Winter of Puzzles - Logic

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December 2021

1 Introduction

Ehrenfeucht-Fraisse games are a way to differentiate between structures that follow first order logic. They are two player games played between Spoiler Samson and Duplicator Delilah.

Suppose we have two structures $\mathcal{A}$ and $\mathcal{B}$, along with two assignments α_0 and β_0 of the two structures respectively of same finite set of variables (say of size t), and we want to know whether there exists a first order formula ϕ with t free variables for which $(\mathcal{A}, \alpha_0) \models \phi$ and $(\mathcal{B}, \beta_0) \not\models \phi$.

The EF game is played by placing pebbles on the universes A, B of the two structures. Initially t pebbles each are placed on the elements corresponding to initial assignments α_0, β_0. Then Samson and Delilah play alternately, with Samson placing a pebble on any element of A or B, and Delilah placing a pebble on any element in the other universe. At the end of each round, we get finite subsets of A and B (of the same size) corresponding to where the pebbles were placed, and a natural map between them (pebbles placed in the same round are mapped to each other). Delilah wins this round if all the relations and functions are preserved by this map, and Samson wins otherwise.

We will see that $(\mathcal{A}, \alpha_0)$ and $(\mathcal{B}, \beta_0)$ can be distinguished by a first order formula if and only if Samson has a winning strategy in this game.

2 Formalising the Games

Let k, m be non-negative integers, and let $\mathcal{G}^k_m(\mathcal{A}, \alpha_0, \mathcal{B}, \beta_0)$ be the game played with k total pebbles per person and m rounds.

Initially both players are given k pebbles numbered 1 through k. Before the game starts pebbles are placed on elements of $\mathrm{dom}(\alpha_0)/\mathrm{const}(\tau)$ and correspondingly numbered pebbles are placed on elements of $\mathrm{dom}(\beta_0)/\mathrm{const}(\tau)$ (e.g., if x_r is a free variable, then pebble of same number is placed on both $\alpha_0(x_r)$ and $\beta_0(x_r)$).

In every move, Samson places one of the pebbles, say number i, on some element in one of the universes of the structures, i.e., at A or B; Delilah responds by placing her pebble numbered i on some element in the other universe.

The situation after move r is given by two functions α_r, β_r to A, B respectively, with same domain D, which is a subset of $\mathrm{const}(\tau) \cup \{x_1, x_2, \dots x_k\}$, where x_i are all the variables that can occur in a formula (including free variables). Here we assume that $\mathrm{const}(\tau) \subseteq D$, and for any constant $c \in \tau$, we have $\alpha_r(c) = c^{\mathcal{A}}$ and $\beta_r(c) = c^{\mathcal{B}}$.

α_r and β_r evolve as follows: Initially it's just assignments to free variables and constants. If at move r pebble i is played on $a \in A$ and $b \in B$, then $\mathrm{dom}(\alpha_r) = \mathrm{dom}(\alpha_{r-1}) \cup \{x_i\}$ and $\mathrm{dom}(\beta_r) = \mathrm{dom}(\beta_{r-1}) \cup \{x_i\}$, with $\alpha_r(x_i) = a$, $\beta_r(x_i) = b$, and agreement with α_{r-1} and β_{r-1} everywhere else.

After r moves have been played, we get a relation $\beta_r \circ \alpha_r^{-1}$ from the range of α_r to range of β_r. This can be extended in the natural way to a map between the induced substructures $\langle \mathrm{range}(\alpha_r) \rangle^{\mathcal{A}}$ to $\langle \mathrm{range}(\beta_r) \rangle^{\mathcal{B}}$. Delilah wins the round if this map is an isomorphism. She wins the game if she wins every round. Otherwise, Samson wins.

Note that that this is a finite, perfect information game, so for fixed structures and assignments, one of the two players has a guaranteed winning strategy. We write $(\mathcal{A}, \alpha_0) \sim^k_m (\mathcal{B}, \beta_0)$ if Delilah has a winning strategy for $\mathcal{G}^k_m(\mathcal{A}, \alpha_0, \mathcal{B}, \beta_0)$. We also write $(\mathcal{A}, \alpha_0) \sim_m (\mathcal{B}, \beta_0)$ if the above holds for all k, and $(\mathcal{A}, \alpha_0) \sim^k (\mathcal{B}, \beta_0)$ if it holds for all m.

3 Examples of a Game

Consider the two structures $\mathcal{N}$ and $\mathcal{Z}$: the natural numbers and integers, over the signature $\tau = \langle <^2 \rangle$. Suppose Samson plays 0 in $\mathcal{N}$ on his first turn, and Delilah responds with some $k \in \mathbb{Z}$. Then Samson can play $k-1$ in $\mathcal{Z}$, and he wins on the second move because $k-1 < k$, while no number is less than 0 in $\mathbb{N}$.

Now consider the same signature but the structures $\mathcal{Z}$ and $\mathcal{Q}$: integers and rational numbers. Samson plays 0 in $\mathcal{Z}$ in his first turn, and 1 in $\mathcal{Z}$ in his second turn. Suppose Delilah plays a, b respectively in $\mathcal{Q}$. Then Samson wins on his third round by playing a number between a and b in $\mathcal{Q}$, as there is no integer between 0 and 1.

Finally consider the same signature but the structures $\mathcal{Q}$ and $\mathcal{R}$: rationals and reals. This time Delilah can always win because both of these sets are dense in themselves (and each other) and both are unbounded, so no matter what Samson plays Delilah can respond with a "close enough" number. We will see that this implies rational numbers and real numbers cannot be distinguished by a first-order sentence using only ordering properties.

4 Exercise 6.5

Prove that $\sim_m^k$ is an equivalence relation.

Proof. Symmetry is obvious because definition of the relation is symmetric w.r.t. the structures. To prove that $(\mathcal{A}, \alpha_0) \sim_m^k (\mathcal{A}, \alpha_0)$, note that Delilah can play a mirroring strategy, i.e., if Samson plays a, she also plays a but in the other structure. After each round the induces substructures stay identical.

Now assume $(\mathcal{A}, \alpha_0) \sim_m^k (\mathcal{B}, \beta_0)$ and $(\mathcal{B}, \beta_0) \sim_m^k (\mathcal{C}, \gamma_0)$. For the game $\mathcal{G}_m^k(\mathcal{A}, \alpha_0, \mathcal{C}, \gamma_0)$, Delilah keeps $(\mathcal{B}, \beta_0)$ in her mind. If Samson plays $a \in A$, then Delilah in her mind plays corresponding winning $b \in B$, and corresponding to that plays $c \in C$. Thus after every round there is an isomorphism between induced substructures of $\mathcal{A}$ and $\mathcal{B}$, and one between those of $\mathcal{B}$ and $\mathcal{C}$. Composing these two, we get the isomorphism between induced substructures of $\mathcal{A}$ and $\mathcal{C}$. Thus Delilah wins every round, and so $(\mathcal{A}, \alpha_0) \sim_m^k (\mathcal{C}, \gamma_0)$, as required. $\square$

5 The Main Result

Let τ be any signature, and let $\mathcal{L}(\tau)$ be the language over τ. We write $\mathcal{L}^k(\tau)$ to be the language in which only $x_1, x_2, \ldots x_k$ appear as variables, and we write $\mathcal{L}_m(\tau)$ to be the set of formulas with quantifier depth at most m. Also let $\mathcal{L}_k^m(\tau)$ to be the language with both. For structures $(\mathcal{A}, \alpha_0)$ and $(\mathcal{B}, \beta_0)$ along with their assignments, we write $(\mathcal{A}, \alpha_0) \equiv_m^k (\mathcal{B}, \beta_0)$ if for any $\phi \in \mathcal{L}_m^k(\tau)$ with free variables from the set $\mathrm{dom}(\alpha_0)/\mathrm{const}(\tau)$, we have $(\mathcal{A}, \alpha_0) \models \phi$ iff $(\mathcal{B}, \beta_0) \models \phi$. We also write $(\mathcal{A}, \alpha_0) \equiv_m (\mathcal{B}, \beta_0)$ if the above holds for all k, and $(\mathcal{A}, \alpha_0) \equiv^k (\mathcal{B}, \beta_0)$ if it holds for all m.

Theorem 1 (The Fundamental Theorem)**.** *Let $\mathcal{A}, \mathcal{B}$ be structures on the same finite, relational vocabulary, and let α_0, β_0 be some initial assignments. Then the following are equivalent:*

1. $(\mathcal{A}, \alpha_0) \sim_m^k (\mathcal{B}, \beta_0)$
2. $(\mathcal{A}, \alpha_0) \equiv_m^k (\mathcal{B}, \beta_0)$

Proof. We use induction on m to prove the equivalence. For $m = 0$, there are no moves played, so Delilah wins iff $\beta_0 \circ \alpha_0^{-1}$ is an isomorphism, that is iff $(\mathcal{A}, \alpha_0)$ and $(\mathcal{B}, \beta_0)$ satisfy the same quantifier-free formulas, so the equivalence holds.

Now assume the equivalence holds for some m. Assume $(\mathcal{A}, \alpha_0)$ and $(\mathcal{B}, \beta_0)$ disagree on some formula ϕ with quantifier depth $m+1$, and take the smallest such formula (we write the formula using only $\vee$, $\neg$ and $\exists$). If the leading connective in ϕ is $\vee$ or $\neq$, we can obtain a smaller formula over which the structures disagree. Else it has the form $\exists x\ \psi(x)$ where $\psi \in \mathcal{L}_m^k(\tau)$. Assume $(\mathcal{A}, \alpha_0)$ satisfies this, but $(\mathcal{B}, \beta_0)$ does not. Then Samson places a pebble on a witness for $\exists x\ \psi(x)$. Delilah places the pebble somewhere in B. Then we get a new assignment $(\mathcal{A}, \alpha_1)$ and $(\mathcal{B}, \beta_1)$ with the pebbles just placed, but they disagree on ψ, so Samson wins by induction hypothesis.

Now assume they agree on every formula with quantifier depth $m+1$. Say Samson plays some $a \in A$ with pebble i to get a new assignment $(\mathcal{A}, \alpha_1)$. Let S be the set of all formulas of the form $\exists x\ \psi(x)$ where $\psi \in \mathcal{L}^k_m(\tau)$ such that a is a witness for it. There are only finitely many of these by Exercise 6.11.1. Let ϕ be the conjunction of all such ψ, and consider formula of the form $\exists x_i \phi(x_i)$. Since $(\mathcal{A}, \alpha_0)$ satisfies this, so does $(\mathcal{B}, \beta_0)$. Therefore there exists a common witness for all such ψ, and Delilah places pebble i on such a $b \in B$ to get $(\mathcal{B}, \beta_1)$. These both agree on every formula in $\mathcal{L}^k_m(\tau)$, so by induction hypothesis Delilah wins.

Hence, we are done by induction. $\square$

6 Exercise 6.11

1. For any finite relational vocabulary τ, prove that there are only finitely many inequivalent formulas (with a bounded number of free variables) in $\mathcal{L}_r(\tau)$.

   Proof. Note that any formula can be written using only $\exists$, $\neg$ and $\wedge$, so we will only use these. We use induction on r to prove this.

   When $r = 0$, there are no quantifiers, so formulas in $\mathcal{L}_r(\tau)$ can be thought of as propositional logic formulas with atomic propositions of the form $R(x), R(c), x = c$ and $x = y$, where R is a relation, c is a constant, and x, y are free variables. So there are finitely many atomic propositions (since number of free variables is bounded by $\mathrm{dom}(\alpha_0)$), and there are finitely many truth assignments that can be done to them; the truth values corresponding to each of these assignments determines a formula up to equivalence. Thus there are finitely many inequivalent formulas in $\mathcal{L}_0(\tau)$ (in fact there are at most 2^{2^m} where m is number of atomic propositions).

   Now assume that the statement is true for all $r < k$. Then, by parsing, any formula in $\mathcal{L}_k(\tau)$ can be written as a PL formula where the atomic propositions are of the form $\exists x Q(x)$, where Q has quantifier rank at most $k-1$ (and has one more free variable). Since there are only finitely many such Q, there are finitely many truth assignments that can be done to them, the effects of which determine a formula up to equivalence. Therefore there are finitely many inequivalent formulas in $\mathcal{L}_k(\tau)$, and we are done by induction. $\square$

2. Let τ be a finite vocabulary, and let $\Gamma \subseteq \mathcal{L}(\tau)$ be a first-order theory. Suppose for any model $\mathcal{A}$ of Γ and any finite subset $S \subseteq A$, the induced substructure $\langle S \rangle^{\mathcal{A}}$ is also finite. Then thee are only finitely many inequivalent sentences in $\mathcal{L}_r(\tau)$ admitted by Γ.

   Proof. I couldn't find a rigorous proof of this. But I think the basic idea is that, since induced substructures of finite structures are finite, large nestings of functions are equivalent to some smaller nesting of function w.r.t. Γ. Hence we can proceed as in the above problem by induction on r, and in base case we only get finitely many inequivalent sentences. The rest of the argument is the same as before. $\square$

3. Give a counterexample to the main result when τ consists only of infinitely many unary relation symbols.

   Proof. Take $A = \mathbb{N}$ and $B = \mathbb{N} \cup \{0\}$ (we aren't French, so $0 \notin \mathbb{N}$). Take α_0 and β_0 to both be empty. Interpret the list of unary relations $R_1, R_2, \dots$ in both structures in the same way: $R_i^{\mathcal{A}}(x)$ and $R_i^{\mathcal{B}}(x)$ both hold iff $x = i$ for all $i \in \mathbb{N}$.

   We first show that $\mathcal{A}$ and $\mathcal{B}$ cannot be distinguished by an FO formula. Let ϕ be an FO formula. Then note that there exists an N such that ϕ does not contain any relations of the form R_k for $k > N$. Thus 0 and any $k > N$ are "practically equivalent" in the sense that 0 is a witness for a formula of the form $\exists x Q(x)$ iff any $k > N$ is. Also if $\forall x Q(x)$ is satisfied by $\mathcal{A}$, then any $k > N$ satisfies $Q(x)$, so 0 must satisfy as well; this implies $\mathcal{B} \models \forall x Q(x)$ (the converse is trivial since $A \subset B$). Viewing ϕ as a PL formula with atomic propositions of the above kind, these two statements imply that $\mathcal{A} \models \phi$ iff $\mathcal{B} \models \phi$.

Now we show that Samson has a winning strategy. Indeed, he can just play 0 in $\mathcal{B}$ at the beginning. Then if Delilah plays some n in $\mathcal{A}$, $R_n^{\mathcal{A}}(n)$ is true but $R_n^{\mathcal{B}}(0)$ is false, so Delilah loses. $\square$

4. Give a counterexample to the main result when $\tau = \langle R^1, f^1 \rangle$ consists of one unary relation symbol and one unary function symbol.

   Proof. We will in fact not use the unary relation to get a contradiction, so for example we can just set it to true always. Take $A = 2\mathbb{Z}$ and $B = \mathbb{Z}$. The unary function f is interpreted on both structures in the same way: $f(x) = x + 2$.

   We first show that $\mathcal{A}$ and $\mathcal{B}$ cannot be distinguished by an FO formula. Assume the contrary, that they can be distinguished by some ϕ. Note that if we take a new binary relation F defines as $F(x, y)$ iff $y = f(x)$, we can rewrite ϕ in terms of purely relational vocabulary (namely F) by adding the uniqueness clauses (i.e., for every x there is a unique y with $F(x, y)$). Thus we can use our main result. Assume ϕ has k variables and quantifier depth m. Then we give a brief winning strategy for Delilah in the k pebbles m moves game: Any time Samson places a pebble on an odd number in $\mathcal{B}$, Delilah responds by placing a pebble at some isolated point in $\mathcal{A}$ (unless the pebble is already close to some other odd number in $\mathcal{B}$, in which case she places it near the previously placed pebble). Even numbers in $\mathcal{A}$ get mapped to the corresponding numbers in $\mathcal{B}$ (unless an even number is close to some pebble corresponding to an odd number, in which case it corresponds to the corresponding odd number). Then even numbers in $\mathcal{B}$ whose counterparts in $\mathcal{A}$ are close to pebble corresponding to odd numbers are again corresponded to some isolated point in $\mathcal{A}$, and so on. Here "close" and "isolated" depend on m (e.g. we can take "close" to mean within $2020^m m^{2020}$ of each other, and "isolated" to mean at least $2020^{2020m} m^{20202020}$ away from any other pebble). Basically if two pebbles are far away, they are guaranteed not to interact in any meaningful way w.r.t F (at least in the next m moves). Thus the two structures cannot be distinuished by any such ϕ.

   (Alternately, we can view F as an adjacency relation on the structures, and use the infinite version of Hanf's theorem because there are either 0 or countably infinitely many elements of any type simultaneously in both $\mathcal{A}, \mathcal{B}$.)

   Now, we give a winning strategy for Samson in the general case (with f instead of F). Samson plays 0 in $\mathcal{B}$; say Delilah responds with some $2a \in \mathcal{A}$. Then Samson plays 1 in $\mathcal{B}$, and Delilah responds with some $2b \in \mathcal{A}$. Clearly $a \neq b$ since $0 \neq 1$. So there exists a positive integer n such that either $f^n(a) = b$ or $f^n(b) = a$ (namely $n = |b - a|$), but there clearly does not exist any n for which $f^n(0) = 1$ or $f^n(1) = 0$. Thus the induced substructures are not isomorphic, and Samson wins on the second turn. $\square$

7 First-Order Expressibility

Let τ be any finite relational vocabulary. Let $\mathcal{C}$ be a class of structures of τ. Let $S \subset \mathcal{C}$. S is called **first-order expressible** or **first-order describable** in $\mathcal{C}$ if there exists a first order sentence ϕ which distinguishes S and $\mathcal{C}/S$; i.e., for any structure $\mathcal{M}$, $\mathcal{M} \models \phi$ iff $\mathcal{M} \in S$. We can use EF games to characterise FO expressible classes:

Theorem 2 (Methodology Theorem)**.** *A subclass $S \subset \mathcal{C}$ is* ***not*** *first-order expressible iff for any $r \in \mathbb{N}$, there exists structures $\mathcal{A}_r, \mathcal{B}_r \in \mathcal{C}$ such that*

1. *$\mathcal{A}_r \in S$ and $\mathcal{B}_r \notin S$; and*
2. *$\mathcal{A}_r \sim_r \mathcal{B}_r$*

Proof. First assume there exist such $\mathcal{A}_r, \mathcal{B}_r$. Then by our main result S cannot be distinguished by any formula in $\mathcal{L}_r$ for any $r \in \mathbb{N}$, so is not FO expressible.

Now assume the condition does not hold. Then there exists an $r \in \mathbb{N}$ such that for any $\mathcal{A} \in S$ and $\mathcal{B} \in \mathcal{C}/S$, we have $\mathcal{A} \not\sim_r \mathcal{B}$, i.e., by our main result there exists a FO sentence $\phi = \phi(A, B) \in \mathcal{L}_r$ such that $\mathcal{A} \models \phi$ but $\mathcal{B} \not\models \phi$. Since $\mathcal{L}_r$ has only finitely many inequivalent sentences, for any $\mathcal{A} \in S$, we can define $\psi(A)$ to be the conjunction of all inequivalent $\phi(A, B)$ over $\mathcal{B} \in \mathcal{C}/S$. Again, since there only finitely many inequivalent $\psi(A)$, we can take disjunction of all of them to get a sentence φ. We claim that φ distinguishes S. Indeed for any $\mathcal{A} \in S$, $\mathcal{A} \models \psi(A)$ so $\mathcal{A} \models \varphi$. If $\mathcal{B} \in \mathcal{C}/S$, then $\mathcal{B}$ does not satisfy at least one clause from each $\psi(A)$ (namely $\phi(A, B)$), so $\mathcal{B} \not\models \varphi$, and we are done. □

One application of Methodology Theorem is in graph theory. We will prove a theorem that will allow us to use Methodology Theorem without constructing an explicit winning strategy for Delilah.

But first, a few definitions. We use $\tau_g = \langle E^2 \rangle$ as the vocabulary, where E is a binary relation standing for adjacency. Let $\mathcal{A}$ be any graph. For any $a \in A$ and $d \in \mathbb{N}$, define $N(a, d)$ to be all vertices in A which are a distance of at most d away from a. Note that $N(a, d)$ is a substructure of $\mathcal{A}$ Define **d-type** of a to be the isomorphism type of $N(a, d)$.

If there are pebbles placed (so a function α is already defined), then these must be accounted for in the isomorphism type as well (i.e., an isomorphism must send a pebble to itself).

We can even extend this to arbitrary structures over relational vocabulary, where two vertices are adjacent iff there is some relation in which they appear together, and isomorphism types must preserve relations and constants. Such a graph is called the **Gaifman Graph** of that structure.

Theorem 3 (Hanf's Theorem)**.** *Let $\mathcal{A}, \mathcal{B}$ be structures over a finite, relational vocabulary and $r \geq 0$ such that for each 2^r-type t, $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements of type t. Then $A \sim_r B$.*

Proof. We show that Delilah wins the game with r moves. She does this by ensuring that after move m, for any $0 \leq m \leq r$, $(\mathcal{A}, \alpha_m)$ and $(\mathcal{B}, \beta_m)$ have the same number of elements in each 2^{r-m} type. Indeed, if she does this, then at the end neighbourhoods of distance 1 around each pebble would have an isomorphism between the two structures, and Delilah will win.

To show the above, we use induction on m. The case $m = 0$ is given to us. Now assume the result holds for some m. Suppose Samson plays v on move $m + 1$, and Delilah chooses v' of the same 2^{r-m} type as v. Consider the isomorphism $f : N(v, 2^{r-m}) \to N(v', 2^{r-m})$. For any $w \in N(v, 2^{r-m-1})$, restriction of f is an isomorphism from $N(w, 2^{r-m-1})$ to $N(f(w), 2^{r-m-1})$ because $d(f(w), v') = d(w, v) \leq 2^{r-m-1}$ (so both neighbourhoods lie in the domain of the isomorphism f). Thus neighbourhoods close to v are mapped in a bijective fashion to those close to v', and neighborhoods far from v are unaffected by placing pebble at v. Therefore the number of elements in each 2^{r-m-1}-type is still the same, and we are done by induction. □

Note that Hanf's theorem works even when the number of elements in some type t is infinite, as long as their cardinality in $\mathcal{A}, \mathcal{B}$ is the same.

8 Applications of Methodology Theorem

Example 1. *Acyclicity is not FO expressible.*

Proof. Let $r \in \mathbb{N}$. Let $\mathcal{A}_r$ be a path with 2^{r+3} vertices, and let $\mathcal{B}_r$ be the disjoint union of a cycle of length 2^{r+2} and path having 2^{r+2} vertices. Both these structures have same number of elements in each 2^r-type. This is because we can map the 2^r elements on each end of $\mathcal{A}_r$ to the corresponding elements in the path component of $\mathcal{B}_r$, and map the middle elements to the cycle component. Thus $\mathcal{A}_r \sim_r \mathcal{B}_r$ by Hanf's theorem, and we are done by Methodology Theorem. □

Example 2. 2-*colorability is not FO expressible.*

Proof. We use similar $\mathcal{A}_r$ and $\mathcal{B}_r$ as before, but instead $\mathcal{A}_r$ has $2^{r+3} + 1$ vertices and the cycle component of $\mathcal{B}_r$ has length $2^{r+2} + 1$. (Note that $\mathcal{A}_r$ is a tree, so is 2-colorable, but $\mathcal{B}_r$ is not because it has an odd cycle.) □

Example 3. *Connectivity is not FO expressible.*

Proof. Same $\mathcal{A}_r$, $\mathcal{B}_r$ as in the first example works. We can also use the graphs used in solution of Exercise 6.11.4 if we allow infinite graphs. $\square$

Example 4. *Reachability is not FO expressible. Here we take $\tau = \langle E^2, u, v\rangle$ where u, v are the constant vertices from where to reach the other.*

Proof. We use the same basic graphs as in the first example. However, in $\mathcal{A}_r$ we place u at one of the ends and v in the middle so that it does not lie in $N(u, 2^r)$. Then in $\mathcal{B}_r$ we place u at one of the ends of the path and place v on the cycle. $\square$

Example 5. *A finite graph having an even number of vertices is not FO expressible*

Proof. For any r, let $\mathcal{A}_r$ be the totally disconnected graph with $2r$ vertices, and let $\mathcal{B}_r$ ve the totally disconnected graph with $2r+1$ vertices. $\square$

Example 6. *Whether a connected graph is Hamiltonian or not is not FO expressible*

Proof. For any r, let $\mathcal{A}_r$ be the cycle with length 2^{r+3} with an edge joining two opposite vertices, and let $\mathcal{B}_r$ be a union of two vertex-disjoint cycles of length 2^{r+2} each, with one edge joining one vertex of one cycle to one of the other. Now use Hanf and Methodology. $\square$

Example 7. *A field having a non-zero characteristic is not FO expressible.*

Proof. Note that vocabulary of a field has binary functions, but these can be converted to ternary relational symbols by adding a uniqueness condition if needed. However note that this adds to any formula at least as many quantifiers as there are functions used in that formula. Now, for any $r \in \mathbb{N}$, let $\mathcal{A}_r = \mathbb{F}_p^{\text{alg}}$ and $\mathcal{B}_r = \mathbb{C}$, where p is a prime much larger than r. Intuitively, for any complex number z chosen by Samson, only some local "area" of z can interact with z (e.g. roots of polynomials of degree at most r with coefficients from the constants and pebbles already placed). In this way Delilah can map z to some "isolated" area of $\mathbb{F}_p^{\text{alg}}$ (one that satisfies isomorphic polynomial equations of degree at most r) and map "close" points to similar "close" points. I don't know how to make this argument more rigorous.

However it is well known using model theory that for any FO sentence ϕ, if $\mathbb{C} \models \phi$, then $\mathbb{F}_p^{\text{alg}} \models \phi$ for all sufficiently large primes p. Now since there are only finitely many inequivalent sentences in $\mathcal{L}_r$, for all sufficiently large primes p the above holds for any $\phi \in \mathcal{L}_r$. Thus our example for $\mathcal{A}_r$ and $\mathcal{B}_r$ works. $\square$