Tutorial 4

Exercise 2. *Our examples of Taylor's series have usually been series about the point* 0. *Write down the Taylor series of the polynomial* $x^3 - 3x^2 + 3x - 1$ *about the point* 1.

$$x^3 - 3x^2 + 3x - 1 = \underbrace{(x-1)^3}$$

of the form $p(x-1)$
hence the T.S at $x=1$

Exercise 3. *What is the Taylor series of the function* $1729x^{1729} + 1728x^{1728} + 1000x^{1000} + 729x^{729} + 1$ *about the point* 0*?*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$p(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\underline{|n}} x^n$$

$$= \sum_{n=0}^{\infty} \frac{\underline{|n} \cdot a}{\underline{|n}} x^n$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

$$= f(x)$$

1. Let $f(x) = 1$ if $x \in [0,1]$ and $f(x) = 2$ if $x \in (1,2]$. Show from the first principles that f is Riemann integrable on $[0,2]$ and find $\int_0^2 f(x)dx$.

Consider $P_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{n}{n}, \ldots \frac{2n}{n} \right\}$

$$L(f, P_n) = \sum_{i=0}^{2n-1} m_i \left(\frac{i+1}{n} - \frac{i}{n} \right)$$

$$= \sum_{i=0}^{2n-1} m_i \left(\frac{1}{n} \right)$$

$$= \sum_{i=0}^{n} (1)\left(\frac{1}{n} \right) + \sum_{i=n+1}^{2n-1} (2)\left(\frac{1}{n} \right)$$

$$= (n+1)\left(\frac{1}{n} \right) + (n-1)(2)\left(\frac{1}{n} \right)$$

$$= 3 - \frac{1}{n}$$

$$U(f, P_n) = 3 + \frac{1}{n}$$

Now, $L(f, P_n) \leq L(f) \leq U(f) \leq U(f, P_n)$

At lt $n \to \infty$, $L(f, P_n) = U(f, P_n)$

Hence, $L(f) = U(f) = 3 = \int_0^2 f(x)dx$

2. (a) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

(b) Give an example of a Riemann integrable function on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a,b]$.

(a) As f is Riemann int,

$$L(f,P) \leq \int_a^b f(x)\,dx \leq U(f,P)$$

Now, $L(f,P) = \sum m_i (x_{i+1} - x_i)$

As $f(x) \geq 0$, $m_i \geq 0 \ \forall i$.

Hence $L(f,P) \geq 0 \Rightarrow \int_a^b f(x)dx \geq 0$

Assume $\exists\, c \in (a,b)$ s.t $f(c) > 0$.

Now, $\exists\, \delta$ s.t $f(x) > \dfrac{f(c)}{2}$

$\forall \ |x - c| < \delta$

(No need to write proof. Obvious with $\epsilon = \frac{f(c)}{2}$)

Hence, for any partition P,

$$U(f,P) \geq (2\delta)\,\frac{f(c)}{2} = \delta \cdot f(c)$$

Hence,

$$U(f) = \operatorname{Inf}\{U(f,P)\} \geq \delta \cdot f(c)$$

But if f was Riemann integrable,

$$\int_a^b f(x)\,dx = 0 = U(f) \geq \delta f(c)$$

which is a contradiction.

Hence $f(x) = 0 \ \forall x$.

(b)

$$f : [0,2] \to \mathbb{R}$$

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

3. Evaluate $\lim_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:

(ii) $S_n = \sum_{i=1}^{n} \frac{n}{i^2 + n^2}$

$$S_n = \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2}\left(\frac{i}{n} - \frac{(i-1)}{n}\right)$$

By comparision, it is Riemann sum of $f(x) = \frac{1}{1+x^2}$ over $x \in [0,1]$

Hence, $\lim_{n \to \infty} S_n = \int_0^1 f(x)\,dx$

$$= \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$