This session will be taught in Hybrid format

When joining on Zoom

- When your microphone or camera are switched on you will be able to be heard in the teaching room and may be seen on the screen in the room.
- If the screen is recorded, you may be captured on the recording if you choose to contribute and have your camera switched on.
- Chat messages to everyone and privately to the host may be visible on screen in the room.

When joining in the Teaching room

- Academic teaching staff should let you know when Zoom is connected and you may see the Zoom meeting on the screen.
- Once connected to Zoom, microphones may capture sound from the whole room. Your voice may be shared to Zoom when speaking at a normal talking volume.
- Once connected to Zoom, cameras will capture video images from the room. These may be shared to Zoom. The camera capture will be a wide shot of the whole room or block of seating. It will not capture close-up images of individual students.
- If the session is recorded, video from the room will be captured on the recording. Your voice will be captured on the recording if you choose to contribute.
- If you connect to the Zoom meeting from within the room (for example to use the chat), please keep your audio on mute or use headphones to avoid feedback issues with the room audio.

The recordings may be stored in the cloud ad any personal information within the recording will be processed in accordance to the General Data Protection Regulations 2018.
A substantial contribution is considered to be anything more than merely answering questions or participating in a group discussion. Where you make a substantial contribution to the delivery of the recorded events, a signed consent form will be obtained prior to the recording being made available for viewing. The Consent Form will address your personal informationand any copyright or other intellectual property in the recording.

Dimensional analysis by co-occurrence
LSA

Felipe Orihuela-Espina

1-Nov-2024

Table of Contents

Section 1

A worked solution to SVD

Singular Value Decomposition

Disclaimer

This tutorial is a modified version of the original Example 3 in:

https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

What follows is an *ad-hoc* solution. I'm not certain of how efficient or numerically stable it is, but it has been chosen as it is a very didactic example. The aim is to exemplify a solution to SVD.

Singular Value Decomposition

Example

Find the SVD of $A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$

Singular Value Decomposition

Example

Find the SVD of $A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$

Solution:
We are requested to solve the SVD factorization of A, that is;

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} = U\Sigma V^*$$

...and remember that V^* is the conjugate transpose matrix V. Moreover, recall that V is unitary by the definition of the SVD factorization, that is;

$$V^*V = VV^* = I$$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):
Since $V^*V = I$;

$$\begin{aligned} A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V^*_{n\times n} \quad &\Rightarrow \quad AV = U\Sigma V^* V \\ \Rightarrow \quad AV = U\Sigma I \quad &\Rightarrow \quad A_{m\times n}V_{n\times n} = U_{m\times m}\Sigma_{m\times n} \end{aligned}$$

So far this is familiar; in the eigendecomposition we have the analogous $AP = P\Lambda$.

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Example

Solution (Cont.):
If we "see" the afore equation $AV = U\Sigma\Lambda$. in terms of the columns singular vectors;

If $m < n$

$$A[\mathbf{v}_1 \dots \mathbf{v}_n] = [\mathbf{u}_1 \dots \mathbf{u}_m] \begin{pmatrix} \sigma_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 \end{pmatrix}$$

Or if $m > n$

$$A[\mathbf{v}_1 \dots \mathbf{v}_n] = [\mathbf{u}_1 \dots \mathbf{u}_m] \begin{pmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \\ 0 & \dots & 0 \end{pmatrix}$$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):

When $m > n$, as there are more singular vectors $\mathbf{u}_m$ than singular values σ_n, we can split U into 2 subspaces;

- $[\mathbf{u}_1 \dots \mathbf{u}_n]$ called the column space of A.
- $[\mathbf{u}_{n+1} \dots \mathbf{u}_m]$ called the left null subspace of A.

The analogous when $m < n$ would be;

- $[\mathbf{v}_1 \dots \mathbf{v}_m]$ called the row space of A.
- $[\mathbf{v}_{m+1} \dots \mathbf{v}_n]$ called the right null subspace of A.

In general, if A is of rank r and $r < m \wedge r < n$, then, the four subspaces will look like;

- $[\mathbf{u}_1 \dots \mathbf{u}_r]$
- $[\mathbf{u}_{r+1} \dots \mathbf{u}_m]$
- $[\mathbf{v}_1 \dots \mathbf{v}_r]$
- $[\mathbf{v}_{r+1} \dots \mathbf{v}_n]$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):
Coming back to our previous equation, whether $m > n$ or $m < n$, we can "neglect" the null subspaces (as this will be 0 when multiplied by Σ), and make Σ full rank

$$A[\mathbf{v}_1 \dots \mathbf{v}_r] = [\mathbf{u}_1 \dots \mathbf{u}_r] \begin{pmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_r \end{pmatrix}$$

Note that the product on both sides is now $m \times r$.

$$A_{m\times r} V_{r\times r} = U_{m\times r} \Sigma_{r\times r}$$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):

$$A[\mathbf{v}_1 \dots \mathbf{v}_r] = [\mathbf{u}_1 \dots \mathbf{u}_r] \begin{pmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_r \end{pmatrix}$$

This way of looking at the SVD is interesting because it clearly shows that these basis vectors diagonalize the matrix A;

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i$$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):

Another interesting observation is that by clearing A on the left side again;

$$A_{m\times r} = U_{m\times r}\Sigma_{r\times r}V^*_{r\times r}$$

...and expanding the multiplication;

$$A = U\Sigma V^* = \mathbf{u_1}\sigma_1\mathbf{v}_1^* + \ldots + \mathbf{u_r}\sigma_r\mathbf{v}_r^*$$

Sorting the singular values in descending order, $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0$, the splitting in equation above gives the r rank-one pieces of A in order of importance.

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):
Let's continue;

SVD does not require A to be square but both $A^T A$ and AA^T are square. With this in mind[a];

$$
\begin{aligned}
A^T A &= (U\Sigma V^*)^T (U\Sigma V^*) \\
&\overset{(AB)^T = B^T A^T}{=} V\Sigma^T U^* U\Sigma V^* \\
&\overset{U^* U = I}{=} V\Sigma^T I \Sigma V^* \\
&\overset{XI = X}{=} V\Sigma^T \Sigma V^* \\
&= V \begin{pmatrix} \sigma_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_r^2 \end{pmatrix} V^*
\end{aligned}
$$

But this is an eigendecomposition that we know how to solve!

[a] Also, remember that $(AB)^T = B^T A^T$

Singular Value Decomposition

Example

Solution (Cont.):
...and analogously;

$$
\begin{aligned}
AA^T &= (U\Sigma V^*)(U\Sigma V^*)^T \\
&\overset{(AB)^T = B^T A^T}{=} (U\Sigma V^*)(V\Sigma^T U^*) \\
&= U\Sigma\Sigma^T U^* \\
&= U \begin{pmatrix} \sigma_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_r^2 \end{pmatrix} U^*
\end{aligned}
$$

Yet another eigendecomposition that we know how to solve!!

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Example

Solution (Cont.):

In other words,the squared singular values σ_r^2 are the eigenvalues of either $A^T A$ or AA^T.

...Further, obtaining the eigendecompositions of $A^T A$ and AA^T will give us V and U respectively.

We can exploit this to solve the SVD of A as we already know how to solve the eigendecomposition!

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):
We start by computing A^TA and AA^T.

$$A^TA = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 9 & 12 \\ 12 & 41 \end{pmatrix}$$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):
We solve the eigendecomposition of either $A^T A$ or AA^T, since the solution of the eigenvalues is the same[a].

Regardless of whether we choose to solve the eigendecomposition of either $A^T A$ or AA^T;

$$\lambda_1 = \sigma_1^2 = 45 \quad \Rightarrow \sigma_1 = \sqrt{45}$$
$$\lambda_2 = \sigma_1^2 = 5 \quad \Rightarrow \sigma_2 = \sqrt{5}$$

We got our singular values!

[a]Not the eigenvectors though! ...but we did not claim above any relation between the singular vectors of A and the eigenvectors of $A^T A$ or AA^T.

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Example

Solution (Cont.):
Solving the eigendecomposition of $A^T A$ gives us V, we get:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Just remember that these eigenvectors are not unique!

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):
For sanity, we can check that the eigendecomposition constraint holds for both eigenvectors of $A^T A$

$$A^T A \mathbf{v}_r = \lambda_r \mathbf{v}_r$$

$$\begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 45 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):
Further, having removed the null subspaces, $V^*_{r\times r}$ is full rank, and hence, the remaning singular vectors $\mathbf{v}_r$ are orthogonal.

…which means that the dot product of any two $\mathbf{v}_i$, $\mathbf{v}_{j|j\neq i}$ is 0.

$$\mathbf{v}_i^T \cdot \mathbf{v}_j = 0$$

Hence, we can also check for sanity that; $\mathbf{v}_1^T \cdot \mathbf{v}_2 = 0$.

$$\mathbf{v}_1^T \cdot \mathbf{v}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 \cdot (-1) + 1 \cdot 1 = 0$$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):
It is common practice to present the singular values rescaled to length 1;

Hence, using the classical pythagorean theorem;

$$hypotenuse^2 = side_x^2 + side_y^2 \quad \Rightarrow \quad hypotenuse = \sqrt{side_x^2 + side_y^2}$$

$$\mathbf{v}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \xRightarrow{side_x=1;\ side_y=1 \rightarrow hypotenuse=\sqrt{2}} \quad \mathbf{v}_{1,normalized} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and analogously;

$$\mathbf{v}_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \xRightarrow{side_x=-1;\ side_y=1 \rightarrow hypotenuse=\sqrt{2}} \quad \mathbf{v}_{2,normalized} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):

Finally, to calculate U we do not even need to calculate the second eigendecompostion, i.e. that of AA^T. Instead, we can use the aforementioned relation;

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i \quad \Rightarrow \quad \mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$$

and we already know; A, both $\mathbf{v}_1$ and $\mathbf{v}_2$, and both σ_1 and σ_2. Ergo;

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{45}} \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Singular Value Decomposition

Example

Solution (Cont.):

Putting everything together[a]:

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad V^* \overset{\forall v_{ij}: v_{ij} \in \mathbb{R}}{=} V^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

And finally, the SVD of $A = U \Sigma V^*$;

$$\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

[a] Note that all elements of V in this case are real numbers, ergo, in this case, $V^* = V^T$

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Example

Solution (Cont.):

And if you want to check in MATLAB:

```
A = [3 0; 4 5]
U_r = [1 -3; 3 1]
U_norm = (1/sqrt(10))*U_r
S = [sqrt(45) 0; 0 sqrt(5)]
Vt_r = [1 1; -1 1]
Vt_norm = (1/sqrt(2))*Vt_r
U_norm*S*Vt_norm
```

Source: https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

Thank you! Questions?