Schur-Weyl Duality

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We use the Weyl integration formula to prove the Schur-Weyl duality and find irreducible representations of S_d and $U(N)$ simultaneously.

1 Schur Polynomials

Fix a positive integer N. We will write $\mathbf{z}$ for the tuple $(z_1, z_2, \dots z_N)$ of indeterminates, and we write $\boldsymbol{\delta} = (N-1, N-2, \dots, 1, 0)$. For any tuple $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$ of integers, we define

$$a_{\boldsymbol{\lambda}}(\mathbf{z}) = \sum_{\pi \in S_N} \operatorname{sgn}(\pi) \cdot \mathbf{z}^{\pi(\boldsymbol{\lambda})}$$

where sgn denotes sign of the permutation and $\mathbf{z}^{\boldsymbol{\lambda}} = z_1^{\lambda_1} \cdots z_N^{\lambda_N}$. This is an anti-symmetric Laurent polynomial with integral coefficients. In case all $\lambda_p \geq 0$, these are polynomials in $\mathbf{z}$. From direct expansion of the determinant, we can see that $a_{\boldsymbol{\lambda}}(\mathbf{z}) = \det[z_p^{\lambda_q}]_{1 \leq p,q \leq N}$.

Proposition 1. *As $\boldsymbol{\lambda}$ ranges over non-increasing N-tuples of integers, the $a_{\boldsymbol{\lambda}+\boldsymbol{\delta}}$ form an integral basis of anti-symmetric Laurent polynomials with integer coefficients.*

Proof. If some monomial $\mathbf{z}^{\boldsymbol{\mu}}$ appears, then by anti-symmetry $\mathbf{z}^{\pi(\boldsymbol{\mu})}$ appears as well with sign $\operatorname{sgn}(\pi)$ and the same coefficient, for any $\pi \in S_N$. Note that $\mu_i \neq \mu_j$ for $i \neq j$, otherwise by anti-symmetry the monomial will get cancelled $\implies$ $\boldsymbol{\mu}$ is strictly decreasing. Therefore we can write $\boldsymbol{\mu} = \boldsymbol{\lambda} + \boldsymbol{\delta}$ for some non-increasing $\boldsymbol{\lambda}$, and $a_{\boldsymbol{\lambda}+\boldsymbol{\delta}}$ occurs with the same coefficient. In this way we can write the Laurent polynomial as integer sum of the $a_{\boldsymbol{\lambda}+\boldsymbol{\delta}}$. □

Let $a_{\boldsymbol{\delta}}(\mathbf{z}) = \Delta(\mathbf{z})$. By the Vandermonde determinant expansion, we can see that

$$\Delta(\mathbf{z}) = \det[z_p^{N-q}]_{1 \leq p,q \leq N} = \prod_{1 \leq p < q \leq N} (z_p - z_q)$$

Proposition 2. *Multiplication by Δ establishes a bijection between symmetric and anti-symmetric Laurent polynomials with integer coefficients. Further, it is also a bijection between symmetric and anti-symmetric polynomials with integer coefficients.*

Proof. Multiplication by Δ is clearly injective. Conversely, suppose f is anti-symmetric integer (Laurent) polynomial. Then $f(\mathbf{z}) = 0$ when $z_p = z_q$, so $(z_p - z_q) \mid f$. Therefore by Gauss lemma, $f_1 = \frac{f(\mathbf{z})}{(z_p - z_q)}$ is an integer (Laurent) polynomial, and it is zero when $z_{p'} = z_{q'}$ for some other pair p', q' of indices. Therefore $\frac{f_1(\mathbf{z})}{(z_{p'} - z_{q'})}$ is again an integer (Laurent) polynomial, and we can keep doing this to get $\frac{f(\mathbf{z})}{\Delta(z)}$ is an integer (Laurent) polynomial, and it is clearly symmetric. □

We define the Schur functions as

$$s_{\boldsymbol\lambda}(\mathbf{z}) = \frac{a_{\boldsymbol\lambda+\boldsymbol\delta}(\mathbf{z})}{\Delta(\mathbf{z})}$$

These are symmetric functions since $a_{\boldsymbol\lambda}$ are anti-symmetric. Further, the previous two propositions imply that $s_{\boldsymbol\lambda}$ form an integral basis of symmetric Laurent polynomials with integer coefficients. More particularly, Schur function of a fixed degree span the space of symmetric Laurent polynomials of that degree. Also, if each $\lambda_p \geq 0$, then $s_{\boldsymbol\lambda}$ are actually polynomials, and they span the space of symmetric polynomials: such $\boldsymbol\lambda$ correspond to partitions of non-negative integers into at most N parts.

The main result is the following:

Theorem 1. *The Schur functions are precisely the irreducible characters of $U(N)$.*

The proof is done in two parts: First, showing that every irreducible character is a Schur function, and second, showing that all Schur functions are characters of some representation of $U(N)$.

2 Weyl Integration Formula

Note that every matrix in $U(N)$ is diagonalizable, so we can work with diagonal matrices $\mathrm{diag}[z_1, \ldots, z_N]$. We state the integration formula for $U(N)$ without proof:

Theorem 2 (Weyl Integration Formula for $U(N)$)**.** *For a class function f in $U(N)$,*

$$\int_{U(N)} f(u)du = \frac{1}{N!}\frac{1}{(2\pi)^N}\int_{[0,2\pi]^N} f(diag[z_1, z_2, \ldots, z_N]) \cdot |\Delta(\boldsymbol{z})|^2 \; d\boldsymbol\phi$$

where $d\boldsymbol\phi = d\phi_1 \cdots d\phi_N$ and $z_p = e^{i\phi_p}$.

Proposition 3. *Schur functions are orthonormal in the inner product defined by integration on $U(N)$.*

Proof. Equivalent to proving

$$\begin{aligned}\delta_{\boldsymbol\lambda,\boldsymbol\mu} &= \frac{1}{N!}\frac{1}{(2\pi)^N}\int_{[0,2\pi]^N} s_{\boldsymbol\lambda}(\mathbf{z}) \cdot \overline{s_{\boldsymbol\mu}(\mathbf{z})} \cdot |\Delta(\mathbf{z})|^2 \; d\boldsymbol\phi \\ &= \frac{1}{N!}\frac{1}{(2\pi)^N}\int_{[0,2\pi]^N} a_{\boldsymbol\lambda+\boldsymbol\delta}(\mathbf{z}) \cdot \overline{a_{\boldsymbol\mu+\boldsymbol\delta}(\mathbf{z})} \; d\boldsymbol\phi\end{aligned}$$

for non-increasing N-tuples $\boldsymbol\lambda, \boldsymbol\mu$ where δ is the Kronecker delta.

The integrand is a homogenous degree N polynomial in z_i and $\overline{z_i}$. However, integration of any non-constant monomial over $[0, 2\pi]^N$ is 0, and the constant 1 integrates to $(2\pi)^N$. Therefore we just need to find the constant term.

If $\boldsymbol\lambda \neq \boldsymbol\mu$, then no term from $a_{\boldsymbol\lambda+\boldsymbol\delta}(\mathbf{z})$ can cancel any term from $\overline{a_{\boldsymbol\mu+\boldsymbol\delta}(\mathbf{z})}$. Therefore the constant term is 0. If $\boldsymbol\lambda = \boldsymbol\mu$, then take any term, WLOG $\overline{z_1}^{\lambda_1+N-1} \cdots \overline{z_N}^{\lambda_N}$ in $\overline{a_{\boldsymbol\lambda+\boldsymbol\delta}(\mathbf{z})}$. This can only be cancelled by the term $z_1^{\lambda_1+N-1} \cdots z_N^{\lambda_N}$ in $a_{\boldsymbol\lambda+\boldsymbol\delta}(\mathbf{z})$, and they both have the same sign. Since there are $N!$ such terms, the constant term is $N!$, whcih cancels with the $\frac{1}{N!}$ outside, as required. $\square$

Corollary. *Every irreducible character of $U(N)$ is a Schur function.*

Proof. Since permutation of eigenvalues doesn't change the conjugacy class of a matrix, all characters of $U(N)$ must be symmetric in $\mathbf{z}$. Further, there is a group homomorphism $U(1)^N \to U(N)$ given by $\mathbf{z} \to \mathrm{diag}[\mathbf{z}]$, so every character of $U(N)$ induces a character of $U(1)^N$. Since every continuous character of $U(1)$ is a Laurent polynomial with integer coefficients, the same is true for $U(1)^N$, so it is true for $U(N)$ as well: Every continuous character of $U(N)$ is a symmetric Laurent polynomial with integer coefficients.

Let $\chi(\mathbf{z})$ be an irreducible character of $U(N)$. Since Schur functions form an integral basis for symmetric Laurent polynomials with integer coefficients, we can write $\chi = \sum b_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}}$ for some Schur functions $s_{\boldsymbol{\lambda}}$ and some integers $b_{\boldsymbol{\lambda}}$. Since χ is irreducible, $\langle \chi, \chi \rangle = 1 \implies \sum b_{\boldsymbol{\lambda}}^2 = 1$ because the Schur functions are orthonormal. Therefore there is only one $\boldsymbol{\lambda}$, and the coefficient $b_{\boldsymbol{\lambda}} = 1$, i.e., χ is a Schur function. $\square$

3 Symmetric Group

Recall that every permutation in S_d has a unique decomposition as a product of disjoint cycles, and that two permutations are conjugate if and only if they have the same cycle type, which is the collection of all cycle lengths, with multiplicities. Ordering the lengths decreasingly leads to a partition $\boldsymbol{\mu} = \mu_1 \geq \cdots \geq \mu_n > 0$ of d. Assume that the numbers $1, 2, \ldots, d$ occur $m_1, m_2, \ldots, m_d$ times, respectively in μ; that is, m_k is the number of k-cycles in our permutation. We shall also refer to the cycle type by the notation $(\mathbf{m})$. Thus, the conjugacy classes in S_d are labelled by $(\mathbf{m})$'s such that $\sum_{k=1}^{d} k \cdot m_k = d$.

Proposition 4. *The size of the conjugacy class with cycle type $(\boldsymbol{m})$ is*

$$\frac{d!}{\prod_k m_k! \cdot \prod_k k^{m_k}}$$

We can define the symmetric power sums for a cycle type $(\mathbf{m})$ as

$$p_{(\mathbf{m})}(\mathbf{z}) = \prod_k (z_1^{\mu_k} + \cdots + z_N^{\mu_k}) = \prod_{k=1}^{d} (z_1^k + \cdots + z_N^k)^{m_k}$$

Since degree of $p_{(\mathbf{m})}$ is d, we can write it as an integer linear combination of $s_{\boldsymbol{\lambda}}$ where $\boldsymbol{\lambda}$ is a partition of d into at most N parts. For any such $\boldsymbol{\lambda}$, we define the class function

$$w_{\boldsymbol{\lambda}}(\mathbf{m}) = \text{ the coefficient of } s_{\boldsymbol{\lambda}}(\mathbf{z}) \text{ in the expansion of } p_{(\mathbf{m})}(\mathbf{z})$$

Therefore $p_{(\mathbf{m})}(\mathbf{z}) = \sum_{\boldsymbol{\lambda}} w_{\boldsymbol{\lambda}}(\mathbf{m}) s_{\boldsymbol{\lambda}}(\mathbf{z})$, where the sum is over partitions of d with at most N parts. The main result is the following:

Theorem 3 (Frobenius Character Formula)**.** *The class functions $w_{\boldsymbol{\lambda}}$ are precisely the irreducible characters of S_d*

Again, we prove this in two parts: First, showing that the $w_{\boldsymbol{\lambda}}$ are orthonormal, and second, showing that every $w_{\boldsymbol{\lambda}}$ is the character of some representation of S_d. Since there are precisely as many $w_{\boldsymbol{\lambda}}$ as there are conjugacy classes in S_d (namely, equal to number of partitions of d), there can be no other irreducible characters of S_d, and we will be done.

Lemma 1 (Cauchy Identity)**.**

$$\sum_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}}(\boldsymbol{y}) s_{\boldsymbol{\lambda}}(\boldsymbol{z}) = \prod_{1 \le p,q \le N} \frac{1}{1 - y_p z_q}$$

where the sum is ranging over all partitions $\boldsymbol{\lambda}$ with at most N parts of all non-negative integers.

Proof. We use the following identity without proof:

$$\det[(1 - y_p z_q)^{-1}]_{1 \le p,q \le N} = \Delta(\mathbf{y}) \Delta(\mathbf{z}) \prod_{1 \le p,q \le N} (1 - y_p z_q)^{-1}$$

But, expanding the reciprocals as Taylor series, and using multilinearity of determinant, we get

$$\begin{aligned}
\det[(1 - y_p z_q)^{-1}]_{1 \le p,q \le N} &= \sum_{l_1,\ldots,l_N \ge 0} \det[(y_p z_q)^{l_q}]_{1 \le p,q \le N} \\
&= \sum_{l_1,\ldots,l_N \ge 0} \det[y_p^{l_q}]_{1 \le p,q \le N} \cdot \prod_{q=1}^{N} z_q^{l_q} \\
&= \sum_{l_1,\ldots,l_N \ge 0} a_{\mathbf{l}}(\mathbf{y}) \mathbf{z}^{\mathbf{l}} \\
&= \sum_{l_1 > l_2 > \cdots > l_N \ge 0} a_{\mathbf{l}}(\mathbf{y}) a_{\mathbf{l}}(\mathbf{z})
\end{aligned}$$

due to anti-symmetry of $a_{\mathbf{l}}$. Now multiplying by Δ proves the identity. □

Proposition 5. *The class functions $w_{\boldsymbol{\lambda}}$ ranging over all partitions of d with at most N-tuples are orthonormal in the usual inner product on S_d.*

Proof. This is equivalent to proving

$$\sum_{(\mathbf{m})} \frac{w_{\boldsymbol{\lambda}}(\mathbf{m}) w_{\boldsymbol{\mu}}(\mathbf{m})}{\prod_k m_k! \cdot \prod_k k^{m_k}} = \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}$$

for any partitions $\boldsymbol{\lambda}, \mu$ (with at most N parts), where δ is the Kronecker delta. Using linear independence of the Schur functions, this can be proved simultaneously for all d via the following identity:

$$\sum_{(\mathbf{m}),\boldsymbol{\lambda},\boldsymbol{\mu}} \frac{w_{\boldsymbol{\lambda}}(\mathbf{m}) w_{\boldsymbol{\mu}}(\mathbf{m})}{\prod_k m_k! \cdot \prod_k k^{m_k}} s_{\boldsymbol{\lambda}}(\mathbf{y}) s_{\boldsymbol{\mu}}(\mathbf{z}) = \sum_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}}(\mathbf{y}) s_{\boldsymbol{\lambda}}(\mathbf{z})$$

where the sum is ranging over all partitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ with at most N parts of all non-negative integers d. From definition of w, the LHS is:

$$\sum_{(\mathbf{m})} \frac{p_{(\mathbf{m})}(\mathbf{y}) p_{(\mathbf{m})}(\mathbf{z})}{\prod_k m_k! \cdot \prod_k k^{m_k}}$$

Separating $p_{(\mathbf{m})}$ into products, the LHS is:

$$\sum_{(\mathbf{m})}\prod_{k>0}\frac{(y_1^k+\cdots+y_N^k)^{m_k}(z_1^k+\cdots+z_N^k)^{m_k}}{m_k!\cdot k^{m_k}} = \sum_{(\mathbf{m})}\prod_{k>0}\frac{\left(\sum_{1\le p,q\le N}(y_p^k z_q^k/k)^{m_k}\right)}{m_k!}$$
$$= \prod_{k>0}\exp\left(\sum_{1\le p,q\le N}\frac{y_p^k z_q^k}{k}\right)$$
$$= \exp\left(\sum_{1\le p,q\le N}\sum_{k>0}\frac{y_p^k z_q^k}{k}\right)$$
$$= \exp\left(-\sum_{1\le p,q\le N}\ln(1-y_p z_q)\right)$$
$$= \prod_{1\le p,q\le N}\frac{1}{1-y_p z_q}$$

Now, using the Cauchy identity, we are done. ◻

4 Schur-Weyl Duality

Note that there is an action of S_d on $(\mathbb{C}^N)^{\otimes d}$, namely permuting the $\mathbb{C}^N$s, that commutes with the $U(N)$ action. Thus we have a $(\mathbb{C}^N)^{\otimes d}$ as a natural $S_d \times U(N)$ representation. We prove the following powerful result:

Proposition 6. *The character of $(\mathbb{C}^N)^{\otimes d}$, as a representation of $S_d \times U(N)$, is $(\pi, diag[z_1,\ldots,z_N]) \to p_{(\boldsymbol{m})}(\boldsymbol{z})$, where π has cycle type $(\boldsymbol{m})$.*

Proof. Let $e_1, e_2, \ldots e_N$ be the standard basis of $\mathbb{C}^N$. Then a basis of $(\mathbb{C}^N)^{\otimes d}$ is $e_{i_1}\otimes\cdots\otimes e_{i_d}$ for all d-tuples $\mathbf{i} = (i_1,\ldots,i_d)$ of positive integers not exceeding N. A group element $\pi \times \mathrm{diag}[z_1,\ldots,z_N]$ (where π has cycle type $(\mathbf{m})$) maps the basis vector to $z_{i_1}\cdots z_{i_d} e_{i_{\pi(1)}}\otimes\cdots\otimes e_{i_{\pi(d)}}$. This contributes to the trace only if $\mathbf{i}$ is constant on each cycle of π.

The diagonal contribution for such an $\mathbf{i}$ is product of terms of the form z_i^k for each cycle of length k. Therefore the total trace would be

$$\prod_{k>0}(z_1^k+\cdots+z_N^k)^{m_k} = p_{(\mathbf{m})}(\mathbf{z})$$

as required. ◻

Corollary. *The $w_{\boldsymbol{\lambda}}$ for partitions $\boldsymbol{\lambda}$ of d are precisely the irreducible characters of S_d.*

Corollary. *The $s_{\boldsymbol{\lambda}}$ for non-increasing N-tuples of integers $\boldsymbol{\lambda}$ are precisely the irreducible characters of $U(N)$.*

Corollary (Schur-Weyl Duality). *Under the action of $S_d \times U(N)$, $(\mathbb{C}^N)^{\otimes d}$ decomposes as a direct sum of products*

$$\bigoplus_{\boldsymbol{\lambda}} S^{\boldsymbol{\lambda}} \otimes V_{\boldsymbol{\lambda}}$$

where the sum ranges over all partitions $\boldsymbol{\lambda}$ of d into at most N parts. Here $V_{\boldsymbol{\lambda}}$ is the irreducible representation of $U(N)$ corresponding to character $s_{\boldsymbol{\lambda}}$, and $S^{\boldsymbol{\lambda}}$ is the irreducible representation of S_d corresponding to character $w_{\boldsymbol{\lambda}}$.

Proof. We prove the above three corollaries simultaneously. Writing $(\mathbb{C}^N)^{\otimes d}$ as a sum of irreducible of the form $V_i \times W_i$ for irreducibles V_i of S_d and W_i of $U(N)$, recalling that every character of $U(N)$ is a Schur function, and collecting together the terms corresponding to irreducibles of $U(N)$, we get that character of $(\mathbb{C}^N)^{\otimes d}$ has the form $\sum_{\boldsymbol{\lambda}} \chi_{\boldsymbol{\lambda}}(\mathbf{m}) s_{\boldsymbol{\lambda}}(\mathbf{z})$ for some characters $\chi_{\boldsymbol{\lambda}}$ of S_d, where the sum is over some non-increasing N-tuples $\boldsymbol{\lambda}$. However, comparing that with the expression $p_{(\mathbf{m})}(\mathbf{z}) = \sum_{\boldsymbol{\lambda}} w_{\boldsymbol{\lambda}}(\mathbf{m}) s_{\boldsymbol{\lambda}}(\mathbf{z})$ of the trace, and using linear independence of Schur functions, we see that the $\boldsymbol{\lambda}$ that appear in the first sum are precisely partitions of d into at most N parts, and the corresponding characters of S_d are precisely $w_{\boldsymbol{\lambda}}$, which are irreducible because they are orthonormal. This proves the Schur-Weyl duality.

To prove the first corollary, just take $N > d$ in the above paragraph, so that every partition of d occurs in the sum, which implies that every $w_{\boldsymbol{\lambda}}$ is a character of S_d, as required.

Varying over arbitrary d, the first paragraph proves that $s_{\boldsymbol{\lambda}}$ for non-increasing N-tuples of non-negative integers, i.e., all Schur polynomials, are indeed irreducible characters of $U(N)$. Now, note that $\det^k$ is a one-dimensional representation of $U(N)$ for any integer k, and its character is $z_1^k \cdots z_N^k$. Therefore, any function of the form $(z_1 \cdots z_N)^k s_{\boldsymbol{\lambda}}(\mathbf{z})$ for integer k and Schur polynomial $s_{\boldsymbol{\lambda}}$ is a character of $U(N)$ (just take product with the one-dimensional representation), and this covers all Schur functions, as required. □

5 Dimensions of Irreducible Representations of S_d

For any partition $\boldsymbol{\lambda} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ of d, we can represent it as a Young diagram, namely a left-justified array of square cells with k rows of length $\lambda_1, \lambda_2, \ldots, \lambda_k$. For a cell in position (i, j), i.e. in the i^{th} row and j^{th} column, the hook $H_{\boldsymbol{\lambda}}(i, j)$ is the set of cells (a, b) such that $a = i$ and $b \geq j$ or $b = j$ and $a \geq i$. The number of cells in $H_{\boldsymbol{\lambda}}(i, j)$ is called the hook length $h_{\boldsymbol{\lambda}}(i, j)$ of (i, j). We state the following theorem without proof:

Theorem 4 (Hook Length Formula). *The number of ways to fill the Young diagram of d with numbers in $\{1, 2, \ldots, d\}$, without repetitions, such that the rows and columns are strictly increasing, is*

$$f^{\boldsymbol{\lambda}} = \frac{d!}{\prod h_{\boldsymbol{\lambda}}(i, j)}$$

where the product is over all cells (i, j) in the Young diagram.

$f^{\boldsymbol{\lambda}}$ is also related to dimensions of irreducible representations of S_d:

Theorem 5. *Let $\rho_{\boldsymbol{\lambda}}$ denote the representation of S_d with character $w_{\boldsymbol{\lambda}}$. Then* $\dim(\rho_{\boldsymbol{\lambda}}) = f^{\boldsymbol{\lambda}}$.

Proof. Note that $\dim(\rho_{\boldsymbol{\lambda}}) = w_{\boldsymbol{\lambda}}(1)$, where (1) denotes the identity permutation. We use the Weyl integration formula. Note that $\langle p_{(1)}, s_{\boldsymbol{\lambda}} \rangle = w_{\boldsymbol{\lambda}}(1)$ in the usual inner product on $U(N)$ because the

$s_{\boldsymbol{\lambda}}$ are orthonormal (as long as N is greater than number of parts of $\boldsymbol{\lambda}$). Putting this in the Weyl integration formula, we get:

$$\begin{aligned} w_{\boldsymbol{\lambda}}(1) &= \frac{1}{N!}\frac{1}{(2\pi)^N}\int_{[0,2\pi]^N}(z_1+\cdots+z_N)^d\cdot\overline{s_{\boldsymbol{\lambda}}(\mathbf{z})}\cdot|\Delta(\mathbf{z})|^2\,d\boldsymbol{\phi} \\ &= \frac{1}{N!}\frac{1}{(2\pi)^N}\int_{[0,2\pi]^N}(z_1+\cdots+z_N)^d\cdot\Delta(\mathbf{z})\cdot\overline{a_{\boldsymbol{\lambda}+\boldsymbol{\delta}}(\mathbf{z})}\,d\boldsymbol{\phi} \end{aligned}$$

The integrand is a homogenous degree N polynomial in z_i and $\overline{z_i}$. However, integration of any non-constant monomial over $[0,2\pi]^N$ is 0, and the constant 1 integrates to $(2\pi)^N$. Therefore we just need to find the constant term.

Let $\mu_i=\lambda_i+N-i$. Pick a term, say $\overline{z_1}^{\mu_1}\cdots\overline{z_N}^{\mu_N}$ in $\overline{a_{\boldsymbol{\lambda}+\boldsymbol{\delta}}(\mathbf{z})}$. This will be cancelled by a product of the form $\operatorname{sgn}(\pi)z_1^{\pi(1)}\cdots z_N^{\pi(N)}$ from $\Delta(\mathbf{z})$ and $\frac{d!}{\prod_{i=1}^N(\mu_i-\pi(i)+1)!}z_1^{\mu_1-\pi(1)+1}\cdots z_N^{\mu_N-\pi(N)+1}$ from $(z_1+\cdots+z_N)^d$. Therefore the constant term is

$$\sum_{\pi\in S_d}\operatorname{sgn}(\pi)\frac{d!}{\prod_{i=1}^N(\mu_i-\pi(i)+1)!}=\frac{d!}{\prod_{i=1}^N\mu_i!}\sum_{\pi\in S_d}\operatorname{sgn}(\pi)\left(\prod_{i=1}^N\mu_i(\mu_i-1)\cdots(\mu_i-\pi(i)+2)\right)$$

The sum can be thought of as a determinant:

$$\det\left[\prod_{k=0}^{j-2}(\mu_i-k)\right]_{1\le i,j\le N}$$

By doing column operations from left to right, it transforms into the Vandermonde determinant $\det[\mu_i^{j-1}]=\prod_{1\le i<j\le N}(\mu_j-\mu_i)$. Therefore the constant term coming from this specific term in $\overline{a_{\boldsymbol{\lambda}+\boldsymbol{\delta}}(\mathbf{z})}$ is

$$\frac{d!}{\prod_{i=1}^N\mu_i!}\prod_{1\le i<j\le N}(\mu_j-\mu_i)$$

Now, we can see that

$$\mu_i!=\prod_{j\le\lambda_i}h_\lambda(i,j)\prod_{1\le j<i}(\mu_i-\mu_j)$$

because the "gaps" in $\mu_i!$ are "filled" by the $(\mu_j-\mu_i)$. Since there are $N!$ terms in $\overline{a_{\boldsymbol{\lambda}+\boldsymbol{\delta}}(\mathbf{z})}$, that cancels with the $\frac{1}{N!}$, which proves the theorem. $\square$