Probability Theory

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October 4, 2015

Contents

Chapter 1

Statistical Inference I **Prof. Chin-Tsang Chiang**

Lecture Notes 1

October 4, 2015 **Scribe: Wei-Chang Lee, Chi-Ning Chou**

Questions

1. What's the difference between **probability** and **statistics**?

2. What's the difference between them and **math**?

Prof. Chaing: Mathematics is in a deterministic environment. With math, we can describe a dynamic system in a deterministic way. Probability involves in a probabilistic system, where the events follow certain distribution and act in like flipping a coin. As to statistics, what we face is not beautiful mathematical objects. Instead, we have to deal with data. We assume there is a distribution or general function behind the data. This function might be deterministic or probabilistic. What a statistician aim to do is to find out (formally speaking, estimate) such underlying distribution.

1.1 Probabilities, Random variables, Distribution

1.1.1 Set Theory

In the world of probability and statistics, we use **set** to intuitively model the object we concerned. As a result, we need to define some basic elements and operations. Furthermore, we will derive some basic properties.

Sample space Here we use Ω to represent sample space, which is composed of all the possible **outcomes**. The element in sample space is often denote as ω.

Event What we really care about is the subset of sample space, which is intuitively being an event. With this notion, we can interact with various outcomes and play with events that share the common outcomes etc.

Set operations Here, we use $A, B, \{A_n\}$ to denote events and a sequence of events.

- Union: $A \cup B := \{w : w \in A \ or \ w \in B\}$
- Intersection: $A \cap B := \{w : w \in A \ and \ w \in B\}$ Here, we can drop out the geometric notion of intersection and think of it as **the outcomes that the two events share**.
- Complementation: $A^C := \{w : w \in \Omega \ and \ w \notin A\}$ The outcomes that haven't happened?
- Set difference: $B \backslash A := B \cap A^C$

limsup Intuitively, it's analogous to the smallest upper bound for an infinite sequence.

$$\limsup A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

If we let $B_n := \bigcup_{k=n}^{\infty} A_k$, we can see that $B_1 \supseteq B_2 \supseteq ....$ Formally, we also have

$$\limsup A_n = \{w : w \in A_n, for \ infinitely \ many \ n\}$$

Intuitively, B_k is the outcomes that share by all the events with index greater than k. Or, for any element ω and barrier m, $\exists n \geq m$ such that $\omega \in A_n$.

liminf It's the analogy to the largest lower bound for an infinite sequence.

$$\liminf A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

If we let $C_n := \bigcap_{k=n}^{\infty} A_k$, we can see that $C_1 \subseteq C_2 \subseteq ....$ Formally, we also have

$$\liminf A_n = \{w : w \in A_n \ for \ all \ but \ finite \ many \ n\}$$

Or, for any element ω in $\limsup A_n$, $\exists m$ such that $\forall n \geq m$, $\omega \in A_n$. That is, after some barrier, ω will appear in all A_n afterwards.

Limit of sequence of events We say a sequence of events $\{A_n\}$ converges to event A if

$$\limsup A_n = \liminf A_n = A$$

Property 1 *Let A, B, C be events in sample space Ω, then the following results hold:*

1. *Community: $A \cap B = B \cap A$, $A \cup B = B \cup A$*
2. *Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cup C) = (A \cup B) \cup C$*
3. *Distributed law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*
4. *De Morgan's law: $(A \cap B)^C = A^C \cup B^C$, $(A \cup B)^C = A^C \cap B^C$*

1.2 Small talk

Prof. Chiang: Can you give an example about statistics?
Me: SVM
Prof. Chiang: What is SVM?
Me: blablabla...
Prof. Chiang: For me, SVM and others techniques are just a chance mechanism. They provides a model or platform for us to analysis the data in some way. But deep in to the heart of data analysis, what we deal with is a general function $G(y, x_1, ..., x_k)$. And all techniques are to model the G with certain assumption and structures.

Bibliography

Chapter 2

Statistical Inference I **Prof. Chin-Tsang Chiang**

Lecture Notes 2

October 4, 2015 **Scribe: Wei-Chang Lee, Chi-Ning Chou**

On Monday, we define the idea of set and the operation over it. Intuitively, we think of the element of sets as an **outcome** and think of a set as an **event**. With these notion, further we want to design **chance mechanism** on them to describe real world.

Here comes the important issues:

- What kinds of domain (event space) for us to work on?
- What kind of chance mechanism (probability function) can we choose?

The critical concepts here is that we want to define **axioms** for these two mathematical objects. Moreover, we want such axioms to be

- Complete: Close under any possible operations.
- Compact: Less number of axioms as possible.

2.1 Recall

2.1.1 limsup and liminf

Intuitively, we think of sup and inf as:

- sup: Upper bound. In set theory, we think of it as **union** since it contains every future event.
- inf: Lower bound. In set theory, we think of it as **intersection** since every future event contains it.

As a result, limsup and liminf is analogous to the **asymptotic** upper bound and lower bound for a sequence of sets.

2.1.2 Distributive law & Partition

1. Consider events A, B and sample space Ω,

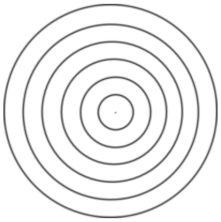
$$B \cap \Omega = B \cap (A \cup A^C) = (B \cap A) \cup (B \cap A^C)$$

As the sets in the union clause are a partition, distributive law tells us that we can divide the event B into two parts. Here we consider a finite partition so you might not get the feeling. But imagine if the partition is infinite...?

2. Consider,

$$A \cup B = (A \cup B) \cap \Omega = (A \cup B) \cap (A \cup A^C) = A \cup (B \cap A^C) = A \cup (B \backslash A)$$

Geometrically, it's like we union another set $B' = (B \backslash A)$ with A where A and B' are **disjoint**. In a special case: $A_1 \supseteq A_2 \supseteq ...$, it can be think of as a concentric circle And thus we can



transform an union into a **disjoint union**.

2.1.3 General De Morgan's Law

The De Morgan's law holds true as the number of sets is infinite! Formally, let $\{A_\alpha : \alpha \in \Gamma\}$, where Γ is an index set and can be infinite, e.g. real line, or time segment. With such notation, we can define something like

$$\bigcup_{\alpha \leq t} A_\alpha$$

Which can be the events that happen before time t. Then, as De Morgan's law holds true, we can flip the union of events and consider the events that haven't show up.

Intuitively we can think of this as the **history** or **filtration** in a *period sense*. And De Morgan's law provides a convenient way for us to do the computation.

2.2 σ-Algebra

2.2.1 Intuitions and basic axioms

By definition, σ-algebra is a collection of subsets of sample space. Intuitively, it is the space of meaningful events. As we mention the **space** of events, σ-algebra must have some nice properties such as closeness under certain operations etc. But here what we concern is not only about what properties (axioms) a σ-algebra has to obey, but also about what's the most compact axioms we can have to capture the sufficient and necessary properties a σ-algebra should follow. And the answer is, there are three of axioms: Consider a σ-algebra $\mathcal{A}$ over sample space Ω,

1. (contain empty event)$\emptyset \in \mathcal{A}$
2. (close under complementation)$\forall A \in \mathcal{A}, A^C \in \mathcal{A}$
3. (close under countable union)$\forall i \in \Gamma$ *countable*, $A_i \in \mathcal{A}$, $\bigcup_{i\in\Gamma} A_i \in \mathcal{A}$

As a quick remark, we can see that with these axioms, we also have

- $\Omega \in \mathcal{A}$ (by 1,2)
- Close under countable intersection (by 2, and De Morgan's law)

2.2.2 Properties

After defining the basic axioms for all σ-algebra, now a more applied issue arises:

$$What's\ the\ smallest\ \sigma - algebra?$$

With some observation, we can easily find out the largest and smallest σ-algebra over sample space Ω is

- Trivial σ-algebra: $\{\emptyset, \Omega\}$
- Total σ-algebra: $\{A : A \subseteq \Omega\}$. That is, the power set of Ω.

To answer this question, we have to clarify some concepts: How do we compare the size of two σ-algebras and to what extent do we want? The first question can be answered with strictly inclusion. If σ-algebra $\mathcal{A}_1$ strictly contains σ-algebra $\mathcal{A}_2$, then we can think of $\mathcal{A}_1$ is larger than $\mathcal{A}_2$.

As to the second question, we can think this in a reverse way: What elements (subsets of Ω) we must have? This question is problem-dependent, so here we imagine that we put all the events that we care into a set A. Then, we got a necessary condition for the σ-algebra that we desire: any σ-algebra that can describe the chance mechanism we're going to use must at least contain all the set in A.

Intuitively, with the above two concepts, it's reasonable to come to a wild guess for the minimal σ-algebra:

$$The\ intersection\ of\ all\ \sigma - algebras\ that\ contains\ A.$$

However, there's still one last thing to check: Is the (countable) intersection of σ-algebras also a σ-algebra? This can be proved in a theorem.

Theorem 1 (σ-algebras are close under countable intersection)
If $\mathcal{A}_\alpha$ is a σ-algebra $\forall \alpha \in \Gamma$, where Γ is a countable index set. Then $\bigcap_{\alpha\in\Gamma} A_\gamma$ is also a σ-algebra.

Proof: We should check three things and it's quite trivial. I believe I will never forget so here I leave it blank. ■

2.3 Probability Function

Finally, it's time for us to construct the **chance mechanism**! That is, we want to design probability function that maps from a σ-algebra to $[0, 1]$. Moreover, such probability function should be analogous to real life. Namely, not every function from σ-algebra $\mathcal{A}$ to $[0, 1]$ can be out candidate. As a result, we want to design rules (axioms) for probability function in advance. And just as the axioms for σ-algebra, we hope such axioms can be both compact and complete.

Komolgorov Axioms Consider sample space Ω and a σ-algebra $\mathcal{A}$ over it. Then, a probability function P should satisfy the following axioms:

- (Lower bound) $P(A) \geq 0, \forall A \in \mathcal{A}$
- (Upper bound) $P(\Omega) = 1$
- (Countably additive) If $A_\alpha \in \mathcal{A}, \forall \alpha \in \Gamma$, where Γ is a countable **mutually exclusive** index set. Then, $P(\bigcup_{\alpha\in\Gamma} A_\alpha) = \sum_{\alpha\in\Gamma} P(A_\alpha)$

Note that this brings us from set functions to **value function**. That is, we no longer playing only with set, but also take account the number, i.e. the probability.

Formally speaking, we say $(\Omega, \mathcal{A}, P)$ is a probability space if $\mathcal{A}$ is a σ-algebra over Ω and P is a probability function.

It's no hard to see that the most important axioms in Komolgorov axioms is countably additive. It allows us to deal with a countably infinite sequence of events. But you must wonder, can we relax the countably additive to finitely additive? If so, do we need to add other constraints? The answer is positive. In fact, we have

$$\text{Countably additive} = \text{Finitely additive} + \text{Axiom of continuity}$$

Here, we state the axiom of continuity without proof (left as exercise)

Remark 1 A measure is continuity from both below and above.

Axiom of continuity If a sequence of events $A_1 \supseteq A_2 \supseteq ...$ and $\lim_{n\to\infty} A_n = \emptyset$, then $\lim_{n\to\infty} P(A_n) = 0$

After defining the axioms of probability function, now we want to design a probability function for our own usage. However, here comes the problem, how to assign probability function for a given usage? A simple theorem tells us that it's sufficient to define an unique probability function with the known probability assignments on **trivial partition**.

Theorem 2 *Let sample space $\Omega = \{w_1, w_2, ...\}$, assign probability value to each outcome as $p_1, p_2, ... \geq 0, \sum_i p_i = 1$ respectively. Define the probability function P as*

$$P(A) := \sum_{i:w_i\in A} p_i$$

With some verification to the axioms of probability function, we can see that P is an unique probability function over Ω and the σ-algebra generated by partition $\{w_1, w_2, ...\}$.

Intuitively, once we want to design a probability function for our chance mechanism, it's sufficient to assign probability values over a sufficient partition.

2.4 Some random notes

2.4.1 Categorical classification/regression

Suppose we are analyzing data with label set $\{A, B, C\}$ where these labels have no numerical meaning. What can we do? There's a simple solution called: **dummy variable**

The idea is quite simple. Originally, we have a variable $s \in \{A, B, C\}$. But we cannot use s in any regression or numerical model. Now, let's change a way of thinking. We no longer focus on **which** label does s take. Instead, we consider

$$\begin{aligned}&\text{Is } s \text{ takes label } A?\\&\text{Is } s \text{ takes label } B?\\&\text{Is } s \text{ takes label } C?\end{aligned}$$

Then, we generate **two** dummy variables to **quantify** the above intuition.

$$x_1 := \begin{cases} 1, s = A \\ 0, else \end{cases}$$
$$x_2 := \begin{cases} 1, s = B \\ 0, else \end{cases}$$

Now, we have two meaning dummy variables that can be fitted into a numerical model!

2.4.2 Boole's inequality

Boole's inequality, a.k.a union bound, provides a simple but useful tools when dealing with asymptotic issues.

Theorem 3 (Boole's inequality) *Consider set of events* $\{A_\alpha : \alpha \in \Gamma\}$. *Then we have*

$$P(\bigcup_{\alpha\in\Gamma} A_\alpha) \leq \sum_{\alpha\in\Gamma} P(A_\alpha)$$

From Boole's inequality, we can derive Borel Cantelli lemma. And from Borel Cantelli lemma, we can define the concept of almost surely convergence.

Chapter 3

Statistical Inference I **Prof. Chin-Tsang Chiang**

Lecture Notes 3

October 4, 2015 **Scribe: Wei-Chang Lee, Chi-Ning Chou**

Today we're going to talk about some properties of probability function and some useful theorems such as Boole's inequality, Bonferroni's inequality, the first Borel-Cantelli lemma. In the next lecture, we will talk about Counting Theory, and today Prof.Chiang also demonstrate some intuitions about counting theory.

3.1 Basic Properties of Probability Function and Their Intuitions

Theorem 4 (properties of probability function) *Suppose $(\Omega, \mathcal{A}, P)$ is a probability space where P satisfies the axioms of probability function. Then,*

1. *(minimizer)* $P(\emptyset) = 0$
2. *(finite partition)* $P(A) = 1 - P(A^C), \forall A \in \mathcal{A}$
3. *(upper bound)* $P(A) \leq 1, \forall A \in \mathcal{A}$
4. *(conditioned on finite partition)* $P(B \backslash A) = P(B \cap A^C) = P(B) - P(B \cap A)$
5. *(union)* $P(A \cup B) = P(A) + P(B) - P(A \cup B)$
6. *(sub-event)* $P(A) \leq P(B), \forall A, B \in \mathcal{A}, A \subseteq B$
7. *(conditioned on countably partition)* $P(B) = \sum_{i=1}^{\infty} P(B \cap A_i)$*, where* $\{A_i\}$ *is a partition of* Ω *and* $A_i \in \mathcal{A}, \forall i$.
8. *(Boole's inequality)* $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$*. Note that here* $\{A_i\}$ *is arbitrary sequence of events.*

The following gives some intuitions behind each result:

1. To prove empty set is a minimizer, we can simply choose a sequence of event as $A_1 = \Omega, A_i = \emptyset, \forall i = 2, 3, ...$ Then the results is obvious as we apply countably mutual exclusive additive axiom.

2. By choosing $A_1 = A, A_2 = A^C, A_i = \emptyset, \forall i = 3, 4, ...,$ countably mutual exclusive additive axiom shows the results.

3. With 2. and the lower bound axiom.

4. Consider $B = B \cap \Omega = B \cap (A \cup A^C)$, by distributive law, $B = (B \cap A) \cup (B \cap A^C)$ where the two terms are mutually exclusive. Thus, we an apply the mutually exclusive additive axiom.

5. By think of $A \cup B$ as $A \cup (B \cap A^C)$ such that the two terms are mutually exclusive. Then consider $B = (B \cap A) \cup (B \cap A^C)$ where the two terms are also mutually exclusive. Apply the mutually exclusive additive axiom, we have $P(A \cup B) = P(A) + P(B \cap A^C) = P(A) + (P(B) - P(B \cap A)) = P(A) + P(B) - P(A \cap B)$.

6. Simply consider $B = A \cup (B \cap A^C)$.

7. The same as 4. by consider $B = \bigcup_{i=1}^{\infty}(B \cap A_i)$ where all the terms are mutually exclusive.

8. Set $A_1^* = A_1, A_2^* = A_2 \backslash A_1, A_3^* = A_3 \backslash (A_1 \cup A_2), ..., A_j^* = A_j \backslash (\bigcup_{i=1}^{j-1} A_i)$ such that $\{A_j^*\}$ is mutually exclusive and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{j=1}^{\infty} A_j^*$, which means that $P(\bigcup_{i=1}^{\infty} A_i) = P(\bigcup_{j=1}^{\infty} A_j^*)$. With an observation on $P(A_j^*) = P(A_j \backslash (\bigcup_{i=1}^{j-1} A_i)) \leq P(A_j)$, we have $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$.

Intuition (properties of probability function)

The key point here is to make the sets become **mutually exclusive** so that we can apply the mutually exclusive additive axiom. On the contrary, it also shows that the mutually exclusive additive axiom is so strong that what we think is reasonable in intuition can be described by it!

3.2 Useful Inequalities, and the First Borel-Cantelli Lemma

In the previous section, Boole's inequality provides an loose upper bound for the probability of the **union** of a sequence of events. Now, we can draw a similar lower bound for the probability of the **intersection** of a sequence of events.

3.2.1 Bonferroni's inequality

Theorem 5 (Bonferroni's inequality)

Let $A_1, A_2, ...$ be a sequence of events, then

$$P(\bigcap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

Proof:

$$P(\bigcap_{i=1}^{n} A_i) = 1 - P(\bigcup_{i=1}^{n} A_i^C) \geq 1 - \sum_{i=1}^{n} P(A_i^C) = 1 - \sum_{i=1}^{n}(1 - P(A_i)) = \sum_{i=1}^{n} P(A_i) - (n-1)$$

■

Just as Boole's inequality, Bonferroni's inequality provides a loose lower bound. What does Bonferroni's inequality brings to us? There's an application in **pairwise comparison**: Suppose we are going to estimate three parameters $\theta_1, \theta_2, \theta_3$ which are highly correlated. How can we lower bound the probability of the event: $(\theta_1 - \theta_2) \in A_{12}, (\theta_1 - \theta_3) \in A_{13}, (\theta_2 - \theta_3) \in A_{23}$? Note that this probability is used for constructing confidence interval.

Now, consider that we have already lower bound the significant level of each parameter, say

$$\begin{aligned} P(\theta_1 - \theta_2 \in A_{12}) &\geq 1 - \alpha_1 \\ P(\theta_1 - \theta_3 \in A_{13}) &\geq 1 - \alpha_2 \\ P(\theta_2 - \theta_3 \in A_{23}) &\geq 1 - \alpha_3 \end{aligned}$$

By Bonferroni's inequality, we have

$$\begin{aligned} & P((\theta_1 - \theta_2) \in A_{12}, (\theta_1 - \theta_3) \in A_{13}, (\theta_2 - \theta_3) \in A_{23}) \\ \geq\; & P(\theta_1 - \theta_2 \in A_{12}) + P(\theta_1 - \theta_3 \in A_{13}) + P(\theta_2 - \theta_3 \in A_{23}) - 2 \\ \geq\; & (1 - \alpha_1) + (1 - \alpha_2) + (1 - \alpha_3) - 2 \\ =\; & 1 - (\alpha_1 + \alpha_2 + \alpha_3) \end{aligned}$$

Intuition (Boole's and Bonferroni's inequalities)

The important concept here is that both Boole's and Bonferroni's inequalities are relatively **loose**. As a result, they can only provide good estimate as the probability of the individual events are **small**. Take a look at the above example, we can see that normally the significant level α_i is relatively small. And that's why here Bonferroni's inequality has some usage.

3.2.2 The first Borel-Cantelli lemma

Before we give the statement of the first Borel-Cantelli lemma, let's consider the following motivation: Imagine after defining the axiom of probability and deduce some nice properties and now you're going to define the concept of **conditional probability** $P(A|B)$. However, an issue coming out:

What if P(B)=0?

This makes sense as now we consider the sample space with countably many outcomes. The probability of a single outcome is measure 0 ($P(\text{outcome}) = 0$). As a result, the conditional probability taught in elementary probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$ is not well-defined!

A simple idea is to relax B or construct a sequence of event $B_1, B_2...$ such that $\lim_{n\to\infty} B_n = B$ and define the conditional probability as $P(A|B) := \lim_{n\to\infty} \frac{P(A\cap B_n)}{P(B_n)}$, which is much more making sense. But here comes a question:

Is the probability of a convergent sequence of events also converge?

Theorem 6 (The first Borel-Cantelli lemma)
Let $\{A_i\}$ be a sequence of events such that $\sum_{i=1}^{\infty} P(A_i) < \infty$, then

$$P(A_{n.i.o}) = 0$$

, where $A_{n.i.o} = \limsup_{n\to\infty} A_n$

Proof: By the definition of limsup, we have $\bigcup_{i=n}^{\infty} A_i \to \limsup_{n\to\infty} A_n$ monotonously. Now, we proof the lemma with two steps:

1. (Continuity axiom) $P(A_{n.i.o}) = \lim_{n\to\infty} P(\bigcup_{i=n}^{\infty} A_i)$
2. (Boole's inequality) $P(\bigcup_{i=n}^{\infty} A_i) \leq \sum_{i=n}^{\infty} P(A_i)$

Since $\sum_{i=1}^{\infty} P(A_i) < \infty$, we now that $\lim_{n\to\infty} \sum_{i=n}^{\infty} P(A_i) = 0$. Thus, $P(A_{n.i.o}) = 0$. ■

Note that there's no independence involves here, which is different from the second Borel-Cantelli's lemma that we'll introduce later.

3.3 Intuitions About Counting Theory

Why do we need counting techniques?

3.3.1 Sampling

What kind of sampling mechanisms do we have? That is, how do we design the mechanism for us to draw samples from a large population? Do we sample periodically? Or, sample from the subset that we believe is more important? Or, we cluster the sample space into a smaller subspace and sample in them respectively?

The chance mechanism we choose will affect the underlying possibility for us to get a certain outcome. That is, the way we analyze or the model we choose will differ as the sampling techniques are distinct. And how do we react to different mechanism is to utilize the tools in counting theory.

3.3.2 Bootstrap

In high-dimensional statistics, the number of parameters are so large that sometimes even the number of samples is less than that of parameters! However, we believe the underlying structure might not be so much. Namely, there might exist some sparse or low rank structure behind the scheme.

But, how can we find out such implicit properties with only a few samples and a great amount of candidate parameters? There's a method called **bootstrap**, which reuses the same samples and generate a subset of new samples from it. With some counting argument, we can see that this will not affect the performance and can increase the number of samples to search for special structure in a high-dimensional setting.

To fulfill such methods, we need to have some tools to analyze the process of reusing samples. And counting theory can provide a great help. For now, this applications give us a sense of motivation to study counting theory.

Chapter 4

Statistical Inference I **Prof. Chin-Tsang Chiang**

Lecture Notes 4

October 4, 2015 **Scribe: Wei-Chang Lee, Chi-Ning Chou**

Today we're going to talk about method of counting to be able to study probability assignments on finite space, we will then give definition about conditional probability on discrete case and intuition about continuous case. After knowing how to calculate conditional probabilities, we can then use Bayes' theorem to connect experimental study to observation study which is called the odds ratio. At the end of the class, Prof.Chiang introduce the concept "independence" and how to interpret it beyond geometric views in a more statistical way.

4.1 Counting Methods

There are two kinds of interpretation of probability measure.

1 The first kind of view is based on the "frequency of occurrence". They do random experiments many times to study how many times event which is interested in would take place. Then, they assign the limit ratio as the probability of certain events. The intuition of this interpretation is that we believe the underlying parameter is invariant so after many repeated experiment the random effect can be cancelled out and the real instinct reveals.

2 The second kind of view is just based on the "subjective belief of interpreter" in the chance of an event occurring. We can just give probability under our faith before any experiment has been done and then do experiment to study whether our prior assumption fit the data or not and then modify it. This procedure can be done by Bayes' theorem.

The counting problem is sometimes sophisticated and along with many restrictions. The way to solve such problem is to break them into series of simple tasks and apply rules to combine it back.

Theorem 7 (Fundamental Theorem of Counting) *Suppose a job consists of k separate tasks, the ith of which can be done in n_i ways, then entire job can be done in $\prod_{i=1}^{k} n_i$ ways.*

Intuition (Classification of counting methods)

The proof of Fundamental Theorem of Counting is quite trivial. Sometimes it is better to think of task as partition criteria such as love or hate, different gender or income level then we just construct a sample space $\Omega = \Omega_1 \times \Omega_2 ... \Omega_k$ which is the cartesian product of k criteria we interested in then we can apply the theorem to calculate there are how many possible outcomes.
In reality, we may face situation such as replacement and unordered. Replacement means that $n_i = n_{i+1}$ and unordered means that we can perform k separate tasks arbitrary without certain order. In such case we have to carefully apply or modify the fundamental counting theorem.

Consider number of possible arrangements of size r drawing from n different subjects. We can divide the case into four categories.

	Without replacement	With replacement
Ordered	$n*(n-1)*(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$	$n*n*n...n = \prod_{i=1}^{r} n = n^r$
Unordered	$\frac{\prod_{i=1}^{r}(n-i+1)}{r!} = \frac{n!}{(n-r)!(r!)} = \binom{n}{r}$	$\binom{n+r-1}{r}$

Proof: Case 1 and case 2 is quite simple just applying the Fundamental Theorem of Counting. For case 3, r different objects can be permuted in $r*(r-1)*(r-2)...1 = r!$ ways but they represent the same arrangement in unordered situation. So we have case 3 equals case 1 divided by r!. Case 4 is the most difficult, we may simply view it as case 2 divided by r! but it will underestimate the possible arrangements since r objects with some of them are of same kind do not construct r! different permutations. A clever way to solve case 4 is to think of r as numbers of coins and place it arbitrary but all into n different box. A coin in ith box means in our ultimate arrangements we have one ith objects. Consider a small case as n=3 and r=3. Then the following figure is the realization of picking two 2th object and one 3th object. And all possible realization cab be expressed as all possible arrangements of 2 | and 3 O which is $\binom{3-1+3}{3}$.

$$\underline{\quad}^1|\underline{OO}^2|\underline{O}^3$$

So for case 4, the answer is $\binom{n-1+r}{r}$. ■

The counting techniques are useful when the sample space is finite and every possible outcomes in S are equally likely. The probability of certain event can be calculated by the number of outcomes in that event times the probability of each outcome from the countably addictive axiom.

Theorem 8 (Enumerating outcomes) *Let* $\Omega = \{w_1, w_2...w_n\}$ *with* $P(\{w_i\}) = \frac{1}{n}\ \forall i$ *then* $P(A) = \sum_{\{w_i\}\in A} P(\{w_i\}) = \sum_{\{w_i\}\in A} \frac{1}{n} = \frac{\sharp(A)}{\sharp(\Omega)}\ \forall A$.

Remark 2 This is also the classical definition of probability from Pierre-Simon Laplace.

4.2 Conditional probability

In reality, we may need to study something like if she is a girl, what is the probability that she wants to get married. Studying these kinds of probabilities under certain situation or restrictions of sample space needs the definition of conditional probability.

Theorem 9 *Let P(.) be a probability measure on σ-algebra F, A and B be events in F and P(B)>0 then, $P(.|B)$, the conditional probability of A given B is denoted by*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

First we need to verify $P(.|B)$ is a probability measure.

1 $P(A|B) \geq 0 \ \forall A \in F$: Since $P(B) > 0$ and $P(A \cap B) \geq 0$ because P(.) is a probability measure, then the ratio $\frac{P(A \cap B)}{P(B)} \geq 0 \ \forall A \in F$.

2 $P(\Omega|B) = 1$: $\frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$.

3 $A_1, A_2 ... A_n ... \in F$, $A_i \cap A_j = \phi$ then $P(\cup_{i=1}^{\infty} A_i|B) = \sum_{i=1}^{\infty} P(A_i|B)$: First notice that $A_1 \cap B, A_2 \cap B ... A_n \cap B ...$ are all disjoint then $P(\cup_{i=1}^{\infty} A_i|B) = \frac{P(\cup_{i=1}^{\infty} A_i \cap B)}{P(B)} = \frac{P(\cup_{i=1}^{\infty} (A_i \cap B))}{P(B)} = \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i|B)$.

$P(.|B)$ is indeed a probability measure, but what is its statistical meaning?

Example 1 Consider a case that we need to study how accurate our new AIDS test is, then the sample space is partitioned by two criteria. The test result is positive/negative and the subject has AIDS or not. We have $\Omega : \{(Test, Disease) : T \in \{+,-\}, D \in \{+,-\}\} = \Omega_T \times \Omega_D$, then

$$P(D_+|T_+) = \frac{P(D_+ \cap T_+)}{P(T_+)} = \frac{P(\{(D_+,T_-),(D_+,T_+)\} \cap \{(D_+,T_+),(D_-,T_+)\})}{P(\{(D_+,T_+),(D_-,T_+)\})}$$

So the conditional probability is actually the study of how sub-sample space Ω_T, Ω_D will affect each other on Ω i.e. what is the relation between each partition criteria, we may not simply think conditional probability in geometric view.

Intuition (Conditional Probability)

Generally speaking, $P(B) > 0$ cannot always be satisfied since we may be interested in continuous data such as heights and weights. We face division by zero when study problem like given father is 1.75 tall what is the probability the son is 1.80 tall. We have $P(\text{father is } 1.75) = 0$ and above definition does not work any more. A natural way to save it is to think of B as a small neighbourhood $(B - \frac{\triangle}{2}, B + \frac{\triangle}{2})$ so we have

$$P(A|B) = \frac{\lim_{\triangle \to 0} P(A \cap (B - \frac{\triangle}{2}, B + \frac{\triangle}{2}))/\triangle}{\lim_{\triangle \to 0} P(B - \frac{\triangle}{2}, B + \frac{\triangle}{2})/\triangle}$$

Noting that both the numerator and denominator are of change rate form so $P(A|B)$ can be thought of as the intensity of the ratio of change rate.
In the previous test-disease example, we can not do medical behavior relying on uncertified test for moral issue. We may only do the reverse direction test. Do the test to patients who have the disease and what is the test result respond to it. So we have only $P(T_+|D_+)$ and also the prevalence rate of the disease $P(D_+)$. How can we know $P(D_+|T_+)$? This is where we need the Bayes' theorem.

Example 2 [Casella 1.3.4] Three prisoners A,B and C are on death row. The governor decides to pardon one of the three prisoners randomly. He informs the warden of his choice but ask warden to keep it secret. A tries to get the warden to tell him who had been pardoned. The warden refuses so A asks another way. A asks which of B and C will be executed. And the following table is warden's possible react.

Pardoned	Warden tells	Probability
A	B dies	r
	C dies	1-r
B	C dies	1
C	B dies	1

The warden tells A that B would be executed, does it reveal any message to A?

Solve: Let W denote the event that warden says B will die then A can update his probability given W.

$$\begin{aligned} P(\text{A pardoned } |W) &= \frac{P(A \cap W)}{P(W)} \\ &= \frac{P(\text{A pardoned B dies})}{P(\text{A pardoned B dies}) + P(\text{C pardoned B dies}) + P(\text{B pardoned B dies})} \\ &= \frac{\frac{1}{3} \cdot r}{\frac{1}{3} \cdot r + \frac{1}{3} + 0} = \frac{r}{r+1} \end{aligned}$$

The answer would rely on warden's behavioral pattern.

	$P(A\|W)$	Message
$r=1$	$\frac{1}{2}$	good news
$r=\frac{1}{2}$	$\frac{1}{3}$	no news
$r=0$	0	bad news

Intuition (Three Prisoners)

When r tends to 1, the probability A would be pardoned reaches $\frac{1}{2}$. This is because when A had been pardoned, the warden would always say B and event W becomes a more good predictor of A pandered and vice versa. When $r = \frac{1}{2}$, the warden has the same chance to tell B or C dies when A had been pardoned so W can not be used to trace back A pardoned. Sometimes people may think $P(A|W) = \frac{1}{2}$ because one of A and C would be saved. But it is $P(A|B^C)$ actually.
Consider $P(C|W)$ it is actually $1 - P(A|W)$. So when $r = \frac{1}{2}$, $P(C|W) = \frac{2}{3}$. Though warden reveals no message to A but it is actually a good news for C. It is like the situation we switch door in the Monty Hall problem. Conditional probability can be really tricky.

■

4.3 Bayes Rule

4.3.1 Bayes rule

Imagine a situation that there are two sample spaces $\Omega_1 = \{D_1, ..., D_p\}, \Omega_2 = \{O_1, ..., O_q\}$ that augments a larger sample space $\Omega = \Omega_1 \times \Omega_2$. What kind of inferences can we draw from the knowledge of outcome O? If I know the conditional probability $P(O|D)$ what can do to inference $P(D|O)$?

Intuitively, there are four categories of probabilities here:

- $P(D)$: probability of getting disease.
- $P(O)$: probability of yielding some symptoms (outcomes).
- $P(D|O)$: probability of getting disease conditioned on having some symptoms.
- $P(O|D)$: probability of having some symptoms conditioned on getting a disease.

The four categories are connected together with the following Bayes rule:

$$P(D=d|O=o) = \frac{P(O=o|D=d)}{\sum_{d\in D} P(D=d)P(O=o|D=d)}$$

The important philosophy here is that, in real life, we won't have all the four categories of probability in hand. That is, we have to draw inference on the probability we care from the probability we known. For example, a doctor want to know the probability of a patient getting disease d conditioned on he has symptom o. If the doctor knows the probability of getting a disease ($P(D)$), and the probability of a outcome to happen conditioned on having disease or not ($P(O|D)$). Then, by applying Bayes rule, he can calculate the conditioned probability $P(D|O)$, which is what he concerns.

Here I a little abuse the notation of D, actually, the event in D must be a **partition**. For example $D = \{\text{disease}, \text{no disease}\}$.

Now, we formally state the Bayes rule as follow:

Theorem 10 (Bayes rule) *Let $A_1, ..., A_n$ be a partition over $\mathcal{A}$ and $B \in \mathcal{A}$, then*

$$P(A_i|B) = \frac{P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$$

Intuition (Bayes rule)

Bayes rule help us update the probability over a **partition** on event space. It utilize what we have:

- The probability of population on the partition.
- The conditioned probability of the observed event conditioned on each event in the partition.

Then calculate the conditioned probability we want to inference: The probability of each event in the partition to happen conditioned on an observation.

4.3.2 Relative risk and Odd ratio

Both relative risk and odd ratio are important statistical concept in epidemiology/experimental study. In this context, we care the following scenario: There's a disease and a treatment, we want to know how well the treatment is but in the meantime hoping the treatment has less side-effect. We can summarize the above scenario in Table 4.1. Here, D refers to the event that the patient has the disease and E is the event that the patient is under treatment (or formally, being exposed to the treatment). In the context of experimental study, see Table 4.2, E can be regarded as the experimental group and E^C can be seen as the control group. Here D and D^C can be simply considered as the presence and absence of an event.

Risk		Treatment	
		Exposed	Not exposed
Disease	Diseased	$P(D \mid E)$	$P(D \mid E^C)$
	Healthy	$P(D^C \mid E)$	$P(D^C \mid E^C)$

Table 4.1: Epodemiology scenario.

Risk		Experiment	
		Experiment	Control
Outcome	Presence	$P(D \mid E)$	$P(D \mid E^C)$
	Absence	$P(D^C \mid E)$	$P(D^C \mid E^C)$

Table 4.2: Experimental study.

With this scenario, we an immediately some intuitive term to help us making inference. The first one is **relative risk (RR)**. Relative risk is the probability ratio of an event to happen under certain exposure or not. In this context, it's simply $\frac{P(D|E)}{P(D|E^C)}$.

Definition 1 (relative risk) *Relative risk (RR) is the probability ratio of an event to happen under an exposure or not. That is*

$$RR := \frac{P_{event\ when\ exposed}}{P_{event\ when\ not\ exposed}}$$

Note that RR can help us inference on the effectiveness of the treatment on the disease. See Table 4.3 for more details on RR.

	Epidemiology	Experimental study
RR>1	Thetreatment is worse than having no treatment.	The event is more likely to happen in the **experimental** group.
RR<1	The treatment is effective.	The event is more likely to happen in the **control** group

Table 4.3: Relative risk.

Now, we can use relative risk to describe the effectiveness of a treatment/experiment on a certain disease/event, we might wander: how about the relation between two diseases/events under the same treatment/experiment? And this is actually the definition of odd ratio.

Definition 2 (odd ratio) *The odd ratio (OD) is the ratio of the relative risk of two events. That is, the odd ratio of event A and B w.r.t a treatment E is*

$$\begin{aligned} OD : &= \frac{RR_A}{RR_B} \\ &= \frac{P(A|E)/P(A|E^C)}{P(B|E)/P(B|E^C)} \end{aligned}$$

The simplest odd ratio is to consider a event and its complement, say D and D^C, then the odd ratio will become

$$\begin{aligned} \frac{P(D|E)/P(D|E^C)}{P(D^C|E)/P(D^C|E^C)} &= \frac{P(D\cap E)/P(D\cap E^C)}{P(D^C\cap E)/P(D^C\cap E^C)} \\ &= \frac{P(D\cap E)/P(D^C\cap E^C)}{P(D^C\cap E)/P(D\cap E^C)} \end{aligned}$$

With this transformation, we can use the four intersection probability to calculate the odd ratio instead of the four conditional probability. In some circumstances, this will be more convenient and intuitive.

4.4 Independence

The initial idea (and the most intuitive concept) of independence is that two event A and B is said to be independent if the probability of A to happen will not change after we have the knowledge of B, and vice versa. Thus, formally we can write

$$P(A) = P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$P(B) = P(B|A) = \frac{P(A \cap B)}{P(A)}$$

And we can see that actually as A and B are independent, $P(A \cap B) = P(A)P(B)$, while the converse is also correct. As a result, this has become the definition of independence.

Definition 3 (independence) *We say two event A and B are independent if $P(A \cap B) = P(A)P(B)$.*

However, note that there are more than one concept about independence in mathematics. For instance, in linear algebra, there's so called linear independence. And even in probability theory, as we consider more than one event, say a group of events, the idea of independence varies. Actually, there are two kinds of independence for more than two events: **pairwise independence** and **mutually independence**.

Definition 4 (pairwise independence) *We say a finite set of events $\{A_i\}$ is pairwise independent if $\forall A_i \neq A_j$, $P(A_i \cap A_j) = P(A_i)P(A_j)$.*

Definition 5 (mutually independence) *We say a finite set of events $\{A_i\}$ is mutually independent if for any subset of $\{A_i\}$, say $\{A'_j\}$, $P(\bigcap_j A'_j) = \prod_j P(A'_j)$.*

Note that mutually independence is **strictly stronger** than pairwise independence. That is, the former implies the latter while the converse is not necessarily true.

Chapter 5

Statistical Inference I		**Prof. Chin-Tsang Chiang**
	Lecture Notes 5	
October 4, 2015		**Scribe: Wei-Chang Lee, Chi-Ning Chou**

After explaining the concept of independence, today we can finally introduce the Second Borel-Cantelli Lemma. A lemma that can lately be used to prove one of the most important result in probability theory, the strong law of large numbers. Also, we talk about another fundamental concept in probability, random variable, which let us change our views from set to the real field.

5.1 The Second Borel-Cantelli Lemma

Also called the reverse Borel-Cantelli Lemma.

Theorem 11 *Let $A_1, A_2, A_3 ... A_n ... \in \mathcal{A}$ be mutually independent. If $\sum_{i=1}^{\infty} P(A_i) = \infty$, then*

$$P(A_{n.i.o}) = 1$$

, where $A_{n.i.o.} = \limsup_{n\to\infty} A_n$.

Proof: First Notice that since $\{A_n\}$ are independent, so $P(\bigcap_{k=n}^{n+j} A_k^c) = \prod_{k=n}^{n+j}(1 - P(A_k))$ and $1 - x \leq e^{-x}$ $\forall x \geq 0$ then

$$P(\bigcap_{k=n}^{n+j} A_k^c) \leq e^{-\sum_{k=n}^{n+j} P(A_k)} \ \forall n$$

And $\sum_{i=1}^{\infty} P(A_i) = \infty$, one has

$$\lim_{j\to\infty} P(\bigcap_{k=n}^{n+j} A_k^c) \leq \lim_{j\to\infty} e^{-\sum_{k=n}^{n+j} P(A_k)} = 0$$

From Boole's inequality, it follows that

$$P(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c) \leq \sum_{n=1}^{\infty} P(\bigcap_{k=n}^{\infty} A_k^c) = 0$$

Then by De Morgan's rule we have,

$$P(\limsup_{n\to\infty} A_n) = P(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k) = 1 - P(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c) = 1$$

■

So when $\{A_n\}$ are mutually independent, combining both Borel-Cantelli lemma, we have a zero-one law: Borel's zero-one law

$$P(A_{n.i.o.}) \text{ is either 0 or 1 with respect to} \sum_{i=1}^{\infty} P(A_i) \text{ converged or not.}$$

This give us a peek of Kolmogorov's zero-one law.

Theorem 12 *Suppose a probability space $(\Omega, \mathcal{A}, P)$and A_n be a sequence of mutually independent σ-algebras contained in $\mathcal{A}$ then the tail events $F \in \bigcap_{n=1}^{\infty} G_n$ where $G_n = \sigma(\bigcup_{k=n}^{\infty} A_k)$ is either zero or one i.e. $P(F) = 1$ or $P(F) = 0$.*

Remark 3 A naive way to understand the theorem is to consider if you have a sequence of independent random variables and an event that is invariant if you ignore finitely many of the variables, then the probability of that event is either 0 or 1 .

Remark 4 Independent assumption is really important in the Borel's zero one law. If a event E with $0 < P(E) < 1$ and let a sequence of events $\{A_n\}$ where each $A_i = E$ then it is clear $\{A_n\}$ are not independent at all and $P(A_{n.i.o.}) = P(E)$ which violates the zero one law.

Intuition (Borel Cantelli Lemma)

The Borel Cantelli Lemma enable us to construct probability in infinite many time/games from probability in sequence of events. If we toss a coin or play a independent game infinite many times, and let A_n be events in nth toss/round. We can first study a pattern of coin or strategy in games has how much chance to win in one toss/round then use Borel Cantelli Lemma to extend it to infinite case that certain patterns will definitely happen or our strategy can finally win. An intersting example is given as below.

Example 3 [Infinite Monkey Theorem] Suppose a monkey has a typewriter, he types any alphabet at random. If he never stops typing, then he can almost surely type any Shakespeare's sonnet.

Solve: Consider the alphabets sequence the monkey types, we can divide this infinitely long string to infinitely many chunks, each chunk has length k where k equals to the number of alphabets in given Shakespeare's sonnet. Let $\{E_n\}$ denotes the monkey successfully types the sonnet in the nth chunk, then we have

$$P(E_n) = \frac{1}{26^k} > 0 \text{ so } \sum_{n=1}^{\infty} P(E_n) = \infty$$

And chunks are independent from one another. Using the Second Borel Cantelli Lemma, we get

$$P(E_{n.i.o.}) = 1$$

There are infinite many chunks, the monkey almost surely types Shakespeare's sonnet. ■

Remark 5 In reality, the monkey can not fulfil this since not only did he has certain behavior pattern but also reality is a very big finite number. If the universe is filled with monkeys and they keep typing, the chance to type Hamlet is less than $\frac{1}{10^{183800}}$. The gap between infinite and finite is bigger than every big finite number we can guess.

Example 4 [Strong Law of large number]We neglect the proof but argue how it can be used in proving it.We want to show that

$$P(\lim_{n\to\infty}\frac{S_n}{n}=\mu)=P(\Omega_0)=1$$

Where $S_n = X_1 + X_2 ... + X_n$ and $E(X_i) = \mu$ then it is equivalent to there exist $\Omega_0 \in \mathcal{A}$ for every $w \in \Omega_0$ it holds that

$$\lim_{n\to\infty}|\frac{S_n}{n}-\mu|=0$$

It suffices to show for any $\epsilon > 0$, $|\frac{S_n}{n} - \mu| > \epsilon$ can occur only a finite number of times. And it can be done by Borel-Cantelli Lemma.

5.2 Random Variables

Before giving formal definition of what is a random variable. Let use first introduce two important terms, Borel sets and Σ-measurable function.

Definition 6 (Borel σ-algebra) *Let S be a topological space then $\mathcal{B}(S)$, the Borel σ-algebra on S, is the σ-algebra generated by the family of open subsets of S.*

$$B(S) := \sigma(open\ sets)$$

And

$$\mathcal{B} := B(R)$$

Every subsets of R we we meet in everyday use is an element of $\mathcal{B}$; but elements of $\mathcal{B}$ can be quite complicated. An easy way to understand $\mathcal{B}$ is by the π system (closed under finite intersection) ξ,

$$\xi = \pi(R) := \{(-\infty, x] : x \in R\}$$

then

$$\mathcal{B} = B(R) = \sigma(\xi)$$

To introduce the concepts of Σ-measurable function and random variable, we should first define the preimage of a set function.

Definition 7 (preimage of set function) *Let $h : \Omega \to \mathcal{X} \subset \mathbb{R}$ be a set function. Then the preimage of $S \subset \mathbb{R}$ over h is*

$$h^{-1} := \{w \in \Omega : h(w) \in S\}$$

Note that the image of the set function is not necessary a really set. That is, it can also be a single point in $\mathbb{R}$ just as the following definition of Σ-measurable function.

Definition 8 (Σ-measurable function) *Suppose that $h : \Omega \to \mathbb{R}$ then h is called a Σ-measurable function if $h^{-1} : \mathcal{B} \to \Sigma$, that is, $h^{-1}(S) \in \Sigma\ \forall S \in \mathcal{B}$. And we write $m\Sigma$ to be the set of all these measurable function on Ω.*

Intuition (measurable function)

A set function h is Σ-measurable if the preimages of every Borel set lies in Σ.

Definition 9 (Random variable) *Let $(\Omega, \mathcal{A})$ be our sample space and family of events. A random variable X is an element of $m\mathcal{A}$ which will satisfied our random mechanism P. Thus,*

$$X : \Omega \to \mathbb{R}, X^{-1} : \mathcal{B} \to \mathcal{A}$$

Intuition (Random variable)

Suppose X is a random variable carried by arbitrary probability triple $(\Omega, \mathcal{A}, P)$ then we have

$$\mathcal{B} \xrightarrow{X^{-1}} \mathcal{A} \xrightarrow{P} [0, 1]$$

Define μ_x by

$$\mu_x := P \circ X^{-1} \text{ i.e. } \mu_x(B) = P(\{w : X(w) \in B\})\ \forall B \in \mathcal{B}$$

And μ_x is actually a probability measure on $(R, \mathcal{B})$, then for any abstract probability space we can use X to change it to $(R, \mathcal{B})$ which we are familiar with.

$$(\Omega, \mathcal{A}, P) \xrightarrow{X} (R, \mathcal{B}, \mu_x)$$

and define the function $F_X : R \to [0, 1]$ as:

$$F_x(c) := \mu_x((-\infty, c]) = P(\{w : X(w) \leq c\}) = P(X \leq c)$$

Now how can we verify any real-valued function X on Ω is indeed a random variable? We first begin from the reverse image of X.

Property 2 (Properties of set function) *Let $X : \Omega \to \mathcal{X} \subset \mathbb{R}$ be a set function, then the following properties hold. (Note that here X is not necessary a random variable)*

1. *(Close under complementation): $X^{-1}(S^C) = (X^{-1}(S))^C$*
2. *(Close under union): $X^{-1}(\bigcup_{\alpha\in\Gamma} S_\alpha) = \bigcup_{\alpha\in\Gamma} X^{-1}(S_i)$, where $\{S_\alpha\}_{\alpha\in\Gamma}$*
3. *(Close under intersection): $X^{-1}(\bigcap_{\alpha\in\Gamma} S_\alpha) = \bigcap_{\alpha\in\Gamma} X^{-1}(S_i)$, where $\{S_\alpha\}_{\alpha\in\Gamma}$*

By definition, a random variable should satisfy the condition that the preimage of every Borel set should in the event space $\mathcal{A}$. However, it's difficult to check since it's hard to enumerate all Borel set and claim the results. Thus, we would like to find a relaxed but necessary condition for a set function to be a random variable. And the following theorem does so.

Theorem 13 (necessary and sufficient condition for random variable) *Let $X : \Omega \to \mathcal{X} \subset \mathbb{R}$ be a set function, then*

$$X \text{ is a random variable} \Leftrightarrow \forall x \in \mathbb{R},\ \{w : X(w) \leq x\} \in \mathcal{A}$$

The proof is left in Appendix .1

5.3 From Set Function to Value Function

Note that random variable is a function that helps us map the event set $\mathcal{A}$ onto the Borel set. The importance here is that know we can do the equivalent operation on **real number** instead of arbitrary σ-algebra. This not only provides us a general and uniform way to play but also gives us the opportunity to operate on an **ordered** set.

Soon, we might wonder can we play with an even more general function: value function instead of just a set function onto real number? And by the construction of random variable, there are two direct value functions that play an important role in probability theory.

Theorem 14 (value functions) *Let X be a random variable w.r.t. $(\Omega, \mathcal{A}, P)$, then we can define*

- *The measure function of X is $\mu_X : \mathcal{B} \to \mathbf{R}^+$ such that*

$$\forall B \in \mathcal{B},\ \mu_X(B) := P\{w : X(w) \in B\}$$

- *The cumulative distribution of X is $F_X : \mathbf{R} \to \mathbf{R}^+$ such that*

$$\forall x \in \mathbf{R},\ F_X(x) := P\{w : X(w) \leq x\}$$

Here, we also have a necessary and sufficient condition for cumulative function. Intuitively, when a function F satisfies the following conditions, then there's a random variable with its unique cumulative distribution being F.

Theorem 15 (necessary and sufficient condition of cumulative distribution) *F is a cumulative distribution iff*

- *(upper and lower bound)* $\lim_{x \to \infty} F(x) = 0,\ \lim_{x \to \infty} F(x) = 1$
- *(non-decreasing)* $\forall x \leq y,\ F(x) \leq F(y)$
- *(right continuous)* $\lim_{x \to x_0^+} = F(x_0)$

The proof is left in Appendix .2.

Intuition (set function and value function)

Careful with the difference of

- (**random variable**): $X : \Omega \to S \subset \mathbb{R}$
- (**measure function**): $\mu_X : \mathcal{B} \to \mathbb{R}^+$
- (**cumulative distribution**): $F_X : \mathbb{R} \to \mathbb{R}^+$

.1 Proof of the necessary and sufficient condition of random variable

Recall that Theorem 13 provides a necessary and sufficient condition for a set function to be a random variable. Here, we prove the correctness of the theorem.

($\Rightarrow$) First, we can see that $\{w : X(w) \leq x\} = X^{-1}[(-\infty, x)]$. Thus, it's sufficient to show that $(-\infty, x)$ is in Borel set $\forall x$. And this is simple since $(-\infty, x) = \bigcup_{n \in \mathbb{N}} [-n, x] \in \mathcal{B}$.

($\Leftarrow$) This direction can be proved in two steps:

1. Show that $\forall a < b,\ X^{-1}([a, b]) \in \mathcal{A}$.

2. Show that $\xi = \{S : \exists w \in \Omega,\ X(w) = S\}$ is a σ-algebra.

With these two results, we can see that the image of X contains $\mathcal{B}$. Thus, X is a random variable. The following show the correctness of these two:

1. $\forall a < b$, consider

$$\begin{aligned} X^{-1}([a, b]) &= X^{-1}((-\infty, b] \backslash (-\infty, a]) = X^{-1}((-\infty, b] \cap (-\infty, a])^C) \\ &= X^{-1}((-\infty, b]) \cap X^{-1}((-\infty, a])^C) \\ &= X^{-1}((-\infty, b]) \cap X^{-1}((-\infty, a]))^C \in \mathcal{A} \end{aligned}$$

2. Check the three axioms of σ-algebra:

   (a) Clearly, $\emptyset \in \xi$.

   (b) $\forall S \in \xi$, $X^{-1}(X^C) = X^{-1}(S)^C \in \mathcal{A}$ by Property 2. Thus $S^C \in \xi$.

   (c) $\forall \{S_\alpha\}_{\alpha \in \Gamma} \in \xi$, $X^{-1}(\bigcup_{\alpha \in \Gamma} S_\alpha) = \bigcup_{\alpha \in \Gamma} X^{-1}(S_\alpha) \in \mathcal{A}$. Thus $\bigcup_{\alpha \in \Gamma} S_\alpha \in \xi$.

.2 Proof of the necessary and sufficient condition of cumulative function

Take $\Omega = (0, 1)$, $\mathcal{A} = \mathcal{B}(0, 1)$ and P be Lebesgue measure.

Remark 6 If $B = (a, b)$ or $[a, b]$ then $Leb(B) = b - a$, i.e. the measure of an interval is its length.

For $w \in (0, 1)$ define

$$X(w) = sup\{y : F(y) < w\}$$

We need to show that X is a random variable and the distribution function $F_x = F$. We must first verify X(w) is well-defined:

1. since $w > 0$ and $\lim_{x \to -\infty} F(x) = 0$, so $\{y : F(y) < w\} \neq \phi$.

2. since $w < 1$ and $\lim_{x \to \infty} F(x) = 1$, so $\{y : F(y) < w\}$ bounded above.

So $X(w) = sup\{y : F(y) < w\}$ exists in $\mathcal{R}$ $\forall w \in (0,1)$, X(w) is well defined.We further claim that:

$$\{w : X(w) \leq c\} = \{w : w \leq F(c)\} = (0, F(c)]$$

Then X is a random variable since

$$\{w : X(w) \leq c\} = (0, F(c)] \in \mathcal{B}(0,1)\text{? need to be verified not almost certain}$$

And

$$F_x(c) = P(X^{-1}(\infty, c)) = Leb(\{w : X(w) \leq c\}) = Leb((0, F(c)]) = F(c)$$

So we can define a random variable X on $(\Omega, \mathcal{B}(0,1), Leb)$ which its distribution function is just given cumulative function. We complete the proof by showing:

$$\{w : X(w) \leq c\} \subset \{w : w \leq F(c)\}, \{w : X(w) \leq c\} \supset \{w : w \leq F(c)\}$$

First observe that

$$F(X(w) + \delta) \geq w, \ \forall \delta > 0$$

Since if $F(X(w) + \delta) < w$ it implies a contradiction:

$$X(w) + \delta \in \{y : F(y) < w\} \leq X(w)$$

And

$$\lim_{\delta \to 0} F(X(w) + \delta) \xlongequal{\text{right-continuous}} F(X(w)) \geq w$$

1. $\subset$ case: $X(w) \leq c \implies F(c) \geq F(X(w)) \geq w$ (non-decreasing) so,
$$\{w : X(w) \leq c\} \subset \{w : w \leq F(c)\}$$

2. $\supset$ case: $F(a) \geq w \implies$ c is an upper bound of $\{y : F(y) < w\}, c \geq \sup\{y : F(y) < w\} = X(w)$
$$\{w : X(w) \leq c\} \supset \{w : w \leq F(c)\}$$

Chi-Ning's version Here, we are going to show that the three conditions in Theorem 15 is necessary and sufficient for a function F to be a cumulative distribution of a random variable.

(Necessary) Let F be the cumulative distribution of random variable X defined on $(\Omega, \mathcal{A}, P)$. Check the three conditions as follow:

1. First, let A_n $(-\infty, -n]$. As $A_n \to \emptyset$ when $n \to \infty$, by the continuity axiom, $\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} F(-n) = 0$. Since, $\lim_{x\to-\infty} F(x) \leq \lim_{x\to-\infty} F(\lfloor x \rfloor) = \lim_{n\to\infty} F(-n) = 0$, we obtain the lower bound. The upper bound can be obtained with the same technique.

2. If $x \leq y$, then $(-\infty, x] \subseteq (= \infty, y]$. Clearly, $F(x) = P((-\infty, x]) \leq P((-\infty, y]) = F(y)$.

3. Let $\{x_n\}$ be a sequence strictly decreases (approaches) to x_0, then $A_n := (-\infty, x_n] \to A = (-\infty, x]$. That is, $F(x) = P(A)$ and $F(x_n) = P(A_n)$, $\forall n$. Now consider $B_n = \mathbb{R} \backslash A_n$, $\forall n$ and $B = \mathbb{R} \backslash A$. For convenience, let $A_0 = \Omega$ and $B_0 = \emptyset$. Then we have
$$\begin{aligned} 1 - P(A) = P(B) = P(\lim_{n\to\infty} B_n) &= P(\bigcup_{n=1}^{\infty} B_n \backslash B_{n-1}) \\ &= \sum_{n=1}^{\infty} P(B_n \backslash B_{n-1}) = \sum_{n=1}^{\infty} P(A_{n-1}) - P(A_n) \\ &= 1 - \lim_{n\to\infty} P(A_n) \end{aligned}$$
Thus, $F(x_0) = P(A) = \lim_{n\to\infty} P(A_n) = \lim_{x \to x_0^+} F(x)$.

(sufficient) Let F be a function satisfies the three conditions in Theorem 15. Take $(\Omega = (0,1), \mathcal{A} = \mathcal{B}(0,1), P = \mathcal{L}(0,1))$ as the probability space. Now, we are going to construct a random variable $X : \Omega \to \mathcal{X} \subset \mathbb{R}$ w.r.t F over $(\Omega, \mathcal{A}, P)$. X is defined as

$$\forall w \in \Omega,\ X(w) := \inf\{x : w \leq F(x)\}$$

To show that X is indeed the random variable with F as the underlying cumulative distribution. We need to first demonstrate that X is a random variable then show the relation between X and F.

1. By Theorem 13, it's sufficient to show that $\forall x \in \mathbb{R}$, $X^{-1}((-\infty, x]) \in \mathcal{A}$. Consider $\forall x \in \mathbb{R}$,

$$\begin{aligned} X^{-1}((-\infty, x]) &= \{w : X(w) \leq (-\infty, x]\} = \{w : \inf\{y : w \leq F(y)\} \in (-\infty, x]\} \\ &= \{w : w \leq F(x)\} = (0, F(x)] \in \mathcal{A} \end{aligned}$$

   Thus, X is a random variable over $(\Omega, \mathcal{A}, P)$.

2. Check that $\forall x \in \mathbb{R}$, $F(x) = P(X \leq x)$. Consider $\forall x \in \mathbb{R}$

$$\begin{aligned} P(X \leq x) &= P(\{w : X(w) \leq x\}) = P(\{w : \inf\{y : w \leq F(y)\} \leq x\}) \\ (\because y \leq x \Rightarrow F(y) \leq F(x)) &= P(\{w : \inf\{F(y) : w \leq F(y)\} \leq F(x)\}) \\ &= P(\{w : w \leq F(x)\}) = P((0, F(x)]) \\ (\because P = \mathcal{L}) &= F(x) \end{aligned}$$

   That is, F is the cumulative distribution of random variable X.