Before we get started

Join the Whatsapp group.

Figure: https://tinyurl.com/ma105tut

Sequences, limits, continuity, differentiability

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MA105 Tutorial Sheet - 1
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Problems to be discussed

Problem -

- 1. (iii)
- 2. (i), (iv)
- 3. (ii)
- 5. (ii)
- 6

Problem 1

1. (iii)

Using the $(\epsilon - N)$ definition of a limit, prove:

$$\lim_{n\to\infty} \frac{n^{\frac{2}{3}} \sin(n!)}{n+1} = 0$$

Problem 1

Solution to 1. (iii) -

We have to show that $\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$,

$$\left| \frac{n^{\frac{2}{3}} \sin(n!)}{n+1} - 0 \right| < \epsilon$$

Note that $|\sin(x)| \leq 1\ \forall x \in \mathbb{R}$, and hence

$$\left| \frac{n^{\frac{2}{3}} \sin(n!)}{n+1} \right| \leq \left| \frac{n^{\frac{2}{3}}}{n+1} \right|$$

Now,

$$\left| \frac{n^{\frac{2}{3}}}{n+1} \right| < \left| \frac{n^{\frac{2}{3}}}{n} \right| = \frac{1}{n^{\frac{1}{3}}} < \epsilon$$

So we can set $n_0 = \lfloor \frac{1}{\epsilon^3} \rfloor + 1$ and be done.

Problem 2

2. (i)

Find the limit:

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\cdots+\frac{n}{n^2+n}\right)$$

2. (iv)

Find the limit:

$$\lim_{n\to\infty} n^{\frac{1}{n}}$$

Problem 2

Solution to 2. (i) -

Note that for $i = 1, 2, \ldots, n$ we have,

$$\frac{n}{n^2 + n} \leq \frac{n}{n^2 + i} \leq \frac{n^2}{n^2}$$

Hence,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2 + n} \leq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2 + i} \leq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n^2}{n^2}$$

Now,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2 + n} = \lim_{n\to\infty} \frac{n^2}{n^2 + n} = \frac{1}{1 + \frac{1}{n}} = 1$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2}=\lim_{n\to\infty}\frac{n^2}{n^2}=1$$

So,

$$1\leq\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i}\leq 1$$

And hence by *Squeeze/Sandwich Theorem* the limit exists and,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i}=1$$

Solution to 2. (iv) -

We see that when $n > 1$, we have $n^{\frac{1}{n}} > 1$.
Hence, let $n^{\frac{1}{n}} = 1 + h_n$ for some sequence $\{h_n\}$.

Now,

$$\begin{aligned} n = (1+h_n)^n &= 1 + nh_n + \frac{n(n-1)}{2}h_n^2 + \dots \\ &\geq \frac{n(n-1)}{2}h_n^2 \end{aligned}$$

And so,

$$h_n^2 \leq \frac{2}{n-1}$$

Now,

$$\lim_{n\to\infty} h_n^2 \leq \lim_{n\to\infty} \frac{2}{n-1} = 0$$

As you will see in Problem 8 of this sheet,

$$\lim_{n\to\infty} h_n^2 = 0 \implies \lim_{n\to\infty} h_n = 0$$

Hence,

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} (1 + h_n) = 1$$

Problem 3

3. (ii)

Show that the following sequence is not convergent:

$$\left\{ a_n = (-1)^n \left(\frac{1}{2} - \frac{1}{n} \right) \right\}$$

Problem 3

Solution to 3. (ii) -

Consider the following subsequences of $\{a_n\}$.

$$\{a_{2n}\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}$$

$$\{a_{2n+1}\} = \left\{-\left(\frac{1}{2} - \frac{1}{n}\right)\right\}$$

Now,

$$\lim_{n\to\infty} a_{2n} = \frac{1}{2}$$

$$\lim_{n\to\infty} a_{2n+1} = -\frac{1}{2}$$

Since the sequence $\{a_n\}$ has 2 subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ converging to different limits, $\{a_n\}$ is not convergent.

Problem 5

5. (ii)

Prove that the following sequence is convergent by showing that it is monotone and bounded. Also find its limit:

$$a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n} \; \forall n \geq 1$$

Problem 5

Solution to 5. (ii) -

Claim 1: $a_n \leq 2 \, \forall n$

We will show this by induction.
Base-case : $n = 1$ as $a_1 = \sqrt{2} \leq 2$
Assuming that inductive hypothesis is true for $k \leq n$,

$$a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2 + 2} \leq 2$$

Claim 2: $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} & a_n \leq a_{n+1} \\ \Leftrightarrow \;\; & a_n \leq \sqrt{2 + a_n} \\ \Leftrightarrow \;\; & a_n^2 \leq 2 + a_n \\ \Leftrightarrow \;\; & (a_n + 1)(a_n - 2) \leq 0 \text{ which is true} \end{aligned}$$

Using both claims, $\{a_n\}$ is bounded and monotonic. Hence, it is convergent and converges to a limit L.

Now to find L, we use the continuity of $f(x) = \sqrt{2+x}$,

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\ L &= \lim_{n\to\infty} \sqrt{2+a_n} \\ L &= \sqrt{2+\lim_{n\to\infty} a_n} \\ L &= \sqrt{2+L} \\ L &= 2 \end{aligned}$$

Hence, $\lim_{n\to\infty} a_n = 2$.

You can also show this using $(\epsilon - N)$ definition of limits by *guessing* the limit as 2.

Problem 6

6

If $\lim_{n\to\infty} a_n = L$ then find $\lim_{n\to\infty} a_{n+1}$ and $\lim_{n\to\infty} |a_n|$.

Problem 6

Solution to 6 -

Claim 1: $\lim_{n\to\infty} a_{n+1} = L$

We have to show that $\forall \epsilon > 0,\ \exists n_1 \in \mathbb{N}$ such that $\forall n > n_1$,

$$|a_{n+1} - L| < \epsilon$$

It is given to us that $\lim_{n\to\infty} a_n = L$, and hence, $\forall \epsilon_0 > 0,\ \exists n_0(\epsilon_0) \in \mathbb{N}$ such that $\forall n > n_0$,

$$|a_n - L| < \epsilon_0$$

Now, set $\epsilon_0 = \epsilon$ in the 2nd equation. We have,

$$|a_n - L| < \epsilon \ \forall n > n_0(\epsilon)$$
$$\Rightarrow |a_{n+1} - L| < \epsilon \ \forall n > n_0(\epsilon)$$

Hence, if we set $n_1 = n_0(\epsilon)$, we have $|a_{n+1} - L| < \epsilon \, \forall n \geq n_1$ which completes the proof.

Claim 2: $\lim_{n \to \infty} |a_n| = |L|$

We have to show that $\forall \epsilon > 0, \; \exists n_1 \in \mathbb{N}$ such that $\forall n > n_1$,

$$||a_n| - |L|| < \epsilon$$

It is given to us that $\lim_{n \to \infty} a_n = L$, and hence, $\forall \epsilon_0 > 0, \; \exists n_0(\epsilon_0) \in \mathbb{N}$ such that $\forall n > n_0$,

$$|a_n - L| < \epsilon_0$$

Now, note that $||a_n| - |L|| \leq |a_n - L|$ by triangle inequality.

Hence, if we again set $\epsilon_0 = \epsilon$ and take $n_1 = n_0(\epsilon_0)$, we get

$$||a_n| - |L|| \leq |a_n - L| < \epsilon \, \forall n > n_1$$

And we are done.

The End

Questions? Comments?

Again

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