \quad (23)$$

$$+\Phi\left[u_{c,t}T_t+u_{c,t}c_t+u_{n,t}n_t-u_{c,0}b_0+[(1-\tau_0^k)F_{k0}+1-\delta]u_{c,0}k_0\right] \quad (24)$$

$$+\theta\left(A_tF(k_{,t}n_t)+(1-\delta)k_t-c_t-g_t-k_{t+1}\right)\Big\} \quad (25)$$

From the FOC we have

$$\frac{\partial\mathcal{L}}{\partial T_t}: \quad \Phi u_{c,t}=0$$

By hypothesis we have $u_{c,t}>0$, thus we necessarily have $\Phi=0$. This implies that the Ramsey plan is strictly equivalent to the social planner objective and therefore reaches the first best !

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