RnD Project Final Report

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April 2024

In this RnD project, we study orbits and orbit closures of complex algebraic groups acting rationally on a finite dimensional vector space. To that end, we study representations of Lie algebras, and use Lie group - Lie algebra correspondence. At the end, we prove the celebrated Hilbert-Mumford theorem about stability of orbits.

All vector spaces, Lie algebras and varieties are assumed to be over $\mathbb{C}$.

1 Solvable Lie Algebras

Definition 1.1. *Let V be a vector space of dimension n. Then $\mathfrak{gl}(V)$ is the Lie algebra on vector space of endomorphisms of V with Lie bracket $[x, y] = xy - yx$.*

Definition 1.2. *Let $\mathfrak{g}$ be a Lie algebra. Then for any $x \in \mathfrak{g}$, define $ad_{\mathfrak{g}}(x)$ as the linear map $ad_{\mathfrak{g}}(x)(y) = [x, y]$.*

Note that $\mathrm{ad}_{\mathfrak{g}}$ is a representation of $\mathfrak{g}$. We just use ad when the Lie algebra is evident.

Recall the Jordan decomposition of a matrix:

Theorem 1.3 (Jordan-Chevalley Decomposition). *Let V be a finite dimensional vector space, and let $x \in \mathfrak{gl}(V)$.*

(a) There exist $x_s, x_n \in \mathfrak{gl}(V)$ such that x_s is diagonalizable, x_n is nilpotent, and $x = x_s + x_n$. Further, this decomposition is unique.

(b) There exist polynomials $P, Q \in \mathbb{C}[t]$ without constant terms such that $P(x) = x_s$ and $P(x) = x_s^*$. (Here x_s^* is the conjugate transpose of x_s.)

(c) The linear map $ad(x)$ has Jordan decomposition $ad(x)_s = ad(x_s)$ and $ad(x)_n = ad(x_n)$.

Proof. Parts (a) and (b) follow from the Jordan Canonical Form for matrices. Now note that the endomorphisms $y \to x_n y$ and $y \to y x_n$ are nilpotent, and they commute, so by binomial theorem, by taking a sufficient large power, we get that their difference $\mathrm{ad}(x_n)$ is also nilpotent. We now prove that $\mathrm{ad}(x_s)$ is diagonalizable. To that end, let $\{v_1, \ldots, v_m\}$ be an eigenbasis of V w.r.t. x_s. Let $e_{i,j}$ be the endomorphism sending v_i to v_j, and everything else to zero. Clearly $e_{i,j}$ is a basis of $\mathfrak{gl}(V)$, and it is easy to see that it must be an eigenbasis w.r.t. $\mathrm{ad}(x_s)$, which proves that it is diagonalizable. Hence Part (c) is proved due to uniqueness of Jordan decomposition. □

Definition 1.4. *Let $\mathfrak{g}$ be a Lie algebra. Define a decreasing sequence of ideals $C_i(\mathfrak{g})$, called the central series of $\mathfrak{g}$ by: $C_0(\mathfrak{g}) = \mathfrak{g}$ and $C_{i+1}(\mathfrak{g}) = [\mathfrak{g}, C_i(\mathfrak{g})]$ for all $i \geq 0$. Then, $\mathfrak{g}$ is called nilpotent if $C_n(\mathfrak{g}) = 0$ for some n.*

We state (without proof) a special case of Engel's theorem, which we will use later.

Theorem 1.5 (Engel's Theorem for Matrix Algebras). *Let V be a finite dimensional vector space and $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Suppose every element of $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is a nilpotent Lie algebra.*

1.1 Cartan's Criterion

Definition 1.6. *Let $\mathfrak{g}$ be a Lie algebra. Define a decreasing sequence of ideals $D_i(\mathfrak{g})$, called the derived series of $\mathfrak{g}$ by: $D_0(\mathfrak{g}) = \mathfrak{g}$ and $D_{i+1}(\mathfrak{g}) = [D_i(\mathfrak{g}), D_i(\mathfrak{g})]$ for all $i \geq 0$. Then, $\mathfrak{g}$ is called solvable if $D_n(\mathfrak{g}) = 0$ for some n.*

The following is the most crucial result in proving solvability of some Lie algebra:

Theorem 1.7 (Cartan's Criterion for Solvability)**.** *Let V be a finite dimensional vector space and $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Suppose for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$, $Tr(xy) = 0$. Then $\mathfrak{g}$ is solvable.*

Proof. Let $x \in [\mathfrak{g}, \mathfrak{g}]$ be arbitrary. Let $x = x_s + x_n$ be the Jordan decomposition of x, where x_s is diagonalizable and x_n is nilpotent. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of x, which are also the eigenvalues of x_s. Write everything as matrices w.r.t. the basis in which x has the Jordan Canonical Form. Then x_s is a diagonal matrix, so xx_s^* is upper triangular with $|\lambda_i|^2$ on the diagonal.

Write $x = \sum_{i=1}^m [y_i, z_i]$ for $y_i, z_i \in \mathfrak{g}$. Thus

$$\sum_{i=1}^n |\lambda_i|^2 = \mathrm{Tr}(xx_s^*) = \sum_{i=1}^m \mathrm{Tr}([y_i, z_i]x_s^*).$$

We rewrite each term in the RHS:

$$\mathrm{Tr}([y_i, z_i]x_s^*) = \mathrm{Tr}(z_i[x_s^*, y_i]) = \mathrm{Tr}(z_i \mathrm{ad}(x_s^*)(y_i)).$$

The first equality holds because $[y_i, z_i]x_s^* - z_i[x_s^*, y_i] = [y_i, z_i x_s^*]$, which has trace 0. Now, by the theorem about Jordan decomposition, there exists a polynomial $Q \in \mathbb{C}[t]$ without a constant term such that

$$\mathrm{ad}(x_s^*) = \mathrm{ad}(x_s)^* = Q(\mathrm{ad}(x)).$$

Write $Q(t) = \sum_{j=1}^k c_j t^j$. Then $\mathrm{ad}(x_s^*)(y_i) = \sum_{j=1}^k c_j \mathrm{ad}(x)^j y_i$. Further, $\mathrm{ad}(x)^j y_i \in [\mathfrak{g}, \mathfrak{g}]$ because $j \geq 1$. Hence $\mathrm{ad}(x_s^*)(y_i) \in [\mathfrak{g}, \mathfrak{g}]$, so by the given condition

$$\mathrm{Tr}(z_i \mathrm{ad}(x_s^*)(y_i)) = \mathrm{Tr}(\mathrm{ad}(x_s^*)(y_i) z_i) = 0.$$

This proves

$$\sum_{i=1}^n |\lambda_i|^2 = 0,$$

so all eigenvalues of x are 0, which implies that x is nilpotent.

Therefore, all elements of $[\mathfrak{g}, \mathfrak{g}]$ are nilpotent, so by Engel's theorem, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent; in particular it is solvable. Hence is $\mathfrak{g}$ is solvable, as required. $\square$

2 Semisimple Lie Algebras

Definition 2.1. *A Lie algebra $\mathfrak{g}$ is called simple if it is non-abelian and has no proper non-zero ideals.*

Definition 2.2. *A Lie algebra $\mathfrak{g}$ is called semisimple if it has no non-zero abelian ideal.*

It is easy to see that sum of any two solvable ideals is also solvable, so the following definition makes sense.

Definition 2.3. *For a Lie algebra $\mathfrak{g}$, the radical $Rad(\mathfrak{g})$ is defined to be its greatest solvable ideal.*

Lemma 2.4. *A non-zero Lie algebra $\mathfrak{g}$ is semisimple iff $Rad(\mathfrak{g}) = 0$.*

Proof. Clearly, any abelian ideal of $\mathfrak{g}$ is contained in $\mathrm{Rad}(\mathfrak{g})$, so the condition is necessary. Conversely, if $\mathrm{Rad}(\mathfrak{g})$ is non-zero, then the last non-zero ideal in its derived series is the required abelian ideal, making $\mathfrak{g}$ not semisimple. $\square$

2.1 Killing Form

Definition 2.5. *Let $\mathfrak{g}$ be a Lie algebra, and let $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a bilinear form. Then β is called invariant if $\beta(x, [y, z]) = \beta([x, y], z)$ for all $x, y, z \in \mathfrak{g}$.*

Note that $\mathrm{Tr}(xy)$ is an invariant bilinear form on $\mathfrak{gl}(V)$. Indeed, trace of $[x, y]$ is always zero, so trace of $[x, y]z - x[y, z] = [y, xz]$ is zero.

Lemma 2.6. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, and let (V, f) be a faithful finite dimensional representation of $\mathfrak{g}$. Then the map $\beta_f(x, y) = Tr(f(x)f(y))$ is an invariant, non-degenerate and symmetric bilinear form on $\mathfrak{g}$.*

Proof. β_f is clearly bilinear and symmetric. Further, since Tr is invariant, so is β_f. So now we prove non-degeneracy. Consider the kernel $\ker(\beta_f)$, i.e., set of all x such that $\beta_f(x, y) = 0$ for all $y \in \mathfrak{g}$. Since β_f is invariant, the kernel is a Lie ideal of $\mathfrak{g}$. Since f is faithful, $f(\ker(\beta_f))$ is a Lie subalgebra of $\mathfrak{gl}(V)$ isomorphic to $\ker(\beta_f)$. Now apply Cartan's criteria, whose condition trivially holds for $f(\ker(\beta_f))$, so it is solvable, and hence so is $\ker(\beta_f)$. But since $\mathfrak{g}$ is semisimple, it has no non-zero solvable ideals, so $\ker(\beta_f) = 0$, which implies β_f is non-degenerate. $\square$

We have an example of a faithful representation: namely, $\mathrm{ad}_{\mathfrak{g}}$ on $\mathfrak{g}/\ker(\mathrm{ad}_{\mathfrak{g}})$. This motivates the following definition:

Definition 2.7 (Killing Form)**.** *The Killing form on $\mathfrak{g}$ is the bilinear form $\kappa_{\mathfrak{g}}(x, y) = Tr(ad_{\mathfrak{g}}(x)ad_{\mathfrak{g}}(y))$.*

Clearly $\kappa_{\mathfrak{g}}$ is symmetric and invariant. Cartan's criterion implies that $\mathfrak{g}$ is solvable if $[\mathfrak{g}, \mathfrak{g}] \subset \ker(\kappa_{\mathfrak{g}})$ (as long as finite dimensional). The Killing form allows us to get a nice criterion for semisimple algebras:

Theorem 2.8 (Cartan-Killing Criterion)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $\mathfrak{g}$ is semisimple iff $\kappa_{\mathfrak{g}}$ is non-degenerate.*

Proof. We first show that $\ker(\kappa_{\mathfrak{g}}) \subset \mathrm{Rad}(\mathfrak{g})$ for any Lie algebra. Indeed, if $\mathfrak{i} = \ker(\kappa_{\mathfrak{g}})$, then $\mathfrak{i}$ is a Lie ideal of $\mathfrak{g}$. The Killing form $\kappa_{\mathfrak{g}}$ restricted to $\mathfrak{i} \times \mathfrak{i}$ is the Killing form on $\mathfrak{i}$; this follows because image of $\mathrm{ad}_{\mathfrak{g}}(x)$ for $x \in \mathfrak{i}$ is in $\mathfrak{i}$. Thus the Killing form $\kappa_{\mathfrak{i}}$ is zero, which implies $\mathfrak{i}$ is solvable. Hence $\mathfrak{i} \subset \mathrm{Rad}(\mathfrak{g})$, as required. This proves that $\mathfrak{g}$ semisimple $\implies$ $\kappa_{\mathfrak{g}}$ is non-degenerate.

For the other direction, we first prove that $\ker(\mathrm{ad}_{\mathfrak{g}}) = 0$. Indeed, $\ker(\mathrm{ad}_{\mathfrak{g}})$ is an abelian ideal of $\mathfrak{g}$, so it is zero since $\mathfrak{g}$ is semisimple. So $\mathrm{ad}_{\mathfrak{g}}$ is a faithful representation of $\mathfrak{g}$, so by Lemma 2.6, $\kappa_{\mathfrak{g}}$ is non-degenerate. $\square$

2.2 Structure of Semisimple Lie Algebras

Definition 2.9. *Let $\mathfrak{g}$ be a semisimple Lie algebra. For any Lie ideal $\mathfrak{i}$, define $\mathfrak{i}^{\perp}$ to be the ideal of all elements y such that $\kappa_{\mathfrak{g}}(x, y) = 0$ for all $x \in \mathfrak{i}$.*

Lemma 2.10. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Then for any ideal $\mathfrak{i}$, both $\mathfrak{i}$ and $\mathfrak{i}^{\perp}$ are semisimple as Lie algebras, and $\mathfrak{i} \oplus \mathfrak{i}^{\perp} = \mathfrak{g}$.*

Proof. Since $\kappa_{\mathfrak{g}}$ is non-degenerate, $\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{i}^{\perp}$ holds. In particular, $[\mathfrak{i}, \mathfrak{i}^{\perp}] \subset \mathfrak{i} \cap \mathfrak{i}^{\perp} = 0$, which implies $[\mathfrak{i}, \mathfrak{g}] = [\mathfrak{i}, \mathfrak{i}]$. Thus any ideal of $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, which proves that the former is also semisimple. $\square$

Note that the above lemma also proves that every quotient of $\mathfrak{g}$ is semisimple: Indeed, $\mathfrak{g}/\mathfrak{i} \cong \mathfrak{i}^{\perp}$.

Theorem 2.11. *Let $\mathfrak{g}$ be a non-zero semisimple Lie algebra. Then*

(a) There exists $t \in \mathbb{N}$ and ideals $\mathfrak{j}_1, \ldots, \mathfrak{j}_t \in \mathfrak{g}$ that are simple as Lie algebras, such that

$$\mathfrak{g} = \bigoplus_{i=1}^{t} \mathfrak{j}_i$$

(b) If s is any ideal of $\mathfrak{g}$ that is a simple Lie algebra, then $\mathfrak{j} = \mathfrak{j}_i$ for some i.

Proof. For Part (a), just choose a simple ideal of $\mathfrak{g}$ (exists because any minimal non-abelian ideal would be simple). Then just use Lemma 2.10 repeatedly to get the required decomposition. This can be done since $\mathfrak{g}$ is finite dimensional.

Now, let $\mathfrak{j}$ be any ideal of $\mathfrak{g}$ that is simple as a Lie algebra. Then $[\mathfrak{j}, \mathfrak{g}]$ is an ideal of $\mathfrak{j}$, and it is non-zero because $\ker(\mathrm{ad}_{\mathfrak{g}}) = 0$. Hence

$$\mathfrak{j} = [\mathfrak{j}, \mathfrak{g}] = \bigoplus_{i=1}^{t} [\mathfrak{j}, \mathfrak{j}_i].$$

So by simplicity of $\mathfrak{j}$ and $\mathfrak{j}$, $\mathfrak{j} = \mathfrak{j}_i = [\mathfrak{j}, \mathfrak{j}_i]$ for some i. $\square$

Corollary 2.12. *If $\mathfrak{g}$ is semisimple Lie algebra, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

Proof. By the proof of Part (b) of Theorem 2.11, $[\mathfrak{j}_i, \mathfrak{j}_i] = \mathfrak{j}_i$, and $[\mathfrak{j}_i, \mathfrak{j}_k] = 0$ for all $i \neq k$. Hence

$$[\mathfrak{g}, \mathfrak{g}] = \bigoplus_{1 \leq i,k \leq t} [\mathfrak{j}_i, \mathfrak{j}_k] = \mathfrak{g} = \bigoplus_{i=1}^{t} \mathfrak{j}_i = \mathfrak{g}.$$

$\square$

3 Complete reducibility of Finite Dimensional Representations

3.1 The Casimir Element

Lemma 3.1. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, β be an invariant, non-degenerate and symmetric bilinear form on $\mathfrak{g}$, and let (V, f) be a representation of $\mathfrak{g}$. Let $(x_1, \ldots, x_n)$ be a basis of $\mathfrak{g}$ and let $(y_1, \ldots, y_n)$ be its dual basis wrt β. Then, the element*

$$c = \sum_{i=1}^{n} f(x_i)f(y_i)$$

is a $\mathfrak{g}$-invariant endomorphism of (V, f).

Proof. Let $z \in \mathfrak{g}$ be arbitrary. Suppose

$$[z, x_i] = \sum_{j=1}^{n} a_{i,j}x_j \text{ and } [z, y_i] = \sum_{j=1}^{n} b_{i,j}y_j$$

for all $1 \leq i \leq n$. By invariance of β,

$$a_{i,j} = \beta([z, x_i], y_j) = -\beta([x_i, z], y_j) = -\beta(x_i, [z, y_j]) = -b_{j,i}.$$

Hence,

$$\begin{aligned}
f(z)c - cf(z) &= [f(z), c] \\
&= \sum_{i=1}^{n}[f(z), f(x_i)f(y_i)] \\
&= \sum_{i=1}^{n}[f(z), f(x_i)]f(y_i) + \sum_{i=1}^{n} f(x_i)[f(z), f(y_i)] \\
&= \sum_{1\leq i,j\leq n} a_{i,j}f(x_j)f(y_i) + \sum_{1\leq i,j\leq n} b_{i,j}f(x_i)f(y_j) \\
&= \sum_{1\leq i,j\leq n} (a_{i,j} + b_{j,i})f(x_j)f(y_i) \\
&= 0.
\end{aligned}$$

This proves c is $\mathfrak{g}$-invariant, as required. □

Such a c is called the Casimir element. Observe that $\mathrm{Tr}(c) = \dim(\mathfrak{g}) \neq 0$. If (V, f) is a faithful and finite dimensional representation of $\mathfrak{g}$, we have a natural invariant non-degenerate symmetric bilinear form $\beta_f(x, y) = \mathrm{Tr}(f(x)f(y))$, as proved in Lemma 2.6. Thus when we mention "Casimir element of a representation", this is the bilinear form we are using.

3.2 Weyl's Theorem

Lemma 3.2. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, and let (V, f) be a finite dimensional representation of $\mathfrak{g}$. If W is a codimension 1 subrepresentation of (V, f), then there exists a subrepresentation X of (V, f) such that $V = W \oplus X$.*

Proof. We proceed by induction on $\dim(V)$. If $\dim(V) = 1$, the claim is trivial. So now assume $\dim(V) > 1$, and let W be a subrepresentation of V with codimension 1. Note that, by going modulo the kernel of f, we may assume (V, f) is a faithful representation.

First assume W is simple. Consider the Casimir element c of V; it is a $\mathfrak{g}$-invariant endomorphism of V. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}$ acts trivially on one-dimensional representations; in particular on V/W. Therefore $f(V) \subset W$, and so $c(V) \subset W$. Further, by Schur's lemma, c is a scalar on W, say multiplication by $\lambda \in \mathbb{C}$. Since $\mathrm{Tr}(c) \neq 0$, $\lambda \neq 0$, and so $c(V) = W$ and c is an isomorphism on W. Therefore $\ker(c)$ is a one-dimensional subrepresentation of V that intersects W only at 0, and so is a direct summand of W.

Now assume W is not simple. Consider a proper non-zero representation W' of W. Let $(V/W', g)$ be the quotient representation, and let $\pi : V \to V/W'$ be the projection map. Then $\pi(W)$ has codimension 1 in V/W', and $\dim(V/W') < \dim(V)$. So by induction hypothesis, there exists a subrepresentation W_1 of V such that $W' \subset W_1$ and $V/W' = \pi(W) \oplus \pi(W_1)$. $\pi(W_1)$ has dimension 1, so $\dim(W_1) = \dim(W') + 1 < \dim(V)$. Hence applying the induction hypothesis to W_1, we get a subrepresentation W_2 of V having dimension 1 such that $W_1 = W' \oplus W_2$. Note that $\dim(W) + \dim(W_2) = \dim(V)$. Further, W and W_2 are disjoint, because $W \cap W_1 = W'$ but W_2 is disjoint from W'. Thus $V = W \oplus W_2$, as required. □

Theorem 3.3 (Weyl's Theorem). *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Then any finite dimensional representation of $\mathfrak{g}$ is completely reducible. More strongly, any subrepresentation is a direct summand.*

Proof. Let (V, ρ) be a finite dimensional representation. If it is simple, we are done. Else, let W be a proper non-zero subrepresentation of V. Consider $\mathrm{Hom}(V, W)$ as a representation of $\mathfrak{g}$, via the map μ given by

$$(\mu(x)(f))(w) = (\rho(x) \circ f)(w) - (f \circ \rho(x))(w).$$

Let $\mathcal{V}$ and $\mathcal{W}$ be subspaces of $\mathrm{Hom}(V, W)$ consisting of maps that are scalar and 0 respectively on W. Then for any $f \in \mathcal{V}$,$x \in \mathfrak{g}$ and $w \in W$, by definition $(\mu(x)(f))(w) = 0$. Therefore $\mathcal{V}, \mathcal{W}$ are subrepresentations of $\mathrm{Hom}(V, W)$, and $\mathfrak{g}$ sends $\mathcal{V}$ to $\mathcal{W}$. Also from definition, $\mathcal{W}$ is a codimension 1 subspace of $\mathcal{V}$.

Now, applying the above lemma, there exists a one dimensional representation of $\mathcal{V}$ that is a direct summand of $\mathcal{W}$. Say the representation is generated by f; WLOG $f|_W = \mathrm{Id}_W$. Then since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}$ acts trivially on $\mathbb{C}f$. Therefore $\ker(f)$ is a subrepresentation of V, and since f is identity on W, $\ker(f) \cap W = (0)$. Therefore by a dimension argument, $\ker(f)$ is a direct summand of W. □

4 Linear Algebraic Groups

In this section, the topology that we will be using is the Zariski topology.

Definition 4.1 (Linear Algebraic Group)**.** *A subgroup $G \subset GL_n(\mathbb{C})$ is called a linear algebraic group if G is an affine variety as viewed as a subset of $\mathbb{C}^{n^2}$.*

It is not hard to prove that e, i.e., the identity is a simple point (in the Zariski topology) for a linear algebraic group G. Thus we can talk about the tangent space at identity, which can be looked at as the space of derivations at e. This has a Lie algebra structure, and this will be the Lie algebra associated with G. We will generally denote groups with capital letters and their corresponding Lie algebras with mathfrak letters. The topology that we use is the standard Zariski topology.

We use the notation $\bigwedge^d V$ to denote the d^{th} exterior power of a vector space V. Since exterior products are quotients of tensor products, there is a natural extension of action (of Lie group or algebra) on V to action on exterior power.

Lemma 4.2 (Action on Exterior Powers)**.** *Let W be a vector space, and let M be a d-dimensional subspace of W. Let $L = \bigwedge^d M$ and $V = \bigwedge^d W$. Then for any $X \in GL(W)$ and $x \in \mathfrak{gl}(W)$,*

(a) $X \cdot L = L$ iff $X \cdot M = M$.

(b) $x \cdot L \subset L$ iff $x \cdot M \subset M$.

Proof. In each case, the"if" part is obvious. For Part (a), choose a basis $\{w_1, \ldots, w_n\}$ of W such that first d vectors form a basis for M, and such that $X \cdot M$ is spanned by $w_{l+1}, \ldots, w_{l+d}$ for some $l \geq 0$. Since L is the one-dimensional vector space spanned by $w_1 \wedge \cdots \wedge w_d$, X sends this vector to some non-zero multiple of itself. However, clearly this image must lie in the vector space spanned by $w_{l+1} \wedge \cdots \wedge w_{l+d}$, which implies $l = 0$.

For Part (b), choose a basis of M such that x maps the first part into a basis of $x \cdot M$, and sends all other basis vectors outside M. Choose $y \in \mathfrak{gl}(W)$ which agrees with x on the first part, while send the other basis vectors to 0. Clearly y leaves M stable, so we can replace x by $x - y$, and assume M intersects $x \cdot M$ trivially. Now, suppose $w_1, \ldots, w_d$ is the chosen basis so that $x \cdot w_1, \ldots, x \cdot w_c$ is a basis of $x \cdot M$, and $x \cdot w_i = 0$ for all $i \geq c + 1$. Note that $w_1, \ldots, w_d, x \cdot w_1, \ldots, x \cdot w_c$ are linearly independent by assumption. Now,

$$x \cdot (w_1 \wedge \cdots \wedge w_d) = \sum_{i=1}^{d} w_1 \wedge \cdots \wedge (x \cdot w_i) \wedge \cdots \wedge w_d.$$

This, by assumption is a multiple of $w_1 \wedge \cdots \wedge w_d$. But the vectors $w_1 \wedge \cdots \wedge (x \cdot w_i) \wedge \cdots \wedge w_d$, for $1 \leq i \leq c$, are linearly independent with $w_1 \wedge \cdots \wedge w_d$, which is impossible unless $c = 0$. Thus $x \cdot M = 0$, which clearly is a subset of M. □

Theorem 4.3 (Chevalley's Theorem)**.** *Let G be a linear algebaric group, and let H be a closed subgroup. Then there is a rational representation $\phi : G \to GL(V)$ and a on-dimensional subspace L of V such that*

$$H = \{X \in G \mid \phi(X)L = L\},$$

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid d\phi(x)L \subset L\}.$$

Proof. The idea is to look at action of G on the space of polynomial functions on G, via $(g \cdot f)(x) = f(g^{-1}x)$ (this is the standard left regular representation). Let I be the ideal of functions that are H-invariant. Since H is Zariski-closed, it is precisely the stabilizer of I. Indeed, H is a subset of the stabilizer trivially, and if $g \notin H$, then by looking at $G \setminus H$ which is a dense open neighborhood of g, we can find a polynomial $f \in I$ such that $f(g) = 1$. Then $(g^{-1} \cdot f)(e) = f(g) = 1 \notin I$, so g cannot stabilize I.

Now, by Hilbert's Basis Theorem, I is finitely generated, say by $f_1, \ldots, f_n$. As we had proved in the course, I can be contained in a finite-dimensional G-stable vector space W (this was proved using the fact that group action can be thought of as an element of the tensor product). Let $M = W \cap I$. Then $f_1, \ldots, f_n \in M$, so if g stabilizes M, then it also stabilizes I, so $g \in H$. The converse is obvious, so H is precisely the stabilizer of M.

All that remains is to compress M into a one-dimensional space. However, this can be done by the above Lemma by taking $V = \bigwedge^d W$ and $L = \bigwedge^d M$, where $d = \dim(M)$. □

Theorem 4.4. *Let G be a linear algebraic group, and let N be a closed normal subgroup of G. Then there is a rational representation $\psi : G \to GL(V)$ such that $N = ker(\psi)$ and $\mathfrak{n} = ker(d\psi)$.*

Proof. Use Chevalley's theorem to construct a morphism $\phi : G \to \mathrm{GL}(V)$ and a line L in V whose stabilizer in G (resp. $\mathfrak{g}$) is N (resp. $\mathfrak{n}$). Let χ be a character of N, i.e., a morphism $N \to \mathbb{C}^*$. Let V_χ be the subspace of V such that N acts on V_χ via χ; i.e., $x \cdot v = \chi(x)v$ for all $x \in N$ and $v \in V_\chi$. Consider the sum of all such V_χ. This sum is direct (because of Dedekind's theorem on characters), and thus only finitely many V_χ appear. Further, since N is normal, it is not hard to see that $\phi(G)$ just permutes the V_χ. So, we can assume V is just the direct sum of V_χ, WLOG.

Let W be the subspace of $\mathrm{End}(V)$ consisting of endomorphisms that stabilize each V_χ. We have a natural isomorphism $W \cong \bigsqcup \mathrm{End}(V_\chi)$. Note that $\mathrm{GL}(V)$ acts on $\mathrm{End}(V)$ via conjugation. Then, $\phi(G)$ stabilizes W, because W stabilizes all V_χ and $\phi(G)$ just permutes the V_χ. Thus we get a representation $\psi : G \to \mathrm{GL}(W)$, which is ϕ, followed by conjugation, followed by restriction to W. This is clearly a rational representation.

It remains to analyze the kernels. If $x \in N$, then $\phi(x)$ acts as a scalar on each V_χ, so conjugating by $\phi(x)$ has no effect on W, i.e., $x \in \ker(\phi)$. Conversely, suppose $\psi(x) = e$. Then $\phi(x)$ stabilizes all V_χ and commutes with $\mathrm{End}(V_\chi)$, which is only possible if $\phi(x)$ acts as a scalar on each V_χ. But, L as defined above lies in one of the V_χ, and $\phi(x)$ acts as a scalar on it, so by Chevalley's theorem, $x \in N$, as required.

To analyze $\ker(d\phi)$, we take differentials. Note that differential of the conjugation map is ad. Hence, if $x \in \mathfrak{n}$, then $d\phi(x)$ acts as a scalar on each V_χ, so $\mathrm{ad}(d\phi(x)) = 0$ on W, i.e., $x \in \ker(d\phi)$. Conversely, if $\mathrm{ad}(d\phi(x)) = 0$ on W, then $d\phi(x)$ stabilizes each V_χ and commutes with $\mathrm{End}(V_\chi)$. Hence it acts as a scalar on each V_χ, so $d\phi(x)L \subset L$, which forces $x \in \mathfrak{n}$ by Chevalley's theorem. □

4.1 Algebraic Group - Lie Algebra Correspondence

Theorem 4.5 (Algebraic Group - Lie Algebra correspondence)**.** *The following are true for a linear algebraic group G:*

(a) If $\phi : G \to G'$ is a morphism of linear algebraic groups, then $ker(d\phi)$ is the Lie algebra associated with $ker(\phi)$.

(b) If I, J are closed subgroups of G, then $\mathfrak{i} \cap \mathfrak{j}$ is the Lie algebra associated with $I \cap J$.

(c) If G is connected, there is a one-to-one and inclusion preserving correspondence between closed connected subgroups of G and their corresponding Lie algebras.

Proof. For Part (a), we can WLOG assume ϕ is surjective, and we can think of it as an isomorphism from $G \to G/\ker(\phi)$. Then, the theorem follows from the construction done in Theorem 4.4.

For Part (b), let $\pi : G \to G/J$ be the canonical morphism. Consider the restriction $\pi' : I \to \pi(I)$. Note that we can identify $\pi(I)$ with $I/I \cap J$. Hence, by Part (a), the Lie algebra associated with $I \cap J$ is $\ker(d\pi') = \mathfrak{i} \cap \ker(d\pi) = \mathfrak{i} \cap \mathfrak{j}$.

Part (c) follows immediately from the first two parts. □

Theorem 4.6. *Let G be a connected linear algebraic group, and let H be a closed connected subgroup. Then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ iff H is normal in G.*

Proof. "If" part follows from Theorem 4.3, since a kernel is always an ideal. Now assume $\mathfrak{h}$ is an ideal. Let $N = \{x \in G \mid x^{-1}Hx = H\}$. Taking differentials, we see that

$$\mathfrak{n} = \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subset \mathfrak{h}\} = \mathfrak{g},$$

since $\mathfrak{h}$ is an ideal. Because G is connected, by the group-algebra correspondence, this forces $G = N$, which implies H is normal. □

4.2 Semisimple Groups

Definition 4.7. *A linear algebraic group G is called semisimple if it has no closed connected commutative normal subgroup except e.*

Theorem 4.8. *A connected linear algebraic group G is semisimple iff $\mathfrak{g}$ is semisimple.*

Proof. Suppose $\mathfrak{g}$ is semisimple. If N is a closed connected commutative normal subgroup of G, then $\mathfrak{n}$ is a commutative ideal. So $\mathfrak{n} = 0$, forcing $N = e$.

Conversely, assume $\mathfrak{n}$ is commutative. Look at the ideal

$$\mathfrak{h} = \{y \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } x \in \mathfrak{n}\}.$$

This contains $\mathfrak{n}$ since it is commutative. Now, Theorem 4.5 implies that H is normal; as a result, the connected component of the center of H that contains the identity is also normal. But G is semisimple, so this is identity, which implies that the corresponding Lie algebra is 0. The corresponding Lie algebra would have been set of all y such that $[x, y] = 0$ for all $x \in \mathfrak{h}$, which contains $\mathfrak{n}$. Hence $\mathfrak{n} = 0$, as required. $\square$

The above theorem implies that we can switch between semisimple groups and their semisimple Lie algebras. This means we can carry over a lot of results, including complete reducibility of representations, to semisimple groups (and rational representations). The groups for which complete reducibilty holds are called reductive groups; thus all semisimple groups are reductive.

5 Hilbert-Mumford Theorem

We present a fairly elementary proof of the Hilbert-Mumford theorem, which uses the following lemma:

Lemma 5.1. *Let $m_{i,j}$ for $1 \leq i \leq r$ and $1 \leq j \leq n$ be integers satisfying the following property: If $b_1, \dots, b_r$ are integers, not all zero, such that*

$$\sum_{i=1}^{r} b_i m_{i,j} = 0 \quad \forall 1 \leq j \leq n,$$

then some two of the b_i have opposite sign. Then, there exist integers c_j such that

$$\sum_{j=1}^{n} m_{i,j} c_j > 0 \quad \forall 1 \leq i \leq r.$$

Proof. It is enough to prove the statement for rationals c_i, and just scale later. Also, the given condition holds for rationals b_i, by scaling. It translates to the kernel of the linear map

$$M : (b_1, \dots, b_r) \to (\sum_{i=1}^{r} b_i m_{i,1}, \dots, \sum_{i=1}^{r} b_i m_{i,n})$$

intersecting the cone of non-negative entries in $\mathbb{Q}^r$ only at the origin. Thus, due to density of $\mathbb{Q}$ in $\mathbb{R}$, it intersects non-negative cone of $\mathbb{R}^r$ only at the origin. Consider the map M^t. Note that $\ker(M)$ and $\mathrm{Im}(M^t)$ are orthogonal complements in $\mathbb{R}^r$.

We show the following: If K is a subspace of $\mathbb{R}^r$ that intersects the non-negative cone only at origin, then $K^\perp$ intersects the interior of the cone. Suppose K has co-dimension $k \geq 2$. Consider the image of the non-negative cone in the projection $\mathbb{R}^r/K \cong \mathbb{R}^k$; call it D. Clearly D is closed. Further, $\mathbb{R}^k \setminus \{0\}$ is connected since $k \geq 2$, so D and $-D$ cannot cover $\mathbb{R}^k \setminus \{0\}$, so there is a non-zero vector $v \in \mathbb{R}^k$ such that $\mathbb{R}v \cap D \setminus \{0\} = \varnothing$. Thus if we add the pull-back of v to K, we get a subspace of one higher dimension, which also intersects the non-negative cone only at 0. Keep doing this until we get K having codimension 1. Then, suppose K is the hyperplane $\sum_{i=1}^{r} \lambda_i x_i = 0$. Since K intersects non-negative cone only at the origin, all the λ_i must be non-zero and have the same sign, WLOG all positive. Then $(\lambda_1, \dots, \lambda_r) \in K^\perp$ is a vector that is in the interior of the cone, as required.

Applying the above property to $\ker(M)$, we get that some vector in $\mathrm{Im}(M^t)$ lies in interior of the non-negative cone. By scaling and using density of rationals, we can assume this vector $(c_1, \dots, c_n)$ to have integer entries. This vector satisfies the precise condition that we want. ☐

The Hilbert-Mumford theorem gives a nice characterization of unstable points in terms of one parameter subgroups.

Theorem 5.2. *Let G be a reductive group acting linearly on a vector space V. Let $v \in V$ be G-unstable, i.e., the closure of the orbit $\overline{G \cdot v}$ contains 0. Then there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot v = 0$.*

Before we prove this theorem, we would need another lemma relating unstable points in G to unstable points in the maximal torus T:

Lemma 5.3. *Let $T \leq G$ be a maximal torus in G, and suppose $v \in V$ is G-unstable. Then the orbit $G \cdot v$ contains a vector u which is T-unstable.*

Proof. We skip the proof here, but the key idea is: In a reductive group, the set of elements conjugate to the maximal torus form an open dense subset of G. If these set of elements is G_s, we can also show that $G_s \cdot v$ is an open dense subgroup of $G \cdot v$, which will prove the lemma. ☐

Now we can finally prove the theorem.

Proof. Moving to the torus, we can assume v is T-unstable. Hence V can be split into different weight classes. More concretely, we can write any $v \in V$ as $\sum_{i=1}^{r} v_i$ where T acts by scalar multiplication on each v_i. Suppose

$$(t_1, t_2, \dots, t_n) \cdot v_i = t_1^{m_{i,1}} \cdots t_n^{m_{i,n}} v_i.$$

for all $1 \leq i \leq r$ and $(t_1, \ldots, t_n) \in T$.

We claim that $m_{i,j}$ satisfy the conditions of Lemma 5.1. Indeed, assume there are integers b_i, not all zero and of the same sign, such that

$$\sum_{i=1}^{r} b_i m_{i,j} = 0 \quad \forall 1 \leq j \leq n.$$

Let $t^{(k)} \in T$ be a sequence such that $t^{(k)} \cdot v \to 0$, which exists since v is T-unstable. This means

$$(t_1^{(k)})^{m_{i,1}} \cdots (t_n^{(k)})^{m_{i,n}} \to 0$$

for all $1 \leq i \leq r$, as $k \to \infty$. WLOG suppose $b_1 \neq 0$. Then we can rewrite

$$m_{1,j} = -\frac{b_2}{b_1} m_{2,j} - \cdots - \frac{b_r}{b_1} m_{r,j}$$

for all j. Thus,

$$t_1^{-m_{1,1}} \cdots t_n^{-m_{1,n}} = (t_1^{m_{2,1}} \cdots t_1^{m_{2,n}})^{\frac{b_2}{b_1}} \cdots (t_1^{m_{r,1}} \cdots t_1^{m_{r,n}})^{\frac{b_r}{b_1}}.$$

Note that each $\frac{b_i}{b_1} \geq 0$. Now let t_i run over $t_i^{(k)}$. As $k \to \infty$, the RHS tends to zero, while the LHS is the reciprocal of a quantity tending to zero, contradiction!.

Hence $m_{i,j}$ satisfy the conditions of Lemma 5.1, so there exist integers c_j such that

$$\sum_{j=1}^{n} m_{i,j} c_j > 0 \quad \forall 1 \leq i \leq r.$$

Now, consider the 1-P.S. $\lambda(t) = \mathrm{diag}(t^{c_1}, \ldots, t^{c_n})$. Now,

$$\lambda(t) \cdot v = \sum_{i=1}^{r} t^{\sum_{j=1}^{n} m_{i,j} c_j} v_i,$$

which tends to 0 as $t \to 0$, as required. $\square$

6 References

1. Linear Algebraic Groups. Author: James E. Humphreys.
2. Complex Semisimple Lie Algebras notes, by Laboratoire d'analyse, géométrie et application. Link.
3. Proof of Cartan's Criterion for Solvability notes, by Alison Miller, Harvard. Link.
4. An Elementary Proof of the Hilbert-Mumford Criterion. Author: B. Sury (2000). Link.