CSIE5002 Prediction, learning, and games

Lecture 8: Aggregating algorithm on a continuum

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Abstract

We have studied the aggregating algorithm when there are finite experts (equivalently, with a finite hypothesis class.) What if there are uncountably many experts? When does this case appear?

Recommended reading

- N. Cesa-Bianchi and G. Lugosi. 2006. *Prediction, Learning, and Games*. Chapter 10.
- T. M. Cover & E. Ordentlich. 1996. Universal portfolios with side information.
- E. Hazan *et al.* 2007. Logarithmic regret algorithms for online convex optimization.
- D. van der Hoeven *et al.* 2018. The many faces of exponential weights in online learning.

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Online portfolio selection

Online portfolio selection (1/2)

Protocol. (Online portfolio selection) Let $T \in \mathbb{N}$. Let $L_0 = 0$. For every $1 \leq t \leq T$, the following happen in order.

1. $\textsc{Learner}$ announces $\gamma_t \in \Delta \subset \mathbb{R}^n$.
2. $\textsc{Reality}$ announces $\omega_t \in \mathbb{R}^n$.
3. Update the cumulative loss as $L_t \leftarrow L_{t-1} + \lambda(\omega_t, \gamma_t)$, where

$$\lambda(\omega, \gamma) := -\log \langle \omega, \gamma \rangle , \quad \forall \omega \in \mathbb{R}^n, \gamma \in \Delta.$$

J. L. Kelly, Jr. A new interpretation of information rate. 1956.

T. M. Cover. Universal portfolios. 1991.

Online portfolio selection (2/2)

Regret.

$$R_T(\gamma) := \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t) - \sum_{t=1}^{T} \lambda(\omega_t, \gamma).$$

Remark. If every ω_t is in the canonical basis of $\mathbb{R}^n$, then we get individual sequence prediction with the logarithmic loss, aka online portfolio selection in the *Kelly market*

J. L. Kelly, Jr. A new interpretation of information rate. 1956.

T. M. Cover. Universal portfolios. 1991.

Interpretations (1/2)

Suppose there are n investment alternatives.

- x_t: Ratios of current wealth invested in the investment alternatives.
- y_t: Price relatives.

Observation. Let $W_0 > 0$ be the initial wealth of $\textsc{Learner}$. Then, after t rounds, the wealth of $\textsc{Learner}$ becomes

$$W_t = W_0 \prod_{\tau=1}^{t} \langle \omega_t, \gamma_t \rangle .$$

Then, we have

$$-\log W_T = \sum_{t=1}^{T} \left[-\log \langle \omega_t, \gamma_t \rangle \right] .$$

Interpretations (2/2)

Suppose $(\omega_t)_{t\in\mathbb{N}}$ is a stochastic process.

Definition. (growth-optimal criterion, aka the Kelly-Latané-Breiman criterion) Choose

$$\gamma_t = \arg\min_{\gamma\in\Delta} \mathsf{E}\left[-\log\langle\omega_t,\gamma\rangle|\omega_1,\ldots,\omega_{t-1}\right].$$

Theorem 1. Let W_t and $\tilde{W}_t$ be the wealth of $\textsc{Learner}$ and any *causal investor*, respectively. Then, $(\tilde{W}_t/w_t)_{t\in\mathbb{N}}$ converges to a finite number almost surely.

J. L. Kelly, Jr. A new interpretation of information rate. 1956.

H. A. Latané. Criteria for choice among risky ventures. 1959.

L. Breiman. Investment policies for expanding business optimal in a long-run sense. 1960.

Algorithm. (Universal portfolio selection) Let π_1 be a probability density function on Δ. For each $1 \leq t \leq T$, announce

$$\gamma_t := \int_\Delta \gamma \pi_t(\gamma) \, \mathrm{d}\gamma,$$

and compute π_{t+1} such that

$$\pi_{t+1}(\gamma) \propto \pi_t(\gamma) \mathrm{e}^{-\lambda(\omega_t, \gamma)}.$$

Question. How do we interpret this algorithm?

T. M. Cover. Universal portfolios. 1991.

T. M. Cover & E. Ordentlich. Universal portfolios with side information. 1996.

Online portfolio selection as learning with expert advice (1/2)

Protocol. (Online portfolio selection, equivalent ver.) Let $T \in \mathbb{N}$. Let the initial cumulative loss $L_0 = 0$. For every $1 \leq t \leq T$, the following happen in order.

1. $\textsc{Expert}$-θ announces $\gamma_t(\theta) := \theta$, for all $\theta \in \Delta$.
2. $\textsc{Learner}$ announces $\gamma_t \in \Delta$.
3. Update the cumulative loss as $L_t \leftarrow L_{t-1} + \lambda(\omega_t, \gamma_t)$, where the loss is the same.

Online portfolio selection as learning with expert advice (2/2)

Regret. (Equivalent ver.)

$$R_T(\theta) \coloneqq \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t) - \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t(\theta)).$$

Observation. If we run the aggregating algorithm, then for each $1 \leq t \leq T$, we need to output some γ_t such that

$$\lambda(\omega, \gamma_t) \leq -\log \int_{\Delta} \mathrm{e}^{-\lambda(\omega,\gamma)} \pi_t(\gamma) \, \mathrm{d}\gamma, \quad \forall \omega \in \mathbb{R}^n.$$

Since $\mathrm{e}^{-\lambda(\omega,\cdot)}$ is linear for every $\omega \in \Omega$, it suffices to set

$$\gamma_t = \int_{\Delta} \gamma \pi_t(\gamma) \, \mathrm{d}\gamma;$$

then, we obtain the universal portfolio selection algorithm.

Regret of universal portfolio selection

Theorem 2. Set π_1 to be the uniform distribution on Δ. Then, the universal portfolio selection algorithm satisfies

$$R_T(\gamma) \leq (n-1)\log(T+1), \quad \forall \gamma \in \Delta.$$

Theorem 3. Set π_1 to be the Dirichlet $(1/2, \dots, 1/2)$ distribution on Δ. Then, the universal portfolio selection algorithm satisfies

$$R_T(\gamma) \leq \frac{n-1}{2}\log T + \log \frac{\Gamma(1/2)^n}{\Gamma(n/2)} + \frac{n-1}{2}\log 2 + o(1).$$

T. M. Cover. Universal portfolios. 1991.

T. M. Cover & E. Ordentlich. Universal portfolios with side information. 1996.

Lemma 1. It holds that

$$\prod_{\tau=1}^{T} \int_{\Delta} \mathrm{e}^{-\lambda(\omega_\tau,\gamma)} \pi_\tau(\gamma)\,\mathrm{d}\gamma = \int_{\Delta} \left(\prod_{\tau=1}^{T} \mathrm{e}^{-\lambda(\omega_\tau,\gamma)} \right) \pi_1(\gamma)\,\mathrm{d}\gamma.$$

Remark. Notice the similarity with the mixture forecaster.

Corollary 1. It holds that

$$\prod_{t=1}^{T} \langle \omega_t, \gamma_t \rangle = \int_{\Delta} \left(\prod_{t=1}^{T} \langle \omega_t, \gamma \rangle \right) \pi_1(\gamma)\,\mathrm{d}\gamma.$$

Proof of Theorem 2 (2/6)

Proof. (Lemma 1) Obviously, the lemma holds for $T = 1$. Suppose the lemma holds for $T = T^*$. Then, for $T = T^* + 1$, we write

$$\left[\int_\Delta \mathrm{e}^{-\lambda(\omega_{T^*+1},\gamma)}\pi_{T^*+1}(\gamma)\,\mathrm{d}\gamma\right]\int_\Delta\left(\prod_{\tau=1}^{T^*}\mathrm{e}^{-\lambda(\omega_\tau,\gamma)}\right)\pi_1(\gamma)\,\mathrm{d}\gamma$$
$$= \int_\Delta\left(\prod_{\tau=1}^{T^*+1}\mathrm{e}^{-\lambda(\omega_\tau,\gamma)}\right)\pi_1(\gamma)\,\mathrm{d}\gamma.$$

Proof of Theorem 2 (3/6)

Lemma 2. Consider the individual sequence prediction problem with the logarithmic loss and alphabet $\mathcal{A} \coloneqq \{1, \dots, n\}$. Consider the static hypotheses and the corresponding mixture forecaster defined by π_1. Denote the resulting regret by S_T. Then, it holds that

$$R_T \leq S_T.$$

To continue, we need the following lemma.

Lemma 3. Let $a_1, \dots, a_m, b_1, \dots, b_m$ be positive real numbers. Then, it holds that

$$\frac{\sum_{1 \leq i \leq m} a_i}{\sum_{1 \leq i \leq m} b_i} \leq \max_{1 \leq i \leq m} \frac{a_i}{b_i}.$$

T. M. Cover & E. Ordentlich. Universal portfolios with side information. 1996.

Proof of Theorem 2 (4/6)

Proof. (Lemma 2) Denote the i-th entry of ω_t and γ by $\omega_t(i)$ and $\gamma(i)$, respectively. Define the joint probability defined by a static forecaster γ as

$$p_\gamma(a_{1:T}) := \prod_{t=1}^{T} \gamma(a_t), \quad \forall a_{1:T} \in \mathcal{A}^T.$$

Notice that

$$\begin{aligned}
\prod_{t=1}^{T} \langle \omega_t, \gamma \rangle &= \sum_{a_{1:T} \in \mathcal{A}^T} \prod_{t=1}^{T} \omega_t(a_t)\gamma(a_t) \\
&= \sum_{a_{1:T} \in \mathcal{A}^T} \left(\prod_{t=1}^{T} \omega_t(a_t) \right) p_\gamma(a_{1:T}).
\end{aligned}$$

Proof of Theorem 2 (5/6)

Proof continued. (Lemma 2) Then we write

$$
\begin{aligned}
\frac{\prod_{t=1}^{T} \langle \omega_t, \gamma \rangle}{\prod_{t=1}^{T} \langle \omega_t, \gamma_t \rangle} &= \frac{\sum_{a_{1:T} \in \mathcal{A}^T} \left(\prod_{t=1}^{T} \omega_t(a_t) \right) p_\gamma(a_{1:T})}{\int_\Delta \left[\sum_{a_{1:T} \in \mathcal{A}^T} \left(\prod_{t=1}^{T} \omega_t(a_t) \right) p_\gamma(a_{1:T}) \right] \pi_1(\gamma) \, \mathrm{d}\gamma} \\
&\le \max_{a_{1:T} \in \mathcal{A}^T} \frac{\left(\prod_{t=1}^{T} \omega_t(a_t) \right) p_\gamma(a_{1:T})}{\int_\Delta \left[\left(\prod_{t=1}^{T} \omega_t(a_t) \right) p_\gamma(a_{1:T}) \right] \pi_1(\gamma) \, \mathrm{d}\gamma} \\
&= \max_{a_{1:T} \in \mathcal{A}^T} \frac{p_\gamma(a_{1:T})}{\int_\Delta p_\gamma(a_{1:T}) \pi_1(\gamma) \, \mathrm{d}\gamma},
\end{aligned}
$$

which proves the lemma. We have used Corollary 1 for the first equality and Lemma 3 for the inequality.

Proof of Theorem 2 (6/6)

Proof. (Theorem 2) It remains to study the regret of the Laplace mixture for the possibly non-binary case.

Theorem 4. Consider the individual sequence prediction problem with the logarithmic loss and alphabet $\mathcal{A} \coloneqq \{1, \ldots, n\}$. Consider the static hypotheses and the Laplace mixture. Then, the regret is bounded from above by

$$\log \binom{T + n - 1}{n - 1} \leq (n - 1) \log(T + 1).$$

Remark. Theorem 3 is proved similarly. The corresponding mixture forecaster is the Krichevsky-Trofimov forecaster.

N. Cesa-Bianchi and G. Lugosi. *Prediction, Learning, and Games*. 2006.

Computational complexity issue

Notice universal portfolio selection requires computing an expectation in each iteration.

Theorem 5. (Kalai-Vempala) There is an algorithm of computational complexity $O(T^{14}n^4)$ for universal portfolio selection with the uniform or Dirichlet $(1/2, \ldots, 1/2)$ prior.

Remark. Developing a faster algorithm for universal portfolio selection, or a fast algorithm with a logarithmic regret for online portfolio selection is an open problem.

A. Kalai & S. Vempala. Efficient algorithms for universal portfolios. 2002.

Online convex optimization with exp-concave losses

Convexity (1/2)

Definition. We say a set $\mathcal{X} \subseteq \mathbb{R}^n$ is *convex*, if and only if

$$(1-\alpha)x + \alpha y \in \mathcal{X}, \quad \forall x, y \in \mathcal{X}, \alpha \in [0,1].$$

Definition. We say a function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is *convex*, if and only if

$$f\left((1-\alpha)x + \alpha y\right) \leq (1-\alpha)f(x) + \alpha f(y), \quad \forall x, y \in \mathcal{X}, \alpha \in [0,1].$$

We say f is *concave* if and only if $-f$ is convex.

Theorem 6. (Jensen's inequality) Let ξ be a random variable on a convex set $\mathcal{X}$. Let f be convex on $\mathcal{X}$. Then, it holds that

$$f(\mathsf{E}\,\xi) \leq \mathsf{E}\,f(\xi).$$

Convexity (2/2)

Proposition 1. Let $f : \mathcal{X} \to \mathbb{R}$ be differentiable. Then, f is convex if and only if

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle , \quad \forall x, y \in \mathcal{X}.$$

Proposition 2. Let $f : \mathcal{X} \to \mathbb{R}$ be twice differentiable. Then, f is convex if and only if

$$\nabla^2 f(x) \geq 0, \quad \forall x \in \mathcal{X}.$$

Online convex optimization

Protocol. (Online convex optimization, OCO) Let $\mathcal{X} \subset \mathbb{R}^n$ be a closed convex set. Let $T \in \mathbb{N}$. Let the initial cumulative loss $L_0 = 0$. For every $1 \leq t \leq T$, the following happen in order.

1. Learner announces $x_t \in \mathcal{X}$.
2. Reality announces a *convex* loss $f_t : \mathcal{X} \to \mathbb{R}$.
3. Compute $L_t \leftarrow L_{t-1} + f_t(x_t)$.

M. Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. 2003.

Examples (1/2)

Example. Individual sequence prediction corresponds to

$$\mathcal{X} = \Delta(\mathcal{A}), \quad f_t : x \mapsto \lambda(\omega_t, x),$$

whenever $\lambda(\omega, \cdot)$ is convex for every $\omega \in \mathcal{A}$, where $\Delta(\mathcal{A})$ denotes the set of probability distributions on $\mathcal{A}$.

Remark. It is easily checked the logarithmic, Brier, and absolute losses are convex.

Example. Online portfolio selection is an online convex optimization problem with

$$\mathcal{X} = \Delta, \quad f_t : x \mapsto -\log \langle \omega_t, x \rangle .$$

Example. Consider empirical risk minimization with a parametric hypothesis class $\mathcal{H} = \{ h_\theta \mid \theta \in \Theta \}$:

$$\hat{\theta}_T \in \arg\min_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \lambda(z_t, h_\theta),$$

for some loss function λ and data $z_1, \ldots, z_T \in \mathcal{Z}$. Suppose $\lambda(z, \cdot)$ is convex for every $z \in \mathcal{Z}$. Then, we may solve an online convex optimization problem, with

$$\mathcal{X} = \Theta, \quad f_t : \theta \mapsto \lambda(z_t, h_\theta),$$

and output

$$\overline{\theta}_T := \frac{\theta_1 + \ldots + \theta_T}{T}$$

as an approximate solution of $\hat{\theta}$. (The optimization error guarantee is algorithm-dependent.)

Exp-concavity

Definition. (Exp-concavity) We say a function f is *η-exp-concave*, if and only if $\mathrm{e}^{-\eta f}$ is concave.

Proposition 3. Suppose f is twice differentiable on $\mathcal{X} \subseteq \mathbb{R}$. Then, f is η-exp-concave if and only if

$$\nabla^2 f(x) \geq \eta \nabla f(x) \left[\nabla f(x)\right]^{\mathrm{T}}, \quad \forall x \in \mathcal{X}.$$

Proof. We write

$$\nabla^2 \left(\mathrm{e}^{-\eta f}\right)(x) = -\eta \nabla^2 f(x) \mathrm{e}^{-\eta f(x)} + \eta^2 \nabla f(x) \left[\nabla f(x)\right]^{\mathrm{T}}.$$

The proposition follows from Proposition 2.

J. Kivinen & M. K. Warmuth. Averaging expert predictions. 1999.

E. Hazan *et al.* Logarithmic regret algorithms for online convex optimization. 2007.

Examples

Example. The loss in online portfolio selection (and hence individual sequence prediction) is obviously 1-exp-concave.

Example. Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex. Define the losses

$$f_t : x \mapsto \frac{1}{2} \left(y_t - \langle a_t, x \rangle \right)^2 ,$$

for some $y_t \in \mathbb{R}$ and $a_t \in \mathbb{R}^n$. *Suppose $f_t(x) \in [0,1]$ for every t and x.* Then, every f_t is $(1/2)$-exp-concave.

Logarithmic regret guarantee

Theorem 7. Consider the online convex optimization problem. Suppose each $f_t : \mathcal{X} \to \mathbb{R}$ is η-exp-concave. Then, there exists an algorithm such that

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x) \leq \frac{1}{\eta} n \left[\log\left(1 + \frac{T}{n}\right) + 1 \right], \quad \forall x \in \mathcal{X}.$$

E. Hazan *et al.* Logarithmic regret algorithms for online convex optimization. 2007.

Exponentially weighted online optimization

Algorithm. (Exponentially weighted online optimization, EWOO)
Let π_1 be the uniform distribution on $\mathcal{X}$. For each $1 \le t \le T$, compute

$$x_t = \int_{\mathcal{X}} x \pi_t(x) \, \mathrm{d}x,$$

and update

$$\pi_{t+1}(x) \propto \pi_t(x) \mathrm{e}^{-\eta f_t(x)}, \quad \forall x \in \mathcal{X}.$$

E. Hazan *et al.* Logarithmic regret algorithms for online convex optimization. 2007.

Proof of Theorem 7 (1/4)

Lemma 4. It holds that

$$\prod_{t=1}^{T} \int_{\mathcal{X}} \mathrm{e}^{-\eta f_t(x)} \pi_t(x)\, \mathrm{d}x = \int_{\mathcal{X}} \left[\prod_{t=1}^{T} \mathrm{e}^{-\eta f_t(x)} \right] \pi_1(x)\, \mathrm{d}x.$$

Remark. Notice the similarity with Lemma 1, or relation to the mixture forecaster.

Remark. One can prove Lemma 4 by induction as for Lemma 1. We provide another proof in the next slide.

Proof of Theorem 7 (2/4)

Proof. (Lemma 4) Notice that $\pi_t(x) \propto \pi_1(x)\mathrm{e}^{-\eta\sum_{\tau=1}^{t-1} f_\tau(x)}$. Then, we write

$$\begin{aligned}
\prod_{t=1}^{T} \int_{\mathcal{X}} \mathrm{e}^{-\eta f_t(x)} \pi_t(x)\,\mathrm{d}x &= \prod_{t=1}^{T} \frac{\int_{\mathcal{X}} \mathrm{e}^{-\eta\sum_{\tau=1}^{t} f_\tau(x)} \pi_1(x)\,\mathrm{d}x}{\int_{\mathcal{X}} \mathrm{e}^{-\eta\sum_{\tau=1}^{t-1} f_\tau(x)} \pi_1(x)\,\mathrm{d}x} \\
&= \int_{\mathcal{X}} \mathrm{e}^{-\eta\sum_{t=1}^{T} f_t(x)} \pi_1(x)\,\mathrm{d}x.
\end{aligned}$$

Proof of Theorem 7 (2.5/4)

Lemma 5. It holds that

$$\mathrm{e}^{-\eta \sum_{t=1}^{T} f_t(x_t)} \geq \frac{\int_{\mathcal{X}} \mathrm{e}^{-\eta \sum_{t=1}^{T} f_t(x)} \, \mathrm{d}x}{\mathrm{vol}(\mathcal{X})}.$$

Proof. By the exp-concavity of f_t and Lemma 4, we write

$$\begin{aligned}
\mathrm{e}^{-\eta \sum_{t=1}^{T} f_t(x_t)} &= \prod_{t=1}^{T} \mathrm{e}^{-\eta f_t(x_t)} \geq \prod_{t=1}^{T} \int_{\mathcal{X}} \mathrm{e}^{-\eta f_t(x)} \pi_t(x) \, \mathrm{d}x \\
&= \int_{\mathcal{X}} \left[\prod_{t=1}^{T} \mathrm{e}^{-\eta f_t(x)} \right] \pi_1(x) \, \mathrm{d}x = \frac{\int_{\mathcal{X}} \mathrm{e}^{-\eta \sum_{t=1}^{T} f_t(x)} \, \mathrm{d}x}{\mathrm{vol}(\mathcal{X})}.
\end{aligned}$$

Proof. (Theorem 7) Let $x^\star$ be a minimizer of $\sum_{t=1}^{T} f_t$ on $\mathcal{X}$. Define

$$\mathcal{X}_\alpha := \{\, \alpha x^\star + (1-\alpha)\, y \mid y \in \mathcal{X} \,\},$$

for some $\alpha \in]0,1[$. Then, it holds that $\mathcal{X}_\alpha \subset \mathcal{X}$,

$$\mathrm{vol}(\mathcal{X}_\alpha) = (1-\alpha)^n \, \mathrm{vol}(\mathcal{X}),$$

and

$$\mathrm{e}^{-\eta f_t(x)} \geq \alpha \mathrm{e}^{-\eta f_t(x^\star)}, \quad \forall x \in \mathcal{X}_\alpha.$$

A. Blum & A. Kalai. Universal portfolios with and without transaction costs. 1999.
E. Hazan *et al.* Logarithmic regret algorithms for online convex optimization. 2007.

Proof of Theorem 7 (4/4)

Proof continued. (Theorem 7) By Lemma 5, we write

$$\begin{aligned}
\mathrm{e}^{-\eta \sum_{t=1}^{T} f_t(x_t)} &\geq \frac{\int_{\mathcal{X}_\alpha} \mathrm{e}^{-\eta \sum_{t=1}^{T} f_t(x)} \, \mathrm{d}x}{\mathrm{vol}(\mathcal{X})} \\
&\geq \frac{\alpha^T \mathrm{e}^{-\eta \sum_{t=1}^{T} f_t(x^\star)} \mathrm{vol}(\mathcal{X}_\alpha)}{\mathrm{vol}(\mathcal{X})} \\
&= (1-\alpha)^n \alpha^T \mathrm{e}^{-\eta \sum_{t=1}^{T} f_t(x^\star)}.
\end{aligned}$$

Setting $\alpha = \frac{T}{T+n}$, we obtain

$$\begin{aligned}
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^\star) &\leq \frac{1}{\eta} \left[T \log\left(1 + \frac{n}{T}\right) + n \log\left(1 + \frac{T}{n}\right) \right] \\
&\leq \frac{n}{\eta} \left[1 + \log\left(1 + \frac{T}{n}\right) \right].
\end{aligned}$$

Exponential weights & online gradient descent

Online optimization

Protocol. (Online optimization) Let $\mathcal{X} \subset \mathbb{R}^n$. Let $T \in \mathbb{N}$. Let the initial cumulative loss $L_0 = 0$. For every $1 \leq t \leq T$, the following happen in order.

1. LEARNER announces $x_t \in \mathcal{X}$.
2. REALITY announces $f_t : \mathcal{X} \to \mathbb{R}$.
3. Compute $L_t \leftarrow L_{t-1} + f_t(x_t)$.

Question. What if the losses are not exp-concave?

Exponential weights algorithm

Algorithm. (Exponential weights) Let $\mathcal{P}_{\mathcal{X}} = \{ \pi \mid \mathsf{E}_{\pi}(x) \in \mathcal{X} \}$. Let $\eta > 0$ and $\pi_1 \in \mathcal{P}_{\mathcal{X}}$. For each $1 \le t \le T$, announce

$$x_t := \int_{\mathcal{X}} x \pi_t(x) \, \mathrm{d}x,$$

and compute

$$\begin{aligned} \tilde{\pi}_{t+1}(x) &\propto \pi_t(x) \mathrm{e}^{-\eta f_t(x)}, \quad \forall x \in \mathcal{X}, \\ \pi_{t+1} &\in \operatorname*{arg\,min}_{\pi \in \mathcal{P}_{\mathcal{X}}} D(\pi, \tilde{\pi}_t), \end{aligned}$$

where $D(\pi, \tilde{\pi}_t) := \mathsf{E}_{\pi} \left[\pi / \tilde{\pi}_t \right]$ denotes the relative entropy.

D. van der Hoeven *et al.* The many faces of exponential weights in online learning. 2018.

Regret bound of the exponential weights

Lemma 6. Define the regret function

$$R_T(x) \coloneqq \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x), \quad \forall x \in \mathcal{X}.$$

Suppose that $\mathcal{P}_{\mathcal{X}}$ is closed (with respect to the total variation distance) and convex. Then, for any $Q \in \mathcal{P}_{\mathcal{X}}$, it holds that

$$\mathsf{E}_{x\sim Q} R_T(x) \le \frac{1}{\eta} D(Q, P_1) + \sum_{t=1}^{T} \delta_t,$$

where the *mixability gap* δ_t is defined as

$$\delta_t \coloneqq f_t(x_t) - \left(\frac{-1}{\eta}\right) \log \mathsf{E}_{x\sim\pi_t} \mathrm{e}^{-\eta f_t(x)}.$$

D. van der Hoeven *et al.* The many faces of exponential weights in online learning. 2018.

Proof of Lemma 6 (1/2)

Theorem 8. Let $\mathcal{C}$ be a closed convex set of probability distributions. Let P be a probability distribution and

$$P_{\mathcal{C}} \in \arg\min_{Q \in \mathcal{P}} D(Q, P),$$

such that $D(P_{\mathcal{C}}, P) < +\infty$. Then, it holds that

$$D(Q, P) \geq D(Q, P_{\mathcal{C}}) + D(P_{\mathcal{C}}, P), \quad \forall Q \in \mathcal{C}.$$

Corollary 2. It holds that

$$D(Q, \pi_{t+1}) \leq D(Q, \tilde{\pi}_{t+1}).$$

I. Csiszár. I-divergence geometry of probability distributions and minimization problems. 1975.

Proof of Lemma 6 (2/2)

Proof. (Lemma 6) We write

$$\begin{aligned}\frac{1}{\eta}\left[D(Q,\pi_t) - D(Q,\pi_{t+1})\right] &\geq \frac{1}{\eta}\left[D(Q,\pi_t) - D(Q,\tilde{\pi}_{t+1})\right] \\ &\geq -\mathsf{E}_Q f_t(x) - \frac{1}{\eta}\log \mathsf{E}_{\pi_t} \mathrm{e}^{-\eta f_t(x)}.\end{aligned}$$

Notice that

$$\sum_{t=1}^{T}\left[D(Q,\pi_t) - D(Q,\pi_{t+1})\right] \leq D(Q,\pi_1).$$

We obtain

$$\mathsf{E}_Q\left[-\sum_{t=1}^{T} f_t(x)\right] \leq \frac{1}{\eta} D(Q,\pi_1) + \sum_{t=1}^{T}\frac{1}{\eta}\log \mathsf{E}_{\pi_t}\mathrm{e}^{-\eta f_t(x)}.$$

D. van der Hoeven *et al.* The many faces of exponential weights in online learning. 2018.

Reduction to online linear optimization (1/3)

Recall that if $f : \mathcal{X} \to \mathbb{R}$ is a convex *differentiable* function, then we have

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle\,, \quad \forall y \in \mathcal{X}.$$

Definition. (Subdifferential) The *subdifferential* of a convex function $f : \mathcal{X} \subseteq \mathbb{R}^p \to \mathbb{R}$ at a point x is defined as

$$\partial f(x) \coloneqq \{\, g \in \mathbb{R}^p \mid \forall y \in \mathcal{X} : f(y) \geq f(x) + \langle g, y - x \rangle \,\}\,.$$

We say that f is *subdifferentiable* at x, if $\partial f(x) \neq \emptyset$. We say that $\nabla f(x)$ is a *subgradient* of f at x, if $\nabla f(x) \in \partial f(x)$.

Reduction to online linear optimization (2/3)

Protocol. (Online linear optimization) Let $\mathcal{X} \subset \mathbb{R}^n$ be a closed convex set. Let $T \in \mathbb{N}$. Let the initial cumulative loss $L_0 = 0$. For every $1 \leq t \leq T$, the following happen in order.

1. Learner announces $x_t \in \mathcal{X}$.
2. Reality announces $y_t \in \mathcal{Y}$.
3. Compute $L_t \leftarrow L_{t-1} + \langle y_t, x_t \rangle$.

Reduction on online linear optimization (3/3)

Proposition 4. Suppose that $\mathcal{X}$ is closed and convex, and f_t are convex and subdifferentiable on $\mathcal{X}$. Then, it holds that

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x) \leq \sum_{t=1}^{T} \langle \nabla f_t(x_t), x_t \rangle - \sum_{t=1}^{T} \langle \nabla f_t(x_t), x \rangle, \quad \forall x \in \mathcal{X}.$$

Proof. As f_t are convex and subdifferentiable on $\mathcal{X}$, we have

$$f_t(x_t) - f_t(x) \leq - \langle \nabla f_t(x_t), x - x_t \rangle, \quad \forall x \in \mathcal{X}.$$

Remark. Therefore, for online convex optimization, it suffices to solve an online linear optimization problem with $y_t = \nabla f_t(x_t)$.

Online gradient descent

Consider applying the exponential weights algorithm to the online linear optimization problem, with $y_t = \nabla f_t(x_t)$ and $\pi_t = \mathcal{N}(x_1, I)$ (the Gaussian distribution with expectation x_1 and identity covariance matrix).

Theorem 9. The resulting algorithm is equivalent to *online gradient descent*. For the original online convex optimization problem, it holds that

$$R_T(x) \le \frac{\|x - x_1\|_2^2}{2\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_2^2, \quad \forall x \in \mathcal{X}.$$

Online gradient descent

Algorithm. (Online gradient descent) Let $x_1 \in \mathcal{X}$. For every $1 \leq t \leq T$, compute

$$\tilde{x}_{t+1} = x_t - \eta \nabla f_t(x_t),$$
$$x_{t+1} \in \arg\min_{x \in \mathcal{X}} \|x - \tilde{x}_{t+1}\|_2^2.$$

Corollary 3. Suppose that $\|\nabla f_t(x_t)\|_2 \leq L$ for some $L > 0$. Define $\gamma := \max_{x \in \mathcal{X}} \|x - x_1\|_2$. Setting $\eta = \gamma/(L\sqrt{T})$, online gradient descent achieves

$$R_T(x) \leq L\gamma\sqrt{T}, \quad \forall x \in \mathcal{X}.$$

Proof of Theorem 9 (1/2)

Lemma 7. Let μ and ν be two Gaussian probability distributions. Then, it holds that

$$D(\mu, \nu) \leq D(\tau, \nu),$$

for any probability distribution τ of the same expectation and covariance matrix as μ.

Sketch of proof. Define the *differential entropy*

$$h(\tau) := \mathsf{E}_\tau \left[\log \tau\right].$$

It is known that among probability distributions of a given expectation and covariance matrix, the Gaussian distribution achieves the largest differential entropy.

S. Ihara. *Information Theory for Continuous Systems*. 1993.

Proof of Theorem 9 (2/2)

Proof. (Theorem 9) Recall that

$$\pi_1(x) \propto \exp\left(-\frac{1}{2}\|x - x_1\|_2^2\right), \quad \forall x \in \mathbb{R}^p.$$

It is easily checked that $\tilde{\pi}_t = \mathcal{N}(\tilde{x}_t, I)$. By Lemma 7, we have $\pi_t = \mathcal{N}(x_t, I)$. (Why???)

The regret bound follows from Lemma 6 with $Q = \mathcal{N}(x, I)$.

D. van der Hoeven *et al.* The many faces of exponential weights in online learning. 2018.

Conclusions

Conclusions

- EWOO may be viewed as the aggregating algorithm with a continuum of experts.
- When the loss is exp-concave, the EWOO yields a logarithmic regret.
- Universal portfolio selection may be viewed as a special case of EWOO, but it has a refined analysis via its connection to individual sequence prediction.
- Online gradient descent is a special case of exponential weights, and achieves a $O\left(\sqrt{T}\right)$ regret for online convex optimization with bounded subgradients.

Next lecture

- Aggregating algorithm continued.