2023～2024 学年第二学期《微积分Ⅱ》期中考试参考答案（2024.5.10）

一、选择题（共 15 分，每小题 3 分）

1. B　　2. D　　3. D　　4. C　　5. A

二、填空题（共 15 分，每小题 3 分）

1. $3x+y-z-3=0$　2. $(2,1,4)$　3. 1　4. $\dfrac{2\sqrt{2}-1}{3}$　5. $\dfrac{\pi}{e}$

三、计算题（共 40 分，每小题 8 分）

1. 解法一：令 $F(x,y,z)=x^2(y+z)-4\sqrt{x^2+y^2+z^2}$，

$$F'_x=2x(y+z)-\frac{4x}{\sqrt{x^2+y^2+z^2}},\quad F'_y=x^2-\frac{4y}{\sqrt{x^2+y^2+z^2}},\quad F'_z=x^2-\frac{4z}{\sqrt{x^2+y^2+z^2}},$$

$$\therefore \left.\frac{\partial z}{\partial x}\right|_P=-\left.\frac{F'_x}{F'_z}\right|_P=\frac{7}{2},\ \left.\frac{\partial z}{\partial y}\right|_P=-\left.\frac{F'_y}{F'_z}\right|_P=-\frac{1}{2},\quad \left.\mathrm{d}z\right|_P=\frac{7}{2}\mathrm{d}x-\frac{1}{2}\mathrm{d}y.$$

解法二：方程两边对 x 求偏导,得 $2x(y+z)+x^2\cdot\dfrac{\partial z}{\partial x}-\dfrac{4}{\sqrt{x^2+y^2+z^2}}\left(x+z\dfrac{\partial z}{\partial x}\right)=0$，

将 $P(-2,2,1)$ 代入上式：$-12+4\dfrac{\partial z}{\partial x}-\dfrac{4}{3}\left(-2+\dfrac{\partial z}{\partial x}\right)=0$，　$\therefore \left.\dfrac{\partial z}{\partial x}\right|_P=\dfrac{7}{2}$.

方程两边对 y 求偏导,得 $x^2\left(1+\dfrac{\partial z}{\partial y}\right)-\dfrac{4}{\sqrt{x^2+y^2+z^2}}\left(y+z\dfrac{\partial z}{\partial y}\right)=0$，

将 $P(-2,2,1)$ 代入上式得 $4\left(1+\dfrac{\partial z}{\partial y}\right)-\dfrac{4}{3}\left(2+\dfrac{\partial z}{\partial y}\right)=0$，$\therefore \left.\dfrac{\partial z}{\partial y}\right|_P=-\dfrac{1}{2}$.

所以，$\left.\mathrm{d}z\right|_P=\left.\dfrac{\partial z}{\partial x}\right|_P\mathrm{d}x+\left.\dfrac{\partial z}{\partial y}\right|_P\mathrm{d}y=\dfrac{7}{2}\mathrm{d}x-\dfrac{1}{2}\mathrm{d}y$.

2. 解法一：$I=\displaystyle\int_0^2\mathrm{d}x\int_{\frac{x^2}{2}}^{x}(x+2y)\mathrm{d}y=\int_0^2(xy+y^2)\Big|_{\frac{x^2}{2}}^{x}\mathrm{d}x=\int_0^2\left(2x^2-\frac{1}{2}x^3-\frac{1}{4}x^4\right)\mathrm{d}x$

$$=\left(\frac{2}{3}x^3-\frac{1}{8}x^4-\frac{1}{20}x^5\right)\Big|_0^2=\frac{26}{15}.$$

解法二：$I=\displaystyle\int_0^2\mathrm{d}y\int_y^{\sqrt{2y}}(x+2y)\mathrm{d}x=\int_0^2\left[y-\frac{1}{2}y^2+2y\left(\sqrt{2y}-y\right)\right]\mathrm{d}y$

$$=\left(\frac{1}{2}y^2-\frac{5}{6}y^3+2\sqrt{2}\cdot\frac{2}{5}y^{\frac{5}{2}}\right)\Big|_0^2=\frac{26}{15}.$$

3. 解法一：选用球坐标系，

$$I=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\mathrm{d}\varphi\int_0^2 r\cos\varphi\cdot r^2\sin^2\varphi\cdot r^2\sin\varphi\,\mathrm{d}r$$

$$=2\pi\int_0^{\frac{\pi}{4}}\sin^3\varphi\cos\varphi\,\mathrm{d}\varphi\int_0^2 r^5\mathrm{d}r=2\pi\cdot\frac{1}{16}\cdot\frac{32}{3}=\frac{4\pi}{3}.$$

解法二：选用柱坐标系，Ω 在 xOy 平面投影区域为 $D_{xy}:x^2+y^2\le 2$.

$$I=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{2}}\rho^3\mathrm{d}\rho\int_\rho^{\sqrt{4-\rho^2}}z\,\mathrm{d}z=2\pi\int_0^{\sqrt{2}}\rho^3(2-\rho^2)\mathrm{d}\rho=\frac{4\pi}{3}.$$

4. 解：$\mathrm{d}s=\sqrt{x_t'^2+y_t'^2+z_t'^2}\,\mathrm{d}t=\sqrt{(-\sin t)^2+\cos^2 t+4}\,\mathrm{d}t=\sqrt{5}\,\mathrm{d}t$，

$$I=\int_0^1\frac{1}{\cos^2 t+\sin^2 t+4t^2}\cdot\sqrt{5}\,\mathrm{d}t=\sqrt{5}\int_0^1\frac{1}{1+4t^2}\mathrm{d}t=\frac{\sqrt{5}}{2}\arctan(2t)\Big|_0^1=\frac{\sqrt{5}}{2}\arctan 2.$$

5. 解法一：取 $\overline{BO}:x=0$，记 L 与 $\overline{BO}$ 所围区域为 D，由格林公式，

$$I=\oint_{L+\overline{BO}}3x^2y\,\mathrm{d}x+(x^3+x-2y)\mathrm{d}y-\int_{\overline{BO}}3x^2y\,\mathrm{d}x+(x^3+x-2y)\mathrm{d}y$$

$$=\iint_D(3x^2+1-3x^2)\mathrm{d}x\mathrm{d}y-\int_2^0-2y\,\mathrm{d}y=S_D-4=\frac{\pi}{2}-4.$$

或：$I=\displaystyle\iint_D 1\mathrm{d}x\mathrm{d}y+\int_2^0 2y\,\mathrm{d}y=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_{2\cos\theta}^{2}\rho\,\mathrm{d}\rho+y^2\Big|_2^0=2\int_0^{\frac{\pi}{2}}\sin^2\theta\,\mathrm{d}\theta-4=\frac{\pi}{2}-4.$

解法二：$OA:x=1+\cos t,y=\sin t$；　$AB:x=2\cos t,y=2\sin t$.

$$I=\int_\pi^0\left[3(1+\cos t)^2\sin t\cdot(-\sin t)+\left((1+\cos t)^3+1+\cos t-2\sin t\right)\cdot\cos t\right]\mathrm{d}t$$

$$+\int_0^{\frac{\pi}{2}}\left[3(2\cos t)^2 2\sin t\cdot(-2\sin t)+\left((2\cos t)^3+2\cos t-4\sin t\right)\cdot 2\cos t\right]\mathrm{d}t$$

$$=-\frac{\pi}{2}+\pi-4=\frac{\pi}{2}-4.$$

解法三：$I=\displaystyle\int_L 3x^2y\,\mathrm{d}x+(x^3-2y)\mathrm{d}y+\int_L x\,\mathrm{d}y$，

其中，$I_1=\displaystyle\int_L 3x^2y\,\mathrm{d}x+(x^3-2y)\mathrm{d}y=\left(x^3y-y^2\right)\Big|_{O(0,0)}^{B(0,2)}=-4$（曲线积分与路径无关），

$$I_2=\int_L x\,\mathrm{d}y=\int_\pi^0(1+\cos t)\cdot\cos t\,\mathrm{d}t+\int_0^{\frac{\pi}{2}}2\cos t\cdot 2\cos t\,\mathrm{d}t=-\frac{\pi}{2}+\pi=\frac{\pi}{2},$$

$\therefore\ I=I_1+I_2=\dfrac{\pi}{2}-4.$

四、解答题（共 24 分，每小题 8 分）

1. 解: $P(x,y)=\dfrac{2x-y}{x^2+y^2}$, $Q(x,y)=\dfrac{x+2y}{x^2+y^2}$, $\quad \dfrac{\partial P}{\partial y}=\dfrac{y^2-x^2-4xy}{(x^2+y^2)^2}=\dfrac{\partial Q}{\partial x}$,

故曲线积分与路径无关, 选取路径为下半圆 $C:\begin{cases}x=\cos\theta,\\ y=\sin\theta,\end{cases}\quad \theta:\ \pi\to 2\pi$,

$$I=\int_{\pi}^{2\pi}\left[(2\cos\theta-\sin\theta)\cdot(-\sin\theta)+(\cos\theta+2\sin\theta)\cos\theta\right]\mathrm{d}\theta=\int_{\pi}^{2\pi}\mathrm{d}\theta=\pi.$$

或: 选取位于 x 轴下方的折线 $\overline{AE}+\overline{EF}+\overline{FB}$ 积分, 其中点 $E(-1,-1),F(1,-1)$,

$$I=\int_0^{-1}\frac{-1+2y}{1+y^2}\mathrm{d}y+\int_{-1}^{1}\frac{2x+1}{x^2+1}\mathrm{d}x+\int_{-1}^{0}\frac{1+2y}{1+y^2}\mathrm{d}y=2\int_{-1}^{0}\frac{1}{1+y^2}\mathrm{d}y+2\int_0^1\frac{1}{x^2+1}\mathrm{d}x=\pi.$$

2. 解法一: 令 $S_0: z=0,\ x^2+y^2\le 4$, 取下侧, 由高斯公式, 得

$$I=\iiint_{\Omega}(3x^2+2y+3y^2)\,\mathrm{d}V-\iint_{S_0}x^3\mathrm{d}y\mathrm{d}z+y^2\mathrm{d}z\mathrm{d}x+3y^2z\,\mathrm{d}x\mathrm{d}y=\iiint_{\Omega}(3x^2+3y^2)\,\mathrm{d}V-0$$

$$=3\int_0^{2\pi}\mathrm{d}\theta\int_0^2\mathrm{d}\rho\int_0^{4-\rho^2}\rho^3\,\mathrm{d}z=6\pi\int_0^2\rho^3(4-\rho^2)\,\mathrm{d}\rho=32\pi.$$

解法二: 设 $S_1: x=\sqrt{4-z-y^2}$ 是 S 在 $x\ge 0$ 部分, S_1 在 yOz 面投影区域为 D_{yz},

$$I_1=\iint_S x^3\mathrm{d}y\mathrm{d}z=2\iint_{S_1}x^3\mathrm{d}y\mathrm{d}z=2\iint_{D_{yz}}(\sqrt{4-z-y^2})^3\mathrm{d}y\mathrm{d}z$$

$$=2\int_{-2}^{2}\mathrm{d}y\int_0^{4-y^2}\left(4-y^2-z\right)^{\frac{3}{2}}\mathrm{d}z=\frac{4}{5}\int_{-2}^{2}\left(4-y^2\right)^{\frac{5}{2}}\mathrm{d}y=16\pi,$$

$$I_2=\iint_S y^2\mathrm{d}z\mathrm{d}x=0,$$

$$I_3=\iint_S 3y^2z\,\mathrm{d}x\mathrm{d}y=\iint_{D_{xy}}3y^2(4-x^2-y^2)\mathrm{d}x\mathrm{d}y=3\int_0^{2\pi}\mathrm{d}\theta\int_0^2\rho^2\sin^2\theta(4-\rho^2)\cdot\rho\,\mathrm{d}\rho=16\pi,$$

所以, $I=I_1+I_2+I_3=32\pi$.

解法三: 利用向量点积法,

$$I=\iint_S\left(x^3,y^2,3y^2z\right)\cdot\left(2x,2y,1\right)\mathrm{d}x\mathrm{d}y=\iint_S\left(2x^4+2y^3+3y^2z\right)\mathrm{d}x\mathrm{d}y$$

$$=\iint_{D_{xy}}\left[2x^4+3y^2(4-x^2-y^2)\right]\mathrm{d}x\mathrm{d}y$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^2\left[2\rho^4\cos^4\theta+3\rho^2\sin^2\theta(4-\rho^2)\right]\cdot\rho\,\mathrm{d}\rho$$

$$=\int_0^{2\pi}\left(\frac{64}{3}\cos^4\theta+16\sin^2\theta\right)\mathrm{d}\theta=32\pi.$$

3. 解: S 在 xOy 平面的投影区域为 $D_{xy}: x^2+y^2\le 1$,

$$I=\iint_{D_{xy}}(2y^2+2-x-y)\sqrt{1+(-1)^2+(-1)^2}\,\mathrm{d}x\mathrm{d}y=2\sqrt{3}\iint_{D_{xy}}(y^2+1)\,\mathrm{d}x\mathrm{d}y$$

$$=2\sqrt{3}\left(\int_0^{2\pi}\mathrm{d}\theta\int_0^1\rho^2\sin^2\theta\cdot\rho\,\mathrm{d}\rho+\pi\right)=2\sqrt{3}\left(\frac{\pi}{4}+\pi\right)=\frac{5\sqrt{3}}{2}\pi.$$

或: $I=\sqrt{3}\iint_{D_{xy}}(2y^2+2)\,\mathrm{d}x\mathrm{d}y=\sqrt{3}\iint_{D_{xy}}(x^2+y^2+2)\,\mathrm{d}x\mathrm{d}y$

$$=\sqrt{3}\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(\rho^2+2\right)\cdot\rho\,\mathrm{d}\rho=\frac{5\sqrt{3}}{2}\pi.$$

五、证明题（共 6 分）

设分片光滑曲面 Σ 是空间有界闭区域 Ω 的边界, 函数 $u(x,y,z),v(x,y,z)$ 在 Ω 上具有连续的二阶偏导数, $\dfrac{\partial v}{\partial \boldsymbol{n}}$ 是函数 v 沿 Σ 的外法线方向 $\boldsymbol{n}$ 的方向导数. 证明:

$$\iiint_{\Omega}u\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\mathrm{d}V=\oiint_{\Sigma}u\frac{\partial v}{\partial \boldsymbol{n}}\mathrm{d}S-\iiint_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right)\mathrm{d}V.$$

证明: 设 Σ 的外法线方向 $\boldsymbol{n}$ 的方向余弦为 $\cos\alpha,\cos\beta,\cos\gamma$, 则

$$\frac{\partial v}{\partial \boldsymbol{n}}=\frac{\partial v}{\partial x}\cos\alpha+\frac{\partial v}{\partial y}\cos\beta+\frac{\partial v}{\partial z}\cos\gamma,$$

$$\oiint_{\Sigma}u\frac{\partial v}{\partial \boldsymbol{n}}\mathrm{d}S=\oiint_{\Sigma}u\left(\frac{\partial v}{\partial x}\cos\alpha+\frac{\partial v}{\partial y}\cos\beta+\frac{\partial v}{\partial z}\cos\gamma\right)\mathrm{d}S$$

$$=\oiint_{\Sigma}u\frac{\partial v}{\partial x}\mathrm{d}y\mathrm{d}z+u\frac{\partial v}{\partial y}\mathrm{d}z\mathrm{d}x+u\frac{\partial v}{\partial z}\mathrm{d}x\mathrm{d}y\quad(\Sigma\text{ 取外侧})$$

$$=\iiint_{\Omega}\left[\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(u\frac{\partial v}{\partial z}\right)\right]\mathrm{d}V$$

$$=\iiint_{\Omega}\left[\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}\right)+\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+u\frac{\partial^2 v}{\partial y^2}\right)+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}+u\frac{\partial^2 v}{\partial z^2}\right)\right]\mathrm{d}V,$$

故等式成立.