This session will be taught in Hybrid format

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- If the screen is recorded, you may be captured on the recording if you choose to contribute and have your camera switched on.
- Chat messages to everyone and privately to the host may be visible on screen in the room.

When joining in the Teaching room

- Academic teaching staff should let you know when Zoom is connected and you may see the Zoom meeting on the screen.
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- Once connected to Zoom, cameras will capture video images from the room. These may be shared to Zoom. The camera capture will be a wide shot of the whole room or block of seating. It will not capture close-up images of individual students.
- If the session is recorded, video from the room will be captured on the recording. Your voice will be captured on the recording if you choose to contribute.
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(Complex) Networks Analysis

Introduction to graph theory and Centrality Analysis (Part I): Tutorial

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Section 1

Introduction to network analysis and graph theory

Exercise

Let be a *simple* i.e. no loops or multi-edges, and *undirected* graph $G = (V, E)$.
Establish the maximum number of edges (graph size) as a function of the number of nodes (graph order)

Network analysis

Exercise

Let be a *simple* i.e. no loops or multi-edges, and *undirected* graph $G = (V, E)$.

Establish the maximum number of edges (graph size) as a function of the number of nodes (graph order)

Answer:

Before building the answer let's get some insight from the base cases:

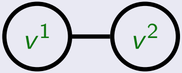
- If $|V| = 1$, then $|E| = 0$ because $E = \emptyset$
  - Remember: Loops are not valid in simple graphs.

Network analysis

Exercise

Answer (Cont.):

- If $|V| = 2$, then $|E| = 1$ because $E = \{(v^1, v^2)\}$.
- Remember: In undirected graphs $(v^1, v^2) = (v^2, v^1)$.

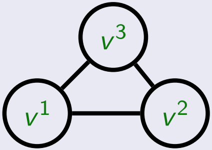


Network analysis

Exercise

Answer (Cont.):

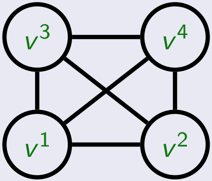
- If $|V| = 3$, then $|E| = 3$ because $E = \{(v^1, v^2), (v^1, v^3), (v^2, v^3)\}$



Exercise

Answer (Cont.):

- If $|V| = 4$, then $|E| = 6$ because $E = \{(v^1, v^2), (v^1, v^3), (v^1, v^4), (v^2, v^3), (v^2, v^4), (v^3, v^4)\}$



Network analysis

Exercise

Answer (Cont.):
Bringing these base cases together; their (lower triangular part of the) adjacency matrices look like:

	v^1
v^1	

	v^1	v^2
v^1		
v^2	✓	

	v^1	v^2	v^3
v^1			
v^2	✓		
v^3	✓	✓	

	v^1	v^2	v^3	v^4
v^1				
v^2	✓			
v^3	✓	✓		
v^4	✓	✓	✓	

Watch out!

Strictly, the adjacency matrix also includes the symmetric upper part. I'm intentionally hiding that here for the sake of clarity.

Network analysis

Exercise

Answer (Cont.):
Bringing these base cases together:

$\lvert V \rvert$	$\lvert E \rvert$
1	0
2	1
3	3
4	6

A couple of additional observations; If loops were permitted i.e. $(v^i, v^j) \in E$ then:

- The number of potential loops are $\lvert V \rvert$
- Further, if in addition the graph was directed i.e. $(v^i, v^j) \neq (v^j, v^i)$, then the largest size would simply be $\lvert E \rvert = \lvert V \rvert^2$.

Network analysis

Exercise

Answer (Cont.):
We are now finally ready to solve the original question;

Starting from a potentially maximum number of edges (for a directed graph) of $|E| = |V|^2$ and removing the loops

$$|E| = |V|^2 - |V| = |V||V| - |V| = |V| \cdot (|V| - 1)$$

...and considering that in undirected graphs $(v^i, v^j) = (v^j, v^i)$, then directed graphs have potentially double number of edges; or in other words, undirected graphs have half the size. Hence;

$$|E| = \frac{|V| \cdot (|V| - 1)}{2}$$

Exercise

Answer (Cont.):

And we can verify:

$\lvert V\rvert$	$\lvert E\rvert$	$\frac{\lvert V\rvert\cdot(\lvert V\rvert-1)}{2}$
1	0	0
2	1	1
3	3	3
4	6	6

Section 2

Centrality Analysis

Exercise

Show that in any group of two or more people, there are always at least two people who have exactly the same number of friends assuming that friendship is reciprocal.

Exercise

Show that in any group of two or more people, there are always at least two people who have exactly the same number of friends assuming that friendship is reciprocal.

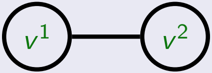
Answer:
Let our friendship graph $G = (V, E)$ has nodes V as people and edges E as frienships.

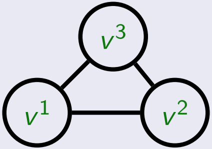
- Since friendship is reciprocal the graph is *undirected* and *simple*.
- Also, in principle there is nothing in the statement that hints that the quality or strength of the friendship has to be considered, hence we can choose to consider the graph *unweighted*

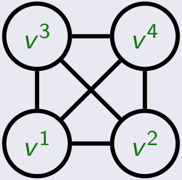
Exercise

Answer (Cont.):
Let's start by constructing some intuition:

If the graph is complete; then all nodes have exactly $|V|-1$ acquaintances, and thus, complying with the statement that there are at least two people (nodes) who have exactly the same number of friends (edges).





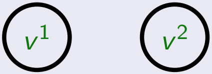


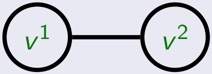
Centrality Analysis

Exercise

Answer (Cont.):

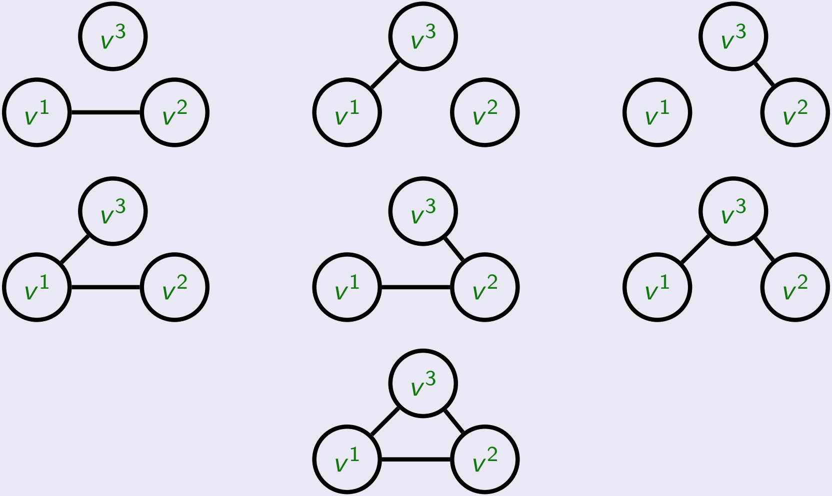
If the graph order is $|V| = 2$ the all possible cases are;





Exercise

Answer (Cont.): If the graph order is $|V| = 3$ the all possible cases are;



Exercise

Answer (Cont.):

Important observation: In all of the above, if any vertex has degree 0, then no other vertex has degree $|V| - 1$.

- If $|V| = 2$, and either $degree(v^1) = 0$ or $degree(v^2) = 0$, then so does the other node has degree 0.
- If $|V| = 3$, and either $degree(v^i) = 0$, then both other nodes $v^j \neq v^i$ have $degree(v^j) = 1$

This makes sense; for a node to have degree $|V| - 1$, it means that it is connected to *all* other nodes, i.e. no node can have degree 0, and the opposite is also true, if a node has degree 0, it means that it is connected to *none* of the other nodes, i.e. no node can have degree $|V| - 1$,

Network analysis

Exercise

Answer (Cont.): Now that we have the intuition, we are ready to prove the statement;

Let's *assume* that there are no pair of nodes v^i and v^j having the same degree; that is, since there are $|V|$ nodes; there ought to be:

- one node with degree 0,
- one node with degree 1,
- ...
- one node with degree $|V| - 1$,

However, if any vertex has degree 0, we know no other vertex can have degree $|V| - 1$ which is a contradiction with our assumption.

Section 3

Learning to test

Random networks

A random graph is a graph in which the distribution of its edges may be described simply by a probability distribution, or by a random process.

A lot of measures in network analysis are based on estimating the deviations from a random graph. Hence, being able to generate a (pseudo-)random graph is critical for testing hypothesis.

Several algorithms exist to generate (pseudo-)random graphs, with Watts–Strogatz model [1] perhaps being the most well known.

Random networks

Algorithm 1: Watts–Strogatz model of (pseudo-)random graphs

Data: N: Graph order
Data: K: Mean degree (even integer)
Data: p: Rewiring probability
Result: G: A random graph characterized by its adjacency matrix A

```
/* Initialization: Construct a regular ring lattice */
/* - a graph with N nodes each connected to K neighbors (K/2 on each side). */
/* Adjacency matrix: Start with no connections */
```
$A \leftarrow zeros(N, N)$;
for $a_{ij} \in A$ *with* $j < i$ `/* Add lattice connections */`
do
  if $mod(|i-j|, N-1-\frac{K}{2}) > 0$ **and** $mod(|i-j|, N-1-\frac{K}{2}) \leq \frac{K}{2}$ **then**
    $a_{ij} \leftarrow 1$;
    $a_{ji} \leftarrow 1$;
  end if
end for
```
/* Rewire: Revisit each connected edge and rewire with proabability p */
/* - Rewiring is done by replacing link (i,j) with link (i,k) where k is chosen uniformly at random from all possible nodes while avoiding self-loops (k ≠ i) and link duplication */
```
for $a_{ij} \in A$ *with* $j < i$ **do**
  if $a_{ij} = 1$ **and** $rand() < p$ **then**
    $k \leftarrow \lceil N * rand() \rceil$;
    while $k = i$ **or** $a_{ik} = 1$ **do**
      $k \leftarrow \lceil N * rand() \rceil$;
    end while
    $a_{ij} \leftarrow 0$;
    $a_{ji} \leftarrow 0$;
    $a_{ik} \leftarrow 1$;
    $a_{ki} \leftarrow 1$;
  end if
end for

Exercise

Implement Watts–Strogatz model in some programming language.
Tip: In this exercise do not aim for efficiency or code elegance; instead focus on closely following the pseudo-code provided.
NOTE: Solution is provided in MATLAB.

Exercise

Implement Watts–Strogatz model in some programming language.
Tip: In this exercise do not aim for efficiency or code elegance; instead focus on closely following the pseudo-code provided.
NOTE: Solution is provided in MATLAB.

Answer:
Please open code:

`IDA2023_0004_CNA_RandomGraphs_WattsAndStrogatz.m`

MATLAB's internal algorihtm is reported here:

https://uk.mathworks.com/help/matlab/math/build-watts-strogatz-small-world-graph-model.html

References I

Duncan J Watts and Steven H Strogatz.
Collective dynamics of 'small-world'networks.
nature, 393(6684):440–442, 1998.

Thank you! Questions?