Game Theory and Applications (博弈论及其应用)

Chapter 4: Continuous Game and Applications

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Recap on Previous Chapter

- Dominant strategy and dominant strategy equilibra
- How to find mixed strategy Nash equilibrium for strictly dominated strategies
- Rationalization and iteration of strictly dominated strategies

Continuous Game

A game $G = \{N, \{A_i\}, \{u_i\}\}$ with complete information is **continuous** if each A_i is non-empty and compact, and $u_i: A \to R$ are continuous.

- Many quantities are essentially continuous: If we're considering how many fish to catch in a season, where the measurement is in millions of tons.
- Cournot game …

How to Find Nash Equilibria

- Finding Nash equilibrium for continuous strategies A_i:

  (1) Find the best response correspondence for each player

  Best response correspondence

$$B_i(a_{-i}) = \{a_i \in \operatorname{argmax}_{a_i \in A_i} u_i(a_i, a_{-i})\}$$

  (2) Find all Nash Equilibria $(a_1^*, a_2^*, \ldots, a_N^*)$ such that

$$a_i^* \in B_i(a_{-i}^*) \text{ for each player}$$

Product Competition Model

- Cournot Model (古偌竞争)
  - Strategy are outputs, prices are decided by outputs
- Bertrand Model (伯特兰德模型)
  - Strategies are prices, outputs are decides by prices

Cournot Competition

- Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

  - Price

$$p(q_1,q_2) = a - q_1 - q_2$$

  - Costs $(i = 1,2)$

$$c_i(q_i) = cq_i$$

收益

  - Payoffs $(i = 1,2)$

$$u_i(q_1,q_2) = (a - q_1 - q_2)q_i - cq_i$$

  - Condition $a > 0, c > 0, q_1 \geq 0, q_2 \geq 0$

Cournot (Non-corporative=competition)

Best Response Correspondence is given by

$$B_i(q_{-i}) = (a - c - q_{-i})/2$$

The Nash equilibria: $\left\{\left(\frac{a-c}{3}, \frac{a-c}{3}\right)\right\}$ Payoff: $\left\{\left(\frac{(a-c)^2}{9}, \frac{(a-c)^2}{9}\right)\right\}$

Proof.

$$u_i(q_1, q_2) = (a - c - q_1 - q_2)q_i$$

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = a - c - q_2 - 2q_1 = 0$$

$$q_1 = (a - c - q_2)/2$$

Solve the equations $q_1 = (a - c - q_2)/2$ and $q_2 = (a - c - q_1)/2$

$q_1 = q_2 = \frac{a-c}{3}$

Cournot (Corporative)

Best Response Correspondence is given by

$$B_i(q_{-i}) = (a-c)/4$$

The Nash equilibria: $\left\{\left(\frac{a-c}{4}, \frac{a-c}{4}\right)\right\}$ Payoff: $\left\{\left(\frac{(a-c)^2}{8}, \frac{(a-c)^2}{8}\right)\right\}$

Proof.

$$u_1(q_1, q_2) + u_2(q_1, q_2) = (a - c - q_1 - q_2)(q_1 + q_2)$$

$$u_1(q_1, q_2) + u_2(q_1, q_2) = (a - c - x)x, x = q_1 + q_2$$

$$x = (a-c)/2$$

The corporative payoffs are better than the competition cases

Bertrand Model

Strategic price rather than output

- Single product produced by 2 firms
- Each firm gives a price strategy q_1 and q_2
- Market price: $\min\{q_1, q_2\}$
- Output demand is $d = a - \min(q_1, q_2)$;
- Cost of firm i is $C_i(q_i) = cq_i \quad (a > c)$
- Payoff:

$a - 2q_1 + c = 0$

$q_1 = \frac{a+c}{2}$

$$u_1(q_1, q_2) = \begin{cases} q_1\overset{d}{(a - q_1)} - c\overset{d}{(a - q_1)} & \text{if } q_1 < q_2 \\ q_1(a - q_1)/2 - c(a - q_1)/2 & \text{if } q_1 = q_2 \\ 0 & \text{if } q_1 > q_2 \end{cases}$$

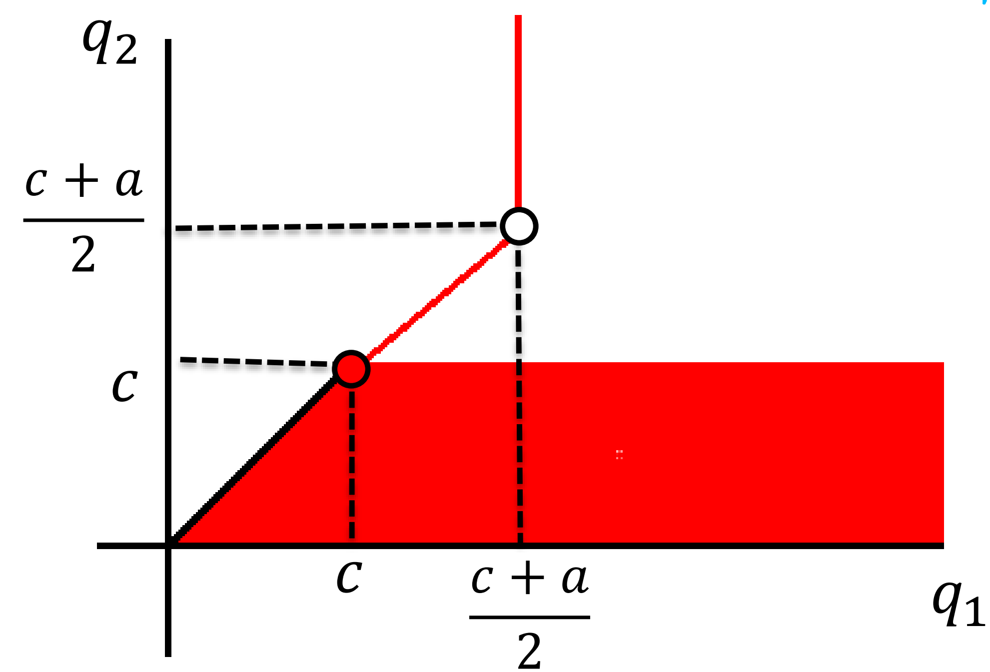
Best Response for $B_1(q_2)$

反应函数

$$B_1(q_2) = \begin{cases} \{q_1 : q_1 > q_2\} & \text{if } q_2 < c \\ \{q_1 : q_1 \geq q_2\} & \text{if } q_2 = c \\ \{q_1 : q_1 = q_2 - \epsilon\} & \text{if } c < q_2 \leq (c+a)/2 \\ \{q_1 : q_1 = (c+a)/2\} & \text{if } q_2 > (c+a)/2 \end{cases}$$

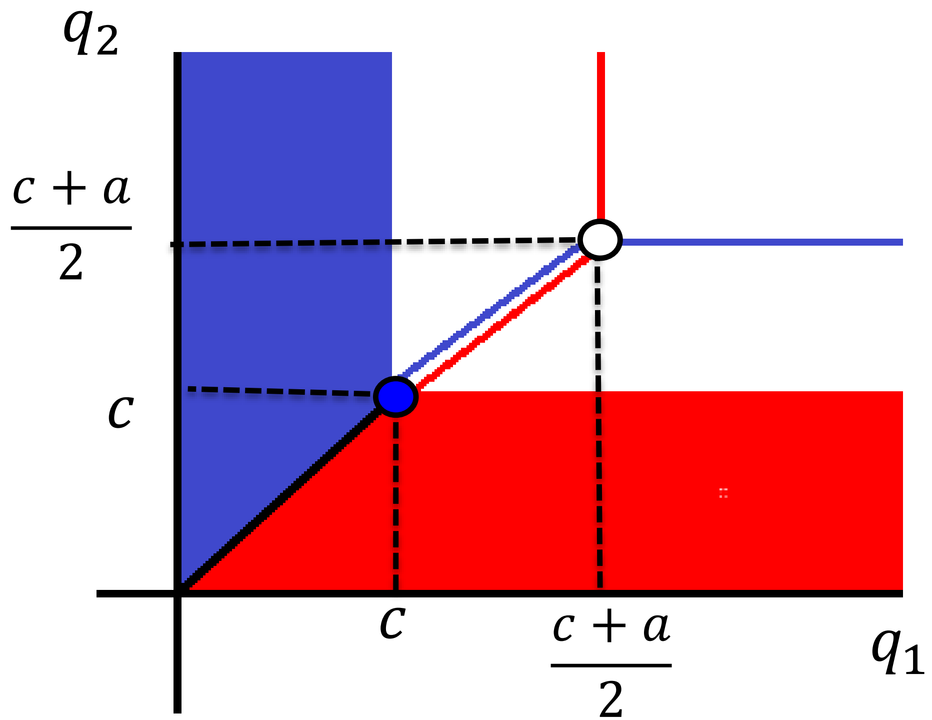
边际成本

垄断price



Nash Equilibrium

(q_1^*, q_2^*) satisfies $q_1^* \in B_1(q_2^*)$ and $q_2^* \in B_2(q_1^*)$
Intersection of the graphs of the best response function



The unique Nash Equilibrium (c, c)

Bertrand Model with Different Product

- $N = \{1,2\}$ with different products
- Price $\{q_1, q_2\}$;
- Demand $(a > 1)$

$$d_1(q_1, q_2) = 10 - aq_1 + q_2 \qquad \times (q_1 - c)$$
$$d_2(q_1, q_2) = 10 - aq_2 + q_1 \qquad \times (q_2 - c).$$

- Cost of firm i is $C_i(x) = cx$

Exercise: Find its Nash equilibrium

$$\begin{cases} \dfrac{\partial d_1}{\partial q_1} = -2aq_1 + q_2 + ac + 10 = 0 \\[2ex] \dfrac{\partial d_2}{\partial q_2} = -2aq_2 + q_1 + ac + 10 = 0. \end{cases}$$

$$\Rightarrow \quad q_1 = q_2 = \frac{ac + 10}{2a - 1}$$

The Hoteling Game

- Two hotels {1,2}
- Choose prizes: $\{p_1, p_2\}$
- Frim 1 and 2 are located at the end of left and right endpoints of the interval [0,1], respectively.
- The cost for each person is c
- Consumers are uniform distributed in [0,1]
- If a consumer is located at x, then the payoff of visting hotel i is

  $$u_c(x,1) = v - p_1 - tx, u_c(x,2) = v - p_2 - t(1-x)$$

Payoffs and Best Responses

What are firms' payoff ?

$$u_1(p_1, p_2) = \left(\frac{1}{2} + \frac{p_2 - p_1}{2t}\right)(p_1 - c)$$

$$u_2(p_1, p_2) = \left(\frac{1}{2} + \frac{p_1 - p_2}{2t}\right)(p_2 - c)$$

What are best response?

$$B_1(p_2) = \frac{c + t + p_2}{2}$$

$$B_2(p_1) = \frac{c + t + p_1}{2}$$

NE: $(c + t, c + t)$

Symmetric Game

A game is symmetric if any player's payoff $u_i(a_i, a_j, a_{-i,j})$ can be converted into any other player's payoff $u_j(a_j, a_i, a_{-i,j})$ simply by re-arranging the player's "names"

Theorem. Any symmetric game has a symmetric NE, where each player uses the same strategy

Prisoner 1 \ Prisoner 2	Confess(c)	Don't confess(d)
Confess(c)	-6 -6	0 -12
Don't confess(d)	-12 0	-1 -1

Partnership Model

Two persons start a firm.

Their efforts $\{e_1, e_2\}$

Equally split the profits of the firm $\pi(e_1, e_2) = se_1e_2$

The cost of effort is c$e_i^2/2$

If they work separately and do not monitor each other

What is the Nash equilibrium if $s > c$?

Existence of Equilibria for Infinite Games

(Nash) Every finite game has a mixed strategy NE

(Debreu, Glicksberg, Fan) Consider a strategic form game $\{N, \{A_i\}, \{u_i\}\}$ such that for each player

- A_i is compact and convex
- $u_i(a_i, a_{-i})$ is continuous in a_{-i}
- $u_i(a_i, a_{-i})$ is continuous and concave in a_i

There exists a pure strategy Nash equilibrium

A More Powerful Theorem

(Glicksberg) Consider a strategic form game $\{N, \{A_i\}, \{u_i\}\}$ such that for each player

- A_i is compact and convex
- $u_i(a)$ is continuous in a

There exists a mixed strategy Nash equilibrium

For continuous pure strategy space, the space of mixed strategy has infinite dimension

War of attribution

- Two players involve in a costly dispute (e.g., two animals fighting over prey)
- Each animal chooses time at which it intends to give up
- Once an animal has given up, the other gets all the prey
- Animals split the prey equally if give up simultaneously
- Each animal prefers as short a fight as possible

- Let time be a continuous variable $[0, +\infty]$
- The value to player i is v_i for the prey
- The unit cost of each player is c for each unit of time

Strategy Game

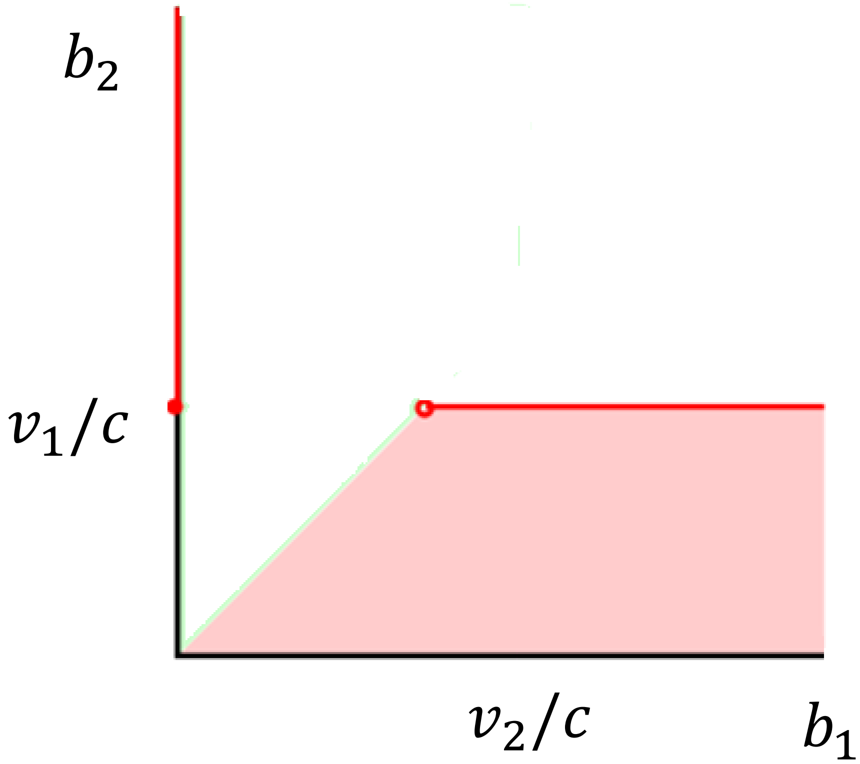
- Two players $N = \{1,2\}$
- Strategies: Each player's strategies $b_i \in [0, +\infty)$
- The payoff for player i is

$$u_i(b_1, b_2) = \begin{cases} -cb_i & \text{if } \ b_i < b_j \\ \frac{1}{2}v_i - cb_i & \text{if } \ b_i = b_j \\ v_i - cb_j & \text{if } \ b_i > b_j \end{cases}$$

where j is another player

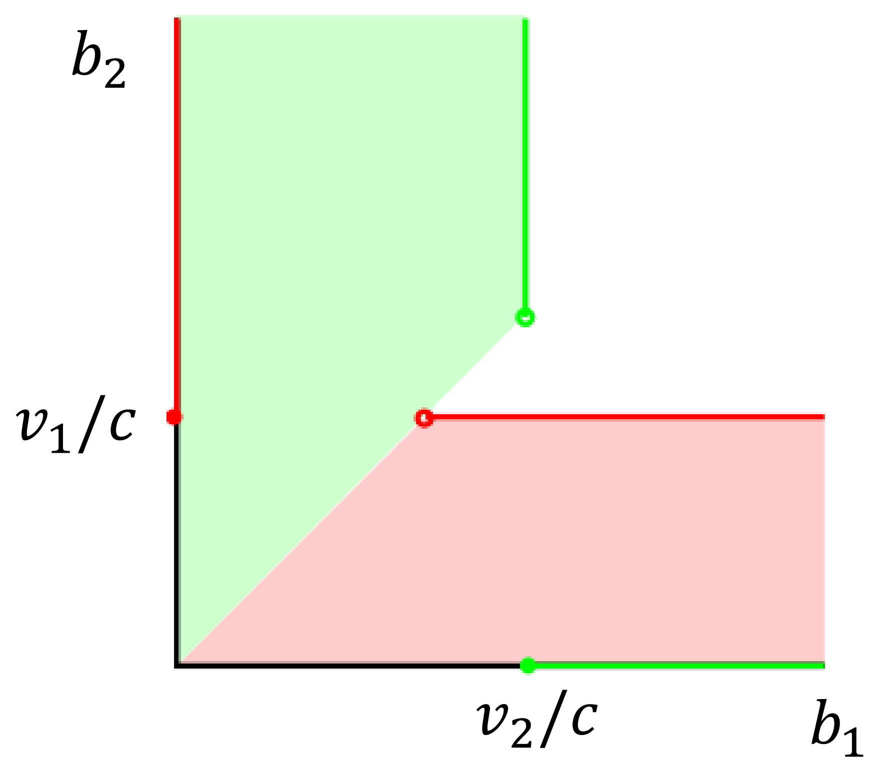
The Best Response

$$B_1(b_2) = \begin{cases} \{b_1 : b_1 > b_2\} & \text{if } b_2 < v_1/c \\ \{b_1 : b_1 = 0 \text{ or } b_1 > b_2\} & \text{if } b_2 = v_1/c \\ \{b_1 : b_1 = 0\} & \text{if } b_2 > v_1/c \end{cases}$$



The Best Response

$$B_i(b_j) = \begin{cases} \{b_i : b_i > b_j\} & \text{if } b_j < v_i/c \\ \{b_i : b_i = 0 \text{ or } b_i > b_j\} & \text{if } b_j = v_i/c \\ \{b_i : b_i = 0\} & \text{if } b_j > v_i/c \end{cases}$$



Nash Equilibrium

$(\boldsymbol{b_1 = 0, b_2 > v_1/c})$

$(\boldsymbol{b_1 > v_2/c, b_2 = 0})$