Lemma 1 (inversion theorem) *Let X be a random variable with characteristic function $\phi_X(t)$ and $a, b \in \mathbb{R}$ with $a < b$, then*

1. *For any random variable X,*

$$P(a < X < b) + \frac{1}{2}P(X = a) + \frac{1}{2}P(X = b) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt$$

2. *If X is a continuous random variable, then*

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t) dt \quad a.e.$$

0.1 Characteristic function

Intuition (inversion theorem)

Inversion theorem provides a *isomorphism* between **distribution** and **characteristic function**.

Proof: The proof is divided into four steps. The first step introduce a integration tool for latter usage. The second step provides a clean form of the original term. The third step uses dominant theorem to show the convergence. The last step gives the density function.

1. **claim**: $\int_0^\infty \frac{\sin \alpha x}{x} dx = \frac{\pi}{2} \text{sign}(\alpha)$

$$\int_0^\infty \frac{\sin \alpha x}{x} dx = \int_0^\infty \int_0^\infty \sin \alpha x \ e^{-ux} du dx$$
$$(\text{by Fubini's thm}) = \int_0^\infty \int_0^\infty \sin \alpha x \ e^{-ux} dx du$$
$$= \dots \text{ some change of integrals } \dots$$
$$= \frac{\pi}{2} \text{sign}(\alpha)$$

2. Consider

$$\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi_X(t)dt = \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty}\frac{e^{-ita}-e^{-itb}}{it}e^{itx}dF_X(x)dt$$

$$= \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty}\frac{\cos t(x-a)+i\sin t(x-a)}{it}dF_X(x)dt$$

$$+ \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty}\frac{\cos t(x-b)+i\sin t(x-b)}{it}dF_X(x)dt$$

$$(\because \text{symmetry}) = \frac{1}{\pi}\int_{0}^{T}\int_{-\infty}^{\infty}\frac{\sin t(x-a)-\sin t(x-b)}{t}dF_X(x)dt$$

$$(\text{by Fubini's thm}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dt dF_X(x)$$

3. By dominant convergence theorem,

$$\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi_X(t)dt \xrightarrow{T\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\sin t(x-a)-\sin t(x-b)}{t}dt dF_X(x)$$

$$= \frac{1}{\pi}\int_{x\in(-\infty,a)}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dt dF_X(x)$$

$$+ \frac{1}{\pi}\int_{x=a}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dt dF_X(x)$$

$$+ \frac{1}{\pi}\int_{x\in(a,b)}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dt dF_X(x)$$

$$+ \frac{1}{\pi}\int_{x=b}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dt dF_X(x)$$

$$+ \frac{1}{\pi}\int_{x\in(b,\infty)}\int_{0}^{T}\frac{\sin t(x-a)-\sin t(x-b)}{t}dt dF_X(x)$$

$$(\text{by the tool in 1.}) = 0 + \frac{1}{2}P(X=a) + P(a<X<b) + \frac{1}{2}P(X=b) + 0$$

4. Suppose X is continuous and $\int_{\mathcal{R}}|\phi_X(t)dt| < \infty$,

$$\int_{a}^{b}f(x)dx = \frac{1}{2\pi}\int_{a}^{b}\int_{\mathcal{R}}e^{-itx}\phi_X(t)dt dx$$

$$= \frac{1}{2\pi}\int_{\mathcal{R}}(\int_{a}^{b}e^{-itx}dx)\phi_X(t)dt$$

$$= \frac{1}{2\pi}\int_{\mathcal{R}}\frac{e^{-ita}-e^{-itb}}{it}\phi_X(t)dt$$

$$= \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi_X(t)dt$$

$$= \frac{1}{2}P(X=a) + P(a<X<b) + \frac{1}{2}P(X=b)$$

$$= P(a<X<b)(\text{ further set b=x, a}\downarrow-\infty)$$

Corollary 1 *For probability measures μ_X and μ_Y on $\mathcal{B}(\mathcal{R})$, the equality $\phi_{\mu_X} = \phi_{\mu_Y}$ implies that $\mu_X = \mu_Y$.*

Proof: From inversion theorem, we have $\mu_X((a,b)) = \mu_Y((a,b))$ $\forall a, b \in C$, where C is the set of all $z \in \mathcal{R}$ such that $\mu_X(\{z\}) = \mu_Y(\{z\}) = 0$. Since C^c is at most countable. The family of $\{(a,b) : a.b \in C\}$ of intervals is a π-system generating $\mathcal{B}(\mathcal{R})$. μ_X and μ_Y agrees on a π-system also agrees on the σ-algebra generated by it. ∎

0.2 Convergence

Theorem 1 *Let $\{X_n\}$ be a sequence of random variables with characteristic functions $\phi_{X_n}(t)$. Suppose that*

- $\lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t)$ *for t in a neighborhood of 0. (pairwise convergence)*

- $\phi_X(t)$ *is a characteristic function of some random variable X.*

Then,

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x), \ \forall x \text{ such that } F_X(x) \text{ is continuous (weakly convergence)}$$

Proof: Let a and b be continuous points of $F_X(x)$ and $F_{X_n}(x)$ for $n \geq N_0$ for some $n \in \mathcal{N}$

$$
\begin{aligned}
F_X(b) - F_X(a) &= \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt \\
&= \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \lim_{n \to \infty} \phi_{X_n}(t) dt \\
&= \lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi_{X_n}(t) dt \text{ (since } |\phi_{X_n}(t)| \leq 1, \text{ dominated)} \\
&= \lim_{n \to \infty} (F_{X_n}(b) - F_{X_n}(a))
\end{aligned}
$$

By setting b=x, a$\downarrow -\infty$ one obtains $\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$. ∎

Remark: We don't have to worry about the discrete points since they must converge to the right value.

> **Intuition (convergence of characteristic function)**
>
> Theorem 1 tells us that if r.v.s converge in characteristic functions, then r.v.s also converge in distribution.

Remark: Tips for calculating MGF: consider the MGF of binomial distributed random variable X such that $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \mathbf{1}_{0,1,\ldots,n}$. We have

$$M_{X_n}(t) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} e^{tx}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} \frac{(pe^t)^x (1-p)^{n-x}}{[(pe^t) + (1-p)]^n} [(pe^t) + (1-p)]^n$$

$$\left(\text{set } p' = \frac{pe^t}{pe^t + (1-p)}\right) = \sum_{x=0}^{n} \binom{n}{x} p'^x (1-p')^{n-x} [(pe^t) + (1-p)]^n$$

$$= [(pe^t) + (1-p)]^n \sum_{x=0}^{n} \binom{n}{x} p'^x (1-p')^{n-x}$$

$$= [(pe^t) + (1-p)]^n$$