Tutorial 6

6. Examine the following functions for the existence of partial derivatives at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero.

(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$

(ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$

(i) $$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly $f_y(0,0) = 0$

(ii) $$\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin^2 h}{|h|} - 0}{h}$$

$$= \lim_{h\to 0} \frac{\sin^2 h}{h\,|h|}$$ which does not exist

Hence f_x does not exist.

By symmetry, f_y does not exist.

7. Let $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \quad \text{for} \quad (x,y) \neq (0,0).$$

Show that f is continuous at $(0,0)$, and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0,0)$.

Continuity shown in 5.3

Partial derivative exist at $(0,0)$ shown in 5.6

$$\frac{\partial f}{\partial x} = \underbrace{2x \sin\left(\frac{1}{x^2+y^2}\right)}_{\text{Finite}} - \underbrace{\frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)}_{\text{Might be unbounded}}$$

We want to show $\frac{\partial f}{\partial x}$ is unbounded in any disc around $(0,0)$.

Suppose we have disc $x^2+y^2 < r^2$.

Consider $(x,y) = \left(\frac{1}{\sqrt{\pi n}}, 0\right)$, $n \in \mathbb{N}$

such that $\left(\frac{1}{\sqrt{\pi n}}\right)^2 + 0^2 < r^2$

$$\Rightarrow \quad n > \frac{1}{\pi r^2}$$

As $\lim_{n\to\infty} f_x\left(\frac{1}{\sqrt{n\pi}}, 0\right) \longrightarrow \infty$ (Check this)

f_x is unbounded. Similarly f_y.

9. Examine the following functions for the existence of directional derivatives and differentiability at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\ xy\frac{x^2-y^2}{x^2+y^2} \quad (ii)\ \frac{x^3}{x^2+y^2} \quad (iii)\ (x^2+y^2)\sin\frac{1}{x^2+y^2}$$

Let $\vec{v} = (a, b)$

Then $\displaystyle \frac{\partial f}{\partial v}\Big|_{(x,y)} = \lim_{h\to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h\sqrt{a^2+b^2}}$

$\displaystyle \Rightarrow \frac{\partial f}{\partial v}\Big|_{(0,0)} = \lim_{h\to 0} \frac{f(ha, hb) - f(0,0)}{h\sqrt{a^2+b^2}}$

$\displaystyle = \lim_{h\to 0} \frac{h^2(a^2+b^2)\sin\left(\frac{1}{h^2(a^2+b^2)}\right)}{h\sqrt{a^2+b^2}}$

$= 0$

Hence all dirn derv exist at $(0,0)$.

10. Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0,0)$, $D_{\underline{u}}f(0,0)$ exists for every vector $\underline{u}$, yet f is not differentiable at $(0,0)$.

Done in class

1. Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3, -1, 0)$ and $(5, 3, 6)$.

$f(x, y, z) = x^2 - y^2 + 2z^2$

$\nabla f = (2x, -2y, 4z)$

Dirn of line $= (5-3, 3-(-1), 6-0)$

$= (2, 4, 6)$

Hence, $\frac{2x}{2} = \frac{-2y}{4} = \frac{4z}{6}$

and $x^2 - y^2 + 2z^2 = 1$

Solutions
$x = -\sqrt{\frac{2}{3}}$, $y = 2\sqrt{\frac{2}{3}}$, $z = -\sqrt{\frac{3}{2}}$
$x = \sqrt{\frac{2}{3}}$, $y = -2\sqrt{\frac{2}{3}}$, $z = \sqrt{\frac{3}{2}}$

2. Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin xy$ at the point $(1, 0)$ has the value 1.

Let dirn be $\hat{u} = (a, b)$; $a^2 + b^2 = 1$.

Since we are given $\frac{\partial f}{\partial u}$ exists, $\frac{\partial f}{\partial u} = \nabla f \cdot \hat{u}$

$\nabla f_{(1,0)} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x + y\cos xy, x\cos xy) = (2, 1)$

$\frac{\partial f}{\partial u}_{(1,0)} = 2a + b = 1$; $a^2 + b^2 = 1$

Solutions
$a = 0$, $b = 1$
$a = \frac{4}{5}$, $b = -\frac{3}{5}$