Change of basis

by **Marco Taboga (with some minor adaptations by FOE)**

- Taboga, Marco (2021). "Change of basis", Lectures on matrix algebra. https://www.statlect.com/matrix-algebra/change-of-basis

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Change of basis

The change of basis is a technique that allows us to express vector coordinates with respect to a "new basis" that is different from the "old basis" originally employed to compute coordinates.

Coordinates

Suppose that a finite-dimensional **vector space** S possesses a basis $E = \{e_1, \ldots, e_K | e_k \in S\}$.

Then, any vector $s \in S$ can be written as a linear combination of the basis:

$$s = \beta_1 e_1 + \ldots + \beta_K e_K$$

where the scalar coefficients $\beta_1, \ldots, \beta_K$ are uniquely determined.

The $K \times 1$ vector

$$[s]_E = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}$$

is called the **coordinate vector** of s with respect to the basis E.

Example

Let S be a vector space and $E = \{e_1, e_2 | e_k \in S\}$ a basis for S, i.e. S is two dimensional. Consider the vector

$$s = 3e_1 + 1e_2$$

Its coordinate vector is

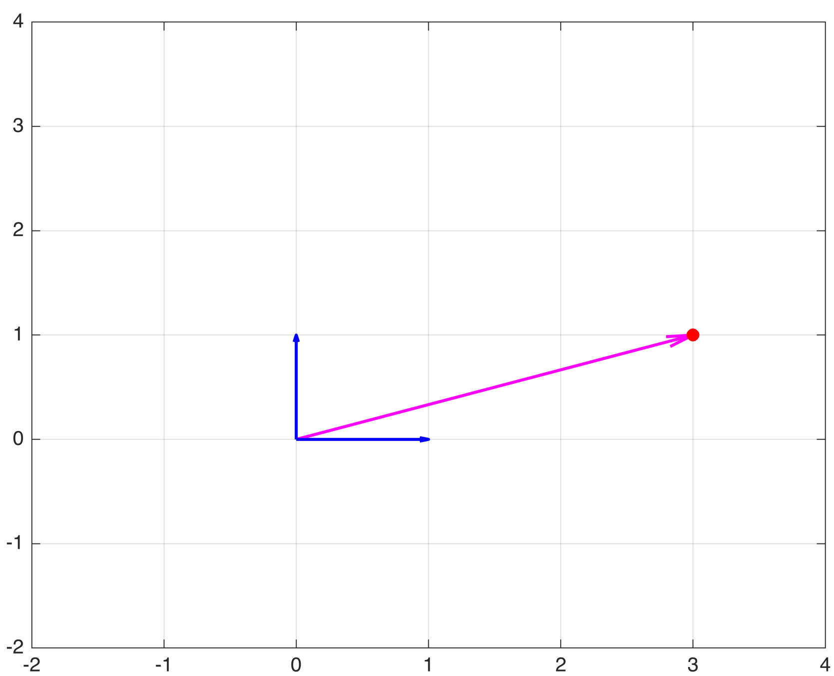
$$[s]_E = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

```
p=[3 1];
figure
hold on
quiver(0,0,p(1),p(2),0,'Color','m','LineWidth',1.5) %Canonical e_2

quiver(0,0,1,0,0,'Color','b','LineWidth',1.5) %Canonical e_1
quiver(0,0,0,1,0,'Color','b','LineWidth',1.5) %Canonical e_2

plot(p(1),p(2),'ro','MarkerFaceColor','red')

%beautify
box on, grid on
set(gca,'XLim',[-2 4])
set(gca,'YLim',[-2 4])
```



The problem of change of basis

The *purpose* of the change-of-basis techniques is to *transform coordinate vectors*. The coordinates of a vector are the coefficients of the linear combination used to represent the vector in terms of the basis.

Now, suppose that we have a second basis $F = \{f_1, f_2 | f_k \in S\}$ also on S. Thus, elements s of Scan also be expressed as:

$s = \gamma_1 f_1 + \ldots + \gamma_K f_K$

By the dimension theorem, E and F have the same number K of vectors. Hence, specifically, in this 2 dimensional space as;

$s = \gamma_1 f_1 + \gamma_2 f_2$

But **what happens to coordinates when we switch from using E as a basis to using F?**

In particular, how do we transform a coordinate vector $[s]_E$ into a vector $[s]_F$ of coordinates with respect to the new basis?

The change-of-basis matrix

The answer to the above porblem is provided by the following proposition.

Proposition Let S be a Kdimensional vector space. Let $E = \{e_1, \ldots, e_K | e_k \in S\}$ and $F = \{f_1, \ldots, f_K | f_k \in S\}$ be two bases for S. Then, there exists a $K \times K$ matrix, denoted here by $A_{E \to F}$ or simply A here for simplicity and called **change-of-basis matrix**, such that,

$\forall s \in S : [s]_F = A_{E \to F}[s]_E$

where $[s]_E$ and $[s]_F$ denote the coordinate vectors of s with respect to E and F respectively.

Proof

Let $s = \beta_1 e_1 + \ldots + \beta_K e_K$ be the representation of s in terms of E. Remember, that $e_k \in S$and hence, each e_k in turn can be expressed by its representation in F i.e.

$e_k = \gamma_1 f_1 + \ldots + \gamma_K f_K$

end viceversa, each f_k in turn can be expressed by its representation in E i.e.

$f_k = \beta_1 e_1 + \ldots + \beta_K e_K$

Therefore, by the rules on the addition and scalar multiplication of coordinate vectors, for an arbitrary $s \in S$we have that

$[s]_F = \beta_1[e_1]_F + \ldots + \beta_K[e_K]_F$

where, remember $[e_k]_F$ are the $K \times 1$ coordinate vectors of the elements of E with respect to F. Adjoin these K vectors so as to form a $K \times K$ matrix

$$A_{E \to F} = \begin{bmatrix} [e_1]_F & \ldots & [e_K]_F \end{bmatrix}$$

Since

$$[s]_E = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}$$

we can rewrite the equation above as

$$[s]_F = A_{E \to F}[s]_E$$

because the product $A_{E \to F}[s]_E$ is equal to a linear combination of the columns of $A_{E \to F}$, with coefficients taken from $[s]_E$ (see the lecture on matrix products and linear combinations). Note that $A_{E \to F}$ does not depend on the particular choice of s, as it depends only on the two bases E and F.

Structure of the change-of-basis matrix

The main take-away from the previous proof is that the columns of the change-of-basis matrix $A_{E \to F}$ are the coordinates of the vectors of the original basis E with respect to the new basis F:

$$A_{E \to F} = \begin{bmatrix} [e_1]_F & \ldots & [e_K]_F \end{bmatrix}$$

Inverse of the change-of-basis matrix

As demonstrated by the next proposition, **the change of basis matrix is invertible**.

Proposition Let S be a vector space. Let $E = \{e_1, \ldots, e_K | e_k \in S\}$ and $F = \{f_1, \ldots, f_K | f_k \in S\}$ be two bases for S. Then, the change-of-basis matrix $A_{E \to F}$ is invertible and its inverse equals $A_{F \to E}$, that is,

$$A_{F \to E} = (A_{E \to F})^{-1}$$

Proof

For any $s \in S$, we have that

$$[s]_F = A_{E \to F}[s]_E$$

and

$$[s]_E = A_{F \to E}[s]_F$$

By combining these two equations, we obtain

$$[s]_F = A_{E\to F}A_{F\to E}[s]_F$$

This can be true for every $s \in S$ only if

$$A_{E\to F}A_{F\to E} = I$$

where I is the $K \times K$ identity matrix. The latter result implies that $A_{F\to E}$ is the inverse of $A_{E\to F}$, which as seen above, this is the case!.

Example

Consider the space S of all 2×1 vectors and let be the two bases; $E = \{e_1, \ldots, e_K | e_k \in S\}$ with

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and $F = \{f_1, \ldots, f_K | f_k \in S\}$ with

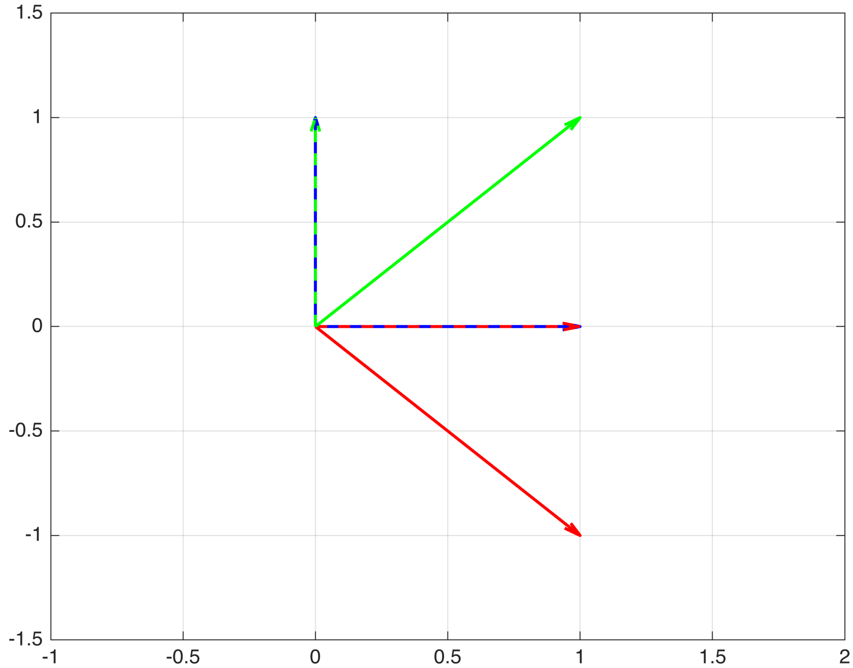
$$f_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; f_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
e = [1 1; 0 -1];
f = [0 1; 1 1];

figure
hold on
quiver(0,0,1,0,0,'Color','b','LineWidth',1.5) %Canonical basis
quiver(0,0,0,1,0,'Color','b','LineWidth',1.5) %Canonical basis

quiver(0,0,e(1,1),e(2,1),0,'Color','r','LineWidth',1.5,'LineStyle','--')
%e_1. Note the superposition with the canonical basis
quiver(0,0,e(1,2),e(2,2),0,'Color','r','LineWidth',1.5) %e_2

quiver(0,0,f(1,1),f(2,1),0,'Color','g','LineWidth',1.5,'LineStyle','--')
%f_1. Note the superposition with the canonical basis
quiver(0,0,f(1,2),f(2,2),0,'Color','g','LineWidth',1.5) %f_2.
%beautify
box on, grid on
set(gca,'XLim',[-1 2])
set(gca,'YLim',[-1.5 1.5])
```



We have

$$\left.\begin{aligned} e_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= -f_1 + f_2 \\ e_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= -2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= -2f_1 + f_2 \end{aligned}\right\}$$

Thus, the coordinate vectors of the elements of E with respect to F are

$$[e_1]_F = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad [e_2]_F = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Therefore, when we switch from E to F, the change-of-basis matrix is

$$A_{E\to F} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

For example, take the vector

$$[s]_{canonical} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Since;

$$s = 3e_1 - 3e_2$$

the coordinates of s with respect to E are

$$[s]_E = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

Its coordinates with respect to F can be easily computed thanks to the change-of-basis matrix:

$$[s]_F = A_{E\to F}[s]_E = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \cdot 3 + -2 \cdot (-3) \\ 1 \cdot 3 + 1 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

We can easily check that this is correct:

$$3f_1 + 0f_2 = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = s_{canonical}$$

Linear operators

Remember that a linear operator on a vector space S is a function $f : S \to S$ such that

$$f(\alpha_1 s_1 + \alpha_2 s_2) = \alpha_1 f(s_1) + \alpha_2 f(s2)$$

for any two vectors $s_1, s_2 \in S$ and any two scalars α_1 and α_2.

Given a basis $E = \{e_1, \dots, e_K | e_k \in S\}$ for S, the matrix of the linear operator with respect to E is the square $K \times K$ matrix $[f]_E$ such that

$$[f(s)]_E = [f]_E[s]_E$$

for any vector $s \in S$.

In other words, if you multiply the matrix of the operator by the coordinate vector of s, then you obtain the coordinate vector of $f(s)$.

Effect on the matrix of a linear operator

What happens to the matrix of the operator when we switch to a new basis? The next proposition provides an answer to this question.

Proposition Let S be a linear space. Let $E = \{e_1, \dots, e_K | e_k \in S\}$ and $F = \{f_1, \dots, f_K | f_k \in S\}$ be two bases for S. Let $f : S \to S$ be a linear operator. Denote by $[f]_E$ and $[f]_F$ the matrices of the linear operator with respect to E and F respectively. Then,

$$[f]_F = A_{F\to E}^{-1}[f]_E A_{F\to E}$$

or, equivalently,

$$[f]_F = A_{E\to F}[f]_E A_{F\to E}$$

where $A_{E \to F}$ and $A_{F \to E}$ are the change-of-basis matrices that allow us to switch from E to F and vice versa.

Proof

Let $s \in S$. We can use the change-of-basis matrix $A_{F \to E}$ to transform the coordinates

$$[s]_E = A_{F \to E}[s]_F$$

and, also to transform the coordinates of the image of s, $f(s)$;

$$[f(s)]_E = A_{F \to E}[f(s)]_F$$

Therefore, the matrix representation of the operator

$$[f(s)]_E = [f]_E[s]_E$$

can be written as

$$A_{F \to E}[f(s)_F] = [f]_E A_{F \to E}[s]_F$$

or

$$[f(s)_F] = A^{-1}_{F \to E}[f]_E A_{F \to E}[s]_F$$

Thus, the matrix

$$[f]_F = A^{-1}_{F \to E}[f]_E A_{F \to E}$$

is the matrix of f with respect to F (which is unique). *cqd*

Change basis, solve, then go back

Since;

$$A^{-1}_{F \to E} = A_{E \to F}$$

we can also re-write

$$[f]_F = A^{-1}_{F \to E}[f]_E A_{F \to E}$$

as

$$[f]_F = A_{E \to F}[f]_E A_{F \to E}$$

Thus, the change-of-basis matrices allow us to easily switch from the matrix of the linear operator with respect to the old basis to the matrix with respect to the new basis.

Note the trick as this is a very common trick in mathematics in general and data analysis in particular; in order to achive a certain operation $[f]_F$ in F, we can change our basis to E, $A_{F \to E}$, solve the operation under basis E, $[f]_E$, where it may be easier, and finally go "back" to the representation given by basis F, $A_{E \to F}$.

Solved exercise

Below you can find some exercises with explained solutions.

Exercise 1

Consider the space S of all 2×1 vectors and let be the two bases; $E = \{e_1, \ldots, e_K | e_k \in S\}$ with

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and $F = \{f_1, \ldots, f_K | f_k \in S\}$ with

$$f_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad f_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In that example, we have shown that the change-of-basis matrix is

$$A_{E \to F} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

Moreover,

$$A_{F \to E} = A_{E \to F}^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

Let f be the linear operator such that

$$\begin{aligned} f(e_1) &= e_1 + e_2 \\ f(e_2) &= -e_2 \end{aligned}$$

Find the matrix $[f]_E$ and then use the change-of-basis formulae to derive $[f]_F$ from $[f]_E$

Solution

The matrix of the linear operator with respect to the basis E is

$$\begin{aligned} [f]_E &= [[f(e_1)]_E f(e_2)]_E] \\ &= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

Now simply applying the change-of-basis formula gives

$$
\begin{aligned}
[f]_F &= A_{E\to F}[f]_E A_{F\to E} \\
&= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -5 & -8 \\ 3 & 5 \end{bmatrix}
\end{aligned}
$$