Integer Arithmetic and Karatsuba Multiplication

□ Irrationals

$\sqrt{2} = 1.414\ 213\ 562\ 373\ 095\ 048 \cdots$

□ Catalan numbers

Set P of banlanced parenteses strings

- $\lambda \in P$ (λ is ~~the~~ empty string)
- If $\alpha, \beta \in P$, then $(\alpha)\beta \in P$

( )	balanced
(	NOT
(( )	balanced

Every nonempty banlanced paren string via rule 2 form a unique α, β pair

eg

(( ))( )( ) obtained by $(\underbrace{(\ )}_{\alpha})\underbrace{(\ )(\ )}_{\beta}$

∘ Enumeration

- C_n = number of balanced paren strings with exactly n pairs of parens, $C_0 = 1$ (empty string)
- $C_0 = 1$ (empty), $C_1 = 1$ ( ) $C_2 = C_0C_1 + C_1C_0$

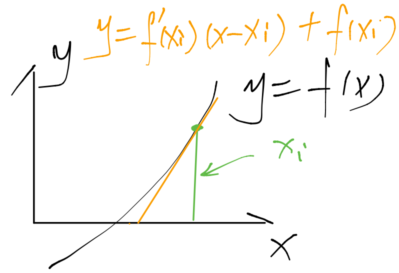
  $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad n \geq 0$

- Catlan numbers: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ⋯

Newton's method

Root of $f(x)=0$ thru successive approx.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



e.g.: calculate $\sqrt{a}$.

$$f(x) = x^2 - a$$

$$x_{i+1} = x_i - \frac{(x_i^2 - a)}{2x_i} = \frac{x_i + a/x_i}{2}$$

How to do high-precision multiplication

if $a = 2$

$x_0 = 1.000\,000\,000$

$x_1 = 1.5\,000\,000\,000$

$x_2 = 1.41666666$

$x_3 = 1.414215686$

$x_4 = 1.414213562$

- Quadratic Convergence

Number of digits of precision double every iteration

High Precision Multiplication

$\sqrt{2}$ to d-digit precision, want integer $\lfloor 10^d \sqrt{2} \rfloor = \lfloor \sqrt{2 \cdot 10^{2d}} \rfloor$

can tell newton's method

Two n-digit numbers (radix (base) $r = 2, 10$)

$0 \le x, y < r^n$,

Divid-and-Conquer:

$x = x_1 r^{n/2} + x_0$ x_1: high half x_0: low half

$y = y_1 r^{n/2} + y_0$

$0 \le x_0, x_1 \le r^{n/2}$ $0 \le y_0, y_1 < r^{n/2}$

let $z_0 = x_0 \cdot y_0$, $z_1 = x_0 y_1 + x_1 y_0$ $z_2 = x_2 \cdot y_2$

$$z = x \cdot y = \underbrace{x_1 y_1}_{z_2} r^n + \underbrace{(x_0 y_1 + x_1 y_0)}_{z_1} r^{n/2} + \underbrace{x_0 y_0}_{z_0}$$

4 multiplies of $n/2$ digits numbers $\Longrightarrow \Theta(n^2)$ time

$T(n) = 4T(n/2) + \Theta(n)$

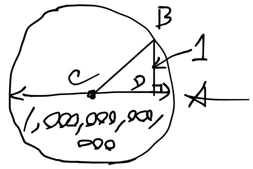
☐ Karatsuba ~~Algorithm~~

$$\left.\begin{aligned} z_0 &= x_0 \cdot y_0 \\ z_2 &= x_2 \cdot y_2 \\ z_1 &= (x_0 + x_1) \cdot (y_0 + y_1) - z_0 - z_2 \end{aligned}\right\} \text{3 multiplies}$$

$\hookrightarrow T(n) = 3T(n/2) + \Theta(n)$

$= \Theta(n^{\log_2 3}) = \Theta(n^{1.58})$

☐ Fun Geometry Problem



$AD = AC - CD$

$= 5000{,}000{,}000{,}00$

$\sqrt{(5000{,}000{,}000{,}00)^2 - 1}$