Tutorial 3

17. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and equals $f'(c)$. Is the converse true ? [Hint: Consider $f(x) = |x|$.]

Given,

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

Hence,

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = f'(c) \quad (1)$$

and

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = f'(c)$$

$$\Rightarrow \lim_{h \to 0^+} \frac{f(c) - f(c-h)}{h} = f'(c) \quad (2)$$

Adding (1) and (2),

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{h} = 2f'(c)$$

For converse, consider $f(x) = |x|$ at $c=0$

$$\lim_{h \to 0^+} \frac{f(h) - f(-h)}{h} = 0$$ but $f'(x)$ does not exist at $x = c$

4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:

(i) $p < 0$.

(ii) f has a local maximum/minimum at $\pm\sqrt{-p/3}$.

(iii) The maximum/minimum values are of opposite signs.

(i) Since $f(x)$ has 3 distinct real roots, $f'(x)$ has 2 dist real roots.

$$f'(x) = 3x^2 + p = 0 \Rightarrow x = \pm\sqrt{\frac{-p}{3}}$$

Hence $p < 0$.

(ii) Since $f'(x) = 0$ at $x = \pm\sqrt{\frac{-p}{3}}$, we need to check $f''(x)$ for max/min.

$$f''(x) = 6x \Rightarrow \begin{array}{l} f''\left(\sqrt{\frac{-p}{3}}\right) > 0 \\ f''\left(-\sqrt{\frac{-p}{3}}\right) < 0 \end{array}$$

Hence max at $-\sqrt{\frac{-p}{3}}$ and min at $\sqrt{\frac{-p}{3}}$

(iii) $$f\left(-\sqrt{\frac{-p}{3}}\right) = q + \sqrt{\frac{-4p^3}{27}}$$

$$f\left(\sqrt{\frac{-p}{3}}\right) = q - \sqrt{\frac{-4p^3}{27}}$$

Now $f'(x) < 0 \quad \forall \quad x \in \left(-\sqrt{\frac{-p}{3}}, \sqrt{\frac{-p}{3}}\right)$.

We can see using graph of $f(x)$ that it has a root in this interval.

(Proof not needed → But try using MVT)

Hence $f\left(-\sqrt{\frac{-p}{3}}\right) f\left(\sqrt{\frac{-p}{3}}\right) < 0$

$$\Rightarrow \quad 4p^3 + 27q^2 < 0$$

10. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

(i) $y = 2x^3 + 2x^2 - 2x - 1$

(ii) $y = 1 + 12|x| - 3x^2$, $x \in [-2, 5]$

$f(x) = 1 + 12|x| - 3x^2$

As $f(x) = f(-x)$, f is even.

Hence, let us draw for $x \geq 0$ where

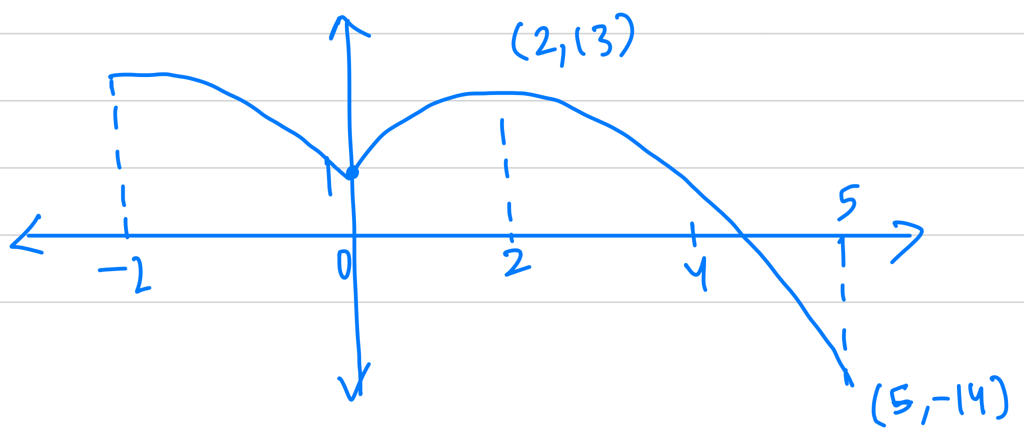
$f(x) = 1 + 12x - 3x^2$

$f(0) = 1$. $f(5) = -14$. $f(-2) = f(2) = 13$

As $f(x)$ is cont, no asymptotes.

$f'(x) = 12 - 6x = 0 \Rightarrow x = 2$

$f''(x) = -6$



13. Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$. Define $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Which of the following statements are true? Why?

(i) If f and g have a local maximum at $x = c$, then so does h.

(ii) If f and g have a point of inflection at $x = c$, then so does h.

(i) Given $\exists \delta_1$ s.t $\forall x \in (c-\delta_1, c+\delta_1)$

$$f(x) \leq f(c)$$

Given $\exists \delta_2$ s.t $\forall x \in (c-\delta_2, c+\delta_2)$

$$g(x) \leq g(c)$$

Let $\delta = \min(\delta_1, \delta_2)$.

Then, $\forall x \in (c-\delta, c+\delta)$,

$$h(x) = f(x)\,g(x) \leq f(c)\,g(c) = h(c)$$

$$h(x) \leq h(c)$$

Hence c is local max.

(ii) $f(x) = g(x) = 1 + \sin x$

is counter example.

(at $c = 0$)

Exercise 1. *Write down the Taylor series for (i)* $\cos x$*, (ii)* $\arctan x$ *about the point* 0*. Write down a precise remainder term* $R_n(x)$ *in each case.*

$$p(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{\underline{\lfloor n}} x^n \; ; \quad R_n(x) = \frac{f^{n+1}(c)}{\underline{\lfloor n+1}} (x-a)^{n+1}$$

$$c \in (a, x)$$

(i) $f(x) = \cos x$

$$f^{n}(0) = \begin{cases} (-1)^{n/2} & \text{when } n = 2k \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\underline{\lfloor 2k}} x^{2k}$$

$R_n(x) =$ write

(ii) $$\int_0^x \left(\frac{d}{dx} \tan^{-1} x\right) dx = \int_0^x \left(\frac{1}{1+x^2}\right) dx$$

$$\tan^{-1} x = \int_0^x (1 - x^2 + x^4 - x^6 - \dots) \, dx$$

$$\Rightarrow p(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

note that since $p(x)$ is a polynomial and $p(x) = \tan^{-1}(x)$, it is Taylor series of $\tan^{-1}(x)$ $\left[\text{for } x^2 < 1 \right]$