Guided Tour of Machine Learning in Finance

Week 3: Unsupervised Learning

Minimum Spanning Trees, Kruskal algorithm, and equity correlation matrices

Igor Halperin

NYU Tandon School of Engineering, 2017

The Kruskal algorithm

Task:
Given a connected weighted graph with N nodes and connection weights d_{ij}, find a minimum spanning tree (MST) - a graph without loops connecting N nodes with $N-1$ links, such that the total weight of all edges is minimized.

Meaning: from $N(N-1)/2$ links, select $N-1$ shortest links that span all the nodes without forming loops.

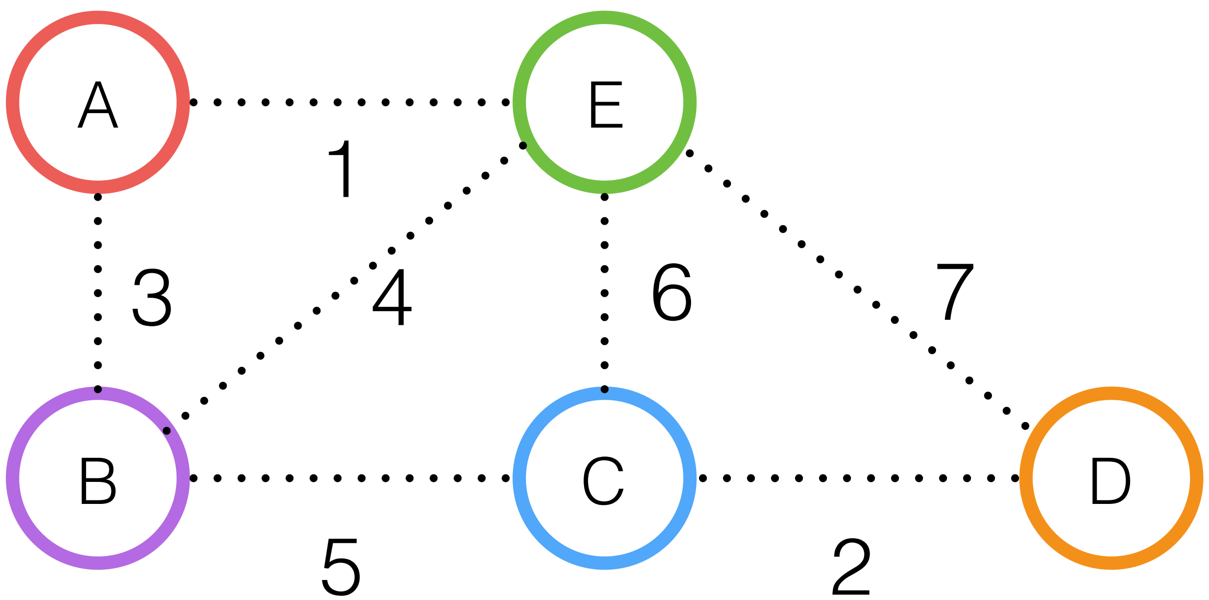
The Kruskal algorithm:

1. Find a minimum weight on a graph. Color it in any color (e.g. red)
2. Find the minimum uncolored edge that does not cross a colored or a red circuit. Mark this edge by a new color.
3. Repeat step 2 until connecting every vertex on the graph. The red edges form a MST.

The Kruskal algorithm

Wikipedia's example $(N = 5)$

Edge	AE	CD	AB	BE	BC	EC	ED
Weight	1	2	3	4	5	6	7



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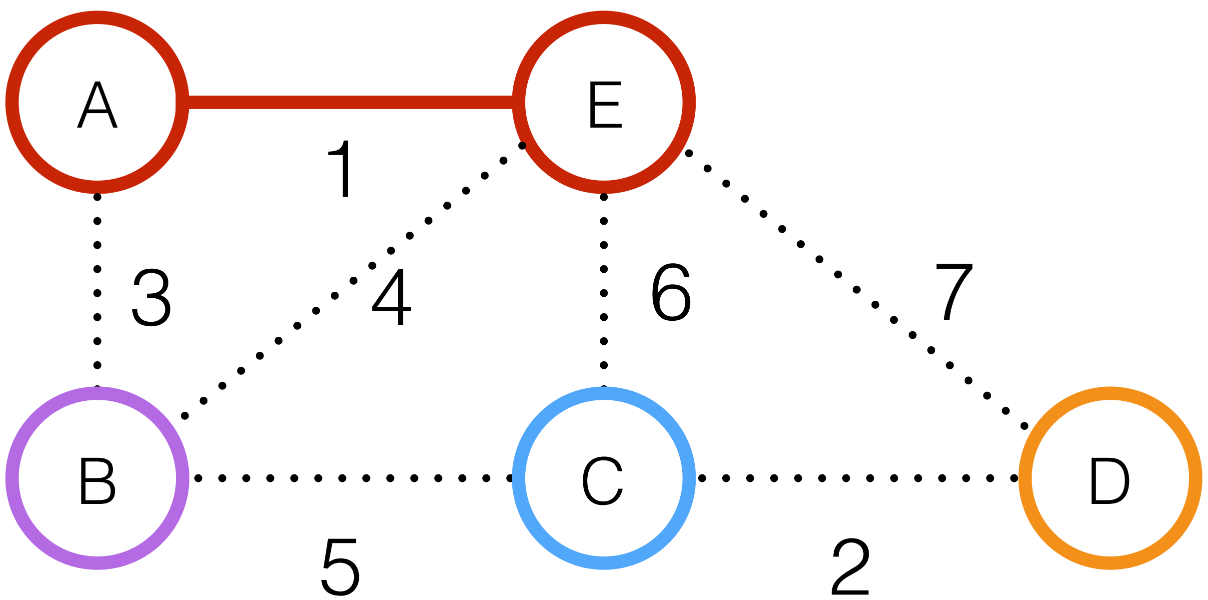

Select the smallest weight

The Kruskal algorithm

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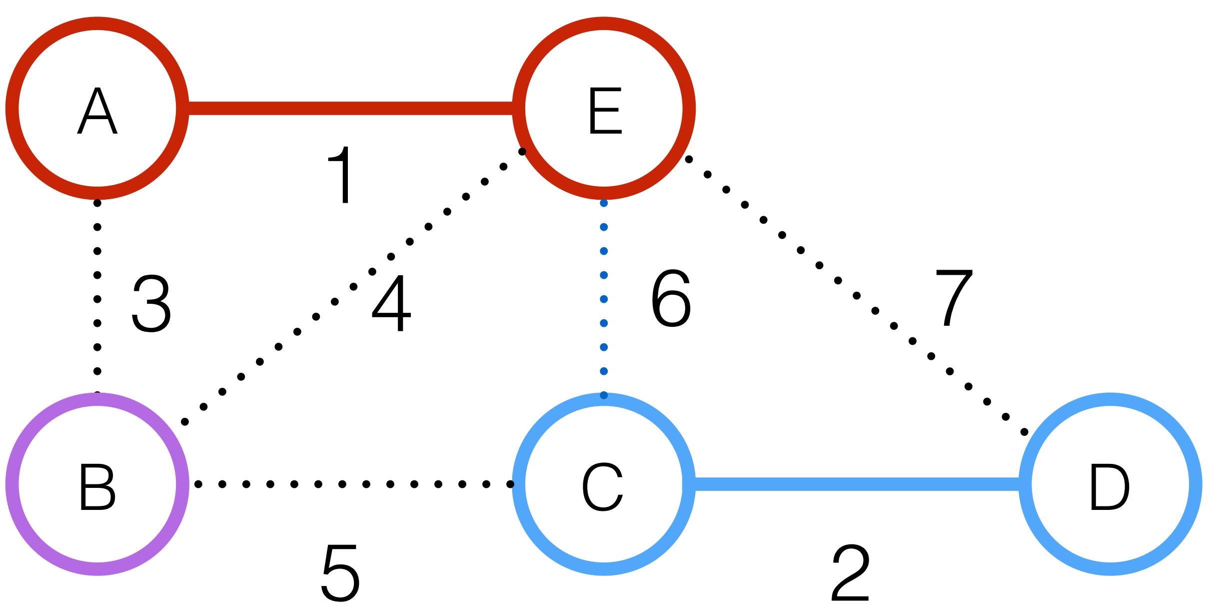
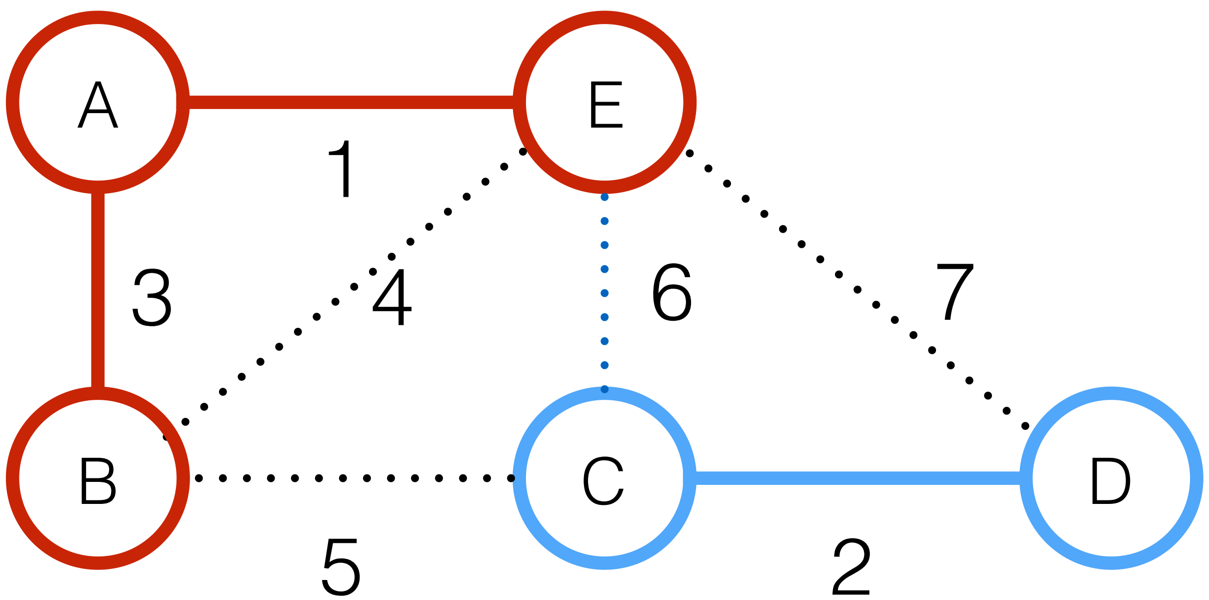


Select another smallest weight

The Kruskal algorithm

Wikipedia's example $(N = 5)$

Edge	AE	CD	AB	BE	BC	EC	ED
Weight	1	2	3	4	5	6	7



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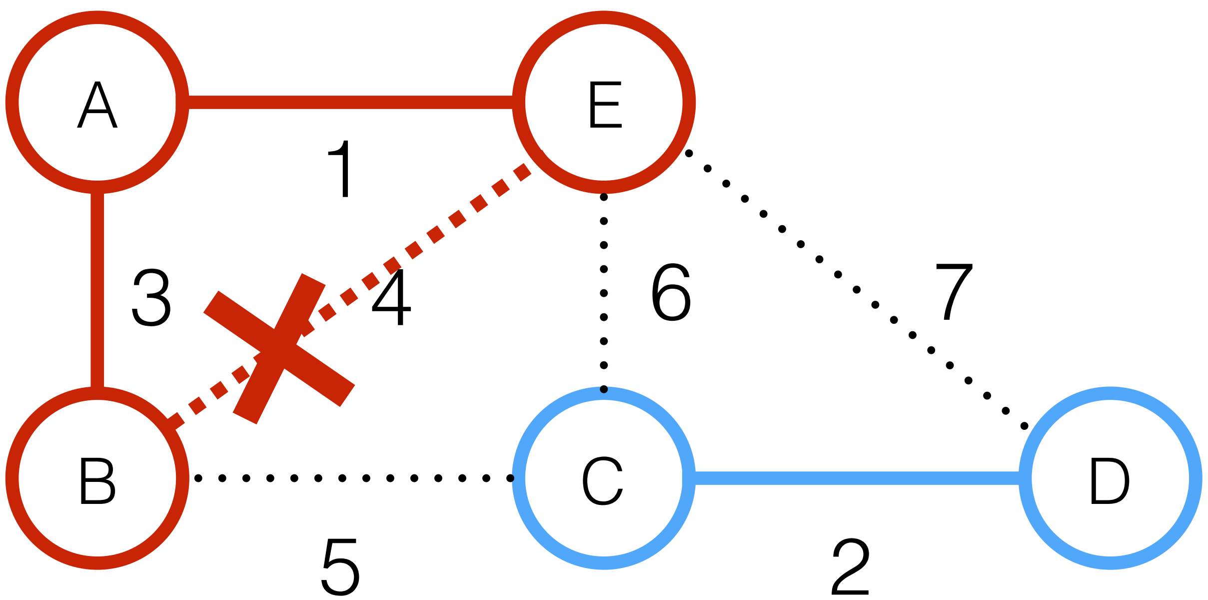

Selecting BE creates a loop!

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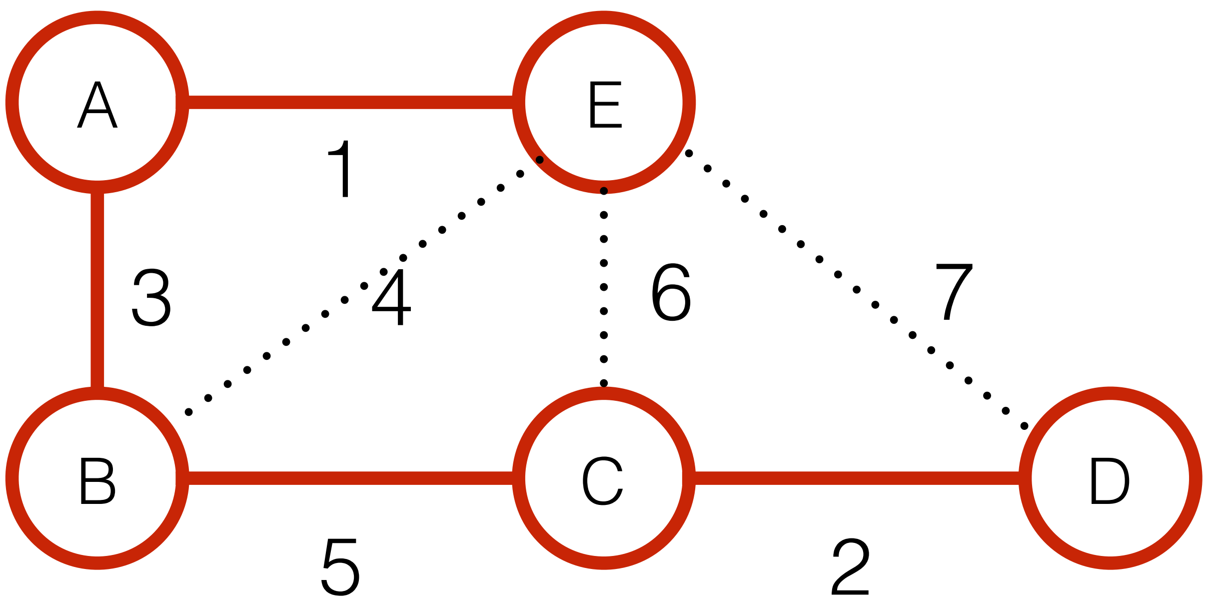

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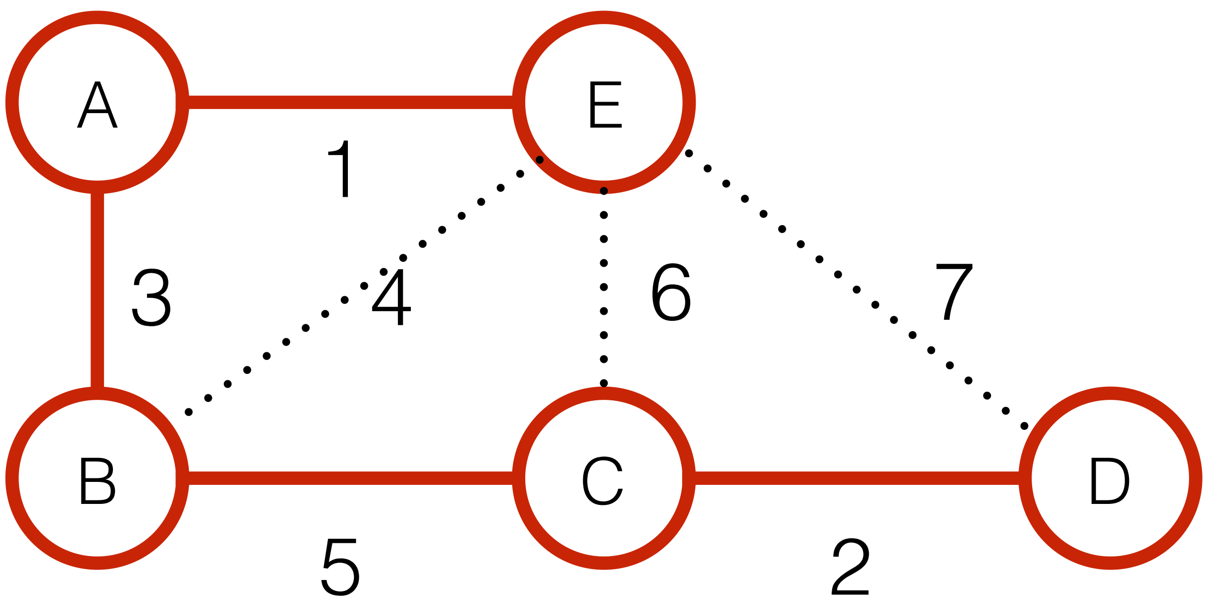

The minimum spanned tree is completed!

The MST compresses the data representation from $N(N-1)/2$ parameters to $N-1$ parameters

Clustering-based filtering

Use the MTS to de-noise the empirical distance matrix $D(\Delta t) = \left\{ d_{ij}(\Delta t) \right\}_{i,j=1}^{N}$ (and the correlation matrix $\mathrm{C}(\Delta \mathrm{t})$!) matrix by replacing

$$d_{ij} \rightarrow d_{ij}^{<}$$

Maximum distance d_{kl} among single steps in a path from i to j in a MST

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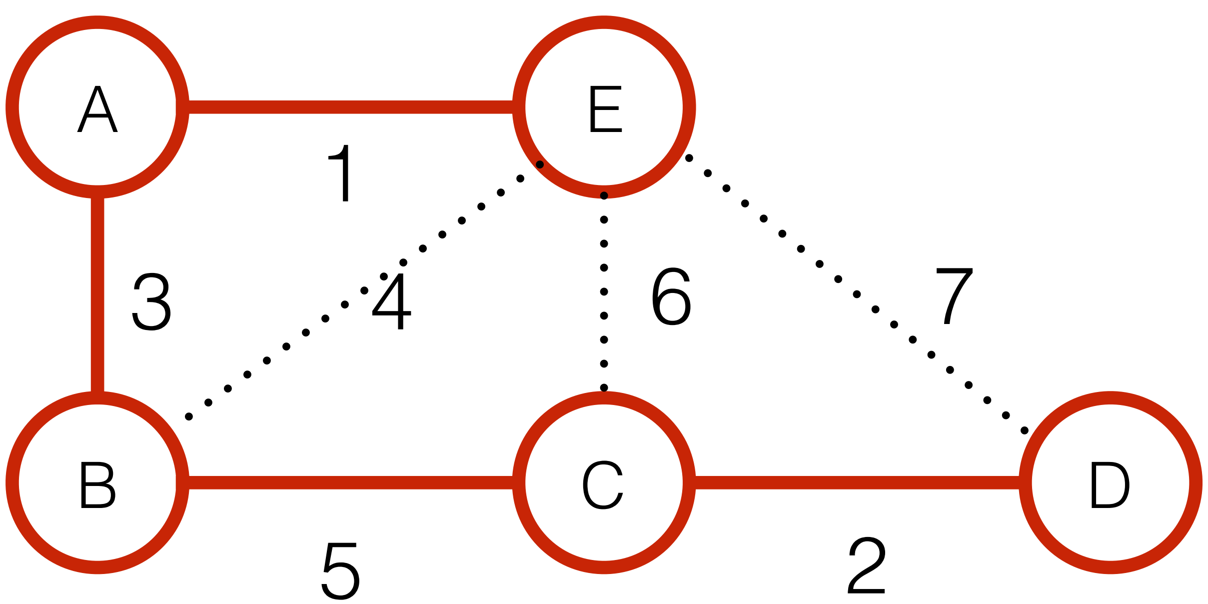

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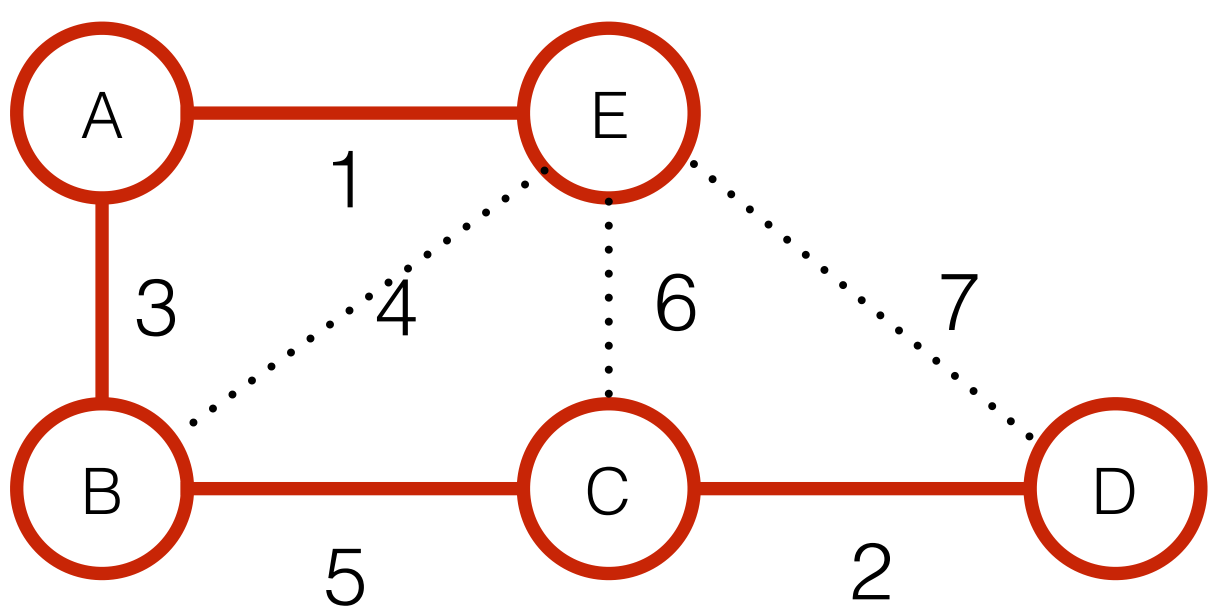
d_{ij}	1	2	3	4	5	6	7
$d_{ij}^{<}$	1	2	3	**3**	5	**5**	**5**



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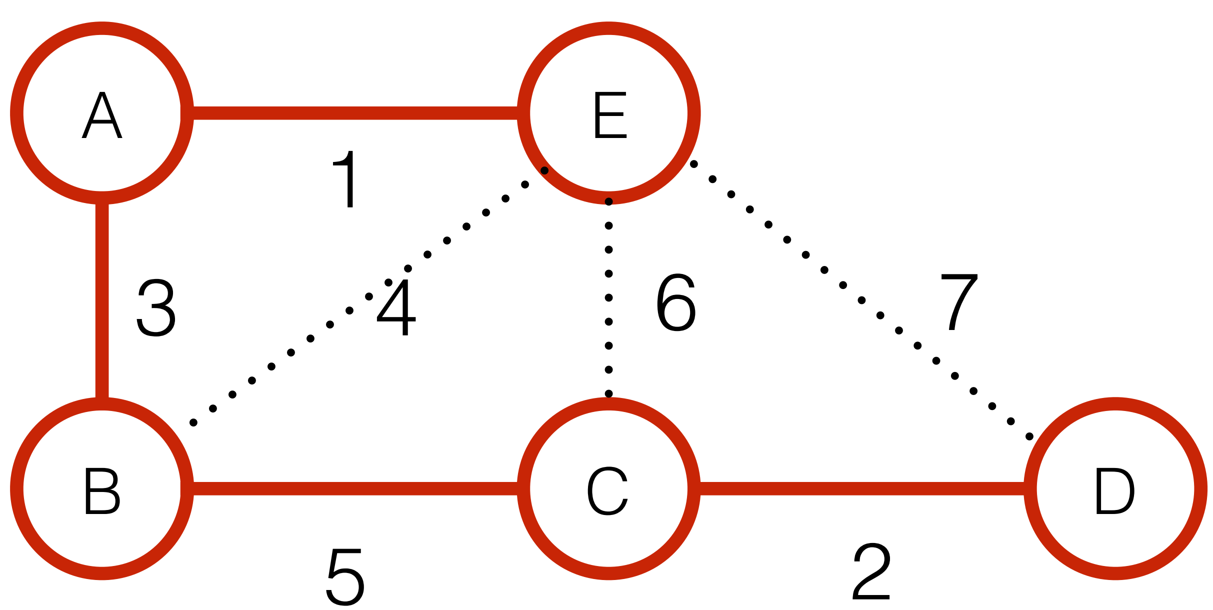
Maximum distance d_{kl} among single steps in a path from i to j in a MST (the Subdominant Ultrametric Distance)

Edge	AE	CD	AB	BE	BC	EC	ED
d_{ij}	1	2	3	4	5	6	7
$d_{ij}^{<}$	1	2	3	**3**	5	**5**	**5**



Ultrametric distance: violates the triangle inequality of an Euclidean metric:

$$d_{ij} \leq \max_{k} \left\{ d_{ik}, d_{kj} \right\}$$

Subdominant Ultrametric - the largest ultrametric among those that are less or equal to d_{ij}

Application for portfolio optimization

Application for portfolio optimization (E. Pantaleo et al, “When Do Improved Covariance Matrix Estimators Enhance Portfolio Optimization? An empirical study of nine estimators”, 2010):

- Applied different filtering techniques to a sample covariance matrix of US stocks
- Compared realized risk for Markowitz-optimal portfolios obtained with different filtered covariance matrices

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- Alternatives used:
    (i) Sample covariance matrix
    (ii) Random Matrix Theory (RMT) based filtering
    (iii) MST and other graph-based filtering

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  (iii) MST and other graph-based filtering

Results:

- Substantially lower-risk optimal portfolios for the regime $T / N > 1$ when short sales are allowed
- No significant improvements over the sample covariance matrix when short sales are not allowed, and/or $T / N < 1$

Control question

Select all correct answers

1. A Minimum Spanning Tree (MTS) constructs a graph with N nodes without loops using $N-1$ links, such that the total weight of all links is **maximized**.
2. A Minimum Spanning Tree (MTS) constructs a graph with N nodes without loops using $N-1$ links, such that the total weight of all links is **minimized**.
3. The MST compresses the parametrization of a correlation matrix from $N(N-1)/2$ parameters to $N-1$ parameters
4. Graph-based filtering of correlation matrices produce lower-risk optimal portfolios for the regime $T/N<1$ when short sales are not allowed
5. Graph-based filtering of correlation matrices produce lower-risk optimal portfolios for the regime $T/N>1$ when short sales are allowed

Correct answers: 2, 3, 5