

# Distributions

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# Chapter 1

Statistical Inference I

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## Lecture Notes 10

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**Lemma 1 (inversion theorem)** Let  $X$  be a random variable with characteristic function  $\phi_X(t)$  and  $a, b \in \mathbb{R}$  with  $a < b$ , then

1. For any random variable  $X$ ,

$$P(a < X < b) + \frac{1}{2}P(X = a) + \frac{1}{2}P(X = b) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt$$

2. If  $X$  is a continuous random variable, then

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t) dt \quad a.e.$$

## 1.1 Characteristic function

### Intuition (inversion theorem)

Inversion theorem provides a *isomorphism* between **distribution** and **characteristic function**.

**Proof:** The proof is divided into four steps. The first step introduce a integration tool for latter usage. The second step provides a clean form of the original term. The third step uses dominant theorem to show the convergence. The last step gives the density function.

1. **claim:**  $\int_0^{\infty} \frac{\sin \alpha x}{x} dx = \frac{\pi}{2} \text{sign}(\alpha)$

$$\begin{aligned}
\int_0^\infty \frac{\sin \alpha x}{x} dx &= \int_0^\infty \int_0^\infty \sin \alpha x e^{-ux} du dx \\
(\text{by Fubini's thm}) &= \int_0^\infty \int_0^\infty \sin \alpha x e^{-ux} dx du \\
&= \dots \text{some change of integrals} \dots \\
&= \frac{\pi}{2} \text{sign}(\alpha)
\end{aligned}$$

2. Consider

$$\begin{aligned}
\frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt &= \frac{1}{2\pi} \int_{-T}^T \int_{-\infty}^\infty \frac{e^{-ita} - e^{-itb}}{it} e^{itx} dF_X(x) dt \\
&= \frac{1}{2\pi} \int_{-T}^T \int_{-\infty}^\infty \frac{\cos t(x-a) + i \sin t(x-a)}{it} dF_X(x) dt \\
&= \frac{1}{2\pi} \int_{-T}^T \int_{-\infty}^\infty \frac{\cos t(x-b) + i \sin t(x-b)}{it} dF_X(x) dt \\
(\because \text{symmetry}) &= \frac{1}{\pi} \int_0^T \int_{-\infty}^\infty \frac{\sin t(x-a) - \sin t(x-b)}{t} dF_X(x) dt \\
(\text{by Fubini's thm}) &= \frac{1}{\pi} \int_{-\infty}^\infty \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x)
\end{aligned}$$

3. By dominant convergence theorem,

$$\begin{aligned}
\frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt &\xrightarrow{T \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^\infty \int_0^\infty \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&= \frac{1}{\pi} \int_{x \in (-\infty, a)} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&\quad + \frac{1}{\pi} \int_{x=a} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&\quad + \frac{1}{\pi} \int_{x \in (a, b)} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&\quad + \frac{1}{\pi} \int_{x=b} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&\quad + \frac{1}{\pi} \int_{x \in (b, \infty)} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
(\text{by the tool in 1.}) &= 0 + \frac{1}{2} P(X = a) + P(a < X < b) + \frac{1}{2} P(X = b) + 0
\end{aligned}$$

4. Suppose  $X$  is continuous and  $\int_{\mathcal{R}} |\phi_X(t)| dt < \infty$ ,

$$\begin{aligned}
\int_a^b f(x) dx &= \frac{1}{2\pi} \int_a^b \int_{\mathcal{R}} e^{-itx} \phi_X(t) dt dx \\
&= \frac{1}{2\pi} \int_{\mathcal{R}} \left( \int_a^b e^{-itx} dx \right) \phi_X(t) dt \\
&= \frac{1}{2\pi} \int_{\mathcal{R}} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt \\
&= \frac{1}{2} P(X = a) + P(a < X < b) + \frac{1}{2} P(X = b) \\
&= P(a < X < b) \text{ (further set } b=x, a \downarrow -\infty)
\end{aligned}$$

■

**Corollary 1** For probability measures  $\mu_X$  and  $\mu_Y$  on  $\mathcal{B}(\mathcal{R})$ , the equality  $\phi_{\mu_X} = \phi_{\mu_Y}$  implies that  $\mu_X = \mu_Y$ .

**Proof:** From inversion theorem, we have  $\mu_X((a, b)) = \mu_Y((a, b)) \forall a, b \in C$ , where  $C$  is the set of all  $z \in \mathcal{R}$  such that  $\mu_X(\{z\}) = \mu_Y(\{z\}) = 0$ . Since  $C^c$  is at most countable. The family of  $\{(a, b) : a, b \in C\}$  of intervals is a  $\pi$ -system generating  $\mathcal{B}(\mathcal{R})$ .  $\mu_X$  and  $\mu_Y$  agrees on a  $\pi$ -system also agrees on the  $\sigma$ -algebra generated by it. ■

### Intuition (relation to moment)

Let  $X$  be a random variable. If  $E[|X^n|] < \infty$ , then  $\frac{d^n}{(dt)^n} \phi_X(t)$  exists for all  $t$  and

$$\frac{d^n}{(dt)^n} \phi_X(t) = E[e^{itX} (iX)^n]$$

so the lower moments are

$$E[X^n] = (-i)^n \frac{d^n}{(dt)^n} \phi_X(0)$$

## 1.2 Convergence

**Theorem 1** Let  $\{X_n\}$  be a sequence of random variables with characteristic functions  $\phi_{X_n}(t)$ . Suppose that

- $\lim_{n \rightarrow \infty} \phi_{X_n}(t) = \phi_X(t)$  for  $t$  in a neighborhood of 0. (pairwise convergence)
- $\phi_X(t)$  is a characteristic function of some random variable  $X$ .

Then,

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x), \forall x \text{ such that } F_X(x) \text{ is continuous (weakly convergence)}$$

**Proof:** Let  $a$  and  $b$  be continuous points of  $F_X(x)$  and  $F_{X_n}(x)$  for  $n \geq N_0$  for some  $n \in \mathcal{N}$

$$\begin{aligned}
F_X(b) - F_X(a) &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \lim_{n \rightarrow \infty} \phi_{X_n}(t) dt \\
&= \lim_{n \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_{X_n}(t) dt \quad (\text{since } |\phi_{X_n}(t)| \leq 1, \text{ dominated}) \\
&= \lim_{n \rightarrow \infty} (F_{X_n}(b) - F_{X_n}(a))
\end{aligned}$$

By setting  $b=x$ ,  $a \downarrow -\infty$  one obtains  $\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x)$ . ■

**Remark:** We don't have to worry about the discrete points since they must converge to the right value.

### Intuition (convergence of characteristic function)

Theorem 1 tells us that if r.v.s converge in characteristic functions, then r.v.s also converge in distribution.

**Remark:** Tips for calculating MGF: consider the MGF of binomial distributed random variable  $X$  such that  $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \mathbf{1}_{0,1,\dots,n}$ . We have

$$\begin{aligned}
M_{X_n}(t) &= \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x} e^{tx} \\
&= \sum_{x=0}^n \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\
&= \sum_{x=0}^n \binom{n}{x} \frac{(pe^t)^x (1-p)^{n-x}}{[(pe^t) + (1-p)]^n} [(pe^t) + (1-p)]^n \\
&= \left( \text{set } p' = \frac{pe^t}{pe^t + (1-p)} \right) \sum_{x=0}^n \binom{n}{x} p'^x (1-p')^{n-x} [(pe^t) + (1-p)]^n \\
&= [(pe^t) + (1-p)]^n \sum_{x=0}^n \binom{n}{x} p'^x (1-p')^{n-x} \\
&= [(pe^t) + (1-p)]^n
\end{aligned}$$