Statistical Inference I	**Prof. Chin-Tsang Chiang**
Lecture Notes 11	
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Lemma 1 (inversion theorem) *Let X be a random variable with characteristic function $\phi_X(t)$ and $a, b \in \mathbb{R}$ with $a < b$, then*

1. *For any random variable X,*

$$P(a < X < b) + \frac{1}{2}P(X = a) + \frac{1}{2}P(X = b) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi_X(t)dt$$

2. *If X is a continuous random variable, then*

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi_X(t)dt \quad a.e.$$

Intuition (inversion theorem)

Inversion theorem provides a *isomorphism* between **distribution** and **characteristic function**.

Proof: The proof is divided into four steps. The first step introduce a integration tool for latter usage. The second step provides a clean form of the original term. The third step uses dominant theorem to show the convergence. The last step gives the density function.

1. **claim**: $\int_0^\infty \frac{\sin \alpha x}{x}dx = \frac{\pi}{2}\text{sign}(\alpha)$

$$\begin{aligned}
\int_0^\infty \frac{\sin \alpha x}{x}dx &= \int_0^\infty \int_0^\infty \sin \alpha x \ e^{-ux}dudx \\
\text{(by Fubini's thm)} &= \int_0^\infty \int_0^\infty \sin \alpha x \ e^{-ux}dxdu \\
&= \text{... some change of integrals ...} \\
&= \frac{\pi}{2}\text{sign}(\alpha)
\end{aligned}$$

2. Consider

$$\begin{aligned}
\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi_X(t)dt &= \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty}\frac{e^{-ita}-e^{-itb}}{it}e^{itx}dF_X(x)dt \\
&= \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty}\frac{\cos t(x-a)+i\sin t(x-a)}{it}dF_X(x)dt \\
&= \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty}\frac{\cos t(x-b)+i\sin t(x-b)}{it}dF_X(x)dt \\
(\because \text{symmetry}) &= \frac{1}{\pi}\int_{0}^{T}\int_{-\infty}^{\infty}\frac{\sin t(x-a)-\sin t(x-b)}{t}dF_X(x)dt \\
(\text{by Fubini's thm}) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dtdF_X(x)
\end{aligned}$$

3. By dominant convergence theorem,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi_X(t)dt &\xrightarrow{T\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\sin t(x-a)-\sin t(x-b)}{t}dtdF_X(x) \\
&= \frac{1}{\pi}\int_{x\in(-\infty,a)}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dtdF_X(x) \\
&+ \frac{1}{\pi}\int_{x=a}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dtdF_X(x) \\
&+ \frac{1}{\pi}\int_{x\in(a,b)}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dtdF_X(x) \\
&+ \frac{1}{\pi}\int_{x=b}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dtdF_X(x) \\
&+ \frac{1}{\pi}\int_{x\in(b,\infty)}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dtdF_X(x) \\
(\text{by the tool in 1.}) &= 0+\frac{1}{2}P(X=a)+P(a<X<b)+\frac{1}{2}P(X=b)+0
\end{aligned}$$

4. Suppose X is continuous and $\int_{\mathcal{R}}|\phi_X(t)dt|<\infty$,

$$\begin{aligned}
\int_a^b f(x)dx &= \frac{1}{2\pi}\int_a^b\int_{\mathcal{R}}e^{-itx}\phi_X(t)dtdx \\
&= \frac{1}{2\pi}\int_{\mathcal{R}}(\int_a^b e^{-itx}dx)\phi_X(t)dt \\
&= \frac{1}{2\pi}\int_{\mathcal{R}}\frac{e^{-ita}-e^{-itb}}{it}\phi_X(t)dt \\
&= \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi_X(t)dt \\
&= \frac{1}{2}P(X=a)+P(a<X<b)+\frac{1}{2}P(X=b) \\
&= P(a<X<b)(\text{ further set b=x, a}\downarrow -\infty)
\end{aligned}$$

■

Corollary 2 *For probability measures μ_X and μ_Y on $\mathcal{B}(\mathcal{R})$, the equality $\phi_{\mu_X} = \phi_{\mu_Y}$ implies that $\mu_X = \mu_Y$.*

Proof: From inversion theorem, we have $\mu_X((a,b)) = \mu_Y((a,b))$ $\forall a, b \in C$, where C is the set of all $z \in \mathcal{R}$ such that $\mu_X(\{z\}) = \mu_Y(\{z\}) = 0$. Since C^c is at most countable. The family of $\{(a,b) : a.b \in C\}$ of intervals is a π-system generating $\mathcal{B}(\mathcal{R})$. μ_X and μ_Y agrees on a π-system also agrees on the σ-algebra generated by it. ■

Intuition (relation to moment)

Let X be a random variable. If $E[|X^n|] < \infty$, then $\frac{d^n}{(dt)^n}\phi_X(t)$ exists for all t and

$$\frac{d^n}{(dt)^n}\phi_X(t) = E[e^{itX}(iX)^n]$$

so the lower moments are

$$E[X^n] = (-i)^n \frac{d^n}{(dt)^n}\phi_X(0)$$

Theorem 3 *Let $\{X_n\}$ be a sequence of random variables with characteristic functions $\phi_{X_n}(t)$. Suppose that*

- $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)$ *for t in a neighborhood of 0. (pairwise convergence)*
- $\phi_X(t)$ *is a characteristic function of some random variable X.*

Then,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),\ \forall x \text{ such that } F_X(x) \text{ is continuous (weakly convergence)}$$

Proof: Let a and b be continuous points of $F_X(x)$ and $F_{X_n}(x)$ for $n \geq N_0$ for some $n \in \mathcal{N}$

$$\begin{aligned}
F_X(b) - F_X(a) &= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)dt \\
&= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \lim_{n\to\infty} \phi_{X_n}(t)dt \\
&= \lim_{n\to\infty} \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi_{X_n}(t)dt \text{ (since } |\phi_{X_n}(t)| \leq 1\text{, dominated)} \\
&= \lim_{n\to\infty} (F_{X_n}(b) - F_{X_n}(a))
\end{aligned}$$

By setting b=x, a$\downarrow -\infty$ one obtains $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$. ■

Remark: We don't have to worry about the discrete points since they must converge to the right value.

Intuition (convergence of characteristic function)

Theorem 3 tells us that if r.v.s converge in characteristic functions, then r.v.s also converge in distribution.

Remark: Tips for calculating MGF: consider the MGF of binomial distributed random variable X such that $f_X(x) = \left(\begin{array}{c} n \\ x \end{array}\right) p^x(1-p)^{n-x}\mathbf{1}_{0,1,\ldots,n}$. We have

$$\begin{aligned}
M_{X_n}(t) &= \sum_{x=0}^{n} \left(\begin{array}{c} n \\ x \end{array}\right) p^x(1-p)^{n-x}e^{tx} \\
&= \sum_{x=0}^{n} \left(\begin{array}{c} n \\ x \end{array}\right) (pe^t)^x(1-p)^{n-x} \\
&= \sum_{x=0}^{n} \left(\begin{array}{c} n \\ x \end{array}\right) \frac{(pe^t)^x(1-p)^{n-x}}{[(pe^t)+(1-p)]^n}[(pe^t)+(1-p)]^n \\
(\text{set } p' = \frac{pe^t}{pe^t+(1-p)}) &= \sum_{x=0}^{n} \left(\begin{array}{c} n \\ x \end{array}\right) p'^x(1-p')^{n-x}[(pe^t)+(1-p)]^n \\
&= [(pe^t)+(1-p)]^n \sum_{x=0}^{n} \left(\begin{array}{c} n \\ x \end{array}\right) p'^x(1-p')^{n-x} \\
&= [(pe^t)+(1-p)]^n
\end{aligned}$$