Tutorial 5

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t) dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin(\lambda(x-t)) dt$$

$$g'(x) = \frac{1}{\lambda} \int_0^x f(t) \cos(\lambda(x-t)) \lambda \, dt$$

$$- (0) \left(\frac{1}{\lambda} f(0) \sin(\lambda(x-0)) \right)$$

$$+ (1) \left(\frac{1}{\lambda} f(x) \sin(\lambda(x-x)) \right)$$

$$= \int_0^x f(t) \cos(\lambda(x-t)) dt$$

$$g''(x) = \int_0^x f(t) - \sin(\lambda(x-t)) \lambda \, dt$$

$$- (0) (f(x) \cos(\lambda(x-0)))$$

$$+ (1) (f(x) \cos(\lambda(x-x)))$$

$$g''(x) = -\lambda \int_0^x f(t)\sin(\lambda(x-t))\,dt + f(x)$$

Hence, $$g''(x) + \lambda^2 g(x) = f(x)$$

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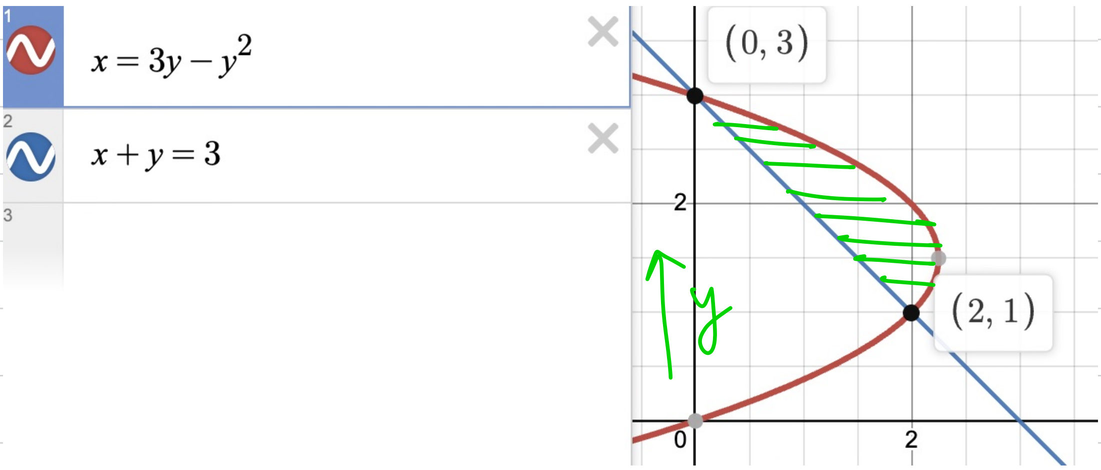
$$\frac{d}{dx}\int_{u(x)}^{v(x)} h(t,x)dt = \int_{u(x)}^{v(x)} \frac{\partial h}{\partial x}(t,x)dt - u'(x)h(u(x),x) + v'(x)h(v(x),x).$$

7. Find the area of the region bounded by the given curves in each of the following cases.

(iii) $x = 3y - y^2$ and $x + y = 3$.



Draw

$$Ar = \int_2^3 \left[(3y - y^2) - (3-y)\right] dy = \frac{4}{3}$$

11. For the following curve

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \le x \le 3,$$

find the arc length as well as the the ~~area of the surface generated~~ by revolving it about the line $y = -1$.

$$\int dl = \int_1^3 \sqrt{1+f'(x)^2}\,dx$$

$$L = \int_1^3 \sqrt{1+\left(x^2 - \frac{1}{4x^2}\right)^2}\,dx$$

$$= \int_1^3 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\,dx$$

$$= \int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \frac{53}{6}$$

3. Using definition, examine the following functions for continuity at $(0,0)$. The expressions below give the value at $(x, y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\ \frac{x^3 y}{x^6 + y^2} \quad (ii)\ xy\frac{x^2 - y^2}{x^2 + y^2} \quad (iii)\ ||x| - |y|| - |x| - |y|.$$

We need to prove $f(x,y) = xy\dfrac{x^2-y^2}{x^2+y^2}$ is cont.

Hence, $\forall \epsilon > 0$ we must have some $\delta > 0$ s.t.

$$\left| xy\frac{x^2-y^2}{x^2+y^2} \right| < \epsilon \quad \forall x, y \ \sqrt{x^2+y^2} < \delta$$

Now,

$$\left| xy \frac{x^2-y^2}{x^2+y^2} \right| = |xy| \frac{|x^2-y^2|}{|x^2+y^2|}$$

By Δ-ineq,

$$|x^2 - y^2| \leq |x^2| + |y^2|$$

Hence $$\frac{|x^2-y^2|}{|x^2+y^2|} \leq 1.$$

$$\Rightarrow |xy| \left| \frac{x^2-y^2}{x^2+y^2} \right| \leq |xy| \leq \frac{1}{2}(x^2+y^2)$$

Now consider $\delta = \sqrt{2\epsilon}$.

We have $\sqrt{x^2+y^2} < \delta$

$$\Rightarrow \frac{1}{2}(x^2+y^2) < \frac{\delta^2}{2} = \epsilon$$

Hence, $$\left| xy \frac{x^2-y^2}{x^2+y^2} \right| \leq \frac{1}{2}(x^2+y^2) < \epsilon$$

and this δ proves continuity.

4. Suppose $f,\, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x, y) \in \mathbb{R}^2$ are continuous:
(i) $f(x) \pm g(y)$ (ii) $f(x)g(y)$ (iii) $\max\{f(x),\, g(y)\}$ (iv) $\min\{f(x),\, g(y)\}$.

$$\min\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}$$

As f, g cont $\Rightarrow$ $f+g$, $f-g$ cont

As $|x|$ cont $\Rightarrow$ $|f-g|$ cont

Hence, $\max\{f,g\}$ is sum of 2 continuous functions.

$\Rightarrow$ $\max\{f, g\}$ is cont.