Square Roots, Newton's Method

Review

Want millionth digit of $\sqrt{2}$

$\lfloor \sqrt{2 \cdot 10^{2d}} \rfloor$ $\quad d = 10^6$ (digits of precision)

- Compute $\sqrt{a}$ via Newton's method
- $x_0 = 1$ (initial guess)
- $x_{i+1} = \dfrac{x_i + a/x_i}{2}$ ← division

Error Analysis of Newton's method

$x_n = \sqrt{a}(1+\varepsilon_n)$ $\quad \varepsilon_n$: error, may $+/-$

$$x_{n+1} = \frac{\sqrt{a}(1+\varepsilon_n) + \frac{a}{\sqrt{a}(1+\varepsilon_n)}}{2} = \sqrt{a}\,\frac{(1+\varepsilon_n) + \frac{1}{1+\varepsilon_n}}{2}$$

$$= \sqrt{a}\left(1 + \frac{\varepsilon_n^2}{2(1+\varepsilon_n)}\right)$$

Therefore:

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2(1+\varepsilon_n)} \Rightarrow \text{Quadratic rate of convergence}$$

- d-digits of precision $\Rightarrow$ $\lg d$ iterations

Multiplication Algorithms

Multiply d-digit number

Divide-and-Conquer

- Naive algorithm (last class) $T(n) = 4T(n/2) + \Theta(n) \Rightarrow \Theta(n^2)$

- Karatsuba: reduce to 3 multiplications

  $T(n) = 3T(n/2) + \Theta(n) \Rightarrow \Theta(n^{\log_2 3}) = \Theta(n^{1.5})$

- Toom-cook:

  $k \geq 2$, break into k parts

  $x \to x_2, x_1, x_0$ } each $d/3$ digits

  $y \to y_2, y_1, y_0$ } Naively there are 9 multiplications

  Toom-3: $T(n) = 5T(n/3) + \Theta(n)$

  $\hookrightarrow \Theta(n^{\log_3 5}) = \Theta(n^{1.463})$

- Schönhage-Strassen

  $\Theta(n \lg n \lg \lg n)$ time, uses FFT, python package GMPY

  (almost $\sim n$)

- Fürer (2007):

  $\Theta(n \log n \, 2^{O(\log^* n)})$

$\log^* n$: iterated logrithm, # of times that log need to apply to get < 1

- fastest asymptoticly

☐ High Precision Division

We want a high-precision rep of $\frac{a}{b}$

〃 〃 〃 〃 〃 $\frac{1}{b}$ first

we will compute $\lfloor \frac{R}{b} \rfloor$ where R is a large value, s.t. it is easy to divide by R ($R = 2^k$, bit shift)

∘ Division

Newton's method for computing $\frac{R}{b}$

$f(x) = \frac{1}{x} - \frac{b}{R}$, has 0 at $x = \frac{R}{b}$ (bit shift $\to \frac{1}{b}$)

$f'(x) = -\frac{1}{x^2}$

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(\frac{1}{x_i} - \frac{b}{R}\right)}{-\frac{1}{x_i^2}}$

$$\boxed{x_{i+1} = x_i + x_i^2\left(\frac{1}{x_i} - \frac{b}{R}\right) = 2x_i - \frac{bx_i^2}{R}}$$

all easy + multiply

e.g. Wan $\frac{R}{b} = \frac{2^{16}}{5} = \frac{65536}{5} = 13107.2$ ← want

Initial guess: $\frac{2^{16}}{4} = 2^{14} = x_0 = 16384$

Newton's : $x_1 = 2(16384) - 5(16384)^2/65536$

$= \underline{12288}$

$x_2 = 2(12288) - 5(12288)^2/65536$

$= \underline{13107}$

...... has quadratic convergence

□ Division: quadratic convergence

d-digits of precision $\Rightarrow$ $\lg d$ iterations

Assume multiplication runs in time $\Theta(n^\alpha)$ $\alpha \geq 1$

Division cost $O(\lg n \cdot n^\alpha)$

Observation:

d-digits of precision $1, 2, 4, \ldots\ldots, d$

$\uparrow$ initial multiplies

The number of operations $= c \cdot 1^\alpha + c \cdot 2^\alpha + c \cdot 4^\alpha + \cdots + c\left(\frac{d}{4}\right)^\alpha + c\left(\frac{d}{2}\right)^\alpha + c d^\alpha$

$< 2c \cdot d^\alpha$

Complexity of divisity $\equiv$ Complexity of multiplication

Complexity of square roots:

$\sqrt{a} \rightarrow \lfloor 10^{2d} a \rfloor$

$\hookrightarrow$ Newton's $(x_i + a/x_i)/2 = x_{i+1}$

$\downarrow$ division

Newton's method $2x_i - bx_i^2/R$

$\downarrow$ many multiplexity

$\equiv$ Complexity of div $\equiv$ Complexity of multi