Set Sizes and Cardinal Numbers

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Introduction

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Bijections and Injections

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- The function $f(a, b) = 2^{a-1}(2b-1)$ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ is a bijection.
- The function $f(x) = (x-1)\left(\frac{1-(-1)^x}{4}\right) - x\left(\frac{1+(-1)^x}{4}\right)$ is a bijection from $\mathbb{N}$ to $\mathbb{Z}$.

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Does this necessarily mean that the two sets have the same size?

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The answer is **yes**.

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For sets A, B, $|A| \leq |B|$ and $|B| \leq |A|$ imply $|A| = |B|$.

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The proof is skipped because it is a bit involved.

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Cantor's Diagonal Argument

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Cardinal Arithmetic

Suppose we have two cardinals u, v corresponding to disjoint sets A, B. Then we can define $u + v$, $u \cdot v$ and u^v as the cardinality of the sets $A \cup B$, $A \times B$ and the set of functions from B to A respectively.

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$$u + v = u \cdot v = \max\{u, v\}$$

Thank You!