Game Theory and Applications (博弈论及其应用)

Chapter 14: Extensive Game with Imperfect Information-III

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Recap on Previous Chapter

- Extensive game with imperfect information
- Formal definition $G = \{N, H, P, I, \{u_i\}\}$
- Information set $I = \{I_1, I_2, \dots I_N\}$
- Pure strategies $A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$
- SPNE and NE
- Perfect recall and imperfect recall

Definition of Mixed and Behavioral Strategies

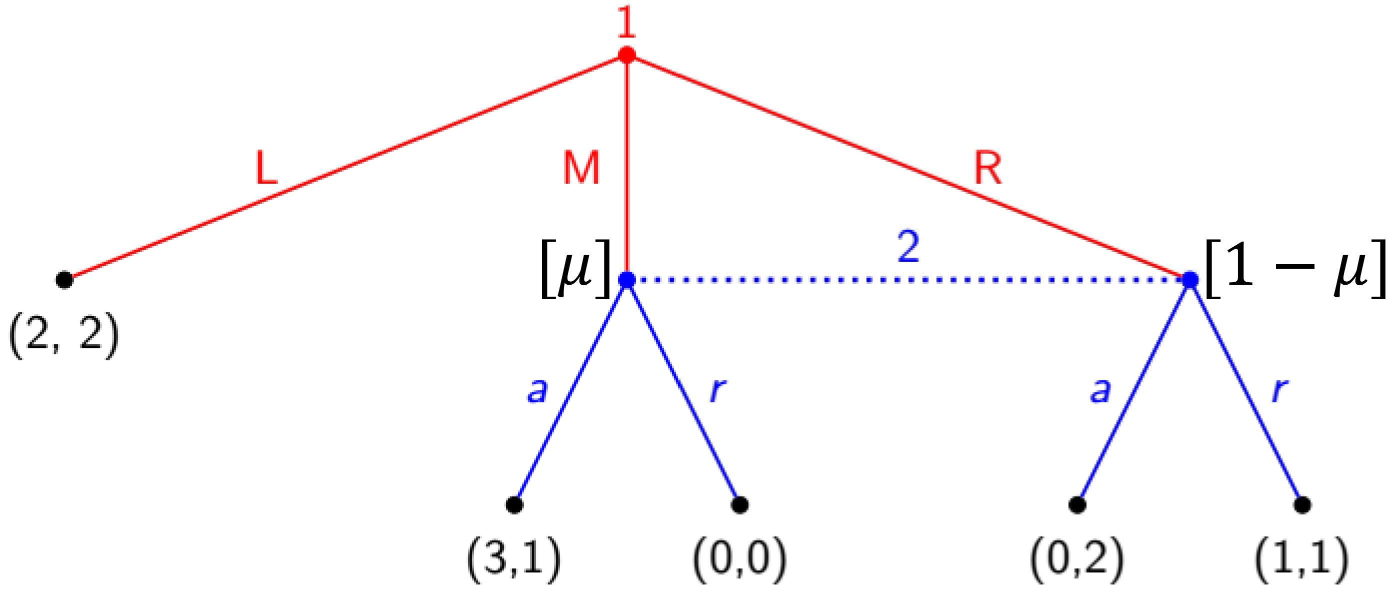
- Mixed Strategies: A mixed strategy of player i is a probability over the set of player i's pure strategy
- Behavioral strategies: A behavior strategy of player i is a collection $\beta_{ik}(I_{ik})_{I_{ik}\in I_i}$ of independent probability measure, where $\beta_{ik}(I_{ik})$ is a probability measure over A(I_{ik})

Theorem In an finite extensive game with perfect recall

- any mixed strategy of a player can be replaced by an equivalent behavioral strategy
- any behavioral strategy can be replaced by an equivalent mixed strategy
- Two strategies are equivalent

Beliefs

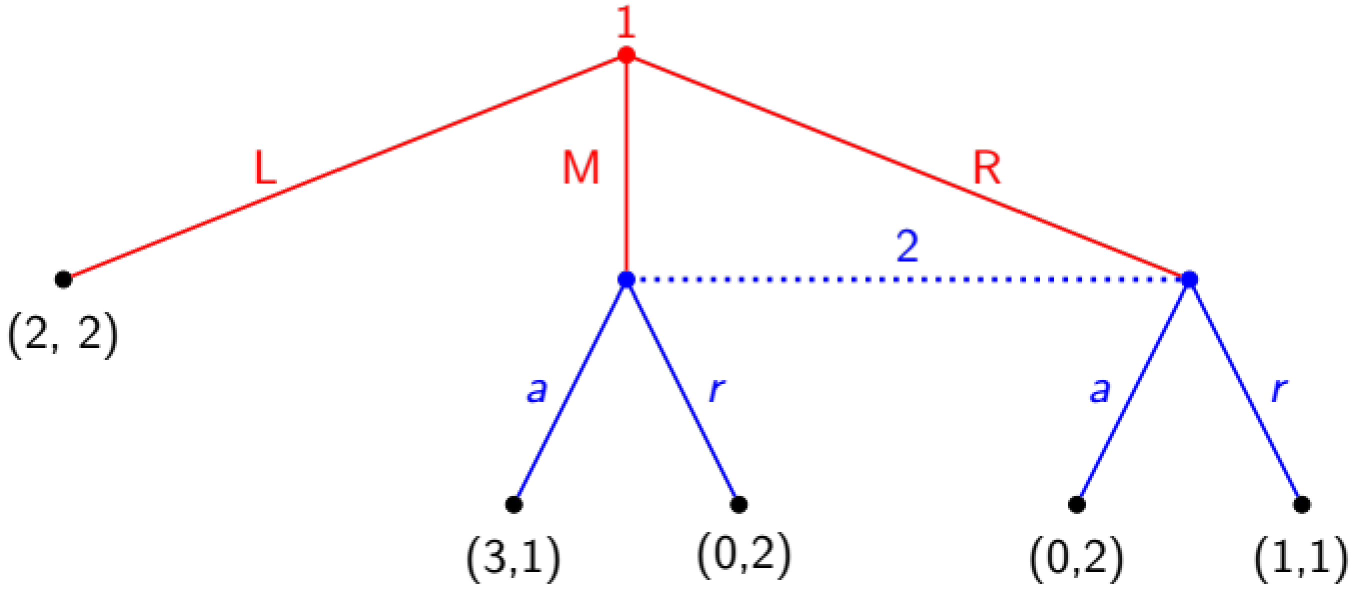
- A belief μ is a function that assigns to every information set a probability measure on the set of histories in the information set
- A behavior strategy β a collection of independent probability measure over the actions after information set



Two Requirements to Beliefs

Bayes consistency: beliefs are determined by Bayes' law in information sets of positive probability; otherwise, beliefs are allowed to be arbitrary for 0 probability.

Consistency: beliefs are determined as a limit of case



1: (L,M,R) with probability $(1-\epsilon, 3\epsilon/4, \epsilon/4)$.

2: belief is well-defined for $\epsilon > 0$, as well as $\epsilon = 0$

Assessment (评估)

- An assessment is a pair (β, μ)
  - β is an outcome of behavioral strategies
  - μ is a belief system
- Assessment (β, μ) is:
  - **Bayesian consistent** if beliefs in information sets reached with positive probability are determined by Bayes' law:

$$\mu_{\mathrm{h,a}}(\mathrm{h}, \mathrm{a}) = \beta_{h,a}(h, a) / \textstyle\sum_a \beta_{h,a}(h, a)$$

    for every information set.
  - **Consistent** if there is a sequence of Bayesian consistent $(\beta^n, \mu^n) \to (\beta, \mu)$ as $n \to \infty$
- (β, μ) is consistent $\to$ (β, μ) Bayesian consistent

Expected Payoffs in Information Sets

Fix assessment (β, μ) and information set I_{ij} of player i. We consider the expected payoff of player i on I_{ij} as

- Given I_{ij}, the belief μ assigns probability over I_{ij} with $\mu(h)$ for $h \in I_{ij}$
- For $h \in I_{ij}$, let $P(e|h, \beta)$ the probability from h to e under the behavioral strategy β, and the payoff is $u_i(e)$

The expected payoff for player i in the information I_{ij} w.r.t. (β, μ), is

$$u_i(\beta_i, \beta_{-i} | I_{ij}, \mu) = \sum_{h \in I_{ij}} \mu(h) \left(\sum_e P(e|h, \beta) u_i(e) \right)$$

Sequential Rational

序列理性.

Assessment (β, μ) is **sequentially rational** if for each information set I_{ij}, player i makes a best response w.r.t. belief μ, that is,

$$u_i(\beta_i, \beta_{-i} | I_{ij}, \mu) \geq u_i(\beta_i', \beta_{-i} | I_{ij}, \mu)$$

for all other behavior strategies β_i' of player i

- Consistency: beliefs have to make sense w.r.t strategies, without requirements on strategies
- Sequential rationality: strategies have to make sense w.r.t. beliefs, without requirements on beliefs

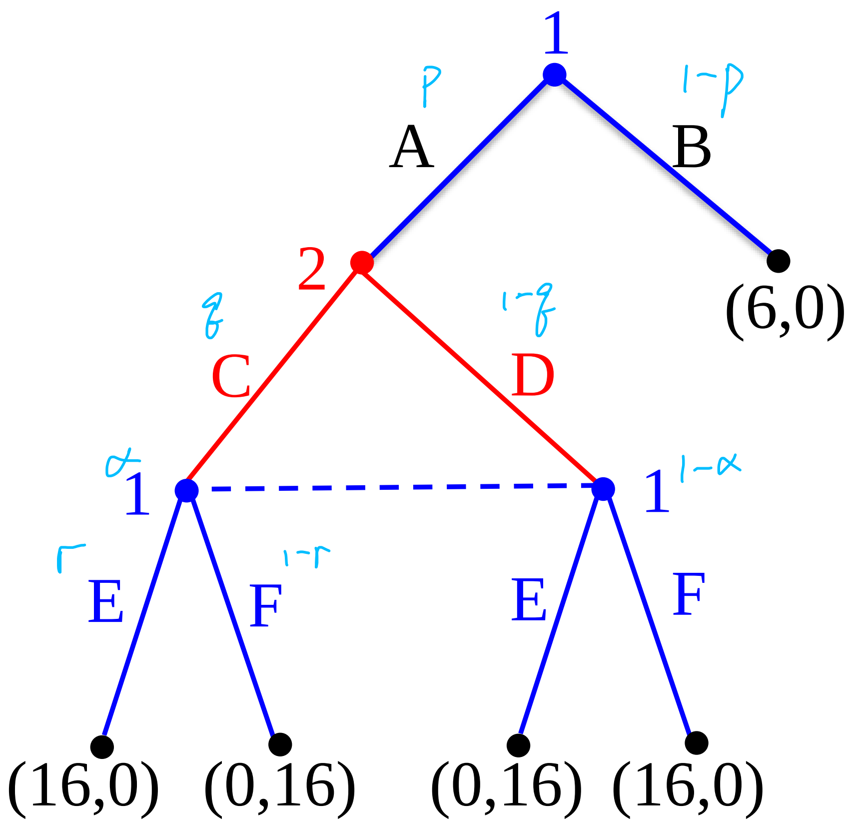
Sequential Equilibrium

An assessment (β, μ) is a **sequential equilibrium** if it is both consistent and sequentially rational.

Theorem

- a) Each finite extensive form game with perfect recall has a sequential equilibrium.
- b) If assessment (β, μ) is a sequential equilibrium, then β is a subgame perfect equilibrium.

Example



$\beta = (\beta_1, \beta_2) = (p, r; q)$.

player 1. select A p

player 1. select E r

player 2. select C q

belief $\mu = (\alpha)$.

player 1 select AC α.

How to calculate the sequential equilibrium?

Example (Consistency)

Behavioral strategies $\beta = (\beta_1, \beta_2) = (p, r; q)$, where

- p: probability that 1 chooses A;
- q: probability that 2 chooses C;
- r: probability that 1 chooses E;

Belief μ can be summarized by one probability α

- α: probability assigns to history AC in inform. set {AC,AD}

- If $p, q, r \in (0,1)$, then Bayes' law gives

$$\alpha = \frac{pq}{pq + p(1-q)} = q$$

For each consistent (β, μ), we have $\alpha = q$

Example (Rationality)

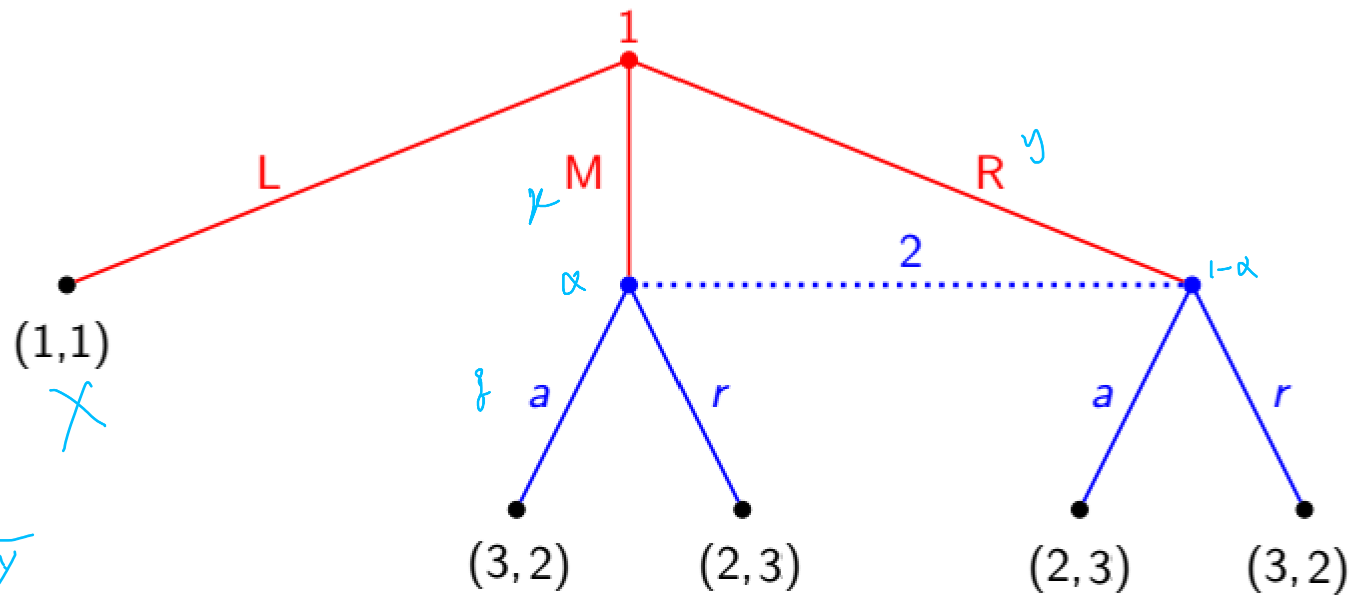
- If $q = 0$, then $\alpha = 0$ and $r = 0$ is player 1's unique best reply in the final info set. But if $r = 0$, then $q = 0$ is not a best reply in 2's info set. Contradiction.
- If $q = 1$, then $\alpha = 1$ and $r = 1$ is player 1's unique best reply in the final info set. But if $r = 1$, then $q = 1$ is not a best reply in 2's info set. Contradiction.
- If $q \in (0,1)$
  - rationality of 2 dictates that both C and D must be optimal and equal, i.e., $16(1-r) \overset{+0\cdot r}{=} 16r^{+0\cdot(1-r)}$, this gives $r = 1/2$
  - In info set (AC,AD), the expected payoff of player 1 is $(1-\alpha), f$

    $\overset{p}{\alpha}16r + (1-\overset{q}{\alpha})16(1-r) = 16 - 16q + 16r(1-2q)$ max.
    - $r = 0$ if $q > 1/2$; $r = 1$ if $q < 1/2$; and $r \in [0,1]$ if $q = 1/2$

    g. > f.

$r = 1/2$ if and only if $q = 1/2$. Finally $p = 1$

Exercise



$\alpha = \frac{x}{x+y}$

if $x=1$. $\alpha=1$. $2: r \Rightarrow x=0$ 矛

if $x=0$. $\alpha=0$. $2: a \Rightarrow x=1$ 矛

if $x\in(0,1)$. $x[3q+2(1-q)] = y[2q+3(1-q)]$ $= 1-x-y$

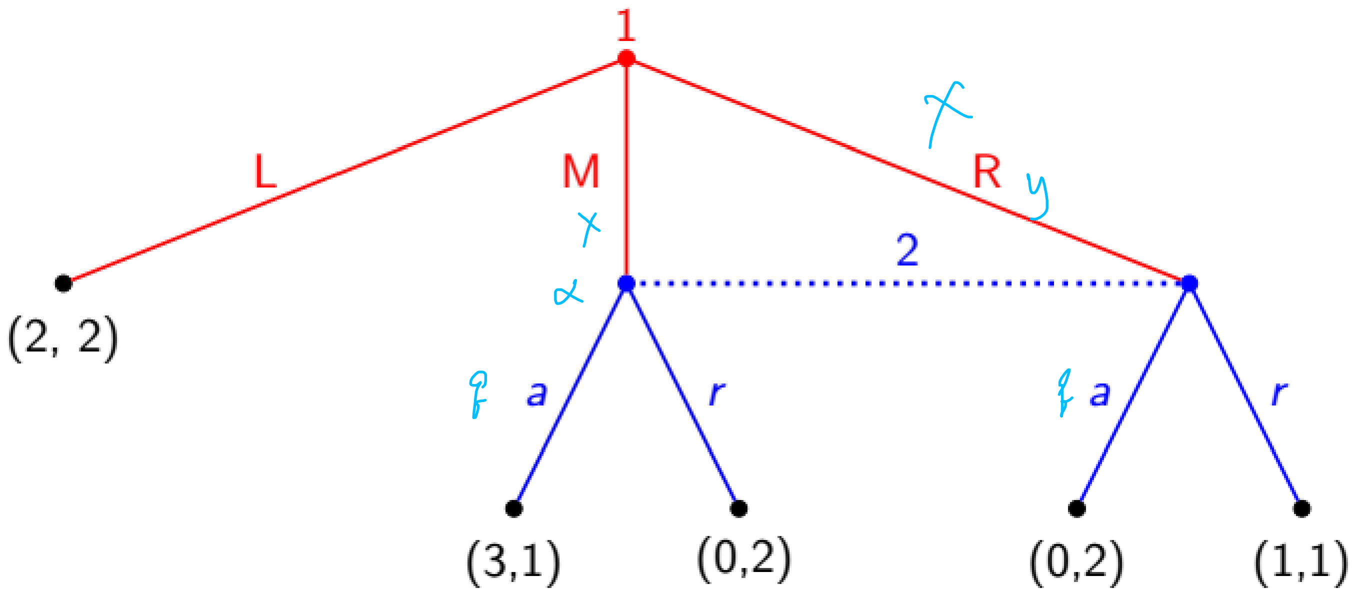
$q+2 = 3-q$

$\Downarrow$

$x(q+2) = y(3-q)$. $q = \frac{1}{2}$.

$\Rightarrow x+y=1$

Exercise



if $x=1$. $\alpha=1$. $\quad q=0. \Rightarrow x=0$ 矛 $\quad y=0$.

if $x=0$ $\alpha=0$ $\quad q=1 \Rightarrow x=1$ 矛 $\quad 3xq=2(1-x)$.

if $x\in(0,1)$. $\quad x[3q+0] = y[0+(1-q)\cdot 1] = 2(1-x-y)$. $\quad q=\frac{2}{3}$

$3xq = y(1-q). = 2(1-x-y)$.

Signaling games (信号传递博弈)

The most interesting class of games that are solved used the sequential Equilibrium concept are signaling games

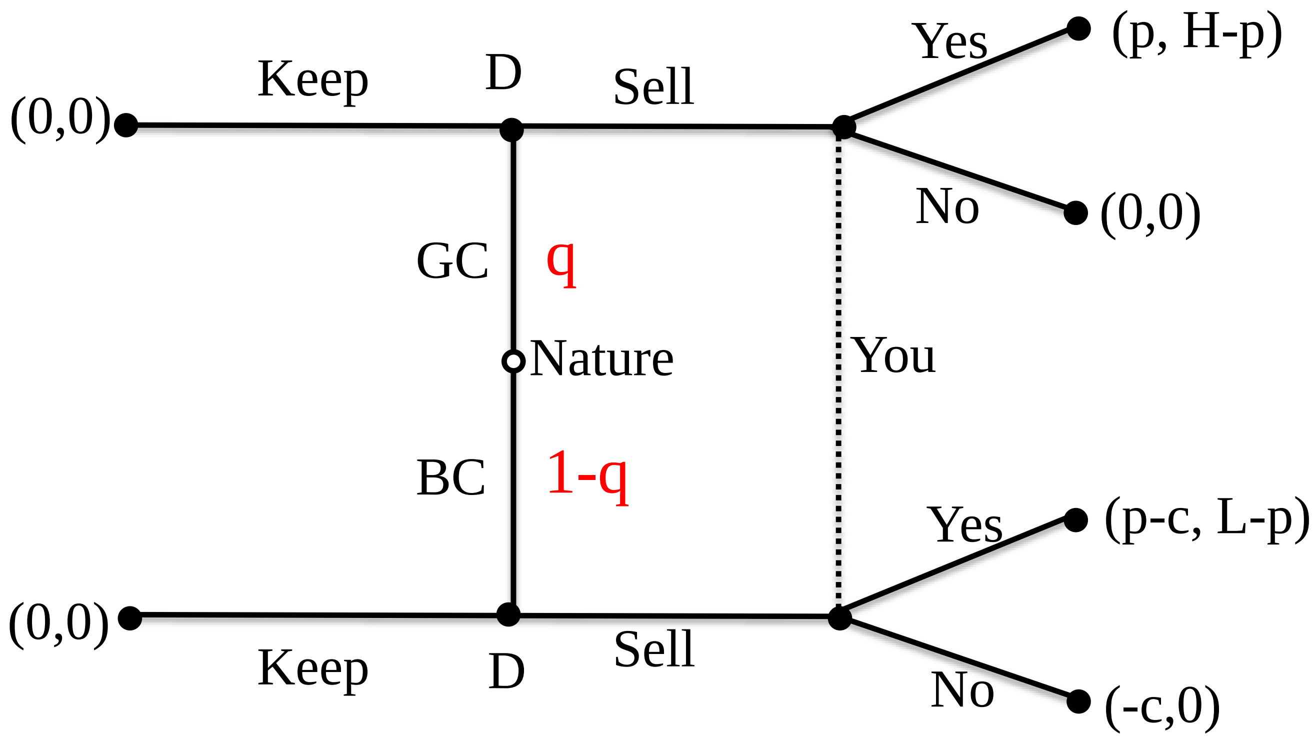
Michael Spence, 2001 Nobel Memorial Prize in economics: job-market signaling model

- A prospective employer can hire an applicant.
- The applicant has high or low ability, but the employer doesn’t know which
- Applicant can give a signal about ability, e.g., education

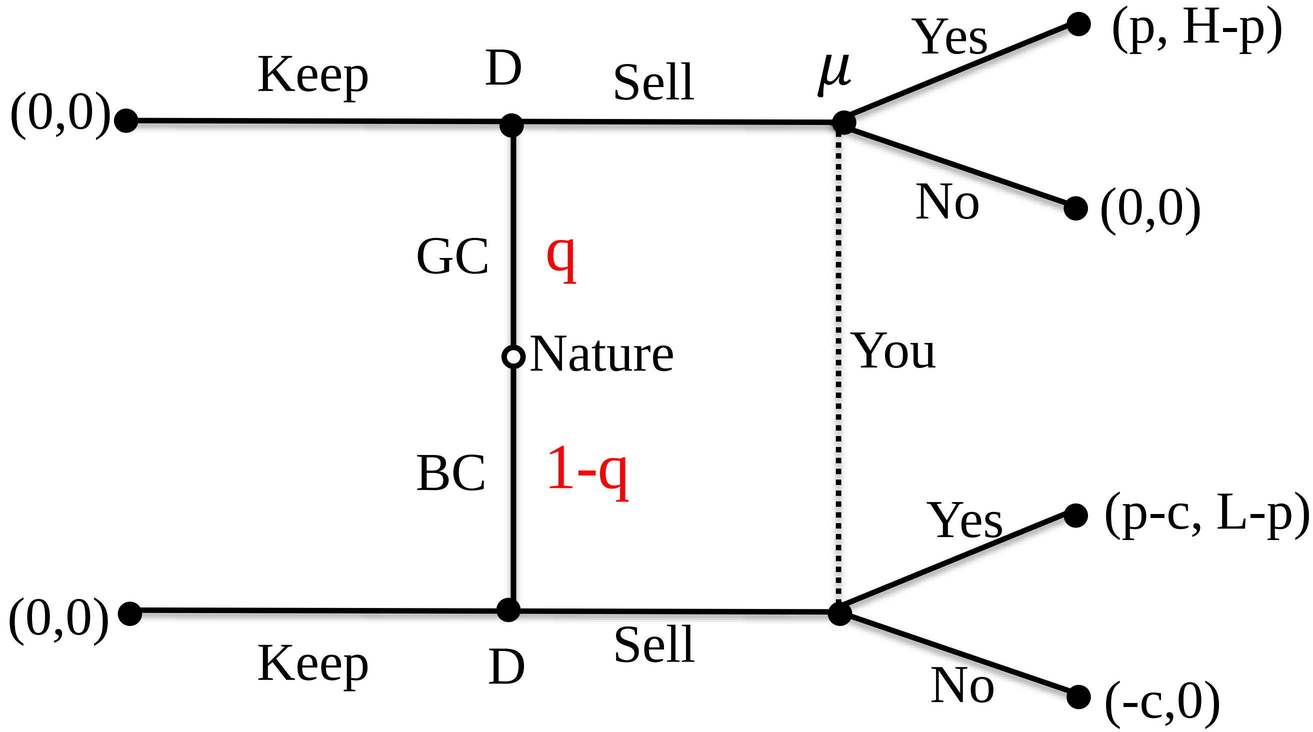
Signaling Games: Used-Car Market

- You want to buy a used-car which may be either good or bad
- A good car is worth H and a bad one L dollars
- You cannot tell a good car from a bad one but believe a proportion q of cars are good
- The car you are interested in has a price p
- The dealer knows quality but you don't
- The bad car needs additional costs c to make it look like good
- The dealer decides whether to put a given car on sale or keep
- You decide whether to buy or not
- Assume $H > p > L$

Signaling Games: Used-Car Market



Belief

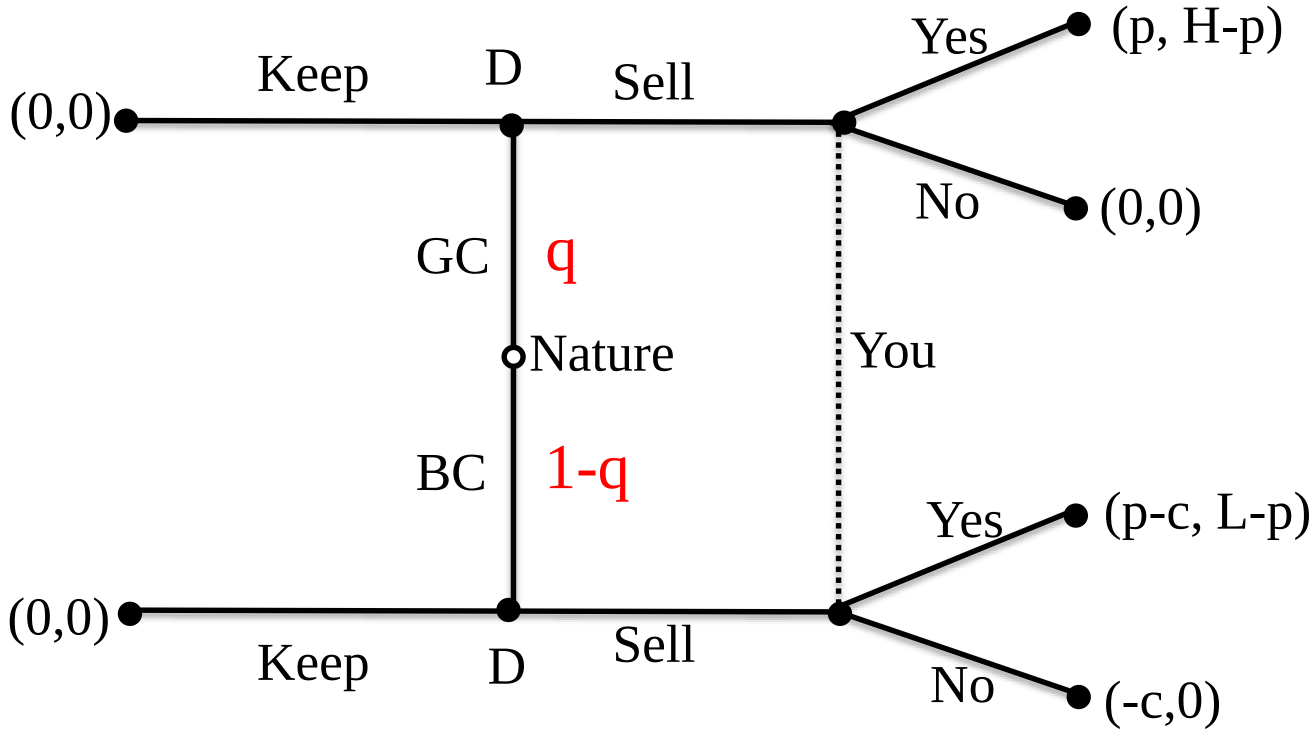


Dealer strategy: Offer if good; Hold if bad

What is your consistent belief if you observe the dealer sell a car?

$$\mu = \frac{P(\text{GC and sell})}{P(\text{sell})} = \frac{q \times 1}{q \times 1 + 0 \times (1-q)} = 1$$

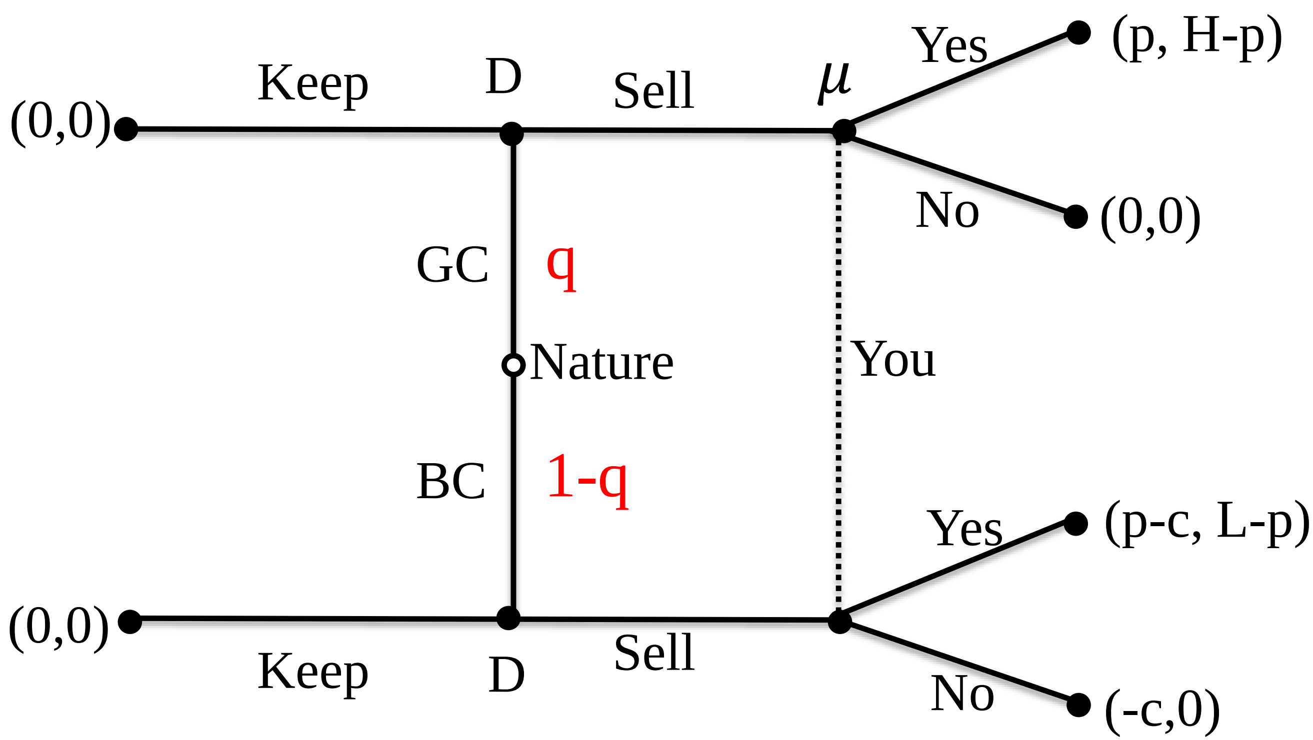
Signaling Games: Used-Car Market



We look for two types of equilibria

1) Pooling Equilibria: GC and BC dealer paly the same strategy
2) Separating Equilibria: GC and BC dealer paly different strategy

Pooling Strategy: Both Sell



Both strategies: Sell

Belief:

$$\mu = \frac{q}{1 \times q + 1 \times \ (1-q)} = q$$

Pooling Strategy: Both Sell

- If Y buys a car with your prior beliefs q your expected payoff is
  $$V = q \times (H - p) + (1 - q) \times (L - p) \geq 0$$
- What does sequential rationality of seller imply?
- You must be buying and it must be the case that $p \geq c$

Pooling Equilibrium I

If $p \geq c$ and $V \geq 0$ the following is a PBE

Behavioral Strategy Profile: (GC: Sell, BC: Sell), (Y: Yes)

Belief System: $\mu = q$

Pooling Equilibria: Both Keep

You must be saying No

- Otherwise Good car dealer would offer

Under what conditions would Ysay No?

$$\mu \times (H - p) + (1 - \mu) \times (L - p) \leq 0$$

So we can set $\mu = 0$

The following is a PBE

Behavioral Strategy Profile: (Good: Hold, Bad: Hold),(You: No)

Belief System: $\mu = 0$

Market failure: a few bad car can ruin a market

Separating Equilibria - Good: Offer and Bad: Hold

- What about your beliefs?

$$\mu = 1$$

- What does you sequential rationality imply?
  - You say Yes
- Is Good car dealer's sequential rationality satisfied?
  - Yes
- Is Bad car dealer's sequential rationality satisfied?
  - Yes if $p \leq c$
- If $p \leq c$ the following is a PBE
  Behavioral Strategy Profile: (Good: Offer, Bad: Hold), (You: Yes)
  Belief System: $\mu = 1$

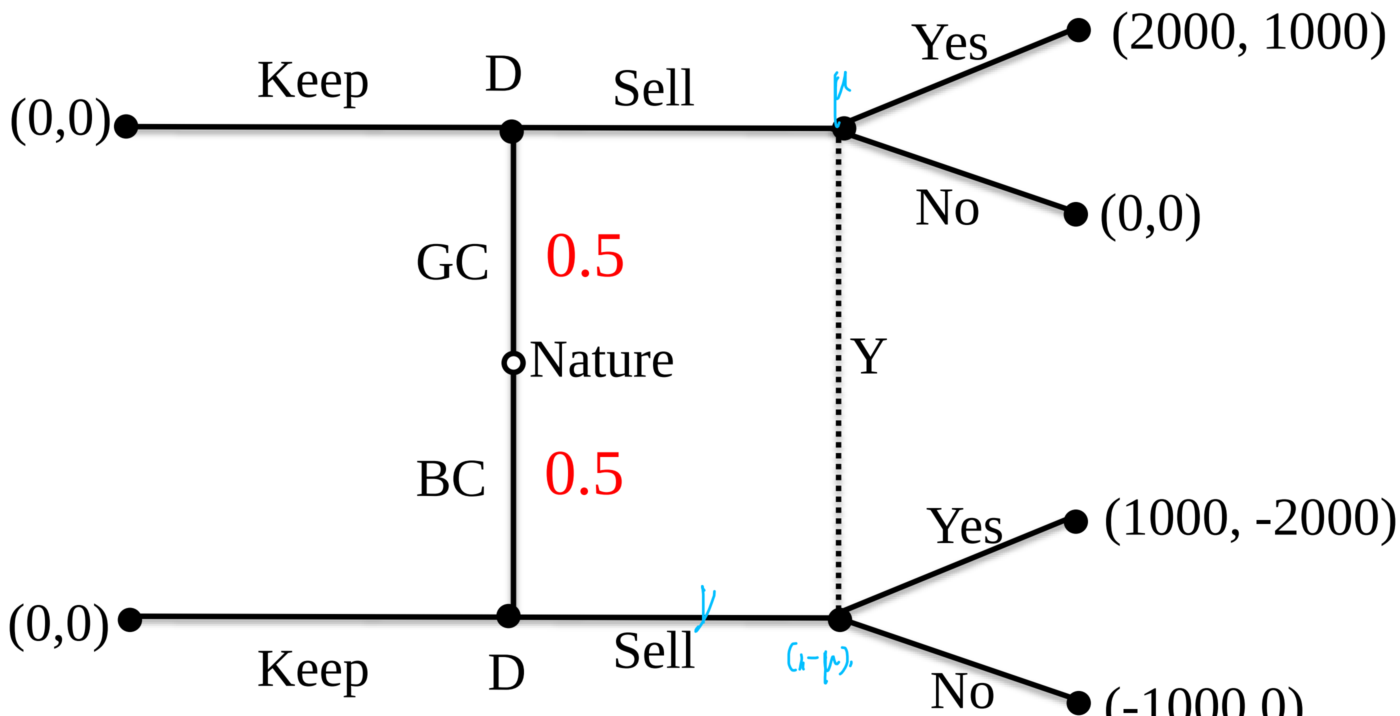
Separating Equilibria - Good: Keep and Bad: Sell

- What does Bayes Law imply about your beliefs?

$$\mu = 0$$

- What does you sequential rationality imply?
  - You say No
- Is Good car dealer’s sequential rationality satisfied?
  - Yes
- Is Bad car dealer’s sequential rationality satisfied?
  - No
- There is no PBE in which Good dealer Holds and Bad dealer Offers

Behavior Strategy



Behavior strategy: Yes Prob. $x \in (0,1)$

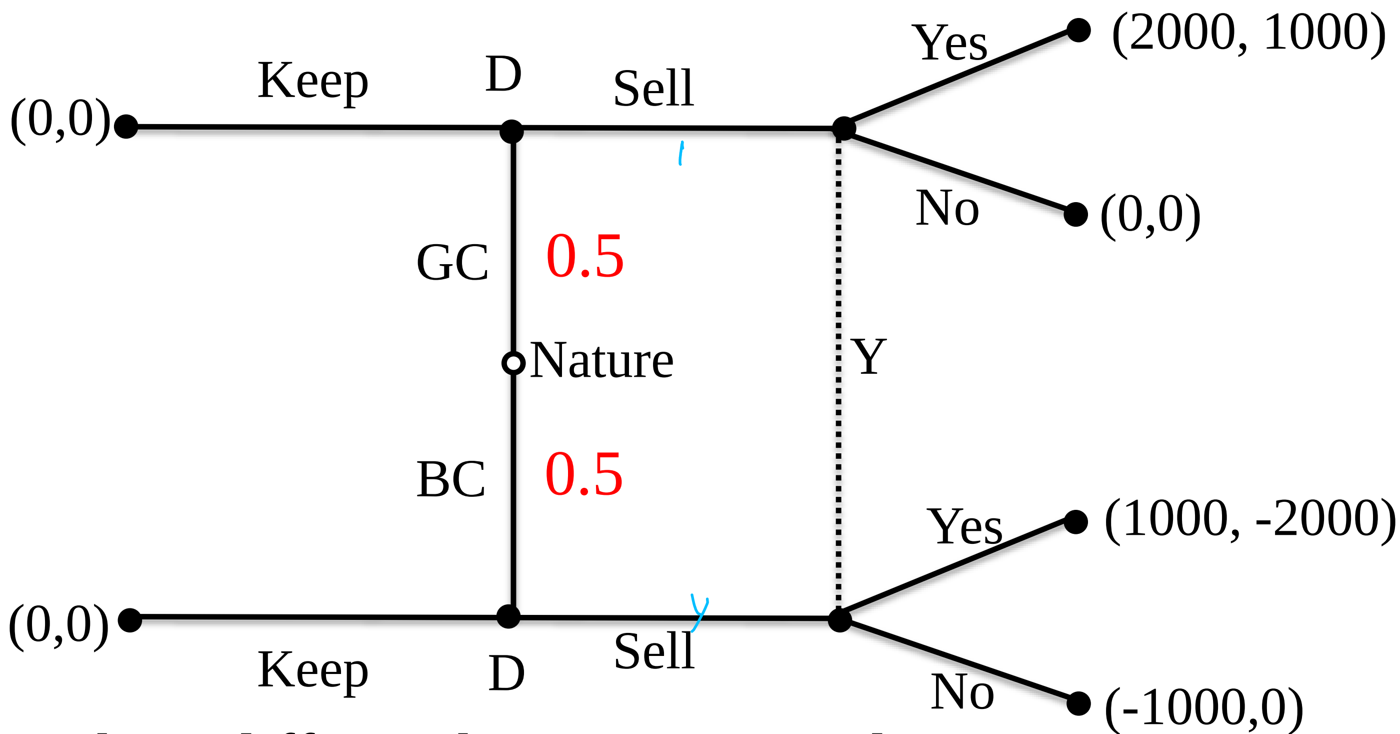
Behavior strategy: BC – sell Prob. y

Belief: GC – sell Prob. μ

You must be indifferent between Yes and No

$$1000\mu - (1-\mu)2000 = 0 \text{ implies } \mu=2/3$$

Behavior Strategy



You must be indifferent between Yes and No

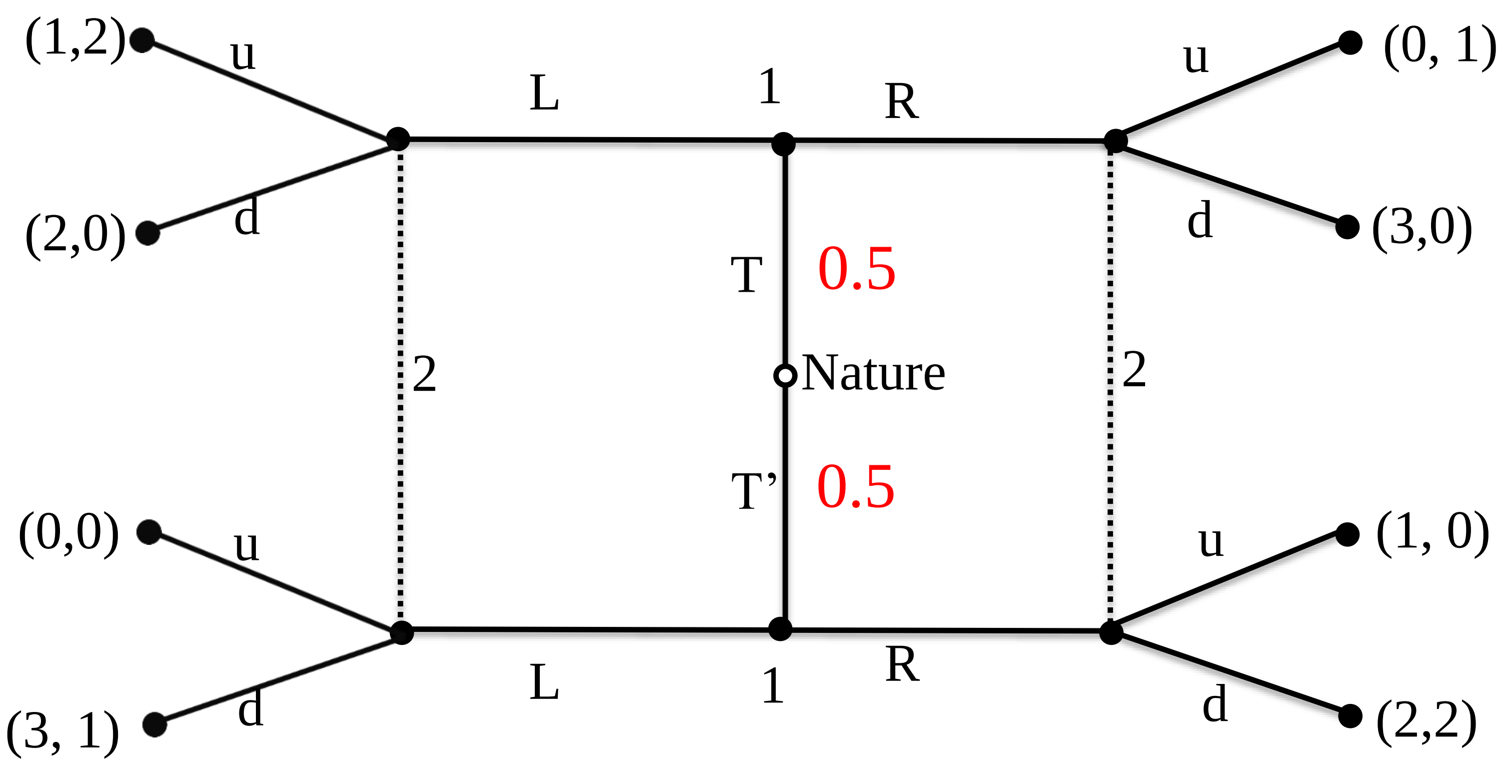
$1000\mu - (1-\mu)2000 = 0$ implies μ=2/3

$\frac{0.5}{0.5+0.5y} = \frac{2}{3}$ implies $y = 0.5$

Bad car dealers must be indifferent between Keep and Sell

$0 = 1000x - 1000(1-x)$ implies $x = 0.5$

Signaling Game: Another Example



1: LL LR RL RR.

2: uu ud du dd.

(a) Find the corresponding strategic form game and its pure-strategy Nash equilibria.

(b) Determine (if any) the game's separating equilibria.

(c) Determine (if any) the game's pooling equilibria.

Signaling Game: Another Example

a) P1's pure strategies are pairs in LL, LR, RL, RR

P2's pure strategies are pairs in uu, ud, du, dd

	uu		ud		Du		dd	
LL	0.5	1	0.5	1	2.5	0.5	2.5	0.5
LR	1	1	1.5	2	1.5	0	2	1
RL	0	0.5	1.5	0	1.5	1	3	0.5
RR	0.5	0.5	2.5	1	0.5	0.5	2.5	1

Nash equilibrium ((R; R); (u; d)).

(b): Separating equilibria must be Nash equilibria:

((R; R); (u; d))

Pooling equilibria, no separating equilibria.

Signaling Game: Another Example

The candidate strategy ((R; R); (u; d))

But what should the belief system be? Let $\alpha_1, \alpha_2 \in [0,1]$ denote the prob. assigned to the top

Bayesian consistency: requires that $\alpha_2 = 1/2$, $\alpha_1 \in [0,1]$

Sequential rationality:

- ((R; R); (u; d)) is a NE
- P2's payoff from u is $2\alpha_1 + 0(1-\alpha_1)$ and from d is $0\alpha_1 + 1(1-\alpha_1)$, so requires $\alpha_1 \geq \frac{1}{3}$

Conclude: Assessments (s1; s2; β) with strategies

- (s1; s2) = ((R; R); (u; d)) and belief system
- $\beta = (\alpha_1, \alpha_2), \alpha_1 \in [1/3,1]$ $\alpha_2 = 1/2$ are pooling equilibria