# Quick proof on univariate OLS estimator

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We suppose that the data consist of $N$ observations $\{y_i, x_i\}_{i=1}^{N}$ where each observation $i$ consist of a dependent variable $y_i$ and an explanatory variable $x_i$. We suppose a linear relationship in our variables such that

$$y_i = b_0 + b_1 x_i + \varepsilon_i \tag{1}$$

Where $b_0$ and $b_1$ are scalars, shared by all observations and $\varepsilon_i$ is the error term.

Our goal is to find $b_0$ and $b_1$ such that the *sum of squared of residuals*(SSR) is minimized. Where the SSR is defined by

$$\begin{aligned} SRR &= \sum_{i=1}^{N} e^2 \\ &= \sum_{i=1}^{N} (y_i - \hat{b}_0 - \hat{b}_1 x_i)^2 \end{aligned}$$

Mathematicaly we seek

$$\min_{\hat{b}_0, \hat{b}_1} \sum_{i=1}^{N} (y_i - \hat{b}_0 - \hat{b}_1 x_i)^2 \tag{2}$$

As we need to minimize a quadratic function, we take the derivative with respect to both argument and equalize to 0.

$$\begin{cases} \dfrac{\partial\, SSR}{\partial \hat{b}_0} = -2 \sum_{i=1}^{N} (y_i - \hat{b}_0 - \hat{b}_1 x_i) \\ \\ \dfrac{\partial\, SSR}{\partial \hat{b}_1} = -2 \sum_{i=1}^{N} x_i (y_i - \hat{b}_0 - \hat{b}_1 x_i) \end{cases}$$

Then we equalize both equations to 0

$$\begin{cases} -2 \sum_{i=1}^{N} (y_i - \hat{b}_0 - \hat{b}_1 x_i) &= 0 \\ -2 \sum_{i=1}^{N} x_i (y_i - \hat{b}_0 - \hat{b}_1 x_i) &= 0 \end{cases}$$

We now have our system of 2 equations with 2 unknown that we need to solve for $\hat{b}_0$ and $\hat{b}_1$

We simplify

$$\begin{cases} \cancel{-2}\sum_{i=1}^{N}(y_i - \hat{b}_0 - \hat{b}_1 x_i) & = 0 \\ \cancel{-2}\sum_{i=1}^{N} x_i(y_i - \hat{b}_0 - \hat{b}_1 x_i) & = 0 \end{cases}$$

Rewriting

$$\begin{cases} \sum_{i=1}^{N} y_i & = \sum_{i=1}^{N} \hat{b}_0 + \hat{b}_1 \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i y_i & = \hat{b}_0 \sum_{i=1}^{N} x_i + \hat{b}_1 \sum_{i=1}^{N} x_i^2 \end{cases}$$

Using the fact that $\sum_{k=1}^{K} a = K \times a$ and that $\hat{b}_0$ do not depends on $i$ we get

$$\begin{cases} \sum_{i=1}^{N} y_i & = N\hat{b}_0 + \hat{b}_1 \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i y_i & = \hat{b}_0 \sum_{i=1}^{N} x_i + \hat{b}_1 \sum_{i=1}^{N} x_i^2 \end{cases}$$

We multiply the first equation by $\frac{1}{N}$

$$\begin{cases} \frac{1}{N}\sum_{i=1}^{N} y_i & = \frac{1}{N}N\hat{b}_0 + \hat{b}_1 \frac{1}{N} \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i y_i & = \hat{b}_0 \sum_{i=1}^{N} x_i + \hat{b}_1 \sum_{i=1}^{N} x_i^2 \end{cases}$$

We define the average $z$ by $\bar{z} := \frac{1}{N}\sum_{i=1}^{N} z_i$.

$$\begin{cases} \bar{y} & = \hat{b}_0 + \hat{b}_1 \bar{x} \\ \sum_{i=1}^{N} x_i y_i & = \hat{b}_0 \sum_{i=1}^{N} x_i + \hat{b}_1 \sum_{i=1}^{N} x_i^2 \end{cases}$$

We rewrite to get an expression of $\hat{b}_0$

$$\begin{cases} \hat{b}_0 & = \bar{y} - \hat{b}_1 \bar{x} \\ \sum_{i=1}^{N} x_i y_i & = \hat{b}_0 \sum_{i=1}^{N} x_i + \hat{b}_1 \sum_{i=1}^{N} x_i^2 \end{cases}$$

We substitute the first equation in the second

$$\begin{cases} \hat{b}_0 & = \bar{y} - \hat{b}_1 \bar{x} \\ \sum_{i=1}^{N} x_i y_i & = (\bar{y} - \hat{b}_1 \bar{x}) \sum_{i=1}^{N} x_i + \hat{b}_1 \sum_{i=1}^{N} x_i^2 \end{cases}$$

Developing

$$\begin{cases} \hat{b}_0 & = \bar{y} - \hat{b}_1 \bar{x} \\ \sum_{i=1}^{N} x_i y_i & = \bar{y} \sum_{i=1}^{N} x_i - \hat{b}_1 \bar{x} \sum_{i=1}^{N} x_i + \hat{b}_1 \sum_{i=1}^{N} x_i^2 \end{cases}$$

We factorize by $\hat{b}_1$

$$\begin{cases} \hat{b}_0 & = \bar{y} - \hat{b}_1 \bar{x} \\ \sum_{i=1}^{N} x_i y_i & = \bar{y} \sum_{i=1}^{N} x_i + \hat{b}_1 \left( \sum_{i=1}^{N} x_i^2 - \bar{x} \sum_{i=1}^{N} x_i \right) \end{cases}$$

Isolating $\hat{b}_1$

$$\begin{cases} \hat{b}_0 & = \bar{y} - \hat{b}_1 \bar{x} \\ \\ \hat{b}_1 & = \frac{\sum_{i=1}^{N} x_i y_i - \bar{y} \sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} x_i^2 - \bar{x} \sum_{i=1}^{N} x_i} \end{cases}$$

After a bit of algebra (see appendix), one can simplify to

$$\begin{cases} \hat{b}_0 & = \bar{y} - \hat{b}_1 \bar{x} \\ \\ \hat{b}_1 & = \frac{\sum_{i=1}^{N} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{N} (x_i - \bar{x})^2} \end{cases}$$

Which is strictly equivalent to

$$\begin{cases} \hat{b}_0 & = \bar{y} - \hat{b}_1 \bar{x} \\ \\ \hat{b}_1 & = \frac{Cov(x, y)}{Var(x)} \end{cases}$$

# Appendix A Simplifying $\hat{b}_1$

## A.1 Finding the covariance

$$\sum_{i=1}^{N} x_i y_i - \bar{y}\sum_{i=1}^{N} x_i = N \times Cov(x,y)$$

This proof need some tricks and heavily play with the fact that $\sum_{i=1}^{N} x_i = \bar{x}$ and that $\sum_{i=1}^{N} \bar{y} = N\bar{y}$. For example, one could write

$$\bar{y}\sum_{i=1}^{N} x_i = N\bar{y}\bar{x} = \sum_{i=1}^{N} \bar{x}\bar{y} = \sum_{i=1}^{N} y_i\bar{x}$$

*Proof.*

$$\begin{aligned}
\sum_{i=1}^{N} x_i y_i - \bar{y}\sum_{i=1}^{N} x_i &= \sum_{i=1}^{N} x_i y_i - N\bar{y}\bar{x} \\
&= \sum_{i=1}^{N} x_i y_i - N\bar{y}\bar{x} - N\bar{y}\bar{x} + N\bar{y}\bar{x} \\
&= \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i\bar{y} - \sum_{i=1}^{N} \bar{x}y_i + \sum_{i=1}^{N} \bar{y}\bar{x} \\
&= \sum_{i=1}^{N} (x_i y_i - x_i\bar{y} - \bar{x}y_i + \bar{y}\bar{x}) \\
&= \sum_{i=1}^{N} (y_i - \bar{y})(x_i - \bar{x}) \\
&= N \times Cov(x,y)
\end{aligned}$$

□

## A.2 Finding the variance

$$\sum_{i=1}^{N} x_i^2 - \bar{x}\sum_{i=1}^{N} x_i = N \times Var(x)$$

This proof is very similar to the on for covariance.

*Proof.*

$$\begin{aligned}
\sum_{i=1}^{N} x_i^2 - \bar{x} \sum_{i=1}^{N} x_i &= \sum_{i=1}^{N} x_i^2 - N\bar{x}\bar{x} \\
&= \sum_{i=1}^{N} x_i^2 - 2N\bar{x}\bar{x} + N\bar{x}\bar{x} \\
&= \sum_{i=1}^{N} x_i^2 - 2\bar{x} \sum_{i=1}^{N} x_i + N\bar{x}^2 \\
&= \sum_{i=1}^{N} (x_i^2 - 2\bar{x}x_i + \bar{x}^2) \\
&= \sum_{i=1}^{N} (x - \bar{x})^2 \\
&= N \times Var(x)
\end{aligned}$$

□