| | |
|---|---|
| **Statistical Inference I** | **Prof. Chin-Tsang Chiang** |
| **Lecture Notes 13** | |
| **October 29, 2015** | **Scribe: Chi-Ning Chou** |

# 1 Poisson distribution

Poisson random variable is defined with a parameter $\lambda$ denoting the rate or intensity of a counting process. As Poisson distribution is **memoryless**, these two notions don't conflict. We define the probability density function of Poisson($\lambda$) as follow:

$$f_X(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}\mathbf{1}_{\{0,1,\ldots\}}(x)$$

The following is the basic properties of Poisson distribution:

- $\mathbb{E}[x|\lambda] = \lambda$
- $var[x|\lambda] = \lambda$
- $M_X(t) = e^{-\lambda(1-e^t)}$

Now, let's consider a theorem that connects the intuition of Poisson process with Poisson distribution.

**Theorem 1 (Poisson process)** *Let $N_t$ be a nondecreasing integer-valued random variable satisfying*

1. $N_0 = 0$
2. $\forall 0 < t_1 < t_2 < t_3 < t_4$, $N_{t_2} - Nt_1 \sim N_{t_2 - t_1}$ (**identical**). $N_{t_2} - N_{t_1}$ *is independent to* $N_{t_4} - N_{t_3}$
3. $\lim_{n\to\infty} \frac{Pr[N_0=1]}{h} = \lambda$ *and* $\lim_{n\to\infty} \frac{Pr[N_0\geq 2]}{h} = 0$

*Then,* $Pr[N_t = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$

**Proof:** First, we consider the case where $k = 0$. Then we use induction to prove the result for all $k$. In the following proof, denote $P_n(t) = Pr[N_t = n]$

1. Suppose $n = 0$, we have $\forall t > 0$

$$\begin{aligned} P_0(t+h) &= Pr[N_t = 0 \text{ and } N_{t+h} - N_t = 0] \\ (\because \text{independent and stationary}) &= P_0(t)P_0(h) \\ &= P_0(t)(1 - \lambda h + o(h)) \end{aligned}$$

Subtract $P_(t)$ on both side and divide by $h$, let $h \to 0$ we have

$$\begin{aligned} P_0'(t) &= \lim_{h\to 0} \frac{P_0(t+h) - P_0(t)}{h} \\ &= \lim_{h\to 0} -\lambda P_0(h) + \frac{o(h)}{h} \\ &= -\lambda P_0(t) \end{aligned}$$

This is equivalent as solving $\frac{d}{dt}\ln P_0(t) = -\lambda$. With the boundary condition $P_0(0) = 0$, we have

$$P_0(t) = e^{-\lambda t}$$

2. Now, consider $n \geq 1$. We have

$$\begin{aligned} P_n(t+h) &= Pr[N_t = n-1 \text{ and } N_{t+h} - N_t = 1] + Pr[N_t = n \text{ and } N_{t+h} - N_t = 0] \\ &+ Pr[N_{t+h} - N_t \geq 2] \\ &= P_{n-1}(t)(\lambda h + o(h)) + P_n(t)(1 - \lambda h + o(h)) + o(h) \end{aligned}$$

Subtract $P_n(t)$ on both side and divide by $h$, let $h \to 0$ we have,

$$\begin{aligned} P_n'(t) &= \lim_{h\to 0} \frac{P_n(t+h) - P_n(t)}{h} \\ &= \lim_{h\to 0} \lambda P_{n-1}(t) - \lambda P_n(t) + \frac{o(h)}{h} \\ &= \lambda P_{n-1}(t) - \lambda P_n(t) \end{aligned}$$

Consider $n = 1$, we have $P_1'(t) = \lambda e^{-\lambda t} - \lambda P_1(t)$, which is equivalent as solving $\frac{d}{dt}(e^{\lambda t}P_1(t)) = \lambda$. With boundary condition $P_1(0) = 0$, we have

$$P_1(t) = \lambda t e^{-\lambda t}$$

With induction hypothesis $P_{n-1}(t) = \frac{(\lambda t)^{n-1}e^{-\lambda t}}{(n-1)!}$, the problem is equivalent as solving $\frac{d}{dt}e^{\lambda t}P_n(t) = \lambda\frac{(\lambda t)^{n-1}}{(n-1)!}$. With boundary condition $P_n(0) = 0$, we have

$$P_n(t) = \frac{(\lambda t)^n e^{\lambda t}}{n!}$$

■

## 1.1 Counting process and Stopping time

In fact, counting process and stopping time are the two side of a coin. The following shows how to interchange from one to another.

**Stopping time $T$ → Counting process $\{N(t), t \geq 0\}$**
For a given stopping $T$, we can define a corresponding zero-one counting process: $N_T(t) := \mathbf{1}_{\{T<t\}}$

**Counting process** $\{N(t), t \geq 0\}$ $\rightarrow$ **Stopping time** $T$
For a counting process $\{N(t), t \geq 0\}$, we can define a stopping time $T$ as $Pr[T > t] = Pr[N(t) = 0]$ so for a Poisson counting process:

$$1 - F_T(t) = e^{-\lambda t}$$
$$f_T(t) = \lambda e^{-\lambda t} \mathbf{1}_{\{0,1,2,\ldots\}}(t)$$

for Gamma distribution:

$$T^* = \sum_{j=1}^{m} T_j^*$$
$$f_{T^*}(t|m, \lambda) = \frac{t^{m-1}\lambda^m e^{-\lambda t}}{\tau(m)} \mathbb{1}_{\{0,\infty\}}(t)$$

## 2 Relationship between distribution

**Example 1** Let X$\sim$Poisson($\lambda$) and Y$\sim$Binomial(n,p), then $f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ and we can expand

$$f_Y(y) = \binom{n}{y} p^y (1-p)^{n-y} = \frac{n-y+1}{y}\frac{p}{1-p}\binom{n}{y-1} p^{y-1}(1-p)^{n-y+1} = \frac{np - yp + p}{y - yp} f_Y(y-1|n,p)$$

so when $p \rightarrow 0, n \rightarrow \infty, np \rightarrow \lambda$, we have $Y \overset{d}{=} X$

$$\begin{aligned} f_Y &= \frac{\lambda}{y} f_Y(y-1|n,p) \\ &= \prod_{i=1}^{y} \frac{\lambda}{i} f_Y(0|n,p) \\ &= \frac{\lambda^y}{y!}(1 - \frac{np}{n})^n \\ &= \frac{\lambda^y e^{-\lambda}}{y!} \end{aligned}$$

**Example 2** Y$\sim$Negative Binomial(r,p), then $f_Y(y) = \binom{y+r-1}{r-1} p^r (1-p)^y$
when $r \rightarrow \infty, p \rightarrow 1, r(1-p) \rightarrow \lambda$, we have $Y \overset{d}{=} Poisson(\lambda)$

$$\begin{aligned} M_Y(t) = E[e^{tY}] &= \sum_{y=0}^{\infty} \binom{y+r-1}{r-1} p^r (1-p)^y e^{ty} \\ &= \sum_{y=0}^{\infty} \binom{y+r-1}{r-1} p^r ((1-p)e^t)^y \\ &= (\frac{p}{1-(1-p)e^t})^r \\ &= (1 + \frac{1}{r}\frac{r(1-p)(e^t-1)}{1-(1-p)e^t})^r \\ &= e^{\lambda(e^t-1)} \end{aligned}$$