工 程 数 学 基 础

习 题 解 答

习　题　一

A

一、判断题

1.√；，2.√；3.×；4.×；5.×；6.×；7.×；8.√；9.√；10.×.

二、填空题

1. $A^C \cap B^C$； **2.** $\mathcal{D}(f)=\{1,2,3,4\}, \mathcal{R}(f)=\{a,b,e\}, f(A_1)=\{a,b,e\}, f^{-1}(B)=\{1,4\}, f^{-1}(b)=\{2,3\}$;

3.满； **4.** $\sup E=\sqrt{2}, \inf E=-3$； **5.** 0； **6.0**； **7.** n； **8.** Y.

B

1. 证　$\forall y\in f(A\cap B)$，$\exists x\in A\cap B$ 使得 $y=f(x)$．由 $x\in A\cap B$，得 $x\in A$，且 $x\in B$ 故 $y=f(x)\in f(A)$ 且 $y\in f(B)$,即 $y\in f(A)\cap f(B)$,因此 $f(A\cap B)\subset f(A)\cap f(B)$.

当 f 是单射时,只需证明 $f(A)\cap f(B)\subset f(A\cap B)$ 即可：

$\forall y\in f(A)\cap f(B)\subset \mathcal{R}(f)$, 由$f$ 是单射知$\exists 1 x\in X$,使得$y=f(x)$.$\because y\in f(A)$,且$y\in f(B)$,

$\therefore x\in A$且$x\in B$,即$x\in A\cap B$,从而$y=f(x)\in f(A\cap B)$, 故 $f(A)\cap f(B)\subset f(A\cap B)$.

是可能的，例如,

$f: x\mapsto x^2$,取$A=[-2,\ 0], B=[-1,\ 3]$,则$A\cap B=[-1,\ 0]$. 于是$f(A\cap B)=f([-1,0])=[0,\ 1]$, 而

$f(A)\cap f(B)=[0,\ 4]\cap[0,\ 9]=[0,\ 4]$. 从而有 .

2. 证（1）$\forall n\in \mathbf{N}$,有$\left[-2+\frac{1}{n},\ 2-\frac{1}{n}\right]\subset(-2,\ 2)$,故$\bigcup_{n=1}^{\infty}\left[-2+\frac{1}{n},\ 2-\frac{1}{n}\right]\subset(-2,\ 2)$.

另一方面,$\forall x\in(-2,\ 2)$,$\exists k\in\mathbf{N}$,使$x\in\left[-2+\frac{1}{k},\ 2-\frac{1}{k}\right]$,故$x\in\bigcup_{n=1}^{\infty}\left[-2+\frac{1}{n},\ 2-\frac{1}{n}\right]$,于是

$(-2,\ 2)\subset\bigcup_{n=1}^{\infty}\left[-2+\frac{1}{n},\ 2-\frac{1}{n}\right]$.

因此，$(-2,\ 2)=\bigcup_{n=1}^{\infty}\left[-2+\frac{1}{n},\ 2-\frac{1}{n}\right]$.

（2）$\forall n\in\mathbf{N}$,有$[-2,\ 2]\subset(-2-\frac{1}{n},\ 2+\frac{1}{n})$,故$[-2,\ 2]\subset\bigcap_{n=1}^{\infty}(-2-\frac{1}{n},\ 2+\frac{1}{n})$.

另一方面，对任意 $x\notin[-2,\ 2]$，即 $|x|>2$，$\exists k\in\mathbf{N}$，使得 $|x|>2+\frac{1}{k}>2$，即

$x \notin (-2-\dfrac{1}{k},\ 2+\dfrac{1}{k})$,从而 $x \notin \bigcap\limits_{n=1}^{\infty}(-2-\dfrac{1}{n},\ 2+\dfrac{1}{n})$，故 $\bigcap\limits_{n=1}^{\infty}(-2-\dfrac{1}{n},\ 2+\dfrac{1}{n}) \subset [-2,2]$.

因此,$[-2,2]=\bigcap\limits_{n=1}^{\infty}(-2-\dfrac{1}{n},2+\dfrac{1}{n})$.

3. 证 设$\sup A=\mu$,且$\sup A=\mu'$,要证$\sup A$唯一,只需证明$\mu=\mu'$即可.

因为$\mu=\sup A$是最小上界,而μ'是A的上界,故$\mu \le \mu'$,又因为$\mu'=\sup A$是最小上界,而μ是A的上界,故$\mu' \le \mu$;因此$\mu=\mu'$.

类似地可以证明$\inf A$是唯一的.

4. 证 设$\{Y_\alpha\}_{\alpha\in D}$是线性空间 X 的一族子空间,要证$\bigcap\limits_{\alpha\in D}Y_\alpha$也是$X$的线性子空间.显然$\bigcap\limits_{\alpha\in D}Y_\alpha \ne \varnothing$，z

只需证明$\bigcap\limits_{\alpha\in D}Y_\alpha$对$X$的线性运算是封闭的.事实上，$\forall x,y\in\bigcap\limits_{\alpha\in D}Y_\alpha$及$\forall\lambda\in K$,,从而对每一个$\alpha\in D$，

有$x,y\in Y_\alpha$,故$x+y\in Y_\alpha$，$\lambda x\in Y_\alpha$.于是,$x+y\in\bigcap\limits_{\alpha\in D}Y_\alpha$，$\lambda x\in\bigcap\limits_{\alpha\in D}Y_\alpha$.因此,$\bigcap\limits_{\alpha\in D}Y_\alpha$是 X 的线性子空间.

5. 证 显然W包含零多项式,故非空；又$\forall f,g\in W$,及$\forall\lambda\in\square$，有

$(f+g)(0)+(f+g)'(0)=f(0)+g(0)+f'(0)+g'(0)=[f(0)+f'(0)]+[g(0)+g'(0)]=0+0=0$,即
$f+g\in W$;$(\lambda f)(0)+(\lambda f)'(0)=\lambda f(0)+\lambda f'(0)=\lambda[f(0)+f'(0)]=\lambda 0=0$,即$\lambda f\in W$.

所以,W是$P_n[0,\ 1]$的线性子空间.

$\forall f\in W\subset P_n[0,\ 1]$,设$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$,则$f'(x)=na_nx^{n-1}+\cdots+2a_2x+a_1$.由$f(0)+f'(0)=0$,得$a_1+a_0=0$,即$a_0=-a_1$,故$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1(x-1)$.

由上可知,$(x-1,x^2,x^3,\cdots,x^n)$是W的一个基,故$\dim W=n$.

6. (1)“$\Rightarrow$”:因为T是线性的,故有$T(0)=0$.于是,若$T(x)=0$,则由T^{-1}存在知T是单射,从而有$x=0$.

“$\Leftarrow$”:要证T^{-1}存在,只需证明T是单射：

$\forall x_1,x_2\in X$,当$T(x_1)=T(x_2)$,即$T(x_1-x_2)=T(x_1)-T(x_2)=0$时,由条件得$x_1-x_2=0$,即$x_1=x_2$,故$T$是单射.

(2)$\forall y_1,y_2\in Y$及$\forall\lambda_1,\lambda_2\in K$,$\exists x_1,x_2\in X$,s.t.$y_1=Tx_1,y_2=Tx_2$,即$x_1=T^{-1}(y_1),x_2=T^{-1}(y_2)$.于是有

$$T^{-1}(\lambda_1y_1+\lambda_2y_2)=T^{-1}[\lambda_1T(x_1)+\lambda_2T(x_2)]=T^{-1}[T(\lambda_1x_1+\lambda_2x_2)]=\lambda_1x_1+\lambda_2x_2=\lambda_1T^{-1}(y_1)+\lambda_2T^{-1}(y_2),$$

故$T^{-1}:Y\to X$是线性的.

7. 解 首先验证$\sigma:\square^{2\times2}\to\square^{2\times2}$是线性的,然后求其在即$B$下的矩阵$A$.

$\forall X_1,X_2\in\square^{2\times2},\forall k_1,k_2\in\square$,由$\sigma$的定义，有

$$B=\left(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$$

$$\sigma(k_1X_1+k_2X_2)=A_0(k_1X_1+k_2X_2)=k_1A_0X_1+k_2A_0X_2=k_1\sigma(X_1)+k_2\sigma(X_2),$$

故$\sigma:\square^{2\times2}\to\square^{2\times2}$是线性的.

关键是求基元$E_{11}=\begin{bmatrix}1&0\\0&0\end{bmatrix},E_{12}=\begin{bmatrix}0&1\\0&0\end{bmatrix},E_{21}=\begin{bmatrix}0&0\\1&0\end{bmatrix},E_{22}=\begin{bmatrix}0&0\\0&1\end{bmatrix}$的像在基$B$下的坐标：

$$\sigma\left(E_{11}\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}=aE_{11}+0E_{12}+cE_{21}+0E_{22},\text{即}\sigma\left(E_{11}\right)=\begin{pmatrix} a & 0 & c & 0 \end{pmatrix}^{T},$$

$$\sigma\left(E_{12}\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}=0E_{11}+aE_{12}+0E_{21}+cE_{22},\text{即}\sigma\left(E_{12}\right)=\begin{pmatrix} 0 & a & 0 & c \end{pmatrix}^{T},$$

$$\sigma\left(E_{21}\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}=bE_{11}+0E_{12}+dE_{21}+0E_{22},\text{即}\sigma\left(E_{21}\right)=\begin{pmatrix} b & 0 & d & 0 \end{pmatrix}^{T},$$

$$\sigma\left(E_{22}\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}=0E_{11}+bE_{12}+0E_{21}+dE_{22},\text{即}\sigma\left(E_{20}\right)=\begin{pmatrix} 0 & b & 0 & d \end{pmatrix}^{T},$$

$$\therefore A=\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

习 题 二

A

一、判断题

1. √；**2.**×；**3.** √；**4.** √；**5.**×；**6.** √；**7.**×；**8.**×；**9.** √；**10.** √；**11.**×；**12.**×.

二、填空题

1. x；**2.** n；**3.** $\lambda,(\lambda-1)^2,\lambda+\mathrm{i},\lambda-\mathrm{i}$；**4.** $\lambda-1,\lambda+1$；**5.** $\begin{bmatrix}2&0&0\\0&0&-4\\0&1&4\end{bmatrix}$；**6.** $\begin{bmatrix}2&0&0\\0&2&0\\0&1&2\end{bmatrix}$；**7.** O；

8. O；**9.** $\lambda-1$；**10.** 6.

三、单项选择题

1.(d)；**2.** (b)；**3.** (b)；**4.** (d)；**5.** (a).

B

1.解

（1）
$$\lambda E-A=\begin{bmatrix}\lambda-2&-1&0\\0&\lambda-2&-1\\0&0&\lambda-2\end{bmatrix}\xrightarrow[{[1,2]\atop[2,3]}]{}\begin{bmatrix}-1&0&\lambda-2\\\lambda-2&-1&0\\0&\lambda-2&0\end{bmatrix}\xrightarrow{[2+1(\lambda-2)]}$$

$$\begin{bmatrix}-1&0&\lambda-2\\0&-1&(\lambda-2)^2\\0&\lambda-2&0\end{bmatrix}\xrightarrow[{[3-1\cdot(\lambda-2)]}]{[3+2\cdot(\lambda-2)]}\begin{bmatrix}-1&0&0\\0&-1&(\lambda-2)^2\\0&0&(\lambda-2)^3\end{bmatrix}\xrightarrow[{[3+2\cdot(\lambda-2)^2]}]{[1\cdot(-1)]}$$

$$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&(\lambda-2)^3\end{bmatrix}\xrightarrow{[2\cdot(-1)]}\begin{bmatrix}1&&\\&1&\\&&(\lambda-2)^3\end{bmatrix},$$

$\therefore d_1(\lambda)=d_2(\lambda)=1,\ \ d_3(\lambda)=(\lambda-2)^3.$

（2）
$$\lambda E-A=\begin{bmatrix}\lambda&-1&-1\\-1&\lambda&-1\\-1&-1&\lambda\end{bmatrix}\xrightarrow{[1,3]}\begin{bmatrix}-1&-1&\lambda\\-1&\lambda&-1\\\lambda&-1&-1\end{bmatrix}\xrightarrow{{[2-1]\atop[3+1(\lambda)]}}$$

$$\begin{bmatrix}-1&-1&\lambda\\0&\lambda+1&-\lambda-1\\0&-\lambda-1&\lambda^2-1\end{bmatrix}\xrightarrow[{[1\cdot(-1)]\atop[2+3]}]{}\begin{bmatrix}1&-1&\lambda-1\\0&\lambda+1&0\\0&-\lambda-1&\lambda^2-\lambda-2\end{bmatrix}\xrightarrow{[3+2]}$$

$$\begin{bmatrix}1&-1&\lambda-1\\0&\lambda+1&0\\0&0&(\lambda+1)(\lambda-2)\end{bmatrix}\xrightarrow[{[2+1]\atop[3-7\cdot(\lambda-1)]}]{}\begin{bmatrix}1&&\\&\lambda+1&\\&&(\lambda+1)(\lambda-2)\end{bmatrix},$$

$\therefore d_1(\lambda)=1, d_2(\lambda)=\lambda+1, d_3(\lambda)=(\lambda+1)(\lambda-2).$

（3）$\lambda E-A=\begin{bmatrix}\lambda & -1 & 0 & 0\\ 0 & \lambda & -1 & 0\\ 0 & 0 & \lambda & -1\\ 5 & 4 & 3 & \lambda+2\end{bmatrix}\to\begin{bmatrix}-1 & 0 & 0 & \lambda\\ \lambda & -1 & 0 & 0\\ 0 & \lambda & -1 & 0\\ 4 & 3 & \lambda+2 & 5\end{bmatrix}$

$$\to\begin{bmatrix}-1 & 0 & 0 & \lambda\\ 0 & -1 & 0 & \lambda^2\\ 0 & \lambda & -1 & 0\\ 0 & 3 & \lambda+2 & 4\lambda+5\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & \lambda\\ 0 & -1 & 0 & \lambda^2\\ 0 & 0 & -1 & \lambda^3\\ 0 & 0 & \lambda+2 & 3\lambda^2+4\lambda+5\end{bmatrix}$$

$$\to\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \lambda^4+2\lambda^3+3\lambda^2+4\lambda+5\end{bmatrix},$$

$\therefore d_1(\lambda)=d_2(\lambda)=d(\lambda)=1,d_4(\lambda)=\lambda^4+2\lambda^3+3\lambda^2+4\lambda+5.$

（4）$\lambda E-A=\begin{bmatrix}\lambda-3 & -1 & 0 & 0\\ 4 & \lambda+1 & 0 & 0\\ -7 & -1 & \lambda-2 & -1\\ 7 & 6 & 1 & \lambda\end{bmatrix}\xrightarrow[{[1,2]}]{}\begin{bmatrix}-1 & \lambda-3 & 0 & 0\\ \lambda+1 & 4 & 0 & 0\\ -1 & -7 & \lambda-2 & -1\\ 6 & 7 & 1 & \lambda\end{bmatrix}$

$$\xrightarrow{\substack{[2+1-(\lambda+1)]\\ [3-1]\\ [4+1-(6)]}}\begin{bmatrix}-1 & \lambda-3 & 0 & 0\\ 0 & \lambda^2-2\lambda+1 & 0 & 0\\ 0 & -\lambda-4 & \lambda-2 & -1\\ 0 & 6\lambda-11 & 1 & \lambda\end{bmatrix}\xrightarrow[\substack{[2+1\cdot(\lambda-3)]\\ [1\cdot(-1)]}]{}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & (\lambda-1)^2 & 0 & 0\\ 0 & -(\lambda+4) & \lambda-2 & -1\\ 0 & 6\lambda-11 & 1 & \lambda\end{bmatrix}$$

$$\xrightarrow[{[2+3]}]{}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & (\lambda-1)^2 & 0 & 0\\ 0 & -6 & \lambda-2 & -1\\ 0 & 6\lambda-10 & 1 & \lambda\end{bmatrix}\xrightarrow{[4+3(\lambda)]}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & (\lambda-1)^2 & 0 & 0\\ 0 & -6 & \lambda-2 & -1\\ 0 & -10 & (\lambda-1)^2 & 0\end{bmatrix}$$

$$\xrightarrow[\substack{[2+4\cdot(-6)]\\ [3+4\cdot(\lambda-2)]}]{[4\cdot(\frac{-1}{10})]}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & (\lambda-1)^2 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 1 & \dfrac{(\lambda-1)^2}{-10} & 0\end{bmatrix}\xrightarrow[{[4\cdot(-1)]}]{[2-4\cdot(\lambda-1)^2]}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & \dfrac{(\lambda-1)^4}{10} & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & \dfrac{(\lambda-1)^2}{-10} & 0\end{bmatrix}$$

$$\xrightarrow[{[3+2\cdot\frac{(\lambda-2^2)}{10}]}]{}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & (\lambda-1)^4 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix}\xrightarrow[{[3,4]}]{[2,4]}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & (\lambda-1)^4\end{bmatrix},$$

$\therefore d_1(\lambda)=d_2(\lambda)=d_3(\lambda)=1,d_4(\lambda)=(\lambda-1)^4.$

2. 解 (1)$\because \det A(\lambda)=-(\lambda+2)^4$，$\therefore D_4(\lambda)=(\lambda+2)^4$,又$\because \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & \lambda+2 \\ 1 & \lambda+2 & 0 \end{vmatrix}=-1\neq 0$,

$\therefore D_3(\lambda)=1$,从而$D_1(\lambda)=D_2(\lambda)=1$.于是不变因子为$d_1(\lambda)=d_2(\lambda)=d_3(\lambda)=1$,

$d_4(\lambda)=(\lambda+2)^4$；初等因子组为$(\lambda+2)^4$.

(2) $B(\lambda)\cong\begin{bmatrix} 1 & 0 & \lambda+\alpha & 0 \\ 0 & 1 & 0 & \lambda+\alpha \\ \lambda+\alpha & 0 & 0 & 0 \\ 0 & \lambda+\alpha & 0 & 0 \end{bmatrix}\cong\begin{bmatrix} 1 & 0 & \lambda+\alpha & 0 \\ 0 & 1 & 0 & \lambda+\alpha \\ 0 & 0 & -(\lambda+\alpha)^2 & 0 \\ 0 & 0 & 0 & -(\lambda+\alpha)^2 \end{bmatrix}$

$$\cong\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & (\lambda+\alpha)^2 & \\ & & & (\lambda+\alpha)^2 \end{bmatrix},$$

故不变因子为 $d_1(\lambda)=d_2(\lambda)=1, d_3(\lambda)=(\lambda+\alpha)^2, d_4(\lambda)=(\lambda+\alpha)^2$;

初等因子组为 $(\lambda+\alpha)^2,(\lambda+\alpha)^2$.

(3)显然 $D_1(\lambda)=1, \det C(\lambda)=(\lambda+1)^3=D_3(\lambda)$,而

$$\mathrm{adj}C(\lambda)=\begin{bmatrix} (\lambda+1)(\lambda+5) & 0 & 8(\lambda+1) \\ 3(\lambda+1) & (\lambda+1)^2 & 6(\lambda+1) \\ -2(\lambda+1) & 0 & (\lambda+1)(\lambda-3) \end{bmatrix},$$

$\therefore D_2(\lambda)=\lambda+1$.

因此$d_1(\lambda)=1, d_2(\lambda)=\lambda+1, d_3(\lambda)=(\lambda+1)^2$; 初等因子组: $\lambda+1,(\lambda+1)^2$.

（4）由第 1 题（4）知$d_1(\lambda)=d_2(\lambda)=d_3(\lambda)=1, d_4(\lambda)=(\lambda+1)^4$.

也可这样解：由行列式的 Laplace 展开定理得

$$\det D(\lambda)=\begin{vmatrix} \lambda-3 & -1 \\ 4 & \lambda+1 \end{vmatrix}\cdot\begin{vmatrix} \lambda-2 & -1 \\ 1 & \lambda \end{vmatrix}=(\lambda-1)^4,$$

故$D_4(\lambda)=(\lambda-1)^4$;又$D(\lambda)$的左下角的三阶子式$\begin{vmatrix} 4 & \lambda+1 & 0 \\ -7 & -1 & \lambda-2 \\ 7 & 6 & 1 \end{vmatrix}=7\lambda^2-24\lambda+37$

与$D_4(\lambda)$是互质的,所以$D_3(\lambda)=1$,从而$D_2(\lambda)=D_1(\lambda)=1$.

因此$d_1(\lambda)=1, d_2(\lambda)=d_3(\lambda)=1, d_4(\lambda)=(\lambda-1)^4$;初等因子组: $(\lambda-1)^4$.

3.解（1）$\because$ $|\lambda E-A|=\begin{vmatrix}\lambda-1 & -2 & 0\\ 0 & \lambda-2 & 0\\ 2 & 1 & \lambda+1\end{vmatrix}=(\lambda+1)(\lambda-1)(\lambda-2)$，$\therefore$ $A\sim J=\begin{bmatrix}1 & & \\ & -1 & \\ & & 2\end{bmatrix}$.

（2）$\because$ $|\lambda E-A|=\begin{vmatrix}\lambda-3 & -7 & 3\\ 2 & \lambda+5 & -2\\ 4 & 10 & \lambda-3\end{vmatrix}=\begin{vmatrix}\lambda & -4 & 3\\ 0 & \lambda+3 & -2\\ \lambda+1 & \lambda+7 & \lambda-3\end{vmatrix}=\begin{vmatrix}\lambda & -4 & 3\\ 0 & \lambda+3 & -2\\ 1 & \lambda+11 & \lambda-6\end{vmatrix}$

$$=\begin{vmatrix}0 & -\lambda^2-11\lambda-4 & -\lambda^2+6\lambda+3\\ 0 & \lambda+3 & -2\\ 1 & \lambda+11 & \lambda-6\end{vmatrix}$$

$$=\lambda^3-\lambda^2+\lambda-1=(\lambda-1)(\lambda-\mathrm{i})(\lambda+\mathrm{i}),$$

$\therefore$ $A\sim J=\begin{bmatrix}1 & & \\ & \mathrm{i} & \\ & & -\mathrm{i}\end{bmatrix}$.

（3）$\because$ $\lambda E-A=\begin{bmatrix}\lambda-3 & -1 & 0 & 0\\ 4 & \lambda+1 & 0 & 0\\ -7 & -1 & \lambda-2 & -1\\ 17 & 6 & 1 & \lambda\end{bmatrix}\xrightarrow[{[1,2]}]{}\begin{bmatrix}-1 & \lambda-3 & 0 & 0\\ \lambda+1 & 4 & 0 & 0\\ -1 & -7 & \lambda-2 & -1\\ 6 & 17 & 1 & \lambda\end{bmatrix}$

$$\xrightarrow[\substack{[2+1\cdot(\lambda+1)]\\ [3+1\cdot(-1)]\\ [4+1\cdot(6)]}]{}\begin{bmatrix}-1 & \lambda-3 & 0 & 0\\ 0 & (\lambda-1)^2 & 0 & 0\\ 0 & -\lambda-4 & \lambda-2 & -1\\ 0 & 6\lambda-1 & 1 & \lambda\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & (\lambda-1)^2 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & -(\lambda-1)^2 & (\lambda-1)^2 & \lambda\end{bmatrix}$$

$$\to\begin{bmatrix}1 & & & \\ & 1 & & \\ & & (\lambda-1)^2 & \\ & & & (\lambda-1)^2\end{bmatrix},$$

$\therefore$初等因子组为$(\lambda-1)^2,(\lambda-1)^2$,于是$J_1=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$，$J_2=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$,故

$$J=\begin{bmatrix}J_1 & \\ & J_2\end{bmatrix}=\begin{bmatrix}1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & 1 & 1\end{bmatrix}.$$

（4）$\lambda E-A=\begin{bmatrix}\lambda & 0 & \cdots & 0 & 0\\ -1 & \lambda & \cdots & 0 & 0\\ \ddots & & & \vdots & \vdots\\ & \ddots & & \vdots & \vdots\\ & & \ddots & -1 & \lambda\end{bmatrix}$，$D_n(\lambda)=\det(\lambda E-A)=\lambda^n$，又有一个$n-1$阶子式

$$\begin{vmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & \ddots & \lambda \\ & & & & -1 \end{vmatrix} = (-1)^{n-1} \neq 0, \therefore D_{n-1}(\lambda) = \cdots = D_1(\lambda) = 1，故$$

$d_1(\lambda) = d_2(\lambda) = \cdots = d_{n-1}(\lambda) = 1, d_n(\lambda) = \lambda^n$;初等因子组为 λ^n,所以

$$A \sim J = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}.$$

(事实上，$\boldsymbol{A}$ 本身就是一个 Jordan 块)

4.解（1）由第 1 题（2）知 $\varphi_1(\lambda) = \lambda + 1, \varphi_2(\lambda) = (\lambda+1)(\lambda-2) = \lambda^2 - \lambda - 2$,所以

$$A \sim C = \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

（2）由第 1 题（3）知 $\varphi(\lambda) = \lambda^4 + 2\lambda^3 + 3\lambda^2 + 4\lambda + 5$，故 B 的有理标准是

$$C = \begin{bmatrix} 0 & 0 & 0 & -5 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

5.解 由 J 立即可知 A 的初等因子组为 $(\lambda-1)^2$, $\lambda - 2$, $(\lambda-2)^2$, 于是不变因子为 $d_1(\lambda) = d_2(\lambda) = d_3(\lambda) = 1, d_4(\lambda) = \lambda - 2, d_5(\lambda) = (\lambda-1)^2(\lambda-2)^2$.即 $\varphi_1(\lambda) = \lambda - 2$，

$$\varphi_2(\lambda) = \lambda^4 - 6\lambda^3 + 13\lambda^2 - 12\lambda + 4，故 C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & -13 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}.$$

6.解 （1）$f(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda-7 & -4 & 4 \\ -4 & \lambda+8 & 1 \\ 4 & 1 & \lambda+8 \end{vmatrix} = \begin{vmatrix} \lambda-7 & -4 & 4 \\ 0 & \lambda+9 & \lambda+9 \\ 4 & 1 & \lambda+8 \end{vmatrix}$

$$=\begin{vmatrix}\lambda-7 & -4 & 8\\ 0 & \lambda+9 & 0\\ 4 & 1 & \lambda+7\end{vmatrix}=(\lambda-9)(\lambda+9)^2.$$

因为$(A-9E)(A+9E)=\begin{bmatrix}-2 & 4 & -4\\ 4 & -17 & -1\\ -4 & -1 & -17\end{bmatrix}\begin{bmatrix}16 & 4 & -4\\ 4 & 1 & -1\\ -4 & -1 & 1\end{bmatrix}=O$,所以最小多项式为$m(\lambda)=(\lambda-9)(\lambda+9)$.

（2）$D_3(\lambda)=\det(\lambda E-B)=\begin{vmatrix}\lambda & -1 & 0\\ 0 & \lambda & -1\\ -2 & -3 & \lambda\end{vmatrix}=\lambda^3-3\lambda-2=(\lambda-2)(\lambda+1)^2$,$\because$有一个二阶子式$\begin{vmatrix}-1 & 0\\ \lambda & -1\end{vmatrix}=1\neq 0$,$\therefore D_1(\lambda)=D_2(\lambda)=1$.

因此,$m(\lambda)=d_3(\lambda)=(\lambda-2)(\lambda+1)^2$.

（3）对$\lambda E-C$施行初等变换得其Smith标准形

$$S(\lambda)=\mathrm{diag}(1,\ 1,\ 1,(\lambda-3)^2,(\lambda-3)^3),$$

$\therefore m(\lambda)=d_5(\lambda)=(\lambda-3)^3$.

7.证　若A可对角化,则A的最小多项式$m(\lambda)$无重零点,必要性得证. 若A有一个无重零点的零化多项式$\varphi(\lambda)$,则因为$\deg m(\lambda)\le\deg\varphi(\lambda)$,故$m(\lambda)$也无重零点,由定理 2.16 知$A$可对角化.

8. 证 (1) $\because A^2+A=2E$, $A^2+A-2E=O$, $\therefore \varphi(\lambda)=\lambda^2+\lambda-2=(\lambda-2)(\lambda+1)$是$A$的一个无重零点的零化多项式,故$A$可对角化.

(2) $\because A^m=E$, $\therefore \lambda^m-1$是A的零化多项式,其零点$\lambda_k=\mathrm{e}^{\frac{2k\pi}{m}\mathrm{i}}$　$(k=0,1,\cdots,m-1)$是互不相同的,故A可对角化.

习　题　三

A

一、判断题

1. √；**2.** √；**3.** √；**4.** √；**5.** √；**6.** √；**7.** √；**8.**×；**9.** √；**10.**×；**11.** √；**12.** √；**13.** √；14.　**15.** √；**16.** √；**17.** √；**18.** √；**19.** √；**20.**×；**21.** √；**22** √；**.23.**×；**24.** √；**25.** √.

二、填空题

1. 0；**2.** y_0；**3.** $\left(1,\frac{1}{2},\cdots,\frac{1}{n}\right)^{\mathrm{T}}$；**4.**　$\frac{1}{2}$；**5.** Banach；**6.** 1；**7.** 3；**8.** $\|A\|_1=5,\|A\|_\infty=2+\sqrt{2},\|A\|_F=\sqrt{14}$；

8. 3.

三、单项选择题

1.(c)；**2.** (c)；**3.** (b)；**4.** (a)；**5.** (b); 6.(c).

B

1. 证　仅验证三角不等式,其余是显然的.

设 $\boldsymbol{x}=(\xi_1,\cdots,\xi_n)^T$, $\boldsymbol{y}=(\eta_1,\cdots,\eta_n)^T$ 是 $\mathbf{R}^n$ 中的任意两个元素.

$$\|\boldsymbol{x}+\boldsymbol{y}\|_1=\sum_{i=1}^n|\xi_i+\eta_i|\le\sum_{i=1}^n(|\xi_i|+|\eta_i|)=\sum_{i=1}^n|\xi_i|+\sum_{i=1}^n|\eta_i|=\|\boldsymbol{x}\|_1+\|\boldsymbol{y}\|_1;$$

$$\|\boldsymbol{x}+\boldsymbol{y}\|_\infty=\max_{1\le i\le n}|\xi_1+\eta_i|\le\max_{1\le i\le n}\{|\xi_i|+|\eta_i|\}\le\max_{1\le i\le n}|\xi_i|+\max_{1\le i\le n}|\eta_i|$$

$$=\|\boldsymbol{x}\|_\infty+\|\boldsymbol{y}\|_\infty.$$

2. 证　因为 $\forall x,y\in C[a,\ b]$ 及 $\forall\alpha\in\mathbf{K}$,有

(N₁) $\|x\|_1=\int_a^b|x(t)|\mathrm{d}t\ge0$,显然若 $x=0$,即 $x(t)\equiv0$,则 $\|x\|_1=0$;反之,若 $\|x\|_1=0$,即 $\int_a^b|x(t)|\mathrm{d}t=0$,则由 $x(t)$ 的连续性,知 $x(t)\equiv0$,即 $x=0$;

(N₂) $\|\alpha x\|_1=\int_a^b|\alpha x(t)|\mathrm{d}t=|\alpha|\int_a^b|x(t)|\mathrm{d}t=|\alpha|\|x\|_1$;

(N₃) $\|x+y\|_1=\int_a^b|x(t)+y(t)|\mathrm{d}t\le\int_a^b|x(t)|\mathrm{d}t+\int_a^b|y(t)|\mathrm{d}t=\|x\|_1+\|y\|_1$;

所以 $\|\cdot\|_1$ 是 $C[a,\ b]$ 上的范数.

3.解　$\|x\|_1=|1|+|-\mathrm{i}|+|1+\mathrm{i}|=2+\sqrt{2},\|x\|_2=\sqrt{|1|^2+|-\mathrm{i}|^2+|1+\mathrm{i}|^2}=2,\|x\|_\infty=\max\{|1|,|-\mathrm{i}|,|1+\mathrm{i}|\}=\sqrt{2}.$

4.解

$$\|A\|_\infty=\max\{|1|+|0|+|-1|,|2|+|1|+|0|,|-\mathrm{i}|+|-1|+|1-\mathrm{i}|\}=\max\{2,3,2+\sqrt{2}\}=2+\sqrt{2},$$
$$\|A\|_1=\max\{|1|+|2|+|-\mathrm{i}|,|0|+|1|+|-1|,|-1|+|0|+|1-\mathrm{i}|\}=\max\{4,2,1+\sqrt{2}\}=4.$$

5.证　(1)设 $\lim\limits_{n\to\infty}x_n=x\in X$,又 $\lim\limits_{n\to\infty}x_n=y\in Y$,只需证明 $x=y$ 即可.

$$\because 0 \le \|x-y\| = \lim_{n\to\infty}\|x-y\| = \lim_{n\to\infty}\|x-x_n+x_n-y\|$$
$$\le \lim_{n\to\infty}\{\|x_n-x\|+\|x_n-y\|\} = \lim_{n\to\infty}\|x_n-x\|+\lim_{n\to\infty}\|x_n-y\| = 0+0=0,$$
$\therefore \|x-y\|=0$,故$x-y=0$,即$x=y$.

（2）设$\lim\limits_{n\to\infty}x_n = x\in X$,则对$\varepsilon=1$,$\exists N\in\square$,使得当$n>N$时,恒有$\|x_n-x\|\le 1$,从而有$\|x_n\|-\|x\|\le\|x_n-x\|\le 1$,即$\|x_n\|\le\|x\|+1$.

取$M=\max\{\|x_1\|,\|x_2\|,\cdots,\|x_N\|,\|x\|+1\}$,则$\forall n\in\square$，有$\|x_n\|\le M$,故$(x_n)$有界.

6.证 设x是X中任意一点,(x_n)是X中收敛于x的任一序列.

由$f:X\to Y$连续,知在Y中有$\lim\limits_{n\to\infty}f(x_n)=f(x)$;又由$g:Y\to Z$连续,知在$Z$中有$\lim\limits_{n\to\infty}g\big(f(x_n)\big)$ $=g\big(f(x)\big)$.即$\lim\limits_{n\to\infty}(g\circ f)(x_n)=(g\circ f)(x_n)$,$\therefore g\circ f:X\to Z$在点$x$处连续.

由$x\in X$的任意性,知$g\circ f:X\to Z$是连续映射.

7. 证 由于(x_n)和(y_n)都是X中的Cauchy序列，则$\forall\varepsilon>0$,$\exists N_1,N_2\in\mathbf{N}$,使得当$n,m>N_1$时，$\|x_n-x_m\|<\dfrac{\varepsilon}{2}$； 当$n,m>N_2$时，$\|y_n-y_m\|<\dfrac{\varepsilon}{2}$.

令$N=\max\{N_1,N_2\}$,则当$m,n>N$时,有

$$\big|\,\|x_n-y_n\|-\|x_m-y_m\|\,\big|\le\|(x_n-y_n)-(x_m-y_m)\|$$
$$\le\|x_nx_m\|+\|y_n-y_m\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

这表明$(\|x_n-y_n\|)$是**R**中Cauchy的序列,由**R**的完备性知,数列$(\|x_n-y_n\|)$收敛.

8.证 (1)$\forall f\in C^1[0,\ 1]$,且$f\ne 0$,即$\exists x_0\in[0,\ 1]$,使得$|f(x_0)|>0$,故$\|f\|_d\ge\max\limits_{0\le x\le 1}|f(x)|\ge|f(x_0)|>0$,
即$\|\cdot\|_d$满足(N_1).

$$\forall f\in C^1[0,\ 1],\forall\lambda\in\square,\|\lambda f\|_d=\max_{0\le x\le1}|\lambda f(x)|+\max_{0\le x\le1}\left|\frac{\mathrm{d}(\lambda f(x))}{\mathrm{d}x}\right|=|\lambda|\max_{0\le x\le1}|f(x)|+|\lambda|\max_{0\le x\le1}\left|\frac{\mathrm{d}(f(x))}{\mathrm{d}x}\right|=|\lambda|\|f\|_d,$$

即$\|\cdot\|_d$满足(N_2).

$$\forall f,g\in C^1[0,\ 1],\|f+g\|_d=\max_{0\le x\le1}|f(x)+g(x)|+\max_{0\le x\le1}\left|\frac{\mathrm{d}(f(x)+g(x))}{\mathrm{d}x}\right|$$
$$\le\max_{0\le x\le1}\big[|f(x)|+|g(x)|\big]+\max_{0\le x\le1}\left[\left|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|+\left|\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right|\right]$$
$$\le\max_{0\le x\le1}|f(x)|+\max_{0\le x\le1}|g(x)|+\max_{0\le x\le1}\left|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|+\max_{0\le x\le1}\left|\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right|$$
$$=\left[\max_{0\le x\le1}|f(x)|+\max_{0\le x\le1}\left|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|\right]+\left[\max_{0\le x\le1}|g(x)|+\max_{0\le x\le1}\left|\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right|\right]=\|f\|_d+\|g\|_d,$$

即$\|\cdot\|_d$满足(N_3).

所以,$\|\cdot\|_d$是$C^1[0,\ 1]$上的范数.

(2) $D: C^1[0,1] \to C[0,1]$ 显然是线性的.因为 $\forall f \in C^1[0,\ 1]$,有

$$\|Df\| = \max_{1\le x\le 1}\left|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right| \le \max_{0\le x\le 1}|f(x)| + \max_{0\le t\le 1}\left|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right| = \|f\|_d,$$

故 D 是有界的.

9. 证　由于 $\|\cdot\|$ 是 $\mathbf{C}^{n\times n}$ 上的方阵范数,故 $\forall A, B \in \mathbf{C}^{n\times n}$ 及 $\forall \alpha \in \mathbf{C}$,有

(1) $\|A\|_* = \|S^{-1}AS\| \ge 0$,并且 $\|A\|_* = \|S^{-1}AS\| = 0 \Leftrightarrow S^{-1}AS = O \Leftrightarrow A = O$;

(2) $\|\alpha A\|_* = \|S^{-1}\alpha AS = O\| = |\alpha|\|S^{-1}AS\| = |\alpha|\|A\|_*$;

(3) $\|A+B\|_* = \|S^{-1}(A+B)S\| = \|S^{-1}AS + S^{-1}BS\| \le \|S^{-1}AS\| + \|S^{-1}BS\| = \|A\|_* + \|B\|_*$;

(4) $\|AB\|_* = \|S^{-1}ABS\| = \|(S^{-1}AS)(S^{-1}BS)\| \le \|S^{-1}AS\|\|S^{-1}BS\| = \|A\|_*\|B\|_*$;

因此,$\|\cdot\|_*$ 是 $\mathbf{C}^{n\times n}$ 上的方阵范数.

10. 解　$\|A\|_F = \sqrt{|1|^2 + |\mathrm{i}|^2 + |-\mathrm{i}|^2 + |1|^2} = 2;$

$$\because f(\lambda) = \det(\lambda E - A) = \begin{vmatrix} \lambda - 1 & -\mathrm{i} \\ -\mathrm{i} & \lambda + 1 \end{vmatrix} = \lambda^2, \therefore \rho(A) = 0;$$

$$\because A^{\mathrm{H}}A = \begin{bmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{bmatrix}\begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2\mathrm{i} \\ -2\mathrm{i} & 2 \end{bmatrix}, |\lambda E - A^{\mathrm{H}}A| = \begin{vmatrix} \lambda - 2 & -2\mathrm{i} \\ 2\mathrm{i} & \lambda - 2 \end{vmatrix} = \lambda(\lambda - 4), \rho(A^{\mathrm{H}}A) = 4,$$

$\therefore \|A\|_2 = \sqrt{\rho(A^{\mathrm{H}}A)} = 2.$

11. 证　显然 $|\lambda| \le \|A\|$.$\because \lambda$ 是可逆阵 A 的特征值,则 $\dfrac{1}{\lambda}$ 是 A^{-1} 特征值,故 $\left|\dfrac{1}{\lambda}\right| \le \|A^{-1}\|$,即

$|\lambda| \ge \dfrac{1}{\|A^{-1}\|}$. $\therefore \dfrac{1}{\|A^{-1}\|} \le |\lambda| \le \|A\|$.

12.证　要证 $x_0 \in \mathcal{N}(T)$, 只需证明 $Tx_0 = 0$.

由$(x_n) \subset \mathcal{N}(T)$,知$Tx_n = 0\left(\forall n \in \square\right)$.于是当$\lim\limits_{n\to\infty} x_n = x_0$,且$T$是有界线性算子时,有

$$Tx_0 = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} 0 = 0,$$

故 $x_0 \in \mathcal{N}(T)$.

习 题 四

A

一、判断题

1.×；**2.**√；**3.**√；**4.**×；**5.**√；**6.**√；**7.**×；**8.**×.

二、填空题

1. $\begin{bmatrix} e^{x_2} & x_1 e^{x_2} & 0 \\ 0 & 1 & \cos x_3 \end{bmatrix}$；**2.** $\dfrac{-2t}{(t^2+1)^2}E$；**3.** 1；**4.** e^{3t}；**5.** $\begin{bmatrix} e^{-2t} & te^{-2t} & \frac{t^2}{2}e^{-2t} \\ & e^{-2t} & te^{-2t} \\ & & e^{-2t} \end{bmatrix}$；

6. $\begin{bmatrix} -\cos t & & \\ & \cos t & \\ & & 2\cos 2t \end{bmatrix}$；**7.** 1；**8.** e^{-3}.

B

1. 解 $\dfrac{dA(t)}{dt}=\begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix}$,

$$\frac{d}{dt}\left[\det A(t)\right]=\frac{d}{dt}\left[\cos^2 t+\sin^2 t\right]=0,$$

$$\det\left(\frac{dA(t)}{dt}\right)=\begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}=\sin^2 t+\cos^2 t=1.$$

2. 解 $f'(x)=\begin{bmatrix} e^{x_2} & x_1 e^{x_2} & 0 \\ 0 & 1 & \cos x_3 \end{bmatrix}$.

3. 解 $\int_0^1 A(t)dt=\begin{bmatrix} \int_0^1 e^t dt & \int_0^1 te^t dt \\ \int_0^1 dt & \int_0^1 2t dt \\ \int_0^1 \sin t dt & \int_0^1 \cos t dt \end{bmatrix}=\begin{bmatrix} e-1 & 1 \\ 1 & 1 \\ 1-\cos 1 & \sin 1 \end{bmatrix}$.

4. 证明（1）$\dfrac{df}{dt}=\dfrac{d}{dt}(x^T Ax)=\dfrac{dx^T}{dt}(Ax)+x^T\dfrac{d}{dt}(Ax)=(\dfrac{dx}{dt})^T Ax+x^T A\dfrac{dx}{dt}$

$$=(x^T A^T\frac{dx}{dt})^T+x^T A\frac{dx}{dt}=x^T A^T\frac{dx}{dt}+x^T A\frac{dx}{dt}=2x^T A\frac{dx}{dt};.$$

（2）$\dfrac{d}{dt}(x^T x)=\dfrac{dx^T}{dt}x+x^T\dfrac{dx}{dt}=(x^T\dfrac{dx}{dt})^T+x^T\dfrac{dx}{dt}=2x^T\dfrac{dx}{dt}$.

5. 证（1）若 $\lim\limits_{k\to\infty}A_k=A$，则 $\lim\limits_{k\to\infty}\|A_k-A\|_2=0$.

∵ $\|A_k^T-A^T\|_2=\|(A_k-A)^T\|_2=\|A_k-A\|_2$（可以证明 $\|A^H\|_2=\|A^T\|_2=\|\overline{A}\|_2=\|A\|_2$[1]），

∴ $\lim\limits_{k\to\infty}\|A_k^T-A^T\|_2=0$，即 $\lim\limits_{k\to\infty}A_k^T=A^T$.

同理可证 $\lim\limits_{k\to\infty}\overline{A_k}=\overline{A}$，由上已证的结果立即可得 $\lim\limits_{k\to\infty}A_k^H=A^H$.

（2） $\sum_{k=0}^{\infty} c_k (A^T)^k = \lim_{N\to\infty}\sum_{k=0}^{N} c_k (A^T)^k = \lim_{N\to\infty}\sum_{k=0}^{N} c_k (A^k)^T = \lim_{N\to\infty}(\sum_{k=0}^{N} c_k A^k)^T$

$= (\lim_{N\to\infty}\sum_{k=0}^{N} c_k A^k)^T = (\sum_{k=0}^{\infty} c_k A^k)^T$

6. 证 令 $\det(\lambda E - A) = \begin{vmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-1 & -1 \\ -1 & 1 & \lambda-3 \end{vmatrix} = (\lambda-2)^3 = 0$ 得 A 的全部特征值均为 **2.** 于是

$B = \frac{1}{3}A$ 的所有特征值都是 $\frac{2}{3}$,故 $\rho(B) = \frac{2}{3} < 1$,因此 $\lim_{k\to\infty} B^k = O$.

7. 证 方法一：当 $t=0$ 时,显然成立,故设 $t \neq 0$.记 $t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} = A$.

$$\det(\lambda E - A) = \lambda^2 + t^2 = (\lambda - \mathrm{i}t)(\lambda + \mathrm{i}t),\quad \lambda_1 = \mathrm{i}t, \lambda_2 = -\mathrm{i}t .$$

对 $\lambda_1 = \mathrm{i}t$,解方程 $(\mathrm{i}tE - A)x = 0$ 可得 $x_1 = \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}$；对 $\lambda_2 = -\mathrm{i}t$ 解方程 $(-\mathrm{i}tE - A)x = 0$

得 $x_2 = \begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix}$.

令 $P = \begin{bmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{bmatrix}$,则 P 可逆且 $P^{-1} = \begin{bmatrix} 1/2 & -\mathrm{i}/2 \\ 1/2 & \mathrm{i}/2 \end{bmatrix}$.

所以 $\mathrm{e}^{t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \mathrm{e}^A = P\mathrm{diag}(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{-\mathrm{i}t})P^{-1} = \begin{bmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{bmatrix}\begin{bmatrix} \mathrm{e}^{\mathrm{i}t} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t} \end{bmatrix}\begin{bmatrix} 1/2 & -\mathrm{i}/2 \\ 1/2 & \mathrm{i}/2 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2}(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}) & \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}) \\ \frac{-1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}) & \frac{1}{2}(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

方法二：记 $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\det(\lambda E - B) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$, $\sigma(B) = \{\mathrm{i}, -\mathrm{i}\}$. B 的最小多项式

$\varphi(\lambda) = \lambda^2 + 1, \deg\varphi(\lambda) = 2$. 故设 $\mathrm{e}^{tB} = a_0(t)E + a_1(t)B$.

∵ $\mathrm{e}^{t\lambda}$ 与 $a_0(t) + a_1(t)\lambda$ 在 $\sigma(B)$ 上的值相等,即

$$\begin{cases} a_0(t) + \mathrm{i}a_1(t) = \mathrm{e}^{\mathrm{i}t} \\ a_0(t) - \mathrm{i}a_1(t) = \mathrm{e}^{-\mathrm{i}t} \end{cases},$$

∴ $a_0(t) = \frac{\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}}{2} = \cos t$， $a_1(t) = \frac{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}} = \sin t$.

因此 $e^{t\begin{bmatrix}0&1\\-1&0\end{bmatrix}}=\cos tE+\sin tB=\begin{bmatrix}\cos t&\sin t\\-\sin t&\cos t\end{bmatrix}$.

8. 解 $\because A$是Jordan块, $\therefore e^{tA}=\begin{bmatrix}e^{-t}&&\\te^{-t}&e^{-t}&\\\frac{t^2}{2}e^{-t}&te^{-t}&e^{-t}\end{bmatrix}$.

9. 解 $\det(\lambda E-A)=\begin{vmatrix}\lambda-2&-1&-4\\0&\lambda-2&0\\0&-3&\lambda-1\end{vmatrix}=(\lambda-2)^2(\lambda-1)$.

$\because (A-2E)(A-E)\neq O$，$\therefore$ A 的最小多项式 $\varphi(\lambda)=(\lambda-2)^2(\lambda-1)$. $\deg\varphi(\lambda)=3$，故设 $f(At)=a_0(t)E+a_1(t)A+a_2(t)A^2=T(At)$. 由 $f(\lambda t)$ 与 $T(\lambda t)$ 在 $\sigma(A)=\{1,2\}$ 上的值相等,于是

(1)对 $f(At)=e^{At}$ 有

$$\begin{cases}a_0(t)+a_1(t)+a_2(t)=e^t\\a_0(t)+2a_1(t)+4a_2(t)=e^{2t}\\a_1(t)+4a_2(t)=te^{2t}\end{cases},\text{解得}\begin{cases}a_0(t)=4e^t-3e^{2t}+2te^{2t}\\a_1(t)=-4e^t+4e^{2t}-3te^{2t}\\a_2(t)=e^t-e^{2t}+te^{2t}\end{cases}$$

所以 $e^{tA}=(4e^t-3e^{2t}+2te^{2t})\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+(-4e^t+4e^{2t}-3te^{2t})\begin{bmatrix}2&1&4\\0&2&0\\0&3&1\end{bmatrix}$

$$+(e^t-e^{2t}+te^{2t})\begin{bmatrix}4&16&12\\0&4&0\\0&9&1\end{bmatrix}=\begin{bmatrix}e^{2t}&12e^t-12e^{2t}+13te^{2t}&-4e^t+4e^{2t}\\0&e^{2t}&0\\0&-3e^t+3e^{2t}&e^t\end{bmatrix}$$

(2)对 $f(At)=\sin(At)$ 有

$$\begin{cases}a_0(t)+a_1(t)+a_2(t)=\sin t\\a_0(t)+2a_1(t)+4a_2(t)=\sin 2t\\a_1(t)+4a_2(t)=t\cos 2t\end{cases}\text{，解得}\begin{cases}a_0(t)=4\sin t-3\sin 2t+2t\cos 2t\\a_1(t)=-4\sin t+4\sin 2t-3t\cos 2t\\a_2(t)=\sin t-\sin 2t+t\cos 2t\end{cases}.$$

$\therefore \sin(At)=a_0(t)E+a_1(t)A+a_2(t)A^2$

$$=\begin{bmatrix}\sin 2t&12\sin t-12\sin 2t+13t\cos 2t&-4\sin t+4\sin 2t\\0&\sin 2t&0\\0&-3\sin t+3\sin 2t&\sin t\end{bmatrix}$$

（注）可利用**(1)**的结果求**(2)**（或 $\cos(At)$）：在**(1)**中分别以 $\mathrm{i}t$ 和 $-\mathrm{i}t$ 替代 t 得 $e^{\mathrm{i}tA}$ 和 $e^{-\mathrm{i}tA}$，

再由公式 $\sin(At)=\dfrac{e^{itA}-e^{-itA}}{2i}$ (或 $\cos(At)=\dfrac{e^{itA}+e^{-itA}}{2}$) 即得.

10. 解 $\det(\lambda E-A)=\begin{vmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & -1 & \lambda+2\end{vmatrix}=\lambda(\lambda+1)^2$ 且 $A(A-E)\neq O$,故 A 的最小多项式

$\phi(\lambda)=\lambda(\lambda+1)^2$， $\deg\varphi(\lambda)=3$，故设 $f(At)=a_0(t)E+a_1(t)A+a_2(t)A^2=T(At)$,即

$$f(At)=a_0(t)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+a_1(t)\begin{bmatrix}0&-1&0\\0&0&-1\\0&1&-2\end{bmatrix}+a_2(t)\begin{bmatrix}0&0&1\\0&-1&2\\0&-2&3\end{bmatrix}$$

$$=\begin{bmatrix}a_0(t) & -a_1(t) & a_2(t)\\ 0 & a_0(t)-a_2(t) & -a_1(t)+2a_2(t)\\ 0 & a_1(t)-2a_2(t) & a_0(t)-2a_1(t)+3a_2(t)\end{bmatrix}.$$

由 $f(\lambda t)$ 与 $T(\lambda t)$ 在 A 上的谱值相等,于是

(1)对 $f(At)=e^{At}$ 有

$$\begin{cases}a_0(t)=1\\ a_0(t)-a_1(t)+a_2(t)=e^{-t}\\ a_1(t)-2a_2(t)=te^{-t}\end{cases},\text{解得}\begin{cases}a_0(t)=1\\ a_1(t)=2-2e^{-t}-te^{-t}\\ a_2(t)=1-e^{-t}-te^{-t}\end{cases}$$

$$\therefore e^{At}=\begin{bmatrix}a_0(t) & -a_1(t) & a_2(t)\\ 0 & a_0(t)-a_2(t) & -a_1(t)+2a_2(t)\\ 0 & a_1(t)-2a_2(t) & a_0(t)-2a_1(t)+3a_2(t)\end{bmatrix}$$
$$=\begin{bmatrix}1 & -2+2e^{-t}+te^{-t} & 1-e^{-t}+te^{-t}\\ 0 & e^{-t}+te^{-t} & -te^{-t}\\ 0 & te^{-t} & e^{-t}-te^{-t}\end{bmatrix}.$$

(2)对 $f(At)=\sin(At)$ 有

$$\begin{cases}a_0(t)=0\\ a_0(t)-a_1(t)+a_2(t)=-\sin t\\ a_1(t)-2a_2(t)=t\cos t\end{cases},\ \text{解得}\begin{cases}a_0(t)=0\\ a_1(t)=2\sin t-t\cos t\\ a_2(t)=\sin t-t\cos t\end{cases}.$$

$$\therefore \sin(At)=\begin{bmatrix}a_0(t) & -a_1(t) & a_2(t)\\ 0 & a_0(t)-a_2(t) & -a_1(t)+2a_2(t)\\ 0 & a_1(t)-2a_2(t) & a_0(t)-2a_1(t)+3a_2(t)\end{bmatrix}$$

$$=\begin{bmatrix}0 & -2\sin t+t\cos t & \sin t-t\cos t\\ 0 & -\sin t+t\cos t & -t\cos t\\ 0 & t\cos t & -\sin t-t\cos t\end{bmatrix}$$

11. 解 $\det(e^A)=e^{\mathrm{tr}A}=e^{2i+3-3}=e^{2i}$.

12. 解 此处 $A=\begin{bmatrix}-7 & -7 & 5\\ -8 & -8 & -5\\ 0 & -5 & 0\end{bmatrix}$, $x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_2(t)\end{bmatrix}$, $C=\begin{bmatrix}3\\ -2\\ 1\end{bmatrix}$.

因为 $\det(\lambda E-A)=\begin{vmatrix}\lambda+7 & 7 & -5\\ 8 & \lambda+8 & 5\\ 0 & 5 & \lambda\end{vmatrix}=(\lambda-5)(\lambda+5)(\lambda+15), \deg\varphi(\lambda)=3$,

故设 $e^{At}=a_0(t)E+a_1(t)A+a_2(t)A^2=T(At)$.

由 $e^{\lambda t}$ 与 $T(\lambda t)$ 在 $\sigma(A)=\{5,-5,-15\}$ 上的值相同, 得方程组

$$\begin{cases}a_0(t)+5a_1(t)+25a_2(t)=e^{5t}\\ a_0(t)-5a_1(t)+25a_2(t)=e^{-5t}\\ a_0(t)-15a_1(t)+225a_2(t)=e^{-15t}\end{cases},$$

解得 $\begin{cases}a_0(t)=\frac{1}{8}(3e^{5t}+6e^{-5t}-e^{-15t})\\ a_1(t)=\frac{1}{10}(e^{5t}-e^{-5t})\\ a_2(t)=\frac{1}{200}(e^{5t}-2e^{-5t}+e^{-15t})\end{cases}$; 于是

$$e^{At}=a_0(t)\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}+a_1(t)\begin{bmatrix}-7 & -7 & 5\\ -8 & -8 & -5\\ 0 & -5 & 0\end{bmatrix}+a_2(t)\begin{bmatrix}105 & 80 & 0\\ 120 & 145 & 0\\ 40 & 40 & 25\end{bmatrix}$$

$$=\frac{1}{10}\begin{bmatrix}2e^{5t}+4e^{-5t}+4e^{-15t} & -3e^{5t}-e^{-5t}+4e^{-15t} & 5e^{5t}-5e^{-5t}\\ -2e^{5t}-4e^{-5t}+6e^{-15t} & 3e^{5t}-e^{-5t}+6e^{-15t} & -5e^{5t}+5e^{-5t}\\ 2e^{5t}-4e^{-5t}+2e^{-15t} & -3e^{5t}-e^{-5t}+2e^{-15t} & 5e^{5t}+5e^{-5t}\end{bmatrix}.$$

所以, 解为 $x(t)=e^{At}C=\frac{1}{10}\begin{bmatrix}17e^{5t}+9e^{-5t}+4e^{-15t}\\ -17e^{5t}-9e^{-5t}+6e^{-15t}\\ 17e^{5t}-9e^{-5t}+2e^{-15t}\end{bmatrix}$, 即

$$\begin{cases}x_1(t)=\dfrac{1}{10}(17e^{5t}+9e^{-5t}+4e^{-15t})\\ x_2(t)=\dfrac{1}{10}(-17e^{5t}-9e^{-5t}+6e^{-15t})\\ x_3(t)=\dfrac{1}{10}(17e^{5t}-9e^{-5t}+2e^{-15t})\end{cases}.$$

习 题 五

A

一、判断题

1. √；**2.** √；**3.** √；**4.** √；**5.** √；**6.**×；**7.** √；**8.** √；**9.**×；**10.** √；**11.** √；**12.**×；**13.** √；14. √ **15.** √.

二、填空题

1. 0 ；**2.** $\{0\}$ ；**3.** $\operatorname{span}A$ ；**4.** 1 ；**5.** 3 ；**6.** O ；**7.** $d_1(\lambda)=1, d_2(\lambda)=\lambda-1, d_3(\lambda)=(\lambda-1)(\lambda-2)$ ；

8. 实；**9.** 0 ；**10.** 1 ；**11.** $a=1, b=-\dfrac{\mathrm{i}}{\sqrt{6}}, c=\dfrac{\mathrm{i}}{\sqrt{3}}$.

三、单项选择题

1.(d)；**2.** (c)；**3.** (c).

B

1.证 (1)$\forall x=(\xi_1,\xi_2,\cdots,\xi_n)^T, y=(\eta_1,\eta_2,\cdots,\eta_n)^T, z=(\varsigma_1,\varsigma_2,\cdots,\varsigma_n)^T\in\mathbb{R}^n$, 及$\forall\lambda,\mu\in\mathbb{R}$，有

$$(\mathrm{I}_1) <\lambda x+\mu y, z>_k=\sum_{k=1}^{n}k(\lambda\xi_k+\mu\eta_k)\varsigma_k=\lambda\sum_{k=1}^{n}k\xi_k\varsigma_k+\mu\sum_{k=1}^{n}k\eta_k\varsigma_k=\lambda<x,z>_k+\mu<y,z>_k;$$

$$(\mathrm{I}_2) <x,y>_k=\sum_{k=1}^{n}k\xi_k\eta_k=\sum_{k=1}^{n}k\eta_k\xi_k=<y,x>_k;$$

$$(\mathrm{I}_3) <x,x>_k=\sum_{k=1}^{n}k\xi_k^2\geq 0, \text{且}$$

$$<x,x>_k=\sum_{k=1}^{n}k\xi_k^2=0\Leftrightarrow\forall k=1,2,\cdots,n,\text{有}\xi_k^2=0\Leftrightarrow\forall k=1,2,\cdots,n,\text{有}\xi_k=0\Leftrightarrow x=0;$$

故 $<\cdot,\cdot>_k$ 是$\mathbb{R}^n$上的一种内积.

(2)$\forall A=\left[a_{ij}\right], B=\left[b_{ij}\right], C=\left[c_{ij}\right]\in\mathbb{C}^{n\times n}$, 及$\forall\lambda,\mu\in\mathbb{C}$, 有

$$(\mathrm{I}_1) <\lambda A+\mu B, C>=\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda a_{ij}+\mu b_{ij})\overline{c_{ij}}=\lambda\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\overline{c_{ij}}+\mu\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}\overline{c_{ij}}=\lambda<A,C>+\mu<B,C>;$$

$$(\mathrm{I}_2) <A,B>=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\overline{b_{ij}}=\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\overline{a_{ij}}b_{ij}}=\overline{\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a_{ij}}b_{ij}}=\overline{<B,A>};$$

$$(\mathrm{I}_3) <A,A>=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\overline{a_{ij}}=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|^2\geq 0, \text{且}$$

$$<A,A>=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|^2=0\Leftrightarrow\forall i,j=1,2,\cdots,n,\text{有}\left|a_{ij}\right|^2=0\text{即}a_{ij}=0\Leftrightarrow A=O;$$

故$<\cdot,\cdot>$是$\mathbb{C}^{n\times n}$上的一种内积. 且$\sqrt{<A,A>}=\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)^{\frac{1}{2}}=\|A\|_F$.

2. 证　右端$=\frac{1}{4}(<x+y,x+y>-<x-y,x-y>)$

$$=\frac{1}{4}(<x,x>+<x,y>+<y,x>+<y,y>$$

$$-<x,x>+<x,y>+<y,x>-<y,y>)$$

$$=\frac{1}{4}(4<x,y>)=\text{左端}.$$

3.证 （1）若$x\in B^{\perp}$,则$\forall y\in B$皆有$x\perp y$,由假设$A\subset B$,于是对每一个$y\in A$皆有$x\perp y$,

即$x\in A^{\perp}$,故$B^{\perp}\subset A^{\perp}$.

（2）若$x\in A$,则$\forall y\in A^{\perp}$皆有$x\perp y$,故$x\in(A^{\perp})^{\perp}$,于是$A\subset(A^{\perp})^{\perp}$.

4.解 显然A是实对称矩阵. $\because \det A_1=2>0,\det A_2=11>0,\det A_3=38>0,\therefore A$正定. 其余略.

5. 证 “$\Rightarrow$”: 若$A\in\mathbb{C}^{n\times n}$正定,则$\det A_n=\det A>0$,故$A$非奇异.

“$\Leftarrow$”: 若A非奇异,则$\det A=\prod_{i=1}^{n}\lambda_i\neq 0$,从而$\lambda_i\neq 0(i=1,2,\cdots,n)$. 又因为$A$半正定,

故有$\lambda_i\geq 0$,于是$\lambda_i>0(i=1,2,\cdots,n)$,所以$A$是正定的.

6.证 先验证A^2是 Hermite 矩阵.

$\because (A^2)^H A^2=(AA)^H AA=A^H A^H AA=A^H AA^H A=AA^H AA^H=AAA^H A^H=A^2(A^2)^H,\therefore A^2$是Hermite矩阵.

再证A^2是正定的.

设$\lambda_1,\lambda_2,\cdots\lambda_n$是$A$的$n$个特征值, 由$A$是Hermite矩阵且可逆知, $\lambda_i\in\mathbb{R}$且$\lambda_i\neq 0(i=1,2,\cdots,n)$.
从而A^2的所有特征值$\lambda_i^2>0(i=1,2,\cdots,n)$,故$A^2$是正定矩阵.

7. 解 （1）令$|\lambda E-A|=\begin{vmatrix}\lambda & -\mathrm{i} & -1\\ \mathrm{i} & \lambda & 0\\ -1 & 0 & \lambda\end{vmatrix}=\lambda^3-2\lambda=0$得$\lambda_1=0$, $\lambda_2=\sqrt{2}$, $\lambda_3=-\sqrt{2}$,由此判定A

不是正定的.

对$\lambda_1=0$解方程组$-Ax=0$,即$\begin{bmatrix}0 & -\mathrm{i} & -1\\ \mathrm{i} & 0 & 0\\ -1 & 0 & \mathrm{i}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$,亦即$\begin{cases}\mathrm{i}\xi_2+\xi_3=0\\ \xi_1=0\end{cases}$,

得$\begin{cases}\xi_1=0\\ \xi_2=\mathrm{i}\xi_3\end{cases}$. 若取$\xi_3=1$,则有$x_1=\begin{bmatrix}0\\ \mathrm{i}\\ 1\end{bmatrix}$.

对 $\lambda_2=\sqrt{2}$ 解 $(\sqrt{2}E-A)x=0$ 可得 $x_2=\begin{bmatrix}\sqrt{2}\\ -\mathrm{i}\\ 1\end{bmatrix}$.

对 $\lambda_3=-\sqrt{2}$ 解 $(-\sqrt{2}E-A)x=0$ 可得 $\boldsymbol{x}_3=\begin{bmatrix}-\sqrt{2}\\ -\mathrm{i}\\ 1\end{bmatrix}$.

由于 x_1,x_2,x_3 分别对应于 A 的不同特征值,故彼此正交.将它们单位化,得

$$\alpha_1=\begin{bmatrix}0\\ \mathrm{i}/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix},\quad \alpha_2=\begin{bmatrix}1/\sqrt{2}\\ -\mathrm{i}/2\\ 1/2\end{bmatrix},\quad \alpha_3=\begin{bmatrix}-1/\sqrt{2}\\ -\mathrm{i}/2\\ 1/2\end{bmatrix}.$$

令 $U=[\alpha_1,\alpha_2,\alpha_3]=\begin{bmatrix}0 & 1/\sqrt{2} & -1/\sqrt{2}\\ \mathrm{i}/\sqrt{2} & -\mathrm{i}/2 & -\mathrm{i}/2\\ 1/\sqrt{2} & 1/2 & 1/2\end{bmatrix}, U^H=\begin{bmatrix}0 & -\mathrm{i}\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & \mathrm{i}/2 & 1/2\\ -1/\sqrt{2} & \mathrm{i}/2 & 1/2\end{bmatrix}$,

则 $U^HAU=\begin{bmatrix}0 & & \\ & \sqrt{2} & \\ & & -\sqrt{2}\end{bmatrix}$.

习 题 六

A

一、判断题

1.×；2.√；3.×；4.×；5.×；6.×；7.×；8.√；9.×.

二、填空题

1. $\begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$；**2.** $\begin{cases} x_1^{(k+1)} = \frac{1}{4}(\quad -3x_2^{(k)} \quad +24) \\ x_2^{(k+1)} = \frac{1}{4}(-3x_1^{(k+1)} \quad +x_3^{(k)}+30)\,(k=0,1,2,\cdots) \\ x_3^{(k+1)} = \frac{1}{4}(\quad x_2^{(k+1)} \quad -24) \end{cases}$；**3.** $(D-L)^{-1}U$；

4. Seidel,Jacobi **.**

B

1. 解（1）$A^{-1}=\begin{bmatrix} -10000 & 10000 \\ 5000.5 & -5000 \end{bmatrix}, \|A\|_\infty = 3.0001, \|A^{-1}\|_\infty = 20000$,

$\therefore\ \mathrm{cond}_\infty A = 60002$.

（2）$B^{-1}=\dfrac{1}{0.2106}\begin{bmatrix} 1.38 & -2.18 \\ -2.79 & 4.56 \end{bmatrix}, \|B\|_1 = 7.35$，$\|B^{-1}\|_1 = 32.00$，

$\therefore\ \mathrm{cond}_1 B = 235.2$.

（3）$C=\begin{bmatrix} 100 & 99 \\ 99 & 98 \end{bmatrix}$是实对称矩阵,故$\mathrm{cond}_2 C = \dfrac{\max\{|\lambda_1|,|\lambda_2|\}}{\min\{|\lambda_1|,|\lambda_2|\}}$(见6-3).令$\begin{vmatrix} \lambda-100 & -99 \\ -99 & \lambda-98 \end{vmatrix}$

$=\lambda^2-198\lambda-1=0$,得$\lambda_1 = 99+\sqrt{9802}, \lambda_2 = 99-\sqrt{9802}, \therefore\ \mathrm{cond}_2 C = \dfrac{|\lambda_1|}{|\lambda_0|} = \dfrac{99+\sqrt{9802}}{99-\sqrt{9802}} \approx 39206.$

2. 解（1）对增广矩阵施行行的初等变换

$$\begin{bmatrix} 2 & 1 & 1 & 4 \\ 3 & 1 & 2 & 6 \\ 1 & 2 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 4 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 4 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

得到等价的上三角方程组$\begin{cases} 2x_1+x_2+x_3=4 \\ -\frac{1}{2}x_2+\frac{1}{2}x_3=0 \\ 3x_3=3 \end{cases}$.

进行回代,得方程组的解为:

$$x_3 = 3/3 = 1,\quad x_2 = -\frac{1}{2}x_3/(-\frac{1}{2}) = 1,\quad x_1 = (4-x_2-x_3)/2 = 1.$$

故解为 $x=(1,1,1)^T$.

（2）对增广矩阵施行初等行变换

$$\begin{bmatrix}1&1&0&3&4\\2&1&-1&1&-1\\3&-1&-1&2&-3\\-1&2&3&-1&4\end{bmatrix}\to\begin{bmatrix}1&1&0&3&4\\0&-1&-1&-5&-9\\0&-4&-1&-7&-15\\0&3&3&2&8\end{bmatrix}\to\begin{bmatrix}1&1&0&3&4\\0&-1&-1&-5&-9\\0&0&3&13&21\\0&0&0&-13&-19\end{bmatrix}$$

得到等价的上三角方程组$\begin{cases}x_1+x_2+3x_4=4\\-x_2-x_3-5x_4=-9\\3x_3+13x_4=21\\-13x_4=-19\end{cases}$.

进行回代,得方程组的解： $x_4=-19/(-13)=\dfrac{19}{13}$, $x_3=(21-13x_4)/3=\dfrac{2}{3}$,

$x_2=-(-9+x_3+5x_4)=\dfrac{40}{39}$, $x_1=4-x_2-3x_4=-\dfrac{55}{39}$,故解为$x=\left(\dfrac{-55}{39},\dfrac{40}{39},\dfrac{2}{3},\dfrac{19}{13}\right)^T$.

3. 解 首先用顺序Gauss消去法.对增广矩阵施行初等行变换：

$$\begin{bmatrix}0.012&0.01&0.167&0.6781\\1&0.8334&5.91&12.1\\3200&1200&4.2&98.1\end{bmatrix}$$

$$\to\begin{bmatrix}0.012&0.01&0.167&0.6781\\0&0.6667\times10^{-4}&-0.8007\times10&-0.4441\times10^{2}\\0&-0.1467\times10^{4}&-0.4453\times10^{5}&-0.1798\times10^{6}\end{bmatrix}$$

$$\to\begin{bmatrix}0.012&0.01&0.167&0.6781\\0&0.6667\times10^{-4}&-0.8007\times10&-0.4441\times10^{2}\\0&0&-0.1762\times10^{9}&-0.9774\times10^{9}\end{bmatrix},$$

经回代得$x_3=5.547$， $x_2=72.43$， $x_1=-81.05$. 此时,$\|Ax-b\|_2=0.1743\times10^6$.

下面用列主元素Gauss消去法.对增广矩阵施行初等行变换（下画横线者为主元素）

$$\begin{bmatrix}0.012&0.01&0.167&0.6781\\1&0.8334&5.91&12.1\\\underline{3200}&1200&4.2&981\end{bmatrix}$$

$$\to\begin{bmatrix}3200&1200&4.2&981\\0&\underline{0.4584}&0.5909\times10&0.1179\times10^{2}\\0&0.5500\times10^{-2}&0.1670&0.6744\end{bmatrix}$$

$$\to\begin{bmatrix}3200&1200&4.2&981\\0&0.4584&0.5909\times10&0.1179\times10^{2}\\0&0&0.9610\times10^{-1}&0.5329\end{bmatrix},$$

经回代得$x_3=5.545$, $x_2=-45.76$, $x_1=17.46$. 此时,$\|\boldsymbol{Ax}-\boldsymbol{b}\|_2=2.289$.

列主元素Gauss消去法比顺序Gauss消去法的精度高.

4. 解 Jacobi迭代格式为

$$\begin{cases} x_1^{(k+1)} = \frac{1}{20}[\qquad -2x_2^{(k)} - 3x_3^{(k)} + 24] \\ x_2^{(k+1)} = \frac{1}{8}[-x_1^{(k)} \qquad - x_3^{(k)} + 12] \\ x_3^{(k+1)} = \frac{1}{15}[-2x_1^{(k)} + 3x_2^{(k)} \qquad + 30] \end{cases} (k = 0,1,2,\cdots).$$

计算结果如下表：

k	1	2	3	4	5	6	7	8
$x_1^{(k)}$	1.200000	0.750000	0.769000	0.768125	0.767330	0.767363	0.767355	0.767354
$x_2^{(k)}$	1.500000	1.100000	1.138750	1.138875	1.138332	1.138414	1.138410	1.138410
$x_3^{(k)}$	2.000000	2.140000	2.120000	2.125217	2.125358	2.125356	2.125368	2.125368

解为 $x_1 = 0.767354$， $x_2 = 1.138410$， $x_3 = 2.125368$.

Seidel 迭代格式与计算结果如下：

$$\begin{cases} x_1^{(k+1)} = \frac{1}{20}[\qquad -2x_2^{(k)} - 3x_3^{(k)} + 24] \\ x_2^{(k+1)} = \frac{1}{8}[-x_1^{(k+1)} \qquad - x_3^{(k)} + 12] \\ x_3^{(k+1)} = \frac{1}{15}[-2x_1^{(k+1)} + 3x_2^{(k+1)} \qquad + 30] \end{cases} (k = 0,1,2,\cdots);$$

k	1	2	3	4	5	6
$x_1^{(k)}$	1.200000	0.748500	0.766421	0.767375	0.767356	0.767354
$x_2^{(k)}$	1.350000	1.142688	1.138105	1.138399	1.138410	1.138410
$x_3^{(k)}$	2.110000	2.128738	2.125432	2.125363	2.125368	2.125368

5. 解 Jacobi 迭代格式为

$$\begin{cases} x_1^{(k+1)} = \frac{1}{20}[\qquad -2x_2^{(k)} - 3x_3^{(k)} + 24] \\ x_2^{(k+1)} = \frac{1}{8}[-x_1^{(k)} \qquad - x_3^{(k)} + 12] \\ x_3^{(k+1)} = \frac{1}{15}[-2x_1^{(k)} + 3x_2^{(k)} \qquad + 30] \end{cases} (k = 0,1,2,\cdots),$$

因为 $M_1 = \begin{bmatrix} 0 & -\frac{3}{4} & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}, \det(\lambda E - M_1) = \begin{vmatrix} \lambda & \frac{3}{4} & 0 \\ \frac{3}{4} & \lambda & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \lambda \end{vmatrix} = \lambda\left(\lambda^2 - \frac{5}{8}\right), \rho(M_1) = \sqrt{\frac{5}{8}} < 1,$

所以 Jacobi 迭代格式收敛.

Seidel 迭代格式为

$$
\begin{cases}
x_1^{(k+1)} = \dfrac{1}{20}[\qquad -2x_2^{(k)} - 3x_3^{(k)} + 24] \\
x_2^{(k+1)} = \dfrac{1}{8}[-x_1^{(k+1)} \qquad - x_3^{(k)} + 12] \\
x_3^{(k+1)} = \dfrac{1}{15}[-2x_1^{(k+1)} + 3x_2^{(k+1)} \qquad + 30]
\end{cases}
(k = 0,1,2,\cdots).
$$

因为系数矩阵 A 对称，且 $\det A_1 = 4 > 0, \det A_2 = 7 > 0, \det A_3 = 24 > 0$,从而$A$正定,故Seidel迭代格式收敛.

6. 解（1）Jacobi迭代矩阵 $M_1 = D^{-1}(L+U) = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$;

$\det(\lambda E - M_1) = \lambda(\lambda^2 + \frac{5}{4})$, $\rho(M_1) = \frac{\sqrt{5}}{2} > 1$.因此,Jacobi迭代格式发散.

Seidel迭代矩阵

$$
M_2 = (D-L)^{-1}U = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix};
$$

$\det(\lambda E - M_2) = \lambda(\lambda + \frac{1}{2})^2$, $\rho(M_2) = \frac{1}{2}$.因此Seidel迭代格式收敛.

（2）Jacobi迭代矩阵 $M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$;

$\det(\lambda E - M_1) = \lambda^3$, $\rho(M_1) = 0$.因此，Jacobi迭代格式收敛.

Seidel迭代矩阵 $M_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$;

$\det(\lambda E - M_2) = \lambda(\lambda - 2)^2$, $\rho(M_2) = 2 > 1$.因此，Seidel迭代格式发散.

***7.用追赶法解线性方程组**

$$
\begin{cases}
3x_1 + x_2 \qquad = -1, \\
2x_1 + 4x_2 + x_3 = 7, \\
\qquad 2x_2 + 5x_3 = 9.
\end{cases}
$$

解 系数矩阵为 $\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix}$.

$u_1 = 3, l_2 = 2/u_1 = 2/3, u_2 = 4-1\cdot l_2 = 10/3, l_3 = 2/u_2 = 3/5, u_3 = 5-1\cdot l_3 = 22/5$;

$y_1 = -1$， $y_2 = 7 - l_2 y_1 = 23/3$， $y_3 = 9 - l_3 y_2 = 22/5$；

$\therefore x_3 = y_3/u_3 = 1$， $x_2 = (y_2 - 1\cdot x_3)/u_2 = 2$， $x_1 = (y_1 - 1\cdot x_2)/u_1 = -1$.即解为

$x = (-1, 2, 1)^T$.

8. 解　把方程组调整为

$$\begin{cases} 3x_3 + x_2 + 2x_1 = 6 \\ 4x_2 + x_1 = 8 \\ x_3 + 2x_1 = 2 \end{cases},$$

此时系数矩阵为

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Seidel 迭代矩阵

$$M = (D-L)^{-1}U = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix},$$

$$\det(\lambda E - M) = \lambda(\lambda - \frac{1}{6} - \frac{1}{6\sqrt{2}}\mathrm{i})(\lambda - \frac{1}{6} + \frac{1}{6\sqrt{2}}\mathrm{i}), \rho(M) = \frac{\sqrt{6}}{12} < 1.$$

因此,此时 Seidel 迭代格式

$$\begin{cases} x_3^{(k+1)} = \frac{1}{3}(6 - x_2^{(k)} - 2x_1^{(k)}) \\ x_2^{(k+1)} = \frac{1}{4}(8 - x_1^{(k)}) \\ x_1^{(k+1)} = \frac{1}{2}(2 - x_3^{(k+1)}) \end{cases}$$

收敛.

习　题　七

A

一、判断题

1.×；**2.**√；**3.**×；**4.**×.

二、填空题

1. $1, n+1$；**2.** 一阶差商$:4,5,5;$二阶差商$:1,\frac{1}{3};$三阶差商$:-\frac{1}{3}$；**3.** $16.640, 0.096, 16.736$.

B

1. 解　因为$x=0.15,$故取$x_0=0.00, x_1=0.10, x_2=0.20.$ 则

$$\begin{aligned}
f(0.15) \approx L_2(0.15) &= \frac{(0.15-0.10)(0.15-0.20)}{(0.00-0.10)(0.00-0.20)} \times 0.000 \\
&+ \frac{(0.15-0.00)(0.15-0.20)}{(0.10-0.00)(0.10-0.20)} \times 0.0998 \\
&+ \frac{(0.15-0.00)(0.15-0.10)}{(0.20-0.00)(0.20-0.10)} \times 0.1987 \\
&= 0+0.07485+0.07451=0.1494.
\end{aligned}$$

$$|R_2(0.15)| \le \frac{1}{3!}|(0.15-0.00)(0.15-0.10)(0.15-0.20)| = 6.25\times 10^{-5}.$$

2.解　对于点$x=76.35$,取$x_0=76, x_1=77, x_2=78, x_3=79$.

作差商表

x	$f(x)$	一阶	二阶	三阶
76	2.83267			
77	2.90256	0.06989		
78	2.97857	0.07601	0.00306	
79	3.06173	0.08316	0.00358	0.00017

于是有

$$\begin{aligned}
(1)\ f(76.35) &\approx N_2(76.35) \\
&= 2.83267+0.0689(76.35-76)+0.00306(76.35-76)(76.35-77) \\
&= 2.83267+0.02412-0.00070 \\
&= 2.85609.
\end{aligned}$$

$$\begin{aligned}
(2)\ f(76.35) &\approx N_3(76.35) = N_2(76.35)+0.00017(76.35-76)(76.35-77)(76.35-78) \\
&= 2.85609+0.00006 \\
&= 2.85615.
\end{aligned}$$

3. 解　选$x_0=0.20, x_1=0.40, x_2=0.60, x_2=0.80$.

作差商表：

x_k	$f(x_k)$	一阶差商	二阶差商	三阶差商
0.20	1.2214			
0.40	1.4918	1.3520		
0.60	1.8221	1.6515	0.7488	
0.80	2.2255	2.0170	0.9138	0.2750

$$f(0.45) \approx N_3(0.45) = 1.2214 + 1.3520(0.45-0.20) + 0.7488(0.45-0.20)(0.45-0.40)$$

$$\begin{aligned} &+0.2750(0.45-0.20)(0.45-0.40)(0.45-0.60) \\ &= 1.2214 + 0.3380 + 0.0094 - 0.0005 \\ &= 1.5683. \end{aligned}$$

或

$$\begin{aligned} f(0.45) \approx L_3(0.45) &= \frac{(0.45-0.40)(0.45-0.60)(0.45-0.80)}{(0.20-0.40)(0.20-0.60)(0.20-0.80)} \times 1.2214 \\ &+ \frac{(0.45-0.20)(0.45-0.60)(0.45-0.80)}{(0.40-0.20)(0.40-0.60)(0.40-0.80)} \times 1.4918 \\ &+ \frac{(0.45-0.20)(0.45-0.40)(0.45-0.80)}{(0.60-0.20)(0.60-0.40)(0.60-0.80)} \times 1.8221 \\ &+ \frac{(0.45-0.20)(0.45-0.40)(0.45-0.60)}{(0.80-0.20)(0.80-0.40)(0.80-0.60)} \times 2.2255 \\ &= 1.5682. \end{aligned}$$

4. 证明 $$\begin{aligned} f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x - x_0} = \frac{u(x_1)v(x_1) - u(x_0)v(x_0)}{x_1 - x_0} \\ &= \frac{u(x_1)v(x_1) - u(x_0)v(x_1) + u(x_0)v(x_1) - u(x_0)v(x_0)}{x - x_0} \\ &= u(x_0)\frac{v(x_1) - v(x_0)}{x_1 - x_0} + \frac{u(x_1) - u(x_0)}{x - x_0} v(x_1) \\ &= u[x_0]v[x_0, x_1] + u[x_0, x_1]v[x_1]. \end{aligned}$$

5. 证明（1）设 $f(x) = x^m$,则当 $m = 0,1,\cdots,n$ 时, $f^{(n+1)}(x) = 0$.因此, $f(x)$ 的 n 次插值多项式

$L_n(x)$ 的插值余项 $R_n(x) = 0$,故有 $L_n(x) \equiv f(x)$,此即 $\sum_{k=0}^{n} x_k^m l_k(x) \equiv x^m, m = 0,1,\cdots,n$.

（2）利用二项式展开公式和已证明的(1)得

$$\sum_{k=0}^{n} (x_k - x)^m l_k(x) = \sum_{k=1}^{n} \sum_{i=1}^{m} C_m^i x_k^{m-i} (-x)^i l_k(x) = \sum_{i=0}^{m} \left[(-1)^i C_m^i x^i \sum_{k=0}^{n} x_k^{m-i} l_k(x) \right]$$

$$= \sum_{i=0}^{m} \left[(-1)^i C_m^i x^i x^{m-i} \right] = x^m \sum_{i=0}^{m} (-1)^i C_m^i$$

而 $\sum_{i=0}^{m} (-1)^i C_m^i = 0$,故有 $\sum_{k=0}^{n} (x_k - x)^m l_k(x) = 0$.

***6.** 解　作变换 $x=\frac{1}{2}(3+t)$，即 $t=3-2x$，则 $f(x)=\frac{2}{3+t}$，记 $\varphi(t)=\frac{2}{3+t}$.对 $\varphi(t)$ 在 $[-1,1]$ 上用 Legendre 多项式作最佳二次平方逼近,设最佳平方逼近函数为

$$\overline{S_2}(t)=a_0\mathrm{p}_0(t)+a_1\mathrm{p}_1(t)+a_2\mathrm{p}_2(t).$$

则

$$a_0=\frac{1}{2}\int_{-1}^{1}\frac{2}{3+t}\mathrm{d}t=\ln 2,$$

$$a_1=\frac{3}{2}\int_{-1}^{1}\frac{2}{3+t}t\mathrm{d}t=6-9\ln 2,$$

$$a_2=\frac{5}{4}\int_{-1}^{1}\frac{2}{3+t}(3t^2-1)\mathrm{d}t=0.5\ln 2-45,$$

因此

$$\overline{S}_2(t)=\ln 2+(6-9\ln 2)t+(65\ln 2-45)(\frac{3}{2}t^2-\frac{1}{2})$$

$$S_2^*(x)=\ln 2+(6-9\ln 2)(2x-3)+(65\ln 2-45)\left[\frac{3}{2}(2x-3)^2-\frac{1}{2}\right]$$

$$=873\ln 2-603+(822-1188\ln 2)x+(390\ln 2-270)x^3$$

$$=2.11749-1.45885x+0.32740x^2.$$

平方误差为

$$\left\|f-S_2^*\right\|_2^2=\int_1^2\left|f(x)-S_2^*(x)\right|^2\mathrm{d}x \quad (\text{令 } x=\tfrac{1}{2}(3+t))$$

$$=\int_{-1}^{1}\left|\varphi(t)-\overline{S}_2(t)\right|^2\frac{1}{2}\mathrm{d}t=\frac{1}{2}\left\|\varphi-\overline{S}_2\right\|_2^2$$

$$=\frac{1}{2}\int_{-1}^{1}\left|\varphi(t)\right|^2\mathrm{d}t-\frac{1}{2}\sum_{k=0}^{2}\frac{2}{2k+1}a_k^2$$

$$=\frac{1}{2}-\frac{1}{2}[2(\ln 2)^2+\frac{2}{3}(6-9\ln 2)^2+\frac{2}{5}(65\ln 2-45)^2]$$

$$=-416.5+1206\ln 2-873\ln^2 2$$

$$=0.0000186.$$

***7.** 解　取 $M=\{1,x,x^2\}$，设拟合曲线为

$$S_2(x)=a_0+a_1x+a_2x^2.$$

因为

$$<\varphi_0,\varphi_0>=7,\quad <\varphi_0,\varphi_1>=28,\quad <\varphi_0,\varphi_2>=140,$$

$$<\varphi_1,\varphi_1>=140,\quad <\varphi_1,\varphi_2>=784,\quad <\varphi_2,\varphi_2>=4676,$$

$$< f,\varphi_0 >= 1,\ < f,\varphi_1 >= -36,\ < f,\varphi_2 >= -308,$$

所以法方程为$\begin{cases} 7a_0 + 28a_1 + 140a_2 = 1 \\ 28a_0 + 140a_1 + 784a_2 = -36 \\ 140a_0 + 784a_1 + 4676a_2 = -308 \end{cases}$,解得 $a_0 = \frac{33}{7}, a_1 = -\frac{2}{3}$， $a_2 = -\frac{2}{21}$，

因此
$$S_2(x) = \frac{33}{7} - \frac{2}{3}x - \frac{2}{21}x^2.$$

$$\|\delta\|_2^2 = \sum_{i=0}^{6} |S_2(x_i) - y_i|^2 = 0.95238.$$

习　题　八

A

一、判断题

1.√;，**2.** ×；**3.**×；**4.** √；**5.** √；**6.** √；**7.** √.

二、填空题

1.1;　**2.**$\frac{3}{8},\frac{3}{8},\frac{1}{8}$;　**3.** $T_8=3.138989, S_1=3.133333, S_2=3.141569, C_1=3.142118,$

$C_2=3.141594, R_1=3.141588$；　**4.** $\frac{2}{n+1}$.

B

1. 解　（1）取 $f(x)$ 为$1,x,x^2$ 得

$$\begin{cases} A_0+A_1+A_2=2 \\ A_1+2A_2=2 \\ A_1+4A_2=\frac{8}{3} \end{cases}$$,解此方程组得 $A_0=\frac{1}{3}, A_1=\frac{4}{3}, A_2=\frac{1}{3}$.

因此求积公式为

$$\int_0^2 f(x)\mathrm{d}x \approx \frac{1}{3}f(0)+\frac{4}{3}f(1)+\frac{1}{3}f(2).$$

当 $f(x)=x^3$ 时,$\int_0^2 f(x)\mathrm{d}x=4$,$\frac{1}{3}f(0)+\frac{4}{3}f(1)+\frac{1}{3}f(2)=4$,求积公式成为等式;而

当 $f(x)=x^4$ 时,$\int_0^2 f(x)\mathrm{d}x=\frac{32}{5}$,$\frac{1}{3}f(0)+\frac{4}{3}f(1)+\frac{1}{3}f(2)=\frac{20}{3}$,求积公式不能成为等式

所以，求积公式的代数精度是 3 次.

（2）取 $f(x)$ 为 $1,x,x^2$,可得

$$\begin{cases} A+B=2 \\ -A+Bx_1=0 \\ A+Bx_1^2=\frac{2}{3} \end{cases}$$,解得 $A=\frac{1}{2}, B=\frac{3}{2}, x_1=\frac{1}{3}$,

因此求积公式为

$$\int_{-1}^1 f(x)\mathrm{d}x \approx \frac{1}{2}f(-1)+\frac{3}{2}f\left(\frac{1}{3}\right).$$

当 $f(x)=x^3$ 时,$\int_{-1}^1 f(x)\mathrm{d}x=0$,$\frac{1}{2}f(-1)+\frac{3}{2}f\left(\frac{1}{3}\right)=-\frac{4}{9}\neq 0$,

所以,求积公式具有 2 次代数精度.

2．解（1）$h=0.2$.

$$T_5 = \frac{0.2}{2}[f(0) + 2\sum_{k=1}^{4} f(x_k) + f(1)]$$

$$= 0.1\left[0 + 2\left(\frac{\ln 1.2}{1+0.2^2} + \frac{\ln 1.4}{1+0.4^2} + \frac{\ln 1.6}{1+0.6^2} + \frac{\ln 1.8}{1+0.8^2}\right) + \frac{\ln 2}{1+1^2}\right]$$

$$= 0.1[2(0.17531 + 0.29006 + 0.34559 + 0.35841) + 0.34657]$$

$$= 0.26853;$$

$$S_5 = \frac{0.2}{6}[f(0) + 4\sum_{k=1}^{4} f(x_{k+\frac{1}{2}}) + 2\sum_{k=1}^{4} f(x_k) + f(1)]$$

$$= \frac{0.4}{3}\left(\frac{\ln 1.1}{1+0.1^2} + \frac{\ln 1.3}{1+0.3^2} + \frac{\ln 1.5}{1+0.5^2} + \frac{\ln 1.7}{1+0.7^2} + \frac{\ln 1.9}{1+1.9^2}\right) + \frac{1}{3}T_5$$

$$= \frac{0.4}{3}\left(0.09437 + 0.24070 + 0.32437 + 0.35613 + 0.35462\right) + \frac{1}{3}\times 0.26853$$

$$= 0.27220.$$

（2）$h = 0.2$.

$$T_6 = \frac{0.2}{2}[f(0) + 2\sum_{k=1}^{5} f(x_k) + f(1.2)]$$

$$= 0.1[0 + 2(\sqrt{0.2}e^{0.2} + \sqrt{0.4}e^{0.4} + \sqrt{0.6}e^{0.6} + \sqrt{0.8}e^{0.8} + \sqrt{1}e^{1}) + \sqrt{1.2}e^{1.2}]$$

$$= 0.1[2(0.54623 + 0.94351 + 1.61141 + 1.99058 + 2.71828) + 3.03701]$$

$$= 1.88570;$$

$$S_6 = \frac{0.2}{6}[f(0) + 4\sum_{k=0}^{5} f(x_{k+\frac{1}{2}}) + 2\sum_{k=1}^{5} f(x_k) + f(1.2)]$$

$$= \frac{0.4}{3}(0.34949 + 0.73935 + 1.16582 + 1.68483 + 2.33338 + 3.15080)$$

$$+ \frac{1}{3}\times 1.88570$$

$$= 1.88506.$$

（3）$h = 0.25$.

$$T_8 = \frac{0.25}{2}[f(-1) + 2\sum_{k=1}^{7} f(x_k) + f(1)]$$

$$= \frac{0.25}{2}[2.71828 + 2\times 9.20715 + 2.71828]$$

$$= 2.98136;$$

$$S_8 = \frac{0.25}{6}[f(-1) + 4\sum_{k=1}^{7} f(x_{k+\frac{1}{2}}) + 2\sum_{k=1}^{7} f(x_k) + f(1)]$$

$$=\frac{0.25}{6}\times 4\times 11.58997+\frac{1}{3}T_8$$
$$=2.92545.$$

3. 解（1）

k	T_{2^k}	S_{2^k}	C_{2^k}	R_{2^k}
0	0.173287	0.274010	0.272222	0.272197
1	0.248829	0.272334	0.272197	0.272198
2	0.266458	0.272206	0.272198	
3	0.270769	0272199		
4	0.271841			

因此 $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x \approx 0.27220.$

（2）

k	T_{2^k}	S_{2^k}	C_{2^k}	R_{2^k}
0	0.6109170	0.6581157	0.6576682	0.6576698
1	0.6463160	0.6576962	0.6576698	0.6576699
2	0.6548512	0.6576715	0.6576700	
3	0.6569664	0.6576700		
4	0.6574941			

因此 $\displaystyle\int_0^{0.8} \mathrm{e}^{-x^2}\mathrm{d}x \approx 0.6576699.$

***4. 解**（1）因为是 Gauss 型求积公式,故其代数精度 $m=2\times1+1=3$. 于是令公式对 $f(x)=1,x,x^2,x^3$ 是准求成立,得

$$\begin{cases} A_0+A_1=2, \\ A_0x_0+A_1x_1=\dfrac{2}{3}, \\ A_0x_0^2+A_1x_1^2=\dfrac{2}{5}, \\ A_0x_0^3+A_1x_1^3=\dfrac{2}{7}. \end{cases} \text{解之得} \begin{cases} x_0=0.1156, & x_1=0.7416 \\ A_0=1.3043, & A_1=0.6957. \end{cases}$$

故 $\displaystyle\int_0^1 \frac{f(x)}{\sqrt{x}}\mathrm{d}x \approx 1.3043f(0.1156)+0.6957f(0.7416).$

（2）因为是 Gauss 型求积公式,故其代数精度 $m=2\times1+1=3$. 于是令公式对 $f(x)=1,x,x^2,x^3$ 是准求成立,得

$$\begin{cases} A_0+A_1=1, \\ A_0x_0+A_1x_1=\dfrac{1}{4}, \\ A_0x_0^2+A_1x_1^2=\dfrac{1}{9}, \\ A_0x_0^3+A_1x_1^3=\dfrac{1}{16}. \end{cases}$$ 解之得 $\begin{cases} x_0=0.602277, & x_1=0.112009, \\ A_0=0.281461, & A_1=0.718539. \end{cases}$ 或 $\begin{cases} x_0=0.112009, & x_1=0.602277, \\ A_0=0.718539, & A_1=0.281461. \end{cases}$

故 $\int_0^1 \ln\dfrac{1}{x} f(x)\mathrm{d}x \approx 0.718539 f(0.112009)+0.281461 f(0.602277).$

*5. 解（1）$\int_0^1 \mathrm{e}^{x^2}\mathrm{d}x=\dfrac{1}{2}\int_{-1}^1 \mathrm{e}^{x^2}\mathrm{d}x$

$$=\frac{1}{2}[0.347855\mathrm{e}^{(-0.861136)^2}+0.652145\mathrm{e}^{(-0.339981)^2}$$

$$+0.652145\mathrm{e}^{(0.339981)^2}+0.347855\mathrm{e}^{(0.861136)^2}]$$

$$=1.462270.$$

（2）$\int_1^3 \mathrm{e}^x \sin \mathrm{d}x \overset{x=t+2}{=} \int_{-1}^1 \mathrm{e}^{t+2}\sin(t+2)\mathrm{d}t$

$$=0.555556\mathrm{e}^{2-0.774597}\sin(2-0.774597)+0.888889\mathrm{e}^2\sin 2$$

$$+0.555556\mathrm{e}^{2+0.774597}\sin(2+0.774597)$$

$$=10.948405.$$

6.解 $\int_a^b f(x)\mathrm{d}x \approx AT+BR=A\left(\dfrac{b-a}{2}[f(a)+f(b)]\right)+B\left((b-a)f\left(\tfrac{a+b}{2}\right)\right).$

令公式对$f(x)=1,x,x^2$准确成立,得

$$\begin{cases} (b-a)(A+B)=b-a, \\ \dfrac{b^2-a^2}{2}(A+B)=\dfrac{b^2-a^2}{2}, \\ \dfrac{a^2+b^2}{2}A+\dfrac{(a+b)^2}{4}B=\dfrac{a^2+b^2+ab}{3}, \end{cases}$$ 即 $\begin{cases} A+B=1, \\ 6A+3B=4. \end{cases}$ 解之得 $\begin{cases} A=\dfrac{1}{3}, \\ B=\dfrac{2}{3}. \end{cases}$

故求积公式为

$$\int_a^b f(x)\mathrm{d}x \approx \frac{1}{3}\left(\frac{b-a}{2}[f(a)+f(b)]\right)+\frac{2}{3}\left((b-a)f\left(\tfrac{a+b}{2}\right)\right),$$

即 $\int_a^b f(x)\mathrm{d}x \approx \dfrac{b-a}{6}[f(a)+4f\left(\tfrac{a+b}{2}\right)+f(b)]$. 这就是梯梯形公式,其代数精度为 3.

7.证 令求积公式对 $f(x)=1,x,x^2,x^3,x^4$ 成立,得方程组 $\begin{cases} A+B+C=4, & ① \\ -A\alpha+C\alpha=0, & ② \\ A\alpha^2+C\alpha^2=\dfrac{16}{3}, & ③ \\ -A\alpha^3+C\alpha^3=0, & ④ \\ A\alpha^4+C\alpha^4=\dfrac{64}{5}. & ⑤ \end{cases}$

由②, ③知 $\alpha \neq 0, A = C$. 由③, ⑤得 $\alpha^2 = \frac{12}{5}$，即 $\alpha = \pm\sqrt{\frac{12}{5}}$. 再由③得 $A = C = \frac{10}{9}$. 最后由①得 $B = \frac{16}{9}$. 故求积公式为

$$\int_{-2}^{2} f(x)\mathrm{d}x \approx \frac{10}{9} f\left(\sqrt{\tfrac{12}{5}}\right) + \frac{16}{9} f(0) + \frac{10}{9} f\left(-\sqrt{\tfrac{12}{5}}\right).$$

因为当 $f(x) = x^5$ 时,左端 $=0$ 右端，于是这个有两个 求积节点的求积公式具有5次(即最高次)代数精度,故是Gauss型求积公式.

习　题　九

A

一、判断题

1.√;，**2.** ×；**3.** √.

二、填空题

1. $\begin{cases} y' = z \\ z' = f(x,y,z) \qquad (a < x \le b) \\ y(a) = y_0, z(a) = y_0^{(1)} \end{cases}$；

2. $\begin{cases} y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ z_{n+1} = z_n + \frac{h}{6}(L_1 + 2L_2 + 2L_3 + L_4) \\ K_1 = z_n, \qquad L_1 = f(x_n, y_n, z_n) \\ K_2 = z_n + \frac{h}{2}L_1,\ L_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1, z_n + \frac{h}{2}L_1) \\ K_3 = z_n + \frac{h}{2}L_2,\ L_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2, z_n + \frac{h}{2}L_2) \\ K_4 = z_n + hL_3,\ L_4 = f(x_n + h, y_n + hK_3, z_n + hL_3) \\ y_0,\ z_0 = y_0^{(1)} \end{cases}$；

3. 数值微分法, 数值积分法, Taylor 展开法；

4. $\begin{cases} y_{n+1} = y_n + \frac{h}{2}(K_1 + K_2) \\ z_{n+1} = z_n + \frac{h}{2}(L_1 + L_2) \\ K_1 = -8y_n + 7z_n, \qquad L_1 = x_n^2 + y_n z_n \\ K_2 = -8(y_n + hK_1) + 7(z_n + hL_1),\ L_2 = (x_n + h)^2 + (y_n + hK_1)(z_n + hL_1) \\ y_0 = 1, z_0 = 0 \end{cases}$

（$n = 0,1,2,\cdots,N-1$.

B

1. 解（1）计算格式为

$$y_{n+1} = y_n + h(x_n^2 + y_n^2),\ n = 0,1,\cdots,9.$$

计算结果列于下表：

x_n	y_n	x_n	y_n	x_n	y_n
0	0.00000	0.4	0.01400	0.8	0.14125
0.1	0.00000	0.5	0.03002	0.9	0.20725
0.2	0.00100	0.6	0.05511	1.0	0.29254
0.3	0.00500	0.7	0.09142		

（2）计算格式为

$$y_{n+1} = y_n + h\frac{1}{x_n}(y_n^2 + y_n),\ n = 0,1,2,3.$$

计算结果列于下表：

x_n	1	1.5	2	2.5	3
y_n	-2.61803	-0.50000	-0.58333	-0.64410	-0.68994

2. 解　计算格式为

$$\begin{cases} y_{n+1} = y_n + \dfrac{h}{2}(K_1 + K_2) \\ K_1 = x_n^2 + y_n^2 \\ K_2 = (x_n + h)^2 + (y_n + hK_1)^2 \\ y_0 = 0 \end{cases}, n = 0,1,\cdots,9 .$$

计算结果列于下表

x_n	y_n	x_n	y_n	x_n	y_n
0	0.00000	0.4	0.02202	0.8	0.17539
0.1	0.00050	0.5	0.04262	0.9	0.25237
0.2	0.00300	0.6	0.07364	1.0	0.35183
0.3	0.00950	0.7	0.11681		

3. 解 标准 Runge – Kutta 格式为

$$\begin{cases} y_{n+1} = y_n + \dfrac{0.1}{3}(K_1 + 2K_2 + 2K_3 + K_4) \\ K_1 = \dfrac{3y_n}{1+x_n} \\ K_2 = \dfrac{3(y_n + 0.1K_1)}{1.1 + x_n} \\ K_3 = \dfrac{3(y_n + 0.1K_2)}{1.1 + x_n} \\ K_4 = \dfrac{3(y_n + 0.2K_3)}{1.2 + x_n} \\ y_0 = 1 \end{cases}, \quad n = 0,1,2,3,4.$$

计算结果列于下表（准确解为 $y = (1+x)^3$）

x_n	K_1	K_2	K_3	K_4	y_n	准确值
0					1.00000	1
0.2	3.00000	3.54545	3.69421	4.34702	1.72755	1.728
0.4	4.31887	4.98331	5.13664	5.90331	2.74295	2.744
0.6	5.87775	6.66145	6.81819	7.69986	4.09418	4.096
0.8	7.67659	8.57972	8.73909	9.73667	5.82921	5.832
1.0	9.71535	10.73802	10.89949	12.01367	7.99601	8

4. 解　　$h = \dfrac{1}{N}$,

$$\begin{cases} y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ z_{n+1} = z_n + \frac{h}{6}(L_1 + 2L_2 2L_3 + L_4) \\ K_1 = -8y_n + 7z_n \\ L_1 = x_n^2 + y_n z_n \\ K_2 = -8(y_n + \frac{h}{2}K_1) - 7(z_n + \frac{h}{2}L_1) \\ L_2 = (x_n + \frac{h}{2})^2 + (y_n + \frac{h}{2}K_1)(z_n + \frac{h}{2}L_1) \\ K_3 = -8(y_n + \frac{h}{2}K_2) + 7(z_n + \frac{h}{2}L_2) \\ L_3 = (x_n + \frac{h}{2})^2 + (y_n + \frac{h}{2}K_2)(z_n + \frac{h}{2}L_2) \\ K_4 = -8(y_n + hK_3) + 7(z_n + hL_2) \\ L_4 = (x_n + h)^2 + (y_n + hK_3)(z_n + hL_3) \\ y_0 = 1, z_0 = 0 \end{cases} \quad (h = \frac{1}{N}, n = 0,1,2,\cdots, N-1).$$

5. 解 令 $y' = z$,初值问题化为

$$\begin{cases} y' = z \\ z' = -\sin y \\ y(0) = 1, z(0) = 1 \end{cases},$$

Euler 格式为

$$\begin{cases} y_{n+1} = y_n + hz_n \\ z_{n+1} = z_n - h\sin y_n \\ y_0 = 1, z_0 = 1 \end{cases}, n = 0,1,2,\cdots, N-1.$$

改进 Euler 格式为

$$\begin{cases} y_{n+1} = y_n + \frac{h}{2}(K_1 + K_2) \\ z_{n+1} = z_n + \frac{h}{2}(L_1 + L_2) \\ K_1 = z_n \\ L_1 = -\sin y_n \\ K_2 = z_n + hL_1 \\ L_2 = -\sin(y_n + hK_1) \\ y_0 = 1, z_0 = 1 \end{cases}, \quad n = 0,1,2,\cdots, N-1$$