Game Theory and Applications (博弈论及其应用)

Chapter 8: Extensive Game

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dynamic

高 尉

Recap on Previous Chapter

- Two-Player Zero-Sum Game
  - Both players do not do too badly
    - For Player 1

$$\max_{a_1\in A_1}\min_{a_2\in A_2} u(a_1,a_2)$$

    - For player 2

$$\min_{a_2\in A_2}\max_{a_1\in A_1} u(a_1,a_2)$$

Player 2

Player 1

	L	M	R
U	1	1	8
M	5	2	4
D	7	0	0

The Minmax Theorem

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^{\top} = \min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^{\top}.$$

Main result: NE for two-player zero-sum game ⟶ LP

Strategy Game

A **strategy game** consists of

- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i: A_1 \times A_2 \times \cdots \times A_N \rightarrow R$ for $i \in N$

$$\boldsymbol{G = \left\{N, \{A_i\}_{i=1}^{N}, \{u_i\}_{i=1}^{N}\right\}}$$

An outcome $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ is a **Nash Equilibrium (NE)** if for each players i

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \text{ for all } a_i \in A_i.$$

How to find Nash Equilibria

1) Calculate directly

  - i) find the best response functions
  - ii) calculate Nash equilibria

2) Eliminate all dominated strategy

3) For two-player zero-sum player, linear programming

Strategy Game

- Every player makes strategy once time **simultaneously** in strategy game
  - Each player make strategy without knowing the strategies of the other players
  - The game does not incorporate any information of sequence, time for players' strategies

Example

In some situation, players can observe others' strategy before they make decision

- Simple Nim game
  - There are n coins
  - Two players select 1 or
  - The winner is the one t

Extensive Game

Strategy Game

- ➢ Set of players
- ➢ Set of strategies
- ➢ Payoff functions

Extensive game provides more information

- ➢ Sequences of players
- ➢ Strategies available at different points in the game

Two variants

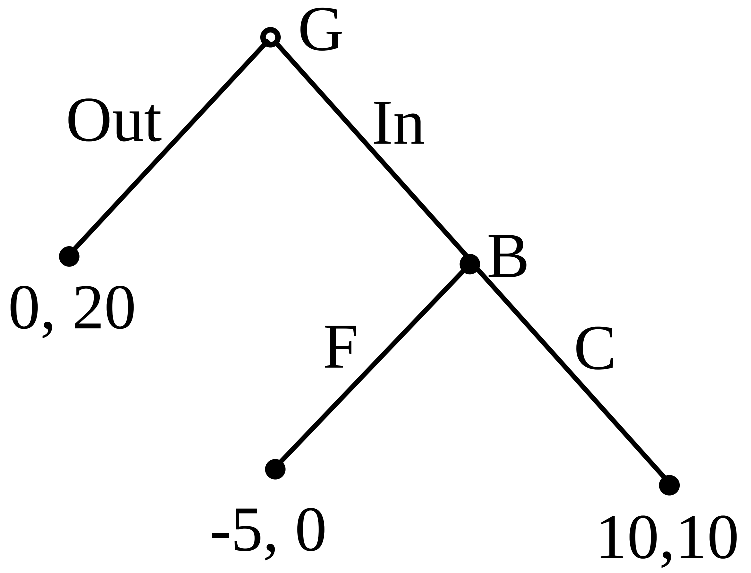
- ✓ perfect information extensive-form games
- ✓ imperfect-information extensive-form games

Entry Game

Google is contemplating entering the Chinese market, and Baidu can either fight the entry or cooperate

Game Tree

- node
  - non-terminal node
  - terminal node
- branches
- players
- strategy
- payoff



Formal Definition of Extensive Game

An extensive game with perfect information includes

- Players N is the set of N players
- Strategies A is a set of all strategies unimportant
- Histories H is a set of strategy sequence (finite or infinite) s.t.
  - The empty sequence $\emptyset \in H$
  - If $a^1 a^2 \dots a^k \in H$ then $a^1 a^2 \dots a^s \in H$ when $s \leq k$
  - If an infinite sequence $\left(a^k\right)_{k=1}^{\infty}$ satisfies $a^1 a^2 \dots a^k \in H$ for each positive k, then $\left(a^k\right)_{k=1}^{\infty} \in H$

Definition of Extensive Game

An extensive game with **perfect information** is defined by

- Players N is the set of N players
- Strategies A is a set of all strategies
- Histories H is a set of sequence (finite or infinite)
  - Each sequence in H is called a history; each component $a^i \in A$ is a strategy
  - Terminal history $a^1 \ldots a^k \in H$ if $k = +\infty$ or $a^1 \ldots a^{k+1} \notin H$ for any $a^{k+1} \in A$.
  - Terminal history set Z={all terminal histories $a^1 \ldots a^k \in H$}

Definition of Extensive Game

An extensive game with perfect information is defined by

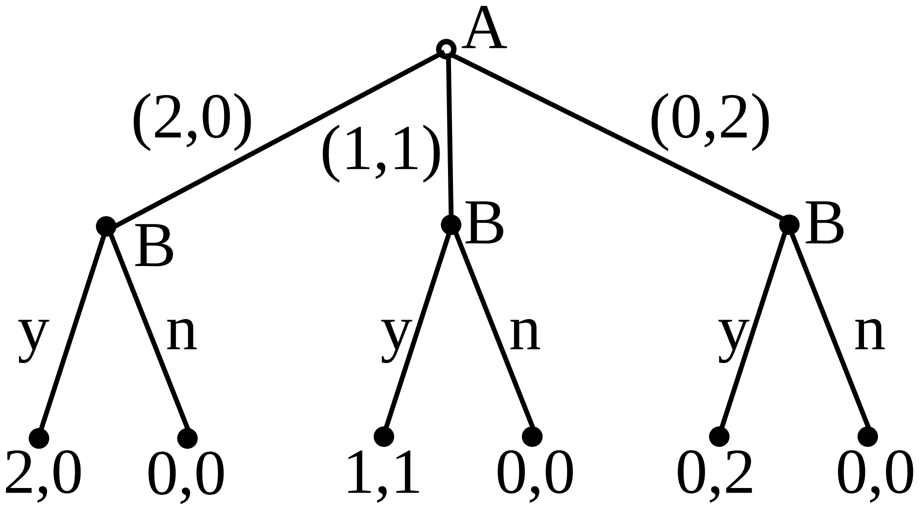
- Players N is the set of N players
- Strategies A is a set of all strategies
- Histories H is a set of sequence (finite or infinite)
- Player function
  - $P: H \setminus Z \rightarrow N$ assigns to **each non-terminal history** a player of N
  - $P(h)$ denotes the player who takes action after the history h
- Payoff function $u_i: Z \rightarrow R$

$$G = \left\{N, H, P, \{u_i\}\right\}$$

Ultimatum Game

$G = \{N, H, P, \{u_i\}\}$

N={A,B}



H={∅,(2,0),(1,1),(0,2), ((2,0),y)}

∪{((2,0),n), ((1,1),y), ((1,1),n)}

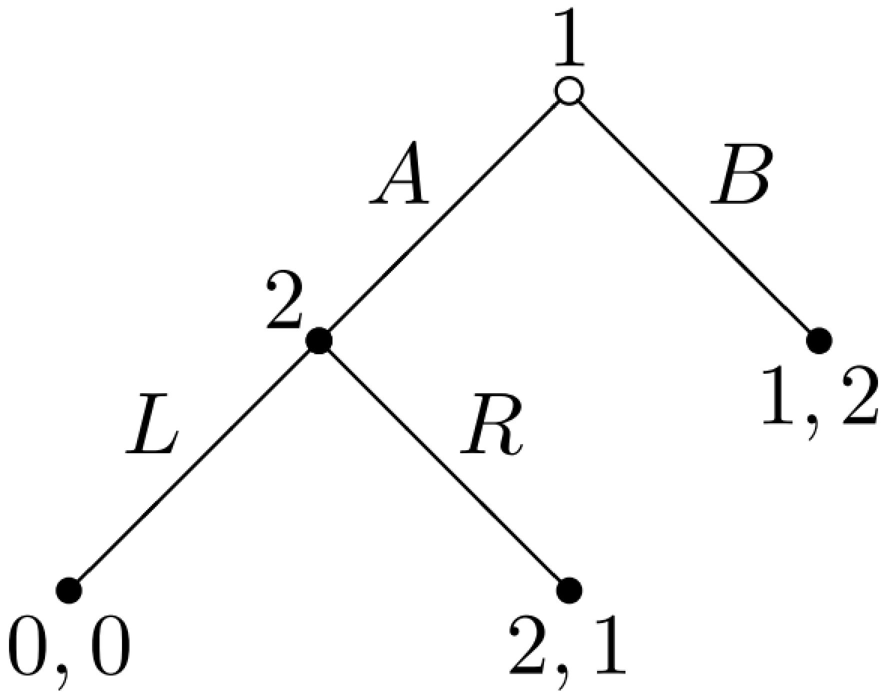
∪{((0,2),y), ((0,2),n)}

P: P(∅)=A; P((2,0))=B; P((1,1))=B; P((0,2))=B

$$u_1\big((2{,}0),\mathrm{y}\big) = 2, u_1\big((2{,}0),\mathrm{n}\big) = 0, u_1\big((1{,}1),\mathrm{y}\big) = 1, u_1\big((1{,}1),\mathrm{n}\big) = 0$$

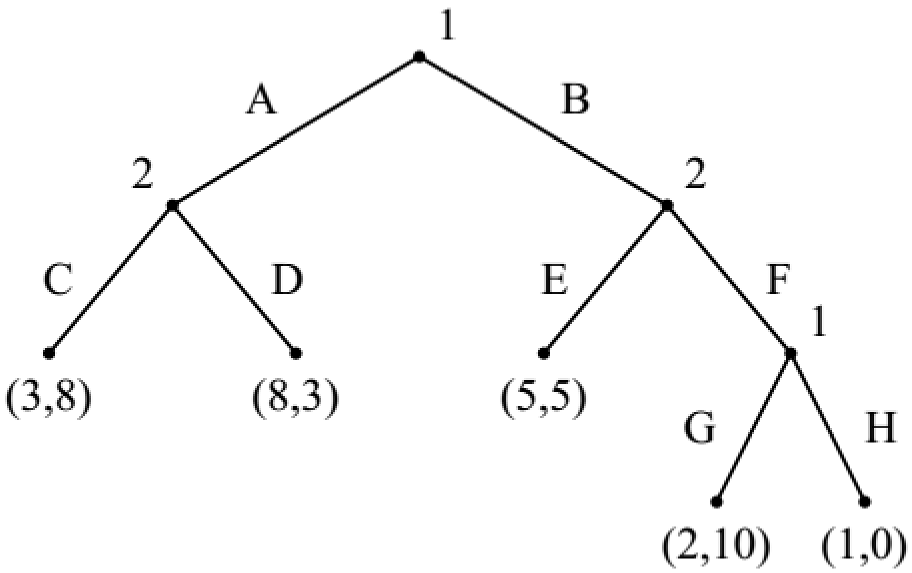
$$u_2\big((2{,}0),\mathrm{y}\big) = 0, u_2\big((2{,}0),\mathrm{n}\big) = 0, u_2\big((1{,}1),\mathrm{y}\big) = 1, u_2\big((1{,}1),\mathrm{n}\big) = 0$$

Example



- $G = \{N, H, P, \{u_i\}\}$
- N={1,2}
- H={∅,A,B,AL,AR}
- P: P(∅)=1; P(A)=2
- $u_1(B) = 1, u_1(AL) = 0, u_1(AR) = 2$
- $u_2(B) = 2, u_2(AL) = 0, u_2(AR) = 1$

Exercise



$N = \{1,2\}$

$H = \{\emptyset, A, B, AC, AD, BE, BF, BFG, BFH\}$

P: $P(\emptyset)=1$ $P(A)=2$ $P(B)=2$ $P(BF)=1$.

$u_1(BFG)=2$ $u_1(BFH)=1$ $u_1(AC)=3$ $u_1(AD)=8$

$u_2(BFG)=10$ $u_2(BFH)=0$ $u_2(AC)=8$ $u_2(AD)=3$.

Pure strategies

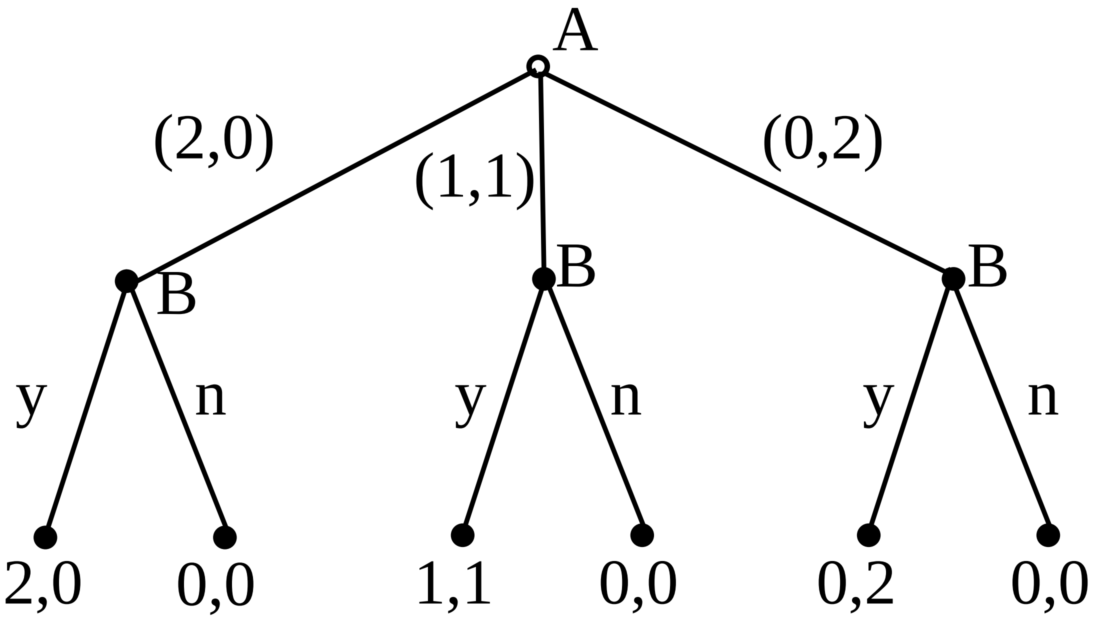
Definition: Given game $G = \{N, H, P, \{u_i\}\}$, the pure strategy for player i is given by the cross product

$$\times_{h \in H} \{a^s : (h, a^s) \in H, p(h) = i\}.$$

结点.

A pure strategy for a player is a complete specification of which deterministic action to take at every node belonging to that player.

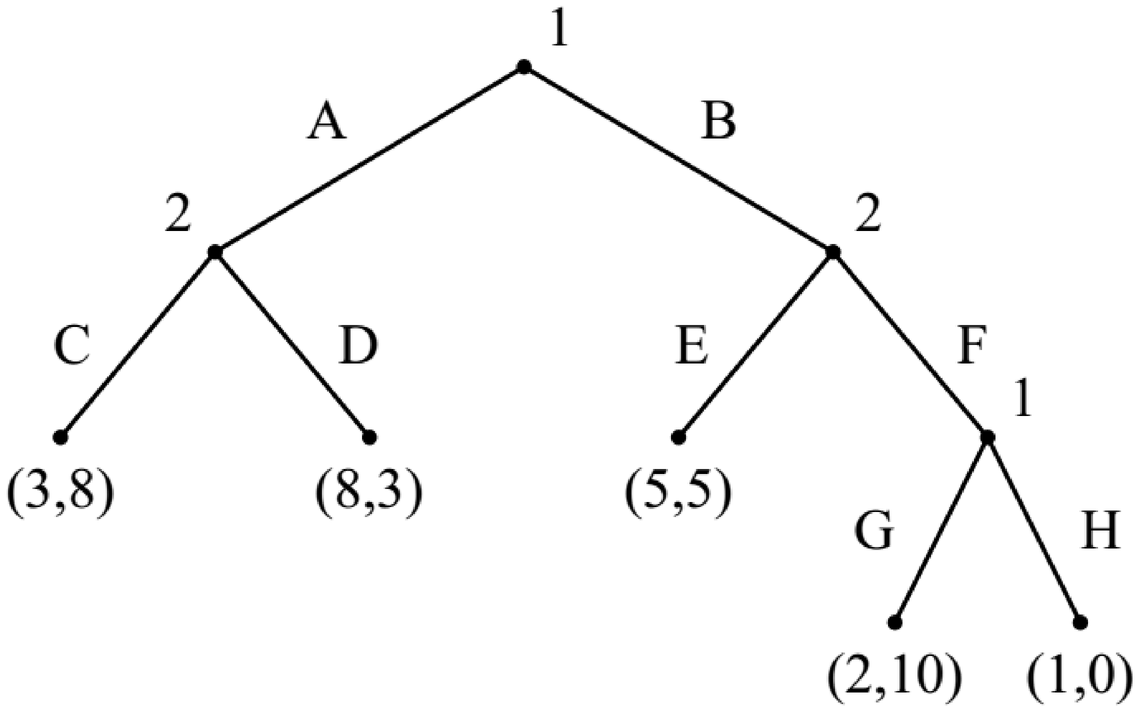
Pure Strategies



- How many pure strategies for each player?
- Player A: {(2,0),(1,1),(0,2)}
- Player B: {yyy,yyn,yny,ynn,nyy,nyn,nny,nnn}

$\{y,n\} \times \{y,n\} \times \{y,n\}$

Pure Strategy Example



What are the pure strategies for players 1 and 2?

1: {A, B} x {G, H}

2: {C, D} x {E, F}

Nash Equilibrium

Based on the definition of pure strategy, we can define

- Mixed strategies
- Best response
- Nash equilibrium

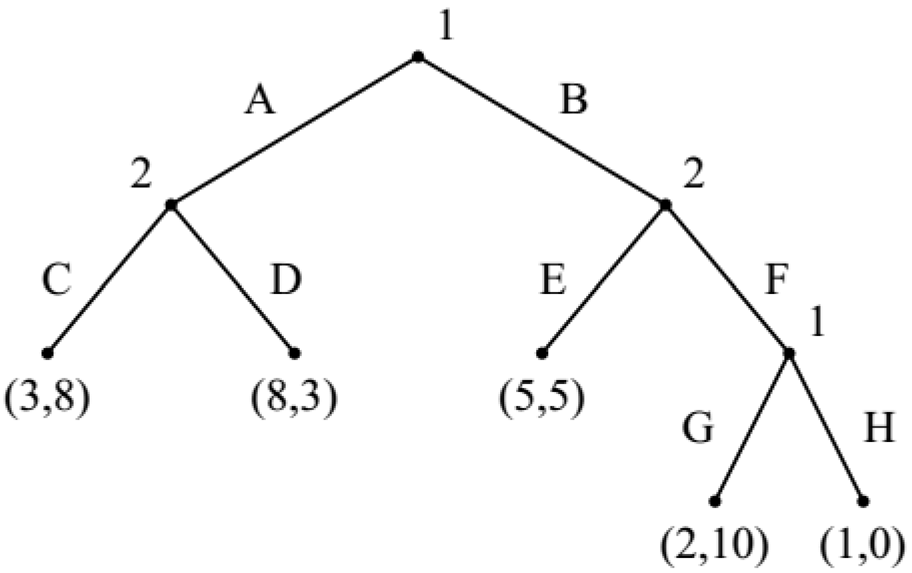
Given extensive $G = \left\{N, H, P, \{u_i\}\right\}$, an strategy outcome $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ is a **Nash equilibrium** if and only if

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \text{ for every } a_i \text{ of player } i$$

How to find Nash Equilibrium: Induced strategy game

Induced Strategy Game

Every extensive game can be converted to a strategy game



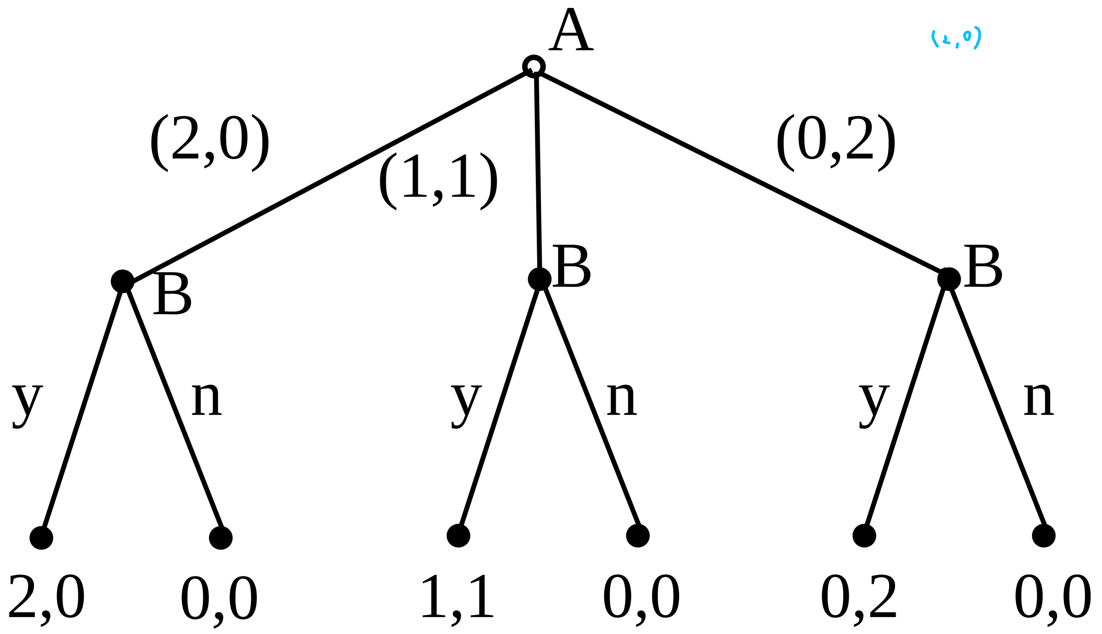
	CE	CF	DE	DF
AG	$3,8$	$3,8$	$8,3$	$8,3$
AH	$3,8$	$3,8$	$8,3$	$8,3$
BG	$5,5$	$2,10$	$5,5$	$2,10$
BH	$5,5$	$1,0$	$5,5$	$1,0$

Remark: This conversion is not reverse

Ultimatum Game

★ 求 N.E 考

引 2 [illegible]



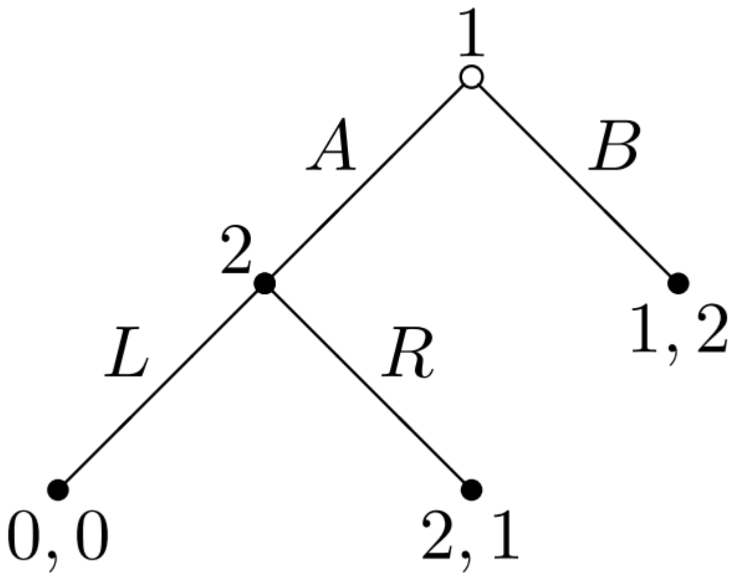
How many Nash Equilibria for ultimatum game?

Kuhn Theorem (1953)

Theorem Every extensive game with perfect information has at least one Pure Strategy Nash Equilibrium (PSNE).

Proof Constructive proof will be introduced later.

Example



	L	R
A	0,0	2,1
B	$\boxed{1,2}$	1,2

Nash Equilibria are (B,L) and (A,R)

- (B,L) is a Nash equilibrium: if player 2 select L, then player 1 select B, and vice verse.
- Is (B,L) reasonable?

(B,L) is an non-credible threat.

非完美了博弈

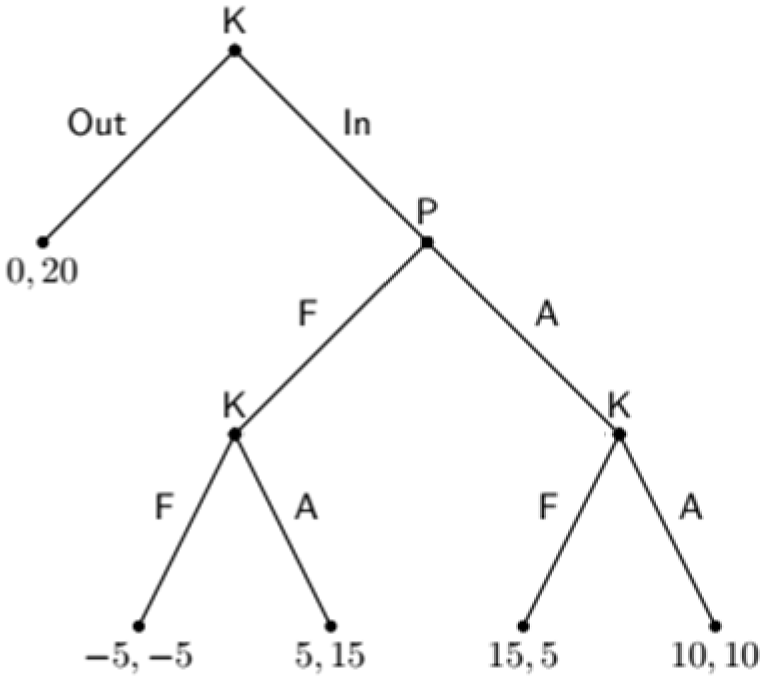
Subgame (子博弈)

Definition A subgame is a set of nodes, strategies and payoffs, following from a single node to the end of game.

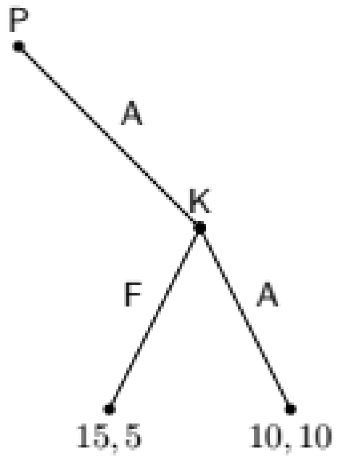
A subgame is a part of the game tree such that

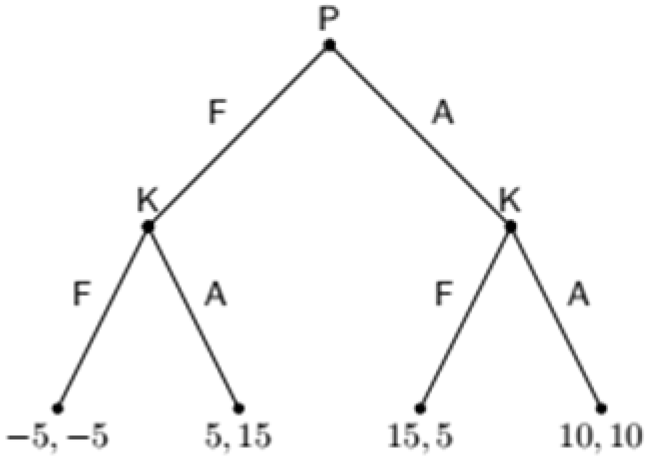
- It starts at a single strategy node
- It contains every successor to this node
- It contains all information in every successor

Example



不是子博弈 →





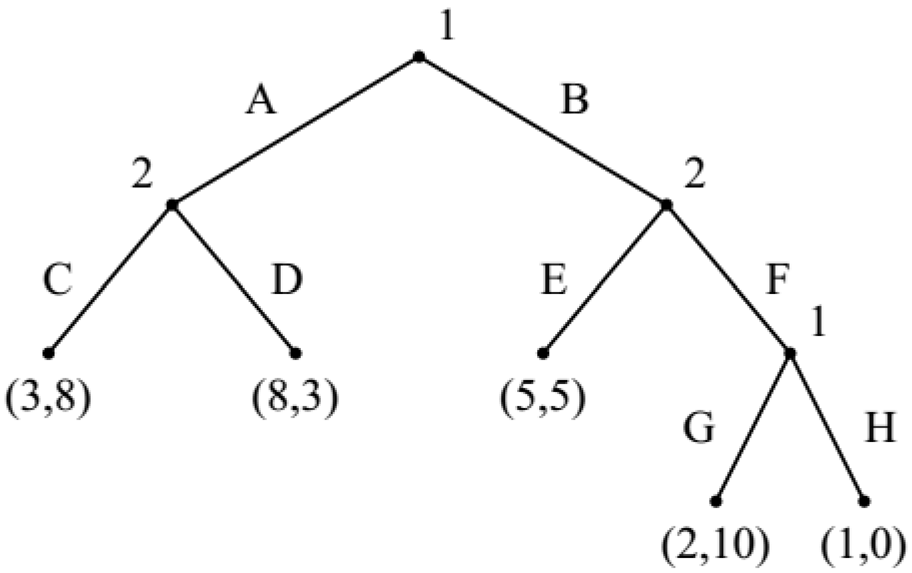
Subgame Perfect Equilibrium

Definition An outcome is $a = (a_1^*, a_2^*, \dots, a_N^*)$ is a subgame perfect (子博弈完美) if it is Nash Equilibrium in every subgame

- Subgame perfect is a Nash Equilibrium
- This definition rules out “non-credible threat”

Theorem Every extensive game with perfect information has a subgame perfect.

Example



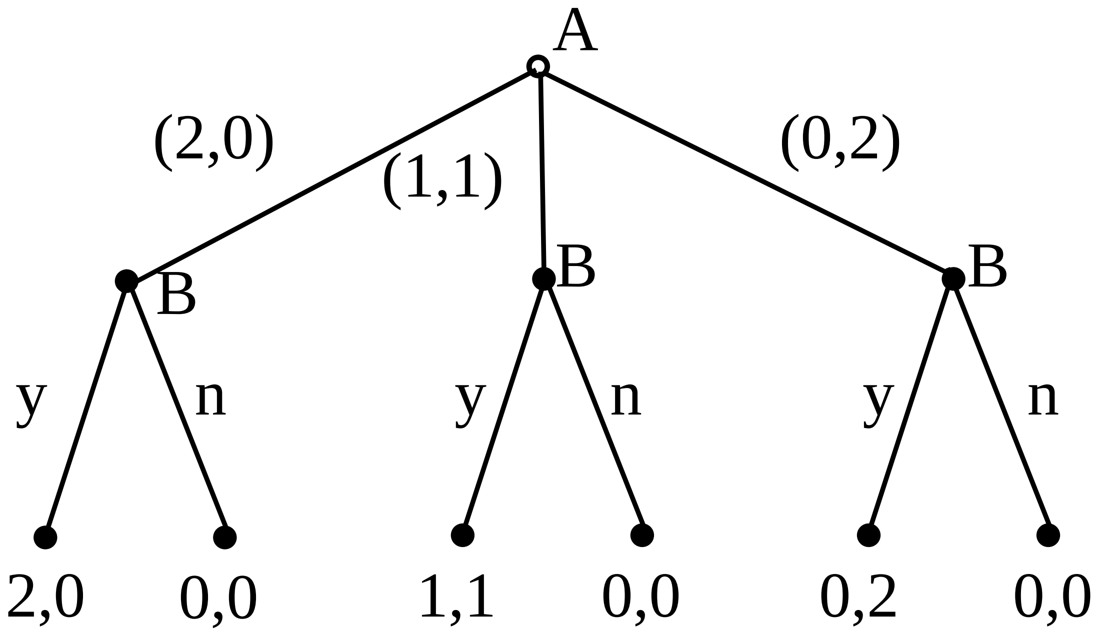
AG . CF ✓

~~AH . CF .~~

~~BH . CE~~

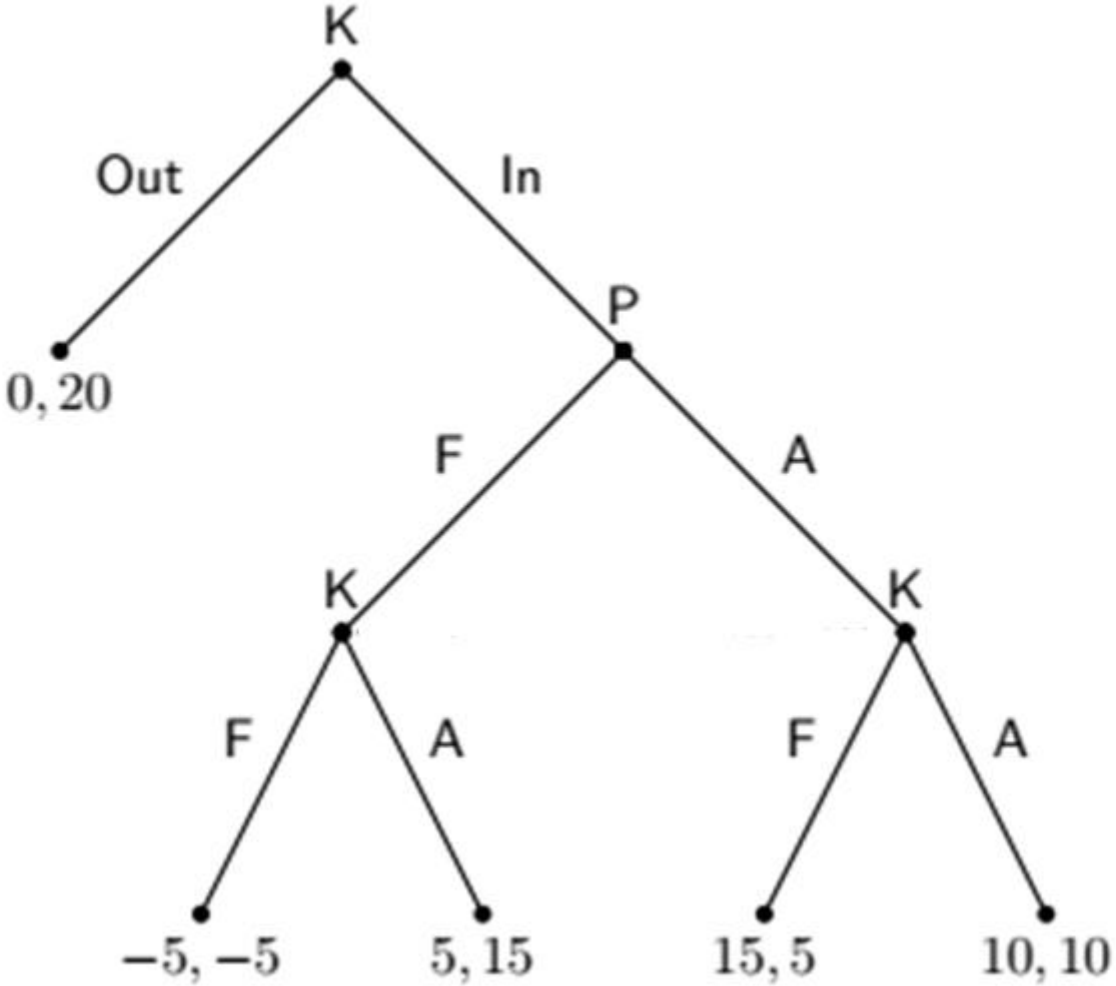
How to find the subgame perfect?

Ultimatum Game

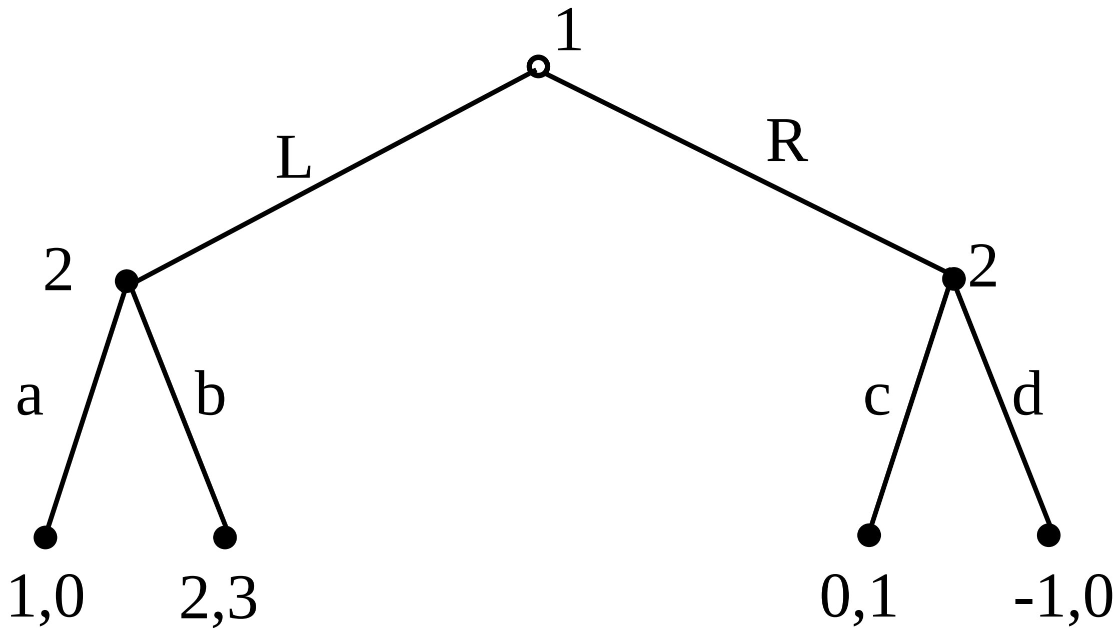


How to find the subgame perfect?

Exercise



Exercise II



How to find the subgame perfect?

Summaries

- Formal definition of extensive game
- Pure strategy for each player and Nash Equilibrium
- How to find Nash Equilibrium
- Subgame
- Subgame Perfect