Statistical Inference I **Prof. Chin-Tsang Chiang**

Lecture Notes 15

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0.1 Logistic distribution

The cumulative distribution function of logistic distribution is

$$F_X(x) = \frac{e^x}{1+e^x}\mathbf{1}_{(-\infty,\infty)}(x)$$

The importance of logistic distribution is that it lies between Cauchy distribution and Gaussian distribution. That is, the decaying rate of logistic distribution is somewhere between $O(2^{-n})$ and $O(2^{-n^2})$.

Another important application of logistic distribution is *logistic regression*. Here, we sketch the formulation of logistic regression:

- Binary response: $Y \in \{0,1\}$
- Explanatory variables: $Z_1,...,Z_p$
- Odds ratio: $\frac{P(Y=1|Z_1,...,Z_p)}{1-P(Y=1|Z_1,...,Z_p)}$
- Logistic transformation:

$$\ln\frac{P(Y=1|Z_1,...,Z_p)}{1-P(Y=1|Z_1,...,Z_p)} = \beta_0+\beta_1Z_1+...+\beta_pZ_p$$

- Positive probability:

$$P(Y=1|Z_1,...,Z_p) = \frac{e^{\beta_0+\beta_1Z_1+...+\beta_pZ_p}}{1+e^{\beta_0+\beta_1Z_1+...+\beta_pZ_p}}$$

To sum up, logistic regression is a special case of generalized linear model and aims to predict the probability of certain outcome. **Generalized linear model**:

$$P(Y|Z_1,..,Z_p) = F_0(\beta_0+\beta_1Z_1+...+\beta_pZ_p)$$

, where F_0 is a cumulative distribution function.

0.2 Beta distribution

Beta distribution is a distribution that describes the battle between 0 and 1. We can create various of distribution in the interval [0,1] with beta distribution. With this abundance property, beta distribution can be used as a prior function in Bayesian analysis. Formally speaking, the density function of beta distribution is

$$f_X(x|\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{Beta}(\alpha,\beta)}\mathbf{1}_{[0,1]}(x)$$

, where $\alpha, \beta > 0$ and $\mathrm{Beta}(\alpha,\beta) = \frac{\Gamma(\alpha)+\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.
Now, let's see some basic properties of beta distribution:

- $\mathbb{E}[X|\alpha,\beta] = \frac{\alpha}{\alpha+\beta}$
- $Var[X|\alpha,\beta] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha\beta+1)}$
- Shape:
  - $\alpha > 1, \beta = 1$, increasing.
  - $\alpha = 1, \beta > 1$, decreasing.
  - $\alpha < 1, \beta < 1$, U shape.
  - $\alpha > 1, \beta > 1$, unimodal.
  - $\alpha = \beta$, symmetric.
  - $\alpha = \beta = 1$, uniform.
- The relation between beta and binomial: $X \sim$Beta(α,β), $Y \sim$Binomial(n,p)

$$P(X \geq p|x+1, n-x) = P(Y \leq x|n,p)$$

0.3 Double exponential distribution (Laplace distribution)

The density function of Laplace distribution is

$$f_X(x|\mu,\sigma^2) = \frac{1}{2\sigma}e^{-|x-\mu|/\sigma}\mathbf{1}_{(-\infty,\infty)}$$

The following is some basic properties:

- $\mathbb{E}[X|\mu,\sigma^2] = \mu$
- $Var[X|\mu,\sigma^2] = 2\sigma^2$

Note that here σ^2 is not variance.

0.4 Log-normal distribution

Let X be a log-normal distribution with parameter (μ, σ^2), then

$$Y = \ln X \sim N(\mu, \sigma^2)$$

The following is some basic properties:

- $\mathbb{E}[X|\mu, \sigma^2] \neq \mu$
- median $= \mu$
- pdf: $f_X(x|\mu, \sigma^2) = (2\pi\sigma^2)^{-1/2}\frac{1}{x}e^{-(\ln x - \mu)^2/2\sigma^2}\mathbf{1}_{(0,\infty)}(x)$
- Have moments but no mgf.
- $\mathbb{E}[X|\mu, \sigma^2] = e^{\mu+\sigma^2/2}$
- $Var[X|\mu, \sigma^2] = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$

0.5 Cauchy distribution

The density function of Cauchy distribution is

$$f_X(x|\mu, \sigma) = \frac{1}{\pi\sigma(1 + (\frac{x-\mu}{\sigma})^2)}\mathbf{1}_{(-\infty,\infty)}(x)$$

The following is some basic properties

- $\mathbb{E}[|X| \mid \mu, \sigma^2] = \infty$
- median $= \mu$
- If X, Y are two independent $N(0, 1)$, then $\frac{X}{Y}$ $\sim$Cauchy(0,1).