Assignment - 0

Problems - 1A, 2, 3, 4

1. (Elegance) This is just for practice, related to Linear Algebra but not Quantum Mechanics. Look at both the Oddtown and Eventown problems (hopefully not the solutions) and solve them. (You might need to go over the field $\mathbb{F}_2$ before this)

Suppose that there are n people who form m clubs. Consider the matrix $M^{n\times m}$ over $\mathbb{F}_2$ such that $M = [m_{ij}]$ and

$$m_{ij} = \begin{cases} 1 & \text{if Person } i \text{ is in Club } j \\ 0 & \text{else} \end{cases}$$

Denote the i^{th} column vector of M as c_i.

Oddtown Problem -

Note that $c_j^T c_i = \delta_{ij} \quad \forall\ i, j \in 1..m$.

Hence, $M^T M = I_{m\times m}$.

Now, $m = \dim(M^T M) \leq \dim(M) \leq n$

$$\Rightarrow m \leq n$$

To show attainability, consider clubs with Club i consisting only of Person i.

Eventown Problem

Now, $c_j^T c_i = 0 \quad \forall\ i, j \Rightarrow M^T M = 0$

2. (Weaker suffices!) Suppose A is any linear operator on a Hilbert space, V. A is said to be Hermitian iff $\langle x, Ay \rangle = \langle Ax, y \rangle$ for all vectors $x, y \in V$.

Now suppose V is finite dimensional. Show that a necessary and sufficient condition for an operator A in V to be Hermitian is:

$$\langle x, Ax \rangle = \langle Ax, x \rangle$$

for all vectors $x \in V$.

We have $\langle t, At \rangle = \langle At, t \rangle \quad \forall\, t$.

Now, set $t = x + y$,

$$\langle x+y, A(x+y) \rangle = \langle A(x+y), x+y \rangle$$

$$\Rightarrow \cancel{\langle x, Ax \rangle} + \langle x, Ay \rangle + \langle y, Ax \rangle + \cancel{\langle y, Ay \rangle} = \cancel{\langle Ax, x \rangle} + \langle Ax, y \rangle + \langle Ay, x \rangle + \cancel{\langle Ay, y \rangle}$$

$$\Rightarrow \langle x, Ay \rangle + \langle y, Ax \rangle = \langle Ax, y \rangle + \langle Ay, x \rangle \quad (1)$$

Similarly set $t = x + iy$ or $y = iy$,

$$\langle x, A\,iy \rangle + \langle iy, Ax \rangle = \langle Ax, iy \rangle + \langle Aiy, x \rangle$$

$$\Rightarrow i\langle x, Ay \rangle - i\langle y, Ax \rangle = i\langle Ax, y \rangle - i\langle Ay, x \rangle$$

$$\Rightarrow \langle x, Ay \rangle - \langle y, Ax \rangle = \langle Ax, y \rangle - \langle Ay, x \rangle \quad (2)$$

Together, (1) and (2) give,

$$\langle x, Ay \rangle = \langle Ax, y \rangle$$

3. (Higher Dimensions!) We're going to give you some matrices, and your job is to find their eigenvalues (no calculator, and write down how you find them, because that is what we want to know)

(a) $\begin{bmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 10 & 12 & 8 \\ 9 & 6 & 6 & 4 \end{bmatrix}$

Consider matrices $M = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

We have (a) $M \otimes N$ (b) $N \otimes M$ (c) $M \otimes M$

Eigenvalues of $A \otimes B$ are $a_i b_j$ where a_i are eigenvalues of A and b_i of B.

Hence, we can calculate the eigenvalues of M and N.

$$\sigma(M) = \frac{7 \pm \sqrt{57}}{2}$$

$$\sigma(N) = \pm 1$$

Hence,

(a) $M \otimes N$ has eigenvalues $\frac{7 \pm \sqrt{57}}{2}, \frac{-7 \pm \sqrt{57}}{2}$

(b) $N \otimes M$ has eigenvalues $\frac{7 \pm \sqrt{57}}{2}, \frac{-7 \pm \sqrt{57}}{2}$

(c) $M \otimes M$ has eigenvalues $-2, \frac{53 \pm 7\sqrt{57}}{2}$

4. (Algebra and Technicalities) Look up the definitions of a norm and a metric. Show that $|x| = \sqrt{\langle x|x\rangle}$ is a valid norm and $d(x,y) = |x-y|$ is a valid metric.

A Hilbert space is just a *complete* inner product space. This means that every Cauchy sequence in the vector space converges to some element **in the vector space**.

Consider the set of all continuous real functions on $[-1,1]$. Confirm that this is a vector space over $\mathbb{R}$ with the normal operations: $(f+g)(t) = f(t)+g(t), (\alpha f)(t) = \alpha f(t)$.

Define the inner product as $\langle f|g\rangle = \int_{-1}^{1} f(t)g(t)dt$. Now take the sequence of functions

$$f_n(t) := \begin{cases} 1 & t \in [-1,0] \\ 1-nt & t \in [0,\frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

Compute $\langle f_n|f_m\rangle$ and show that $\{f_n\}$ is Cauchy. Find where this sequence converges and conclude that this inner product space is not complete.

Skipping few initial steps,

WLOG $m \geq n$,

$$\langle f_m | f_n \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$$

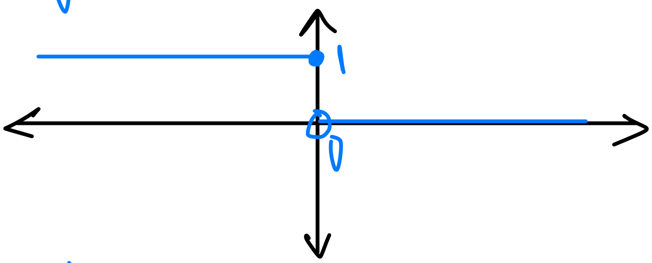
$$= 1 - \frac{n-3m}{6m^2} = 1 + \theta\left(\frac{1}{m}\right)$$

Now, with metric $d(f_m, f_n) = \langle f_m - f_n | f_m - f_n \rangle$,

$$\langle f_m - f_n | f_m - f_n \rangle = \langle f_m | f_m \rangle + \langle f_n | f_n \rangle - 2\langle f_m | f_n \rangle$$

Each of the three terms obviously go to 0 as $m, n \to \infty$ and hence the sequence is Cauchy.

One such function to which $\{f_n\}$ converges is



which is discontinuous and hence not in the inner product space. Hence it is incomplete.

Show the following remarkable result: *Finite dimensional vector space V under field $\mathbb{F}$ is a Hilbert space under any valid inner product on V for $\mathbb{F} = \mathbb{R}/\mathbb{C}$.*

Consider an orthonormas basis $\{e_i\}$ for space V of dimension d.
To show completeness, we consider a Cauchy seq in V such that $\langle v_m - v_n | v_m - v_n \rangle \to 0$ as $m, n \to \infty$.

Now, $$v_m - v_n = \sum_{i=1}^{d} (\alpha_{m_i} - \alpha_{n_i}) e_i$$

Hence, $$\langle v_m - v_n | v_m - v_n \rangle = \sum (\alpha_{m_i} - \alpha_{n_i})^2$$

As $\sum (\alpha_{m_i} - \alpha_{n_i})^2 \to 0$ with $m, n \to \infty$,
$\Rightarrow (\alpha_{m_i} - \alpha_{n_i}) \to 0$

$\Rightarrow \{\alpha_{n_i}\}$ is a Cauchy sequence $\forall i \in 1..d$
$\Rightarrow \{\alpha_{n_i}\}$ converges to some $\alpha_i \in \mathbb{R}$ or $\mathbb{C}$

Now, consider the vector $v = \sum \alpha_i e_i \in V$ and it is trivial to note that $\{v_n\}$ converges to v.
Hence, the vector space is complete and is a Hilbert space by defn.