Morse Theory and its Applications

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What is Morse Theory?

Morse theory is the study of topology of differentiable manifolds from the critical points of real-valued smooth functions on them. More specifically, if we have a smooth manifold M and a "nice" function $f : M \to \mathbb{R}$, we can gauge the homotopy type of M from just the indices of the critical points of f. It is a way of relating the local behaviour of f to the global structure of M.

Note: "Smooth" would always mean C^∞.

Definition

Let M be a smooth manifold, $f : M \to \mathbb{R}$ be a smooth function. Let $p \in M$, and let U is a neighborhood of p with a local coordinate system $(x_1, x_2, \dots x_n)$.

- p is called a critical point of f if $\frac{\partial f}{\partial x_1}(p) = \frac{\partial f}{\partial x_2}(p) = \cdots = \frac{\partial f}{\partial x_n}(p) = 0$. Equivalently, the induced map $df : TM_p \to T\mathbb{R}_{f(p)}$ on tangent spaces is zero.
- The value $f(p)$ is called the critical value of f at p.
- A critical point p is called non-degenerate if the Hessian matrix at p: $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is non-singular.
- The index of f at a non-degenerate critical point is the maximal dimension of a subspace of $\mathbb{R}^n$ on which the Hessian of f at p is negative-definite.

Behaviour near critical points

We show that the behaviour of f near p is completely determined by the index of f at p.

Lemma

Let f be a smooth function in a convex neighborhood V of 0 in $\mathbb{R}^n$ with $f(0) = 0$. Then there exist smooth functions g_i on V such that

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g(x_1, \ldots, x_n)$$

and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$ for each i.

Proof.

$$f(x_1, \ldots, x_n) = \int_0^1 \frac{df(tx_1, \ldots, tx_n)}{dt} dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) \cdot x_i \, dt$$

So we can take $g_i(x_1, \ldots, x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) dt$ $\square$

Behaviour near critical points

Lemma (Morse Lemma)

Let p be a non-degenerate critical point of f with index k. Then there exists a local coordinate system $(y_1, \dots y_n)$ in a neighborhood U of p such that $y_i(p) = 0$ for all i and

$$f = f(p) - y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots y_n^2$$

holds throughout U.

Proof.

We first show that if f has this form, then k is indeed the index of f at p. If f has this form, then Hessian H of f at p is a diagonal matrix with k -2s and $(n-k)$ 2s. Therefore H has a negative-definite subspace of dimension k, and a positive-definite subspace of dimension $n-k$, which proves that the index is k. □

Proof of Morse Lemma (continued)

Proof.

We now prove the existence of such local coordinates. By suitable shifting we can assume p is the origin of $\mathbb{R}^n$ in the local coordinates and $f(p) = f(0) = 0$. Applying the previous lemma twice we get

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g(x_1, \ldots, x_n)$$
$$g_i(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_j h_{ij}(x_1, \ldots, x_n)$$

for some smooth h_{ij} because $g_i(0) = \frac{\partial f}{\partial x_i} = 0$. Hence $f = \sum x_i x_j h_{ij}$. We can assume $h_{ij} = h_{ji}$, by replacing them both by $\frac{h_{ij}+hji}{2}$. Moreover the matrix $(h_{ij}(0))$ is equal to $\frac{1}{2}H$.

Now we want to "diagonalize" the above expression for f. □

Proof of Morse Lemma (continued)

Proof.

We proceed by induction: Suppose there are some local coordinates $u_1, \dots u_n$ in some neighborhood U_1 of p such that

$$f = \pm u_1^2 + \cdots \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_j G_{ij}$$

where G_{ij} are smooth functions with $G_{ij} = G_{ji}$. By a suitable linear transformation in the last $n - r + 1$ coordinates we can assume $G_{rr}(0) \neq 0$. Let g be the square root of $|G_{rr}|$ in some neighborhood $U_2 \subset U_1$. Define new smooth functions v_i in U_2 as $v_i = u_i$ for $i \neq r$, and

$$v_r(u_1, \dots u_n) = g(u_1, \dots u_n) \left(u_r + \sum_{i > r} u_i \frac{G_{ir}}{G_{rr}} \right)$$

Note that $\frac{\partial v_r}{\partial u_r}(0) = g(0) \neq 0$. □

Proof of Morse Lemma (continued)

Proof.

Hence the determinant of the Jacobian of v_i is non-zero, and so they form a local coordinate system in some small neighborhood U_3 of p (by inverse function theorem). We can also check that f can be expressed as:

$$f = \sum_{i \leq r} \pm v_i^2 + \sum_{i,j > r} v_i v_j G'_{ij}$$

for some smooth G'_{ij}. Thus proceeding by induction we can diagonalize the expression for f, and we are done. □

Corollary

Non-degenerate critical points are isolated.

Homotopy Type using Critical Values

Let $f : M \to R$ be a smooth function, and let $M_a = f^{-1}(-\infty, a]$ for all $a \in \mathbb{R}$. Note that, if a is not a critical value, then using implicit function theorem, M_a is a smooth manifold with boundary.

Theorem (Fundamental Theorems)

- *Suppose $a \leq b$ are real numbers such that $f^{-1}[a, b]$ is compact and contains no critical points of f. Then M_a is diffeomorphic to M_b. Furthermore, M_a is a deformation retract of M_b.*
- *Let p be a non-degenerate critical point of f with index k. Setting $f(p) = c$, suppose $f^{-1}[c - \varepsilon, c + \varepsilon]$ is compact and contains no critical points of f other than p, for some $\varepsilon > 0$. Then for all sufficiently small ε, $M_{c+\varepsilon}$ has the homotopy type of $M_{c-\varepsilon}$ with a k-cell attached.*
- *If f has no degenerate critical points, and each M_a is compact, then M has the homotopy type of a CW complex with a k-cell for every index k critical point of f. (Such functions are called Morse functions).*

Existence of Morse Functions

Theorem

Let M be a smooth manifold embedded in $\mathbb{R}^n$. Then for almost all $p \in \mathbb{R}^n$, the distance function $L_p : M \to \mathbb{R}$ given by $L_p(q) = ||p - q||^2$ has no degenerate critical points on M.

Theorem

Any bounded smooth function $f : M \to \mathbb{R}$ can be uniformly approximated by smooth functions with no degenerate critical points.

Proof.

Choose an embedding $h : M \to \mathbb{R}^n$ such that the first projection is the function f. Let $c > 0$ be large and for some small ε_i choose $p = (-c + \varepsilon_1, \varepsilon_2, \ldots \varepsilon_n)$ such that L_p doesn't have degenerate critical points. Then $g(x) = \frac{L_p(x) - c^2}{2c}$ uniformly approximates f. □

Example: Torus

Applications

Theorem

If M is a compact manifold and f is a smooth function on M with exactly two critical points, both of which are non-degenerate, then M is homeomorphic to a sphere.

Proof.

Since M is compact, f is bounded, and the two critical points must correspond to the absolute minimum and absolute maximum of f; say they are 0 and 1 respectively. The indices of critical points corresponding to the absolute minimum and maximum are 0 and n respectively. Thus by Morse lemma, for small enough $\varepsilon > 0$, the sets $M_\varepsilon = f^{-1}[0, \varepsilon]$ and $f^{-1}[1-\varepsilon, 1]$ are closed n-cells. But, M_ε is homeomorphic (in fact diffeomorphic) to $M_{1-\varepsilon}$. Thus M is the union of two n-cells, $M_{1-\varepsilon}$ and $f^{-1}[1-\varepsilon, 1]$, attached along their boundary. Thus M is homeomorphic to S^n. $\square$

High-level Applications

Theorem (h-Cobordism Theorem)

Let W be a compact smooth manifold having two boundary components V and V' which are both deformation retracts of W. If V, V' are both simply connected and have dimension ≥ 5, then W is diffeomorphic to $V \times [0, 1]$.

Theorem (Generalized Poincare Conjecture for $n \geq 5$)

If a smooth closed manifold M is homotopy equivalent to S^n for $n \geq 5$, then M is homeomorphic to S^n.