CSIE5002 Prediction, learning, and games

Lecture 10: Defensive forecasting & quantile bound

Yen-Huan Li (yenhuan.li@csie.ntu.edu.tw)

27.05.2019

Department of Computer Science and Information Engineering
National Taiwan University

Abstract

The $O\left(\sqrt{T \log K}\right)$ regret achieved by the hedge is indeed optimal. The optimality, however, is with respect to the worst case. In practice, we may not encounter the worst case. Can we develop an algorithm that achieves a significantly smaller regret when the situation is easy?

Recommended reading

- A. Chernov & V. Vovk. Prediction with advice of unknown number of experts. 2010.
- A. Chernov *et al.* Supermartingales in prediction with expert advice. 2010.
- W. M. Koolen & T. van Erven. Second-order quantile methods for experts and combinatorial games. 2015.
- V. Vovk. Prediction as statements and decisions. 2006.

Table of contents

Defensive forecasting

Recap: Learning with expert advice

Protocol. (Learning with expert advice) Let $T, K \in \mathbb{N}$. Let Γ and Ω be given sets. Let the loss function $\lambda : \Omega \times \Gamma \to \mathbb{R}$. For every $1 \le t \le T$, the following happen sequentially.

- $\textsc{Expert}$-i announces $\gamma_t(i) \in \Gamma$, $1 \le i \le K$.
- $\textsc{Learner}$ announces $\gamma_t \in \Gamma$.
- $\textsc{Reality}$ announces $\omega_t \in \Omega$.
- $\textsc{Learner}$ suffers the loss $\lambda(\omega_t, \gamma_t)$.

Regret. The regret is given by

$$R_T(i) \coloneqq \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t) - \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t(i)), \quad 1 \le i \le K.$$

Recap: Aggregating algorithm

Algorithm. (Aggregating algorithm) Let $(w_1(i))_{1 \le i \le K}$ be a probabilitfy vector. For every $1 \le t \le T$, announce any γ_t such that

$$\lambda(\omega, \gamma_t) \le \frac{-1}{\eta} \log \sum_{i=1}^{K} \frac{w_t(i)}{\sum_{j=1}^{K} w_t(j)} \mathrm{e}^{-\eta \lambda(\omega, \gamma_t(i))}, \quad \forall \omega \in \Omega,$$

and after observing ω_t, compute

$$w_{t+1}(i) = w_t(i) \mathrm{e}^{-\eta \lambda(\omega_t, \gamma_t(i))}, \quad \forall 1 \le i \le K.$$

Mixability condition revisited (1/2)

The AA requires for every $1 \leq t \leq T$, there exists some γ_t such that

$$\lambda(\omega, \gamma_t) \leq \frac{-1}{\eta} \log \sum_{i=1}^{K} \frac{w_t(i)}{\sum_{j=1}^{K} w_t(j)} \mathrm{e}^{-\eta \lambda(\omega, \gamma_t(i))}, \quad \forall \omega \in \Omega.$$

Proposition 1. The condition above is equivalent to requiring for every $1 \leq t \leq T$, there exists some γ_t such that

$$\begin{aligned} &\sum_{i=1}^{K} w_1(i) \mathrm{e}^{\eta \left\{ [\lambda(\omega,\gamma_t) - \lambda(\omega,\gamma_t(i))] + \sum_{\tau=1}^{t-1} [\lambda(\omega_\tau,\gamma_\tau) - \lambda(\omega_\tau,\gamma_\tau(i))] \right\}} \\ &\qquad \leq \sum_{i=1}^{K} w_1(i) \mathrm{e}^{\eta \sum_{\tau=1}^{t-1} [\lambda(\omega_\tau,\gamma_\tau) - \lambda(\omega_\tau,\gamma_\tau(i))]}, \quad \forall \omega \in \Omega. \end{aligned}$$

A. Chernov *et al.* Supermartingales in prediction with expert advice. 2010.

Proof of Proposition 1

Proof. (Proposition 1) The condition under consideration is equivalent to requiring

$$\sum_{i=1}^{K} w_t(i) \geq \sum_{i=1}^{K} w_t(i) \mathrm{e}^{-\eta\lambda(\omega,\gamma_t(i))} \mathrm{e}^{\eta\lambda(\omega,\gamma_t)}.$$

Multiplying both sides by $\mathrm{e}^{\eta \sum_{\tau=1}^{t-1} \lambda(\omega_\tau,\gamma_\tau)}$ and noticing that

$$w_t(i) = w_1(i) \mathrm{e}^{-\eta \sum_{\tau=1}^{t-1} \lambda(\omega_\tau,\gamma_\tau(i))},$$

the proposition follows.

Mixability revisited (2/2)

Corollary 1. The AA satisfies

$$R_T(i) \leq \frac{1}{\eta} \log \frac{1}{w_1(i)}.$$

Proof. By Proposition 1, we obtain

$$\sum_{i=1}^{K} w_1(i) \mathrm{e}^{\eta \sum_{t=1}^{T} [\lambda(\omega_t, \gamma_t) - \lambda(\omega_t, \gamma_t(i))]} \leq \sum_{i=1}^{K} w_1(i) = 1.$$

Then, every summand in the inequality above is bounded above by 1. The corollary follows.

Defensive forecasting

Prototypical algorithm. (Defensive forecasting) For every $1 \leq t \leq T$, define $f_t : \Omega \times \Gamma \to \mathbb{R}$, and announce any γ_t such that

$$f_t(\gamma_t, \omega) \leq f_{t-1}(\gamma_{t-1}, \omega_{t-1}), \quad \forall \omega \in \Omega.$$

Remark. We set f_0 as a constant, and expect a regret bound can be derived via the inequality $f_T(\gamma_T, \omega_T) \leq f_0$.

Question. When is a defensive forecasting algorithm well-defined?

V. Vovk *et al.* Defensive forecasting. 2005.

(Game theoretic) supermartingale (1/2)

Definition. (~~Stochastic~~ process) A *~~stochastic~~ process* is an operator S that for any given sequence

$$(\pi_\tau, \omega_\tau)_{1 \leq \tau \leq t-1} \in (\Delta(\Omega) \times \Omega)^{t-1},$$

assigns a function $S_t : \Delta(\Omega) \times \Omega \to \mathbb{R}$.

Definition. (Supermartingale) A *(game theoretic) supermartingale* is a process S such that for any $T \in \mathbb{N}$, $(\pi_t, \omega_t)_{t \in \mathbb{N}}$, and $\pi \in \Delta(\Omega)$,

$$\mathsf{E}_{\omega \sim \pi} S_T(\pi, \omega) \leq S_{T-1}(\pi_{T-1}, \omega_{T-1}).$$

Remark. Notice that this is not the probabilistic supermartingale.

A. Chernov *et al.* Supermartingales in prediction with expert advice. 2010.

(Game theoretic) supermartingale (2/2)

Theorem 1. (Levin) Let Ω be a compact metric space. Let $q : \Delta(\Omega) \times \Omega \to \mathbb{R}$ such that $q(\cdot, \omega)$ is continuous for every $\omega \in \Omega$. Suppose that

$$\mathsf{E}_{\omega \sim \pi}\, q(\pi, \omega) \leq C, \quad \forall \pi \in \Delta(\Omega).$$

for some $C \in \mathbb{R}$. Then, there exists some $\pi \in \Delta(\Omega)$ such that

$$q(\pi, \omega) \leq C, \quad \forall \omega \in \Omega.$$

Proof sketch. Apply Sperner's lemma or Ky Fan's minimax theorem.

L. A. Levin. Uniform tests of randomness. 1976.

A. Chernov *et al.* Supermartingales in prediction with expert advice. 2010.

Example

Consider individual binary sequence prediction with the logarithmic loss, in which the prediction space is $[0, 1]$. For each prediction $\gamma \in [0, 1]$, associate with it $\pi \in \Delta(\{ 0, 1 \})$ such that $\pi(1) = \gamma$. Consider the process S given by

$$S_t(\pi, \omega) := \sum_{i=1}^{K} \mathrm{e}^{\eta[\lambda(\omega,\pi(1))-\lambda(\omega,\gamma_t(i))]+\eta\sum_{\tau=1}^{t-1}[\lambda(\omega_\tau,\gamma_\tau)-\lambda(\omega_\tau,\gamma_\tau(i))]}.$$

Proposition 2. The process S defined above is a supermartingale for every $\eta \in]0, 1]$.

Proof sketch. Show that $\mathsf{E}_{\omega\sim\mathrm{Bernoulli}(p)}\mathrm{e}^{\eta[\lambda(\omega,p)-\lambda(\omega,p')]} \leq 1$ for every p, p'.

A. Chernov *et al.* Supermartingales in prediction with expert advice. 2010.

Example (2/2)

Remark. By Theorem 1, for every $t \in \mathbb{N}$, there exits some $\pi_t \in \Delta(\Omega)$ such that

$$S_t(\pi_t, \omega) \leq S_{t-1}(\pi_{t-1}, \omega_{t-1}), \quad \forall \omega \in \Omega.$$

Therefore, defensive forecasting with

$$f_t(\gamma, \omega) := S_t((\gamma, 1-\gamma), \omega)$$

is well-defined with $\gamma_t = \pi_t(1)$.

Remark. Indeed, the corresponding algorithm is AA.

A. Chernov *et al.* Supermartingales in prediction with expert advice. 2010.

Recap: Decision theoretic online learning

Protocol. (Decision theoretic online learning, DTOL) Let $T \in \mathbb{N}$. Let $\mathcal{A} = \{ 1, \ldots, K \}$ for some $K \in \mathbb{N}$. For every $1 \leq t \leq T$, the following happen sequentially.

1. Learner announces $\gamma_t \in \Delta(\mathcal{A})$.
2. Reality announces $\omega_t \in \mathbb{R}^K$.
3. Learner suffers the loss

$$\lambda(\omega_t, \gamma_t) := \langle \omega_t, \gamma_t \rangle \,.$$

Defensive forecasting for decision theoretic online learning

Theorem 2. Defensive forecasting with

$$f_t(\gamma, \omega) := \sum_{i=1}^{K} \frac{1}{K} \mathrm{e}^{\eta[L_{t-1} - L_{t-1}(i)] - \frac{\eta^2}{2}} \times \mathrm{e}^{\eta[\lambda(\omega,\gamma) - \lambda(\omega,\gamma_t(i))] - \frac{\eta^2}{2}}$$

is well-defined, where L_t and $L_t(i)$ denote the cumulative losses of $\textsc{Learner}$ and $\textsc{Expert}$-i, respectively. Moreover, it achieves

$$L_T - L_T(i) \leq \sqrt{2T \log K}, \quad \forall 1 \leq i \leq K.$$

A. Chernov & V. Vovk. Prediction with advice of unknown number of experts. 2010.

Multi-valued supermartingale

In general, it may be difficult to associate $(f_t)_{t\in\mathbb{N}}$ with a supermartingale. Consider a *multi-valued process* given by

$$S_t(\pi) := \{\, g : \Omega \to \mathbb{R} \mid \exists\gamma \in G(\pi)\, \forall\omega \in \Omega\; g(\omega) = f_t(\gamma, \omega) \,\},$$

where

$$G(\pi) := \arg\min_{\gamma\in\Gamma} \mathsf{E}_{\omega\sim\pi}\, \lambda(\gamma, \omega), \quad \forall\pi \in \Delta(\Omega).$$

Definition. We say that S_t is a *multi-valued supermartingale*, if for every $T \in \mathbb{N}$, $(\pi_t, \omega_t)_{t\in\mathbb{N}}$, and $\pi \in \Delta(\Omega)$,

$$\sup\{\, \mathsf{E}_\pi g(\omega) \mid g \in S_T(\pi) \,\} \le \inf S_{T-1}(\omega_{T-1}).$$

A. Chernov & V. Vovk. Prediction with advice of unknown number of experts. 2010.

Levin's theorem for multi-valued supermartingale

Theorem 3. Let $\Omega = [0,1]^K$. Let $\mathcal{F}$ be a set of functions $f : \{ 1, \ldots, K \} \to \mathbb{R}$. Let $R : \Delta(\Omega) \to 2^{\mathcal{F}}$. Suppose that $\mathsf{E}_\pi \omega = \mathsf{E}_{\pi'} \omega$ implies $R(\pi) = R(\pi')$. Suppose that R is closed, and $R(\pi) \neq \emptyset$ and $R(\pi) + [0, +\infty]^\Omega$ is convex for every $\pi \in \Delta(\Omega)$. If

$$\mathsf{E}_\pi g(\omega) \leq C, \quad \forall \pi \in \Delta(\Omega), g \in R(\pi),$$

for some $C \in \mathbb{R}$, then there exists some $g^\star \in \cup_{\pi \in \Delta(\Omega)} R(\pi)$ such that

$$g^\star(\omega) \leq C, \quad \forall \omega \in \Omega.$$

Remark. That is, under some technical conditions, defensive forecasting by a multi-valued supermartingale is well-defined.

A. Chernov & V. Vovk. Prediction with advice of unknown number of experts. 2010.

Proof of Theorem 2 (1/2)

The following lemma shows S_T is a multi-valued supermartingale.

Lemma 1. For every $\pi \in \Delta(\Omega)$, $\gamma \in G(\pi)$, $\gamma' \in \Gamma$, and $\eta \in \mathbb{R}$, it holds that

$$\mathsf{E}_\pi \mathrm{e}^{\eta[\lambda(\omega,\gamma)-\lambda(\omega,\gamma')]-\left(\eta^2/2\right)} \leq 1.$$

Proof. Notice the random variable $\xi := \lambda(\omega, \gamma) - \lambda(\omega, \gamma')$ takes values in $[-1, 1]$. By Hoeffding's inequality, we write

$$\log \mathsf{E}_\pi \mathrm{e}^{\eta\xi} \leq \frac{\eta^2}{2}.$$

Remark. We call the martingale defined via $(S_t)_{t\in\mathbb{N}}$ a *Hoeffding martingale*.

Proof of Theorem 2 (2/2)

Proof sketch. (Theorem 2) It can be checked that Theorem 3 holds for the Hoeffding martingale under consideration. Therefore, the corresponding defensive forecasting algorithm is well-defined. Then, we write

$$f_T(\gamma_t, \omega_T) = \sum_{i=1}^{K} \frac{1}{K} \mathrm{e}^{\eta[L_T - L_T(i)] - \left(\eta^2/2\right)} \leq 1.$$

It remains to bound f_T from below by any of the summands and optimizing over η.

A. Chernov & V. Vovk. Prediction with advice of unknown number of experts. 2010.

Quantile regret bound

Possibility of refinements

Consider the decision theoretic online learning problem. We know there are algorithms achieving

$$\sum_{t=1}^{T} \lambda(\omega_t, \gamma_t) - \sum_{t=1}^{T} \lambda(\omega_t, \gamma) = O\left(\sqrt{T \log K}\right).$$

Possibility of refinements.

- What if we would like to compete with the cumulative loss by time-varying predictions?
- What if the data is easy?
- *What if there are many good experts?*

Possibility of a quantile bound

Let $\tilde{\mathcal{A}} \subset \mathcal{A}$ be given. Suppose we want to compete with the actions in $\tilde{\mathcal{A}}$.

Proposition 3. The hedge algorithm with learning rate

$$\eta = \sqrt{\frac{8}{T} \log \frac{1}{\pi_1(\tilde{\mathcal{A}})}}$$

achieves

$$\sum_{t=1}^{T} \lambda(\omega_t, \gamma_t) - \max_{i \in \tilde{\mathcal{A}}} \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t(i)) \leq \sqrt{\frac{T}{2} \log \frac{1}{\pi_1(\tilde{\mathcal{A}})}}.$$

Proof of Proposition 3

Proof. (Proposition 3) Recall the hedge achieves

$$\sum_{t=1}^{T} \lambda(\omega_t, \gamma_t) \leq \frac{-1}{\eta} \log \sum_{i=1}^{K} \pi_1(i) \mathrm{e}^{-\eta \sum_{t=1}^{T} \omega_t(i)} + \frac{\eta T}{8}, \quad \forall 1 \leq i \leq K.$$

We write

$$\begin{aligned}
\sum_{t=1}^{T} \lambda(\omega_t, \gamma_t) &\leq \frac{-1}{\eta} \log \sum_{i \in \tilde{\mathcal{A}}} \pi_1(i) \mathrm{e}^{-\eta \sum_{t=1}^{T} \omega_t(i)} + \frac{\eta T}{8} \\
&\leq \frac{-1}{\eta} \log \left[\mathrm{e}^{-\eta \max_{j \in \tilde{\mathcal{A}}} \sum_{t=1}^{T} \omega_t(j)} \sum_{i \in \tilde{\mathcal{A}}} \pi_1(i) \right] + \frac{\eta T}{8} \\
&= \max_{j \in \tilde{\mathcal{A}}} \sum_{t=1}^{T} \omega_t(j) + \frac{1}{\eta} \log \frac{1}{\pi_1(\tilde{\mathcal{A}})} + \frac{\eta T}{8}.
\end{aligned}$$

Issue of the quantile bound derived

Observation. In Proposition 3, the set $\tilde{\mathcal{A}}$ has to be given, and the learning rate η has to be set accordingly.

Question. Is there an algorithm that achieves a quantile bound simulataneously for all possible $\tilde{\mathcal{A}} \subset \mathcal{A}$? That is, is there an algorithm that, for any $\tilde{\mathcal{A}} \subset \mathcal{A}$ such that $|\tilde{\mathcal{A}}| = \delta K$, achieves the following regret bound?

$$\sum_{t=1}^{T} \lambda(\omega_t, \gamma_t) - \max_{i \in \tilde{\mathcal{A}}} \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t(i)) = \tilde{O}\left(\sqrt{T \log \frac{1}{\delta}}\right).$$

Observation. The difficulty lies in tuning the learning rate η.

Aggregation of learning rates (1/2)

Recall decision theoretic online learning can be also solved by defensive forecasting with a Hoeffding supermartingale. The corresponding regret bound was obtained via the inequality

$$\sum_{i=1}^{K} \frac{1}{K} \mathrm{e}^{\eta[L_T - L_T(i)] - \left(\eta^2/2\right)} \leq 1.$$

Bounding the sum from below by a summation over $\tilde{\mathcal{A}}$, we obtain a quantile bound. However, the optimal learning rate η also depends on $\tilde{\mathcal{A}}$.

Aggregation of learning rates (2/2)

Observation. If we find an optimal learning rate $\eta^\star$, then using another learning rate $\alpha\eta^\star$ for some $\alpha > 0$ results only a constant (hence negligible) loss in the regret.

Aggregation of learning rates. Consider aggregating all possible $\eta > 0$ via a probability distribution on $]0, \overline{\eta}]$ for some $\overline{\eta} > 0$, such that the probability assigned to the interval $[\eta, \alpha\eta]$ *is independent of* η.

Question. What is the desired probability distribution on $]0, \overline{\eta}]$?

W. M. Koolen and T. van Erven. Second-order quantile methods for experts and combinatorial games. 2015.

Quantile regret bound via defensive forecasting (1/2)

Theorem 4. Defensive forecasting with

$$\overline{f}_t(\gamma, \omega) := \int_0^{1/\mathrm{e}} f_t(\gamma, \omega; \eta) \frac{1}{\eta \left(\log \frac{1}{\eta}\right)^2} \mathrm{d}\eta,$$

is well-defined, where

$$f_t(\gamma, \omega; \eta) \sum_{i=1}^{K} \frac{1}{K} \mathrm{e}^{\eta[L_{t-1} - L_{t-1}(i)] - \frac{\eta^2}{2}} \times \mathrm{e}^{\eta[\lambda(\omega,\gamma) - \lambda(\omega,\gamma_t(i))] - \frac{\eta^2}{2}}.$$

L_t and $L_t(i)$ denoting the cumulative losses of LEARNER and EXPERT-i, respectively.

A. Chernov & V. Vovk. Prediction with advice of unknown number of experts. 2010.

Quantile regret bound via defensive forecasting (2/2)

Theorem 4 continued. Moreover, for every $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ such that $|\tilde{\mathcal{A}}| = \varepsilon|\mathcal{A}|$ and $\delta \in (0, 1/4)$, it achieves

$$
\begin{aligned}
& L_T - \max_{i \in \tilde{\mathcal{A}}} L_T(i) \\
& \quad \leq \frac{2}{\sqrt{2-\delta}} \sqrt{T \log \frac{1}{\varepsilon} + \frac{1}{2} T \log \frac{1}{\delta} + 2T \log\log T} + O\left(\log \frac{1}{\varepsilon}\right) \\
& \quad = O\left(\sqrt{T \log \frac{1}{\varepsilon} + T \log\log T}\right).
\end{aligned}
$$

A. Chernov & V. Vovk. Prediction with advice of unknown number of experts. 2010.

Proof sketch. (Theorem 4) By Lemma 1, the corresponding process is a multi-valued supermartingale. By Theorem 3, the corresponding defensive forecasting algorithm is well-defined. Then, we write

$$\int_0^{1/\mathrm{e}} f_T(\gamma_T, \omega_T; \eta) \frac{1}{\eta \left(\log \frac{1}{\eta}\right)^2} \mathrm{d}\eta \leq 1.$$

Bounding the sum in f_T from below by the summation over $i \in \tilde{\mathcal{A}}$.

A. Chernov & V. Vovk. Prediction with advice of unknown number of experts. 2010.

Conclusions

Conclusions

- The idea of defensive forecasting can be motivated from the AA. Indeed, AA is a special case of defensive forecasting.
- The notion of a supermartingale yields a regret bound for defensive forecasting. Levin's theorems ensure a defensive forecasting algorithm is well-defined.
- The main difficulty in developing a quantile regret guarantee lies in tuning the learning rate.
- By the scale invariability of the optimal learning rate (up to a constant loss in the regret), we aggregate the learning rates by an appropriate probability distribution in defensive forecasting, and obtain a quantile regret bound.

Next lecture

One of the following.

- Second-order regret bounds. Adaptive regret.
- Calibrated forecasting.