Game Theory and Applications (博弈论及其应用)

Chapter 5.1： Applications II

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Recap on Previous Chapter

Continous game $G = \left\{N, \{A_i\}, \{u_i\}\right\}$

Every continuous game has at least one mixed strategy NE

If $u_i(a_i, a_{-i})$ is continuous and concave in a_i for a continuous game $\{N, \{A_i\}, \{u_i\}\}$, then there exists a pure strategy NE

Applications

① Product Competition Model (Cournot and Bertrand)

② War of attribution

Meeting Problem

- Persons A and B chat very well today, and they decide to meet again between 1:00 and 2:00 tomorrow
- However, they forget to decide the specific time and they do not have the contact information
- Rule: One person will wait at most 10 minutes, and then leave if he do not meet the other

- Problem: do the two persons will meet

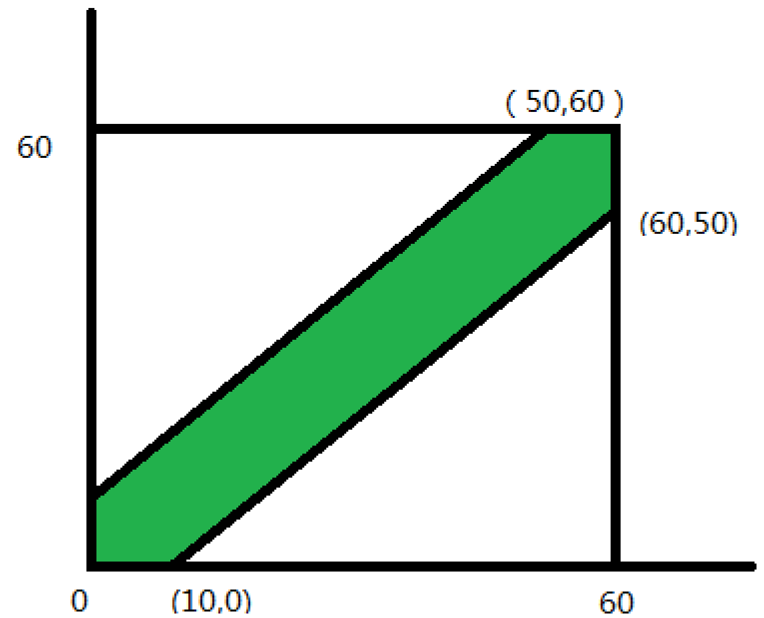
General Persons

A arrives: $x \in [0,60]$ B arrives: $y \in [0,60]$

If A and B meet, then

$$|x - y| \leq 10$$

which implies $x - y \leq 10$ and $x - y \geq -10$



Probability is 11/36 (<1/3)

Smart Persons

- If A arrives 1:00, then B meets [1:00, 1:10], prob. 10/60
- If A arrives 1:01, then B meets [1:00, 1:11], prob. 11/60
- …
- If A arrives 2:10, then B meets [2:00, 2:20], prob. 20/60
- Both A and B are very smart, they will select [2:10-2:50]
- Repeat this process, they will select [2:20 2:40]
- Repeat this process, they will select [2:30 2:30]
- The NE is {2:30,2:30}

Election

- Several candidates vote for political office
- Each candidate chooses a policy position
- Each citizen, who has preferences over policy positions, votes for one of the candidates
- Candidate who obtains the most votes wins.

Strategic game:

- Players: candidates
- Set of actions of each candidate: set of possible positions
- Payoff is 1 for winner; is 0.5 for ties; and is 0 for loser

- Note: Citizens are not players in this game

Example

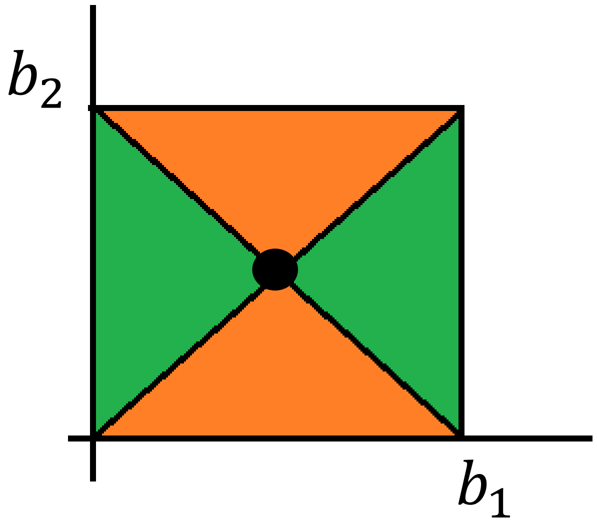
- Two candidates $N = \{1,2\}$
- Set of possible position: $b_1, b_2 \in [0,1]$
- Citizens are continuous, and are distributed uniformly on [0,1], and vote for the candidate with closet position.
- Payoff

$$u_i(b_1, b_2) = \begin{cases} 1 & \text{if } i \text{ wins} \\ 0.5 & \text{if } i \text{ ties} \\ 0 & \text{if } i \text{ loses} \end{cases}$$

Best Response

The best response function $B_i(b_j)$ is give as follows:

- If $b_j < 1/2$, then $B_i(b_j) = \{b_i : b_j < b_i < 1 - b_j\}$
- If $b_j = 1/2$, then $B_i(b_j) = \{b_i : b_i = 1/2\}$
- If $b_j > 1/2$, then $B_i(b_j) = \{b_i : 1 - b_j < b_i < b_j\}$



The Nash Equilibrium (1/2,1/2)

Auction

- Open bid auctions
  - Ascending-bid auction
    - Price is raised until only one bidder remains, who wins and pays the final prize
  - Descending-bid auction
    - Price is lowered until someone accepted, who wins the product at the current prize
- Sealed bid auctions
  - First/second prize auction
    - Highest bidder wins, pays the first/second highest bid

First Price Auction (Two players)

$N = \{1,2\}$: players bid a building

$v_i \geq 0$: the true value for player i $(v_1 > v_2 > 0)$

$b_i \geq 0$: the bid price for player i

Player 1 bids successfully if $b_1 = b_2$

The payoff functions for player i

$$u_1(\mathrm{b}_1, \mathrm{b}_2) = \begin{cases} v_1 - b_1 & \text{if } b_1 \geq b_2 \\ 0 & \text{otherwise} \end{cases}$$

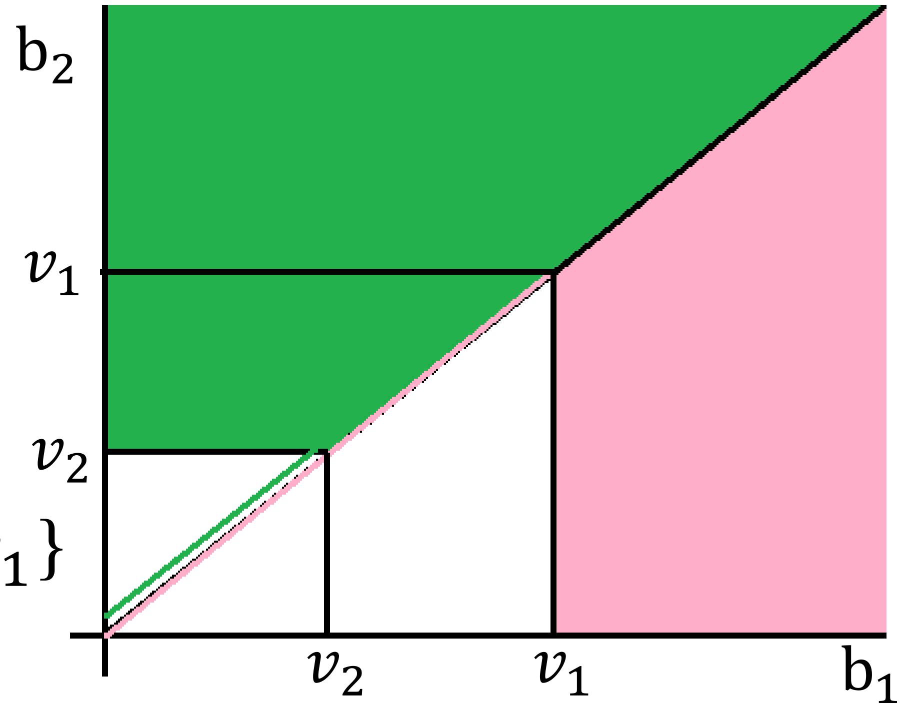
$$u_2(\mathrm{b}_1, \mathrm{b}_2) = \begin{cases} v_2 - b_2 & \text{if } b_2 > b_1 \\ 0 & \text{otherwise} \end{cases}$$

First Price Auction (Two players)

The best response functions

$$B_1(\mathrm{b}_2) = \begin{cases} \{b_1 : b_1 < b_2\} & \text{if} \quad b_2 \geq v_1 \\ \{b_1 : b_1 = b_2\} & \text{if} \quad b_2 < v_1 \end{cases}$$

$$B_2(\mathrm{b}_1) = \begin{cases} \{b_2 : b_2 \leq b_1\} & \text{if} \quad b_1 \geq v_2 \\ \{b_2 : b_2 = b_1 + \epsilon\} & \text{if} \quad b_1 < v_2 \end{cases}$$



The Nash Equilibrium

$\{(b_1^*, b_2^*) : v_2 \leq b_1^* = b_2^* \leq v_1\}$

First Price Auction (N players)

$N = \{1, 2, ..., \mathrm{N}\}$: players bid a building

$v_1 > v_2 > \cdots > v_N > 0$: the true value for player i

$b_i \geq 0$: the bid price for player i

The payoff functions for player i

$$u_1(\mathrm{b}_1, ..., \mathrm{b}_\mathrm{N}) = \begin{cases} v_1 - b_1 & \text{if } \; b_1 \geq \max\left\{b_j\right\}_{j \neq 1} \\ 0 & \text{otherwise} \end{cases}$$

$$u_i(\mathrm{b}_1, ..., \mathrm{b}_\mathrm{N}) = \begin{cases} v_i - b_i & \text{if } \; b_i > \max\left\{b_j\right\}_{j \neq i} \\ 0 & \text{otherwise} \end{cases}$$

Necessary Condition

Theorem If $(b_1^*, \dots, b_N^*)$ is a NE, then $b_1^* \geq b_i^*$ and $b_1^* \geq v_2$

Pf. Assume $\mathrm{b}^* = (b_1^*, \dots, b_N^*)$ is a NE, and there is $b_i^* > b_1^*$.

If $b_i^* > v_2$, then $u_i(b^*) < 0 < u_i(b_1^*, \dots b_{i-1}^*, 0, b_{i+1}^*, \dots b_N^*)$, and b^* is not a NE.

If $b_i^* \leq v_2$, then $u_1(b^*) = 0 < u_i(v_2^*, b_2^*, \dots b_N^*)$ and b^* is not a NE.

If $b_1^* < v_2$, then $u_2(b^*) = 0 < u_2(b_1^*, b_1^* + (v_2 - b_1^*)/2,$ $b_3^* \dots b_N^*)$

First Price Auction (N players)

There are many Nash equilibria

$$\left\{ \begin{array}{c} (b_1^*, \dots, b_N^*) \text{: i) } v_1 \geq b_1^* \geq v_2; \quad \text{ii) } b_1^* \geq b_i^* \text{ for all } i; \\ \text{iii) } b_1^* = b_k^* \text{ for some } k \end{array} \right\}$$