CSIE5410 Optimization algorithms

Lecture 11: Follow-the-leader-type methods

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Abstract

In previous lectures, we have introduced online convex optimization from the perspective of mirror descent and learning in games.

In this lecture, we introduce another line of research, providing another perspective of online convex optimization.

Recommended reading

- A. Kalai and S. Vempala. 2005. Efficient algorithms for online decision problems.
- S. Shalev-Shwartz. 2011. Online learning and online convex optimization. Section 2.
- E. Hazan. 2016. Introduction to online convex optimization. Section 5.
- *J. Abernethy *et al.* 2016. Perturbation techniques in online learning and optimization.

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Follow-the-leader

Protocol of online optimization

Protocol. Let $T \in \mathbb{N}$ and $L_0 = 0$. For $t = 1, \dots, T$, the following happen sequentially.

1. $\textsc{Learner}$ announces $x_t \in \mathcal{X}$.
2. $\textsc{Reality}$ announces $f_t : \mathcal{X} \to \mathbb{R}$.
3. $\textsc{Learner}$ updates their loss: $L_t \leftarrow L_{t-1} + f_t(x_t)$.

Definition. The *regret* is given by

$$R_T := \sum_{t=1}^{T} f_t(x_t) - \min_x \left\{ \sum_{t=1}^{T} f_t(x) \,\middle|\, x \in \mathcal{X} \right\}.$$

Fictitious play in a two-player zero-sum game: One player's perspective

Protocol. Let $T \in \mathbb{N}$ and $x_0 \in \Delta_p$. Let $A \in \mathbb{R}^{p \times q}$ For $t = 1, \ldots, T$, the following happen sequentially.

1. LEARNER announces $x_t \in \Delta_p$.
2. REALITY announces $f_t : x \mapsto \langle x, Ay_t \rangle$ for some $y_t \in \Delta_q$.
3. LEARNER outputs their loss: $L_t \leftarrow L_{t-1} + f_t(x_t)$.

Fictitious play. Choose $x_1 \in \Delta_p$, and

$$x_t \in \arg\min_x \left\{ \sum_{\tau=1}^{t-1} f_\tau(x) \;\middle|\; x \in \Delta_p \right\}, \quad \forall t \in \{ 2, \ldots, T \}.$$

Remark. If both players do the fictitious play, then the time average of their strategies converges to a Nash equilibrium.

Follow-the-leader in an online optimization problem

Protocol. Let $T \in \mathbb{N}$ and $L_0 = 0$. For $t = 1, \dots, T$, the following happen sequentially.

1. $\textsc{Learner}$ announces $x_t \in \mathcal{X}$.
2. $\textsc{Reality}$ announces $f_t : \mathcal{X} \to \mathbb{R}$.
3. $\textsc{Learner}$ outputs their loss: $L_t \leftarrow L_{t-1} + f_t(x_t)$.

Follow the leader (FTL). Choose $x_1 \in \mathcal{X}$, and

$$x_t \in \arg\min_x \left\{ \sum_{\tau=1}^{t-1} f_\tau(x) \;\middle|\; x \in \mathcal{X} \right\}, \quad \forall t \in \{ 2, \dots, T \}.$$

A. Kalai and S. Vempala. 2005. Efficient algorithms for online decision problems.

FTL may fail. (1/2)

Consider the following *decision theoretic online learning* problem.

Protocol. Let $T \in \mathbb{N}$ and $L_0 = 0$. For $t = 1, \ldots, T$, the following happen sequentially.

1. LEARNER announces $x_t \in \Delta_2$.
2. REALITY announces $f_t : x \mapsto \langle x, y_t \rangle$ for some $y_t \in \mathbb{R}^2$.
3. LEARNER updates their loss: $L_t \leftarrow L_{t-1} + f_t(x_t)$.

Remark. Notice that LEARNER may always choose experts in an almost sure manner.

FTL may fail. (2/2)

Proposition. Suppose that

$$y_t = \begin{cases} (0, 1/2) & \text{, if } t = 1, \\ (1, -1) & \text{, if } t \text{ is even}, \\ (-1, 1) & \text{, otherwise.} \end{cases}$$

Then the regret of FTL is not sub-linear.

Proof. The FTL achieves a cumulative loss larger than or equal to $(T-1)$. However, if one always choose $x_t = (1, 0)$, then the cumulative loss is bounded by 1. Hence, we have $R_T \geq T - 2$.

A. Kalai and S. Vempala. 2005. Efficient algorithms for online decision problems.

Be-the-leader

Suppose that $\textsc{Learner}$ can access f_t in the t-th round.

Be-the-leader. Choose

$$x_t \in \arg\min_x \left\{ \sum_{\tau=1}^{t} f_\tau(x) \,\middle|\, x \in \mathcal{X} \right\}, \quad \forall t \in \{ 1, \dots, T \} .$$

Theorem. Be-the-leader achieves

$$R_T \coloneqq \sum_{t=1}^{T} f_t(x_{t+1}) - \min_x \left\{ \sum_{t=1}^{T} f_t(x) \,\middle|\, x \in \mathcal{X} \right\} \leq 0, \quad \forall T \in \mathbb{N}.$$

Remark. Therefore, it is desirable to do one-step look-ahead.

Proof of the theorem

Proof. We prove by induction. The inequality obviously holds for $T = 1$. Suppose that the inequality holds for $T - 1$. Then, we write

$$\begin{aligned}
\sum_{t=1}^{T} f_t(x_{t+1}) &\leq f_T(x_{T+1}) + \sum_{t=1}^{T-1} f_t(x_{T+1}) \\
&= \sum_{t=1}^{T} f_t(x_{T+1}) \\
&\leq \sum_{t=1}^{T} f_t(x), \quad \forall x \in \mathcal{X}.
\end{aligned}$$

FTL vs. be-the-leader

The following result immediately follows.

Corollary. Suppose that LEARNER adopts the FTL approach. Then, it holds that

$$R_T \leq \sum_{t=1}^{T} \left[f_t(x_t) - f_t(x_{t+1}) \right], \quad \forall T \in \mathbb{N}.$$

Remark. Therefore, if the sequence $(x_t)_{t \in \mathbb{N}}$ does not change rapidly, then the regret is small.

FTL with strongly convex losses

Theorem. If the losses f_t are μ-strongly convex for some $\mu > 0$, then FTL achieves

$$R_T \leq \frac{2}{\mu} \sum_{t=1}^{T} \frac{\|\nabla f_t(x_t)\|_2^2}{t}.$$

Corollary. If in addition, $\|\nabla f_t(x_t)\|_2 \leq L$ for some $L \geq 0$, then

$$R_T \leq \frac{2L^2}{\mu} \left(1 + \log T\right).$$

Question. Why is it possible to achieve a sub-linear regret now?

S. M. Kakade and S. Shalev-Shwartz. 2008. Mind the duality gap: Logarithmic regret algorithms for online optimization.

Proof of the theorem (1/3)

Proof. We start with the comparison with follow-the-leader:

$$
\begin{aligned}
R_T &\leq \sum_{t=1}^{T} [f_t(x_t) - f_t(x_{t+1})] \\
&\leq \sum_{t=1}^{T} \langle \nabla f_t(x_t), x_t - x_{t+1} \rangle \\
&\leq \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_2 \|x_t - x_{t+1}\|_2.
\end{aligned}
$$

Then we prove that $\|x_{t+1} - x_t\|_2$ is small when t is large (why?).

Proof of the theorem (2/3)

Proof continued. For convenience, we define

$$f_{1:t} \coloneqq f_1 + \cdots + f_t.$$

By the optimality condition and monotonicity of the subgradient, we have

$$\langle \nabla f_{1:t-1}(x_t), x_{t+1} - x_t \rangle \geq 0,$$

and

$$\begin{aligned} \langle \nabla f_{1:t-1}(x_{t+1}), x_t - x_{t+1} \rangle &\geq - \langle \nabla f_t(x_{t+1}), x_t - x_{t+1} \rangle \\ &\geq \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle . \end{aligned}$$

Summing up the two inequalities, we obtain

$$\begin{aligned} &\langle \nabla f_{1:t-1}(x_{t+1}) - \nabla f_{1:t-1}(x_t), x_{t+1} - x_t \rangle \\ &\quad \leq \| \nabla f_t(x_t) \|_2 \| x_t - x_{t+1} \|_2. \end{aligned}$$

Proof of the theorem (3/3)

Proof continued. Notice that $f_{1:t-1}$ is $[\mu(t-1)]$-strongly convex. We write

$$\begin{aligned}\mu\,(t-1)\,\|x_{t+1}-x_t\|_2^2 &\leq \langle \nabla f_{1:t-1}(x_{t+1}) - \nabla f_{1:t-1}(x_t), x_{t+1}-x_t\rangle \\ &\leq \|\nabla f_t(x_t)\|_2 \|x_{t+1}-x_t\|_2.\end{aligned}$$

Therefore, we obtain

$$\|x_{t+1}-x_t\|_2 \leq \frac{1}{\mu(t-1)}\|\nabla f_t(x_t))\|_2.$$

It remains to notice that $t \leq 2(t-1)$ for all $t \geq 2$.

Remark. The proof here is different from the original one.

Follow-the-regularized-leader

How do we deal with possibly non-strongly convex losses?

Follow-the-regularized-leader (FTRL). Let h be a function 1-strongly convex on $\mathcal{X}$ with respect to a norm $\|\cdot\|$. FTRL iterates as follows:

$$x_1 \leftarrow \arg\min_x \{\, \mu h(x) \mid x \in \mathcal{X} \,\},$$

$$x_t \leftarrow \arg\min_x \{\, (f_1 + \cdots + f_{t-1})(x) + \mu h(x) \mid x \in \mathcal{X} \,\}, \quad \forall t \geq 2,$$

for some $\mu > 0$.

Remark. Let $x_0 \in \mathcal{X}$. FTRL is basically FTL with $f_0 := \mu h$.

S. Shalev-Shwartz. 2007. *Online Learning: Theory, Algorithms, and Applications*.

Relating FTRL and FTL

Lemma. Suppose that

$$\max_x \{ h(x) \mid x \in \mathcal{X} \} - h(x_1) \le R^2,$$

for some positive real R. Then, FTRL achieves

$$R_T \le \mu R^2 + \sum_{t=1}^{T} \left[f_t(x_t) - f_t(x_{t+1}) \right].$$

Observation. We know that if μ is large (and f_t have bounded gradients), then the second term in the bound should be small; however, then the first term becomes large. Therefore, there is a trade-off regarding the value of μ.

Proof of the lemma

Proof. Let $x_0 \in \mathcal{X}$. Notice that FTRL is FTL with $f_0 \coloneqq \mu h$. We write

$$\sum_{t=0}^{T} f_t(x_t) - \sum_{t=0}^{T} f_t(x) \leq \sum_{t=0}^{T} \left[f_t(x_t) - f_t(x_{t+1}) \right], \quad \forall x \in \mathcal{X}.$$

Then, we get

$$\begin{aligned}
&\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x) \\
&\quad \leq \mu \left[h(x) - h(x_0) \right] + \left\{ \mu \left[h(x_0) - h(x_1) \right] + \sum_{t=1}^{T} \left[f_t(x_t) - f_t(x_{t+1}) \right] \right\} \\
&\quad = \mu \left[h(x) - h(x_1) \right] + \sum_{t=1}^{T} \left[f_t(x_t) - f_t(x_{t+1}) \right], \quad \forall x \in \mathcal{X}.
\end{aligned}$$

Regret bound for FTRL

Theorem. Suppose that f_t are convex on $\mathcal{X}$. Then, FTRL achieves

$$R_T \leq \mu R^2 + \frac{1}{\mu} \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_*^2.$$

Corollary. Suppose that in addition, $\|\nabla f_t(x_t)\|_* \leq L$ for some $L \geq 0$. Then, setting

$$\mu = \frac{L\sqrt{T}}{R},$$

FTRL achieves

$$R_T \leq 2RL\sqrt{T}.$$

Proof of the theorem (1/2)

The proof is similar to that for FTL with strongly convex losses.

Proof. By the optimality condition and monotonicity of the subgradient, we have

$$\langle \nabla(\mu h + f_{1:t-1})(x_t), x_{t+1} - x_t \rangle \geq 0,$$

and

$$\begin{aligned} \langle \nabla(\mu h + f_{1:t-1})(x_{t+1}), x_t - x_{t+1} \rangle &\geq - \langle \nabla f_t(x_{t+1}), x_t - x_{t+1} \rangle \\ &\geq \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle . \end{aligned}$$

Summing up the two inequalities, we obtain

$$\begin{aligned} &\langle \nabla(\mu h + f_{1:t-1})(x_{t+1}) - \nabla(\mu h + f_{1:t-1})(x_t), x_{t+1} - x_t \rangle \\ &\quad \leq \|\nabla f_t(x_t)\|_* \|x_{t+1} - x_t\|. \end{aligned}$$

Proof of the theorem (2/2)

Proof continued. Notice that $\mu h + f_{1:t-1}$ is μ-strongly convex. We have

$$\mu \|x_{t+1} - x_t\|^2 \leq \|\nabla f_t(x_t)\|_* \|x_{t+1} - x_t\|;$$

which implies

$$\|x_{t+1} - x_t\| \leq \frac{1}{\mu} \|\nabla f_t(x_t)\|_*.$$

Therefore, we write

$$\begin{aligned} \sum_{t=1}^{T} [f_t(x_t) - f_t(x_{t+1})] &\leq \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_* \|x_{t+1} - x_t\| \\ &\leq \frac{1}{\mu} \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_*^2. \end{aligned}$$

It remains to apply the lemma relating FTRL and FTL with strongly convex losses.

Linearized FTRL and lazy online mirror descent

Linearized FTRL (1/2)

Define $g_t(x) \coloneqq \langle \nabla f_t(x_t), x \rangle$. Then we have

$$\begin{aligned} R_T &= \max_x \left\{ \sum_{t=1}^{T} [f_t(x_t) - f_t(x)] \;\middle|\; x \in \mathcal{X} \right\} \\ &\leq \max_x \left\{ \sum_{t=1}^{T} \langle \nabla f_t(x_t), x_t - x \rangle \;\middle|\; x \in \mathcal{X} \right\} \\ &= \max_x \left\{ \sum_{t=1}^{T} [g_t(x_t) - g_t(x)] \;\middle|\; x \in \mathcal{X} \right\}. \end{aligned}$$

Observation. Therefore, it suffices to minimize the regret in the *online linear optimization problem* with losses given by g_t.

Linearized FTRL (2/2)

Linearized FTRL.

$$x_1 \leftarrow \arg\min_x \{\, \mu h(x) \mid x \in \mathcal{X} \,\},$$

$$x_t \leftarrow \arg\min_x \left\{ \left[\sum_{\tau=1}^{t-1} \langle \nabla f_\tau(x_\tau), x \rangle\right] + \mu h(x) \;\middle|\; x \in \mathcal{X} \right\}, \quad \forall t \geq 2.$$

Corollary. Suppose that h is 1-strongly convex with respect to a norm $\|\cdot\|$. Suppose that f_t are convex and $\|\nabla f_t(x_t)\|_* \leq L$ for all t. Define R as in previous slides. Then, setting $\mu = R^{-1}L\sqrt{T}$, the linearized FTRL achieves

$$R_T \leq 2RL\sqrt{T}.$$

Equivalence with the lazy online mirror descent (1/2)

Recall the online mirror descent (OMD): Let $x_1 \in \mathcal{X}$. Set

$$x_{t+1} \in \arg\min_x \left\{ \eta \left\langle \nabla f_t(x_t), x - x_t \right\rangle + D_h(x, x_t) \mid x \in \mathcal{X} \right\}, \quad \forall t \in \mathbb{N}.$$

Recall that the OMD can be written as: Let $x_1 \in \mathcal{X}$. Set

$$\begin{aligned} y_{t+1} &\leftarrow (\nabla h)^{-1} \left[\nabla h(x_t) - \eta \nabla f_t(x_t) \right], \quad \forall t \in \mathbb{N}, \\ x_{t+1} &\in \arg\min_x \left\{ D_h(x, y_{t+1}) \mid x \in \mathcal{X} \right\}, \quad \forall t \in \mathbb{N}. \end{aligned}$$

Equivalence with the lazy online mirror descent (2/2)

Lazy OMD. Let y_1 be such that $\nabla h(y_1) = 0$. Set

$$x_1 \in \arg\min_x \{ D_h(x, y_1) \mid x \in \mathcal{X} \},$$

$$y_{t+1} \leftarrow (\nabla h)^{-1} \left[\nabla h(y_t) - \eta \nabla f_t(x_t) \right], \quad \forall t \in \mathbb{N},$$

$$x_{t+1} \in \arg\min_x \{ D_h(x, y_{t+1}) \mid x \in \mathcal{X} \}, \quad \forall t \in \mathbb{N}.$$

Theorem. The linearized FTRL is equivalent to the lazy OMD with $\mu = \eta^{-1}$.

Yu. Nesterov. 2009. Primal-dual subgradient methods for convex problems.

E. Hazan and S. Kale. 2008. Extracting certainty from uncertainty: Regret bounded by variation in costs.

L. Xiao. 2010. Dual averaging methods for regularized stochastic learning and online optimization.

Proof of the equivalence (1/2)

Proof. By definition, we have

$$x_1 \in \arg\min_x \{ \, h(x) \mid x \in \mathcal{X} \, \} \,,$$

which coincides with our choice of the first iterate in the linearized FTRL.

Notice that by definition, we have

$$\begin{aligned}
\nabla h(y_{t+1}) = \nabla h(y_t) - \eta \nabla f_t(x_t) = \cdots &= \nabla h(y_1) - \eta \sum_{\tau=1}^{t} \nabla f_\tau(x_\tau) \\
&= -\eta \sum_{\tau=1}^{t} \nabla f_\tau(x_\tau).
\end{aligned}$$

Proof of the equivalence (2/2)

Proof. The optimality condition for x_{t+1} gives

$$\langle \nabla h(x_{t+1}) - \nabla h(y_{t+1}), x - x_{t+1} \rangle \geq 0, \quad \forall x \in \mathcal{X}.$$

Plugging in our expression of $\nabla h(y_{t+1})$, we obtain

$$\langle \nabla h(x_{t+1}) + \eta \sum_{\tau=1}^{t} \nabla f_\tau(x_\tau), x - x_{t+1} \rangle \geq 0, \quad \forall x \in \mathcal{X},$$

which is the optimality condition for the convex optimization problem:

$$x_{t+1} \in \arg\min_x \left\{ \left[\sum_{\tau=1}^{t} \langle \nabla f_\tau(x_\tau), x \rangle \right] + \eta^{-1} h(x) \;\middle|\; x \in \mathcal{X} \right\}$$

Implications of the equivalence

Observation. The linearized FTRL is computationally as easy as OMD.

Corollary. Suppose that h is 1-strongly convex with respect to a norm $\|\cdot\|$. Suppose that f_t are convex and $\|\nabla f_t(x_t)\|_* \leq L$ for all t. Define R as in previous slides. Then, setting $\eta = \frac{R}{L\sqrt{T}}$, the lazy OMD achieves

$$R_T \leq 2RL\sqrt{T}.$$

Remark. The linearized FTRL is also known as Nesterov's dual averaging.

Follow-the-perturbed-leader

Follow-the-perturbed-leader

Consider the *decision-theoretic online learning problem*.

Follow-the-perturbed-leader (FTPL).

- In the t-th round, pick random numbers $\nu_1, \ldots, \nu_m$, and choose $\textsc{Expert}$

  $$i_t \in \arg\min_i \left\{ L_{i,t-1} + \frac{1}{\eta}\nu_i \;\middle|\; i \in \{1, \ldots, m\} \right\},$$

  where $L_{i,t-1}$ denotes the cumulative loss of $\textsc{Expert}$ i up to the $(t-1)$-th round.

J. Hannan. 1957. Approximation to Bayes risk in repeated play.

D. Fudenberg and D. M. Kreps. 1993. Learning mixed equilibria.

A. T. Kalai and S. Vempala. 2005. Efficient algorithms for online decision problems.

Gumbel distribution

Gumbel distribution. The probability distribution of the standard Gumbel random variable ξ is given by

$$\mathsf{P}\left(\xi \le x\right) = \mathrm{e}^{-\mathrm{e}^{-x}}, \quad \forall x \in \mathbb{R}.$$

Theorem. Let $v_1, \ldots, v_m \in \mathbb{R}$. Let $\xi_1, \ldots, \xi_m$ be independent standard Gumbel random variables. Define

$$i^\star = \arg\min_i \left\{ v_i - \eta^{-1}\xi_i \;\middle|\; i \in \{ 1, \ldots, m \} \right\},$$

for some $\eta > 0$. Then,

$$\mathsf{P}\left(i^\star = i\right) = \frac{\mathrm{e}^{-\eta v_i}}{\sum_{j=1}^m \mathrm{e}^{-\eta v_j}}.$$

S. Arora *et al.* 2012. The multiplicative weights update method: A meta-algorithm and applications.

Proof of the theorem (1/3)

Definition. The probability distribution function of an exponential random variable ζ of parameter $\lambda > 0$ is given by

$$\mathsf{P}\left(\zeta \leq y\right) = \begin{cases} 1 - \mathrm{e}^{-\lambda y} & \text{, if } y > 0 \\ 0 & \text{, otherwise.} \end{cases}$$

Lemma. Let ξ be a standard Gumbel random variable. Then $\zeta := \mathrm{e}^{-\xi}$ is an exponential random variable of parameter $\lambda = 1$.

Proof. We write

$$\mathsf{P}\left(\zeta \leq y\right) = \mathsf{P}\left(\xi \geq -\log y\right) = 1 - \mathrm{e}^{-\mathrm{e}^{\log y}} = 1 - \mathrm{e}^{-y}, \quad \forall y > 0.$$

Obviously, $\mathsf{P}(\zeta \leq y) = 0$ for all $y \leq 0$.

Proof of the theorem (2/3)

Lemma. Let $\zeta_1, \ldots, \zeta_m$ be independent exponential random variables of parameter 1. Let $w_1, \ldots, w_m \in]0, +\infty[$. Then,

$$\mathsf{P}\left(\arg\max_i \frac{w_i}{\zeta_i} = j\right) = \frac{w_j}{\sum_{i=1}^m w_i}, \quad \forall j \in \{\, 1, \ldots, m \,\}.$$

Proof. By independence of the random variables, we write

$$\mathsf{P}\left(\frac{w_i}{\zeta_i} \leq \frac{w_j}{\zeta_j},\ \forall i \neq j \middle| \zeta_j\right) = \prod_{i \neq j} \left(\mathrm{e}^{-\frac{w_i \zeta_j}{w_j}}\right) = \mathrm{e}^{-\sum_{i \neq j} \frac{w_i \zeta_j}{w_j}}.$$

Then,

$$\mathsf{P}\left(\frac{w_i}{\zeta_i} \leq \frac{w_j}{\zeta_j},\ \forall i \neq j\right) = \int_{[0,+\infty[} \mathrm{e}^{-\sum_{i \neq j} \frac{w_i \zeta_j}{w_j}} \mathrm{e}^{-\zeta_j} \, \mathrm{d}\zeta_j.$$

Proof of the theorem (3/3)

Proof of the theorem. By monotonicity of the exponential function, we have

$$\begin{aligned}
&\arg\min_i \left\{ v_i - \eta^{-1}\xi_i \;\middle|\; i \in \{ 1, \ldots, m \} \right\} \\
&\quad = \arg\min_i \{ \eta v_i - \xi_i \mid i \in \{ 1, \ldots, m \} \} \\
&\quad = \arg\max_i \left\{ \frac{\mathrm{e}^{-\eta v_i}}{\mathrm{e}^{-\xi_i}} \;\middle|\; i \in \{ 1, \ldots, m \} \right\}.
\end{aligned}$$

Then, the theorem follows from the lemma in the previous slide.

Implication

Corollary. Suppose that we choose $\nu_1, \ldots, \nu_m$ as independent negative standard Gumbel random variables. Then FTPL is equivalent to the hedge algorithm, and achieves

$$R_T = O\left(\sqrt{T \log m}\right).$$

Remark. Then, FTPL provides a computationally easy approach to implementing the hedge algorithm.

Remark. There is a principled approach to analyzing FTPL algorithms based on *randomized smoothing*.

J. Abernethy *et al.* 2016. Perturbation techniques in online learning and optimization.

Warning

The well-known version of FTPL does not use Gumbel random variables.

Follow-the-perturbed-leader (FTPL, well-known version).

- In the t-th round, pick *uniform random numbers* $\nu_1, \ldots, \nu_m \in [0, 1]$, and choose $\textsc{Expert}$

$$i_t \in \arg\min_i \left\{ L_{i,t-1} + \frac{1}{\eta}\nu_i \;\middle|\; i \in \{1, \ldots, m\} \right\},$$

where $L_{i,t-1}$ denotes the cumulative loss of $\textsc{Expert}$ i up to the $(t-1)$-th round.

A. Kalai and S. Vempala. 2005. Efficient algorithms for online decision problems.

Conclusions

Summary

- We started with FTL. An important result is the comparison between FTL and be-the-leader.

- By the comparison result (aka FTL-BTL lemma), we have shown that FTRL achieves a sublinear regret.

- FTRL is equivalent to the lazy OMD.

- A version of FTPL is equivalent to the hedge algorithm.

Next lecture

- TBD.