Sequences, limits, continuity, differentiability

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MA105 Tutorial Sheet - 1
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Problems to be discussed

Problem -

- 5
- 7
- 9
- 10
- 12
- 13. (i), (ii)

Problem 5

5. (ii)

Prove that the following sequence is convergent by showing that it is monotone and bounded. Also find its limit:

$$a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n} \; \forall n \geq 1$$

Problem 5

Solution to 5. (ii) -

Claim 1: $a_n \leq 2 \; \forall n$

We will show this by induction.
Base-case : $n = 1$ as $a_1 = \sqrt{2} \leq 2$
Assuming that inductive hypothesis is true for $k \leq n$,

$$a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2 + 2} \leq 2$$

Claim 2: $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} & a_n \leq a_{n+1} \\ \Leftrightarrow \;\; & a_n \leq \sqrt{2 + a_n} \\ \Leftrightarrow \;\; & a_n^2 \leq 2 + a_n \\ \Leftrightarrow \;\; & (a_n + 1)(a_n - 2) \leq 0 \text{ which is true} \end{aligned}$$

Using both claims, $\{a_n\}$ is bounded and monotonic. Hence, it is convergent and converges to a limit L.

Now to find L, we use the continuity of $f(x) = \sqrt{2+x}$,

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\
L &= \lim_{n\to\infty} \sqrt{2+a_n} \\
L &= \sqrt{2+\lim_{n\to\infty} a_n} \\
L &= \sqrt{2+L} \\
L &= 2
\end{aligned}$$

Hence, $\lim_{n\to\infty} a_n = 2$.

You can also show this using $(\epsilon - N)$ definition of limits by *guessing* the limit as 2.

Problem 7

7.

If $\lim_{n\to\infty} a_n = L \neq 0$, then show that $\exists n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \quad \forall n \geq n_0$$

Problem 7

Solution to 7 -

It is given to us that $\lim_{n\to\infty} a_n = L$, and hence, $\forall \epsilon_0 > 0,\ \exists n_0(\epsilon_0) \in \mathbb{N}$ such that $\forall n > n_0$,

$$|a_n - L| < \epsilon_0$$

Now, set $\epsilon = \frac{|L|}{2}$.
We get that $\exists n_0 \in \mathbb{N}$ such that

$$|a_n - L| < \frac{|L|}{2}$$
$$\Rightarrow ||a_n| - |L|| \leq |a_n - L| < \frac{|L|}{2}$$

Which gives us $-\frac{|L|}{2} < |a_n| - |L|$ and hence $|a_n| \geq \frac{|L|}{2}\ \forall n > n_0$.

Problem 9

9.

For given sequences $\{a_n\}$ and $\{b_n\}$, prove or disprove the following,

(i) $\{a_n b_n\}$ is convergent if $\{a_n\}$ is convergent

(ii) $\{a_n b_n\}$ is convergent if $\{a_n\}$ is convergent and $\{b_n\}$ is bounded

Problem 9

Solution to 9. (i) -

Consider the following sequences $\forall n \geq 1$ -

$$a_n = \frac{1}{n}$$

$$b_n = n^2$$

Here, $\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = 0$, but $\{a_n b_n\} = \{n\}$ is divergent.

Hence this statement is false.

Problem 9

Solution to 9. (i) -

Consider the following sequences $\forall n \geq 1$ -

$$a_n = 1$$

$$b_n = (-1)^n$$

Here, $\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = 1$, and $\{b_n\}$ is bounded as $|b_n| \leq 1$, but $\{a_n b_n\} = \{(-1)^n\}$ is divergent.

Hence this statement is false.

Check that these sequences are also a counterexample for 9. (i).

Problem 10

10

Prove that the sequence $\{a_n\}$ is convergent **iff** the sequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are convergent to the same limit.

Problem 10

Solution to 10 -

Let L be the common limit.

Forward implication: $\{a_n\}$ convergent $\Rightarrow \{a_{2n}\}, \{a_{2n+1}\}$ convergent to same limit.

We are given that $\forall \epsilon_0 > 0,\ \exists n_0(\epsilon_0)$ such that $|a_n - L| < \epsilon_0\ \forall n > n_0$.

We need to prove that $\forall \epsilon_1 > 0,\ \exists n_1$ such that $|a_{2n} - L| < \epsilon_1\ \forall n > n_1$.

Set $\epsilon = \epsilon_1$ in the given statement, and check that having

$$n_1 = n_0(\epsilon) = n_0(\epsilon_1)$$

works. Hence, the sequence $\{a_{2n}\}$ is convergent to L.

Similarly, the sequence $\{a_{2n+1}\}$ is convergent to L.

Problem 10

Reverse implication: $\{a_{2n}\}, \{a_{2n+1}\}$ convergent to same limit $\Rightarrow$ $\{a_n\}$ is convergent.

We are given that $\forall \epsilon_1 > 0,\ \exists n_1(\epsilon_1)$ such that $|a_{2n} - L| < \epsilon_1\ \forall n > n_1$ and $\forall \epsilon_2 > 0,\ \exists n_2(\epsilon_2)$ such that $|a_{2n+1} - L| < \epsilon_2\ \forall n > n_2$.

We need to prove $\forall \epsilon > 0,\ \exists n_0$ such that $|a_n - L| < \epsilon\ \forall n > n_0$.

Again, set $\epsilon_1 = \epsilon_2 = \epsilon$ and check that having

$$n_0 = max(2n_1(\epsilon), 2n_2(\epsilon) + 1)$$

works.

Hence, the sequence $\{a_n\}$ is convergent to L.

Problem 12

12.

Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x \to \alpha} f(x)$ exists for some $\alpha \in \mathbb{R}$. Show that,

$$\lim_{h \to 0} [f(\alpha + h) - f(\alpha - h)] = 0$$

Analyze the converse.

Problem 12

Solution to 12 -

Let $\lim_{x\to\alpha} f(x)$ be equal to L.

Then $\lim_{h\to 0} f(\alpha+h) = \lim_{h\to 0} f(\alpha-h) = L$.

We have,

$$0 \leq |f(\alpha+h) - f(\alpha-h)| \leq |f(\alpha+h) - L| + |f(\alpha-h) - L|$$

By Sandwich/Squeeze theorem,

$$\lim_{h\to 0} |f(\alpha+h) - f(\alpha-h)| = 0$$

Now, use converse of Problem 6 to show that,

$$\lim_{h\to 0} [f(\alpha+h) - f(\alpha-h)] = 0$$

The **converse** is false.

Consider this example with $\alpha = 0$,

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{|x|} & \text{otherwise} \end{cases}$$

We can see that $[f(\alpha + h) - f(\alpha - h)] = 0 \; \forall x \neq 0$.

But $\lim_{h \to 0} f(x)$ does not exist.

Problem 13

13. (i)

Discuss the continuity of

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin(\frac{1}{x}) & \text{otherwise} \end{cases}$$

13. (ii)

Discuss the continuity of

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x\sin(\frac{1}{x}) & \text{otherwise} \end{cases}$$

Problem 13

Solution to 13. (i) -

We will use the sequential criterion of (dis)continuity.

Consider the 2 sequences $\{a_n\}$ and $\{b_n\}$,

$$a_n = \frac{1}{(4n+1)\frac{\pi}{2}}$$

$$b_n = 0$$

Note that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$.

But we have,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \sin((4n+1)\frac{\pi}{2}) = \lim_{n\to\infty} 1 = 1$$

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} 0 = 0$$

Since we have found 2 sequences $\{a_n\}$ and $\{b_n\}$, such that

$$\lim_{n\to\infty} a_n = x = 0 \text{ and } \lim_{n\to\infty} b_n = x = 0$$

but

$$1 = \lim_{n\to\infty} f(a_n) = \lim_{x\to 0} f(x) \neq \lim_{n\to\infty} f(b_n) = \lim_{x\to 0} f(x) = 0$$

The function is not continuous at $x = 0$.

Problem 13

Solution to 13. (ii) -

We claim that the function is continuous.

We need to prove that $\forall \epsilon > 0, \ \exists \delta > 0$ such that $\forall |x| < \delta$,

$$|x \sin(\frac{1}{x})| < \epsilon$$

Now, since $|\sin(\frac{1}{x})| \leq 1 \ \forall x \in \mathbb{R} \setminus \{0\}$.

$$|x \sin(\frac{1}{x})| \leq |x| < \delta$$

Hence, setting $\delta = \epsilon$ completes our proof.

The End

Questions? Comments?
Ask on group