

Lecture Notes 10

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Lemma 1 (inversion theorem) Let X be a random variable with characteristic function $\phi_X(t)$ and $a, b \in \mathbb{R}$ with $a < b$, then

1. For any random variable X ,

$$P(a < X < b) + \frac{1}{2}P(X = a) + \frac{1}{2}P(X = b) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt$$

2. If X is a continuous random variable, then

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t) dt \quad a.e.$$

0.1 Characteristic function

Intuition (inversion theorem)

Inversion theorem provides a *isomorphism* between **distribution** and **characteristic function**.

Proof: The proof is divided into four steps. The first step introduce a integration tool for latter usage. The second step provides a clean form of the original term. The third step uses dominant theorem to show the convergence. The last step gives the density function.

1. **claim:** $\int_0^\infty \frac{\sin \alpha x}{x} dx = \frac{\pi}{2} \text{sign}(\alpha)$

$$\begin{aligned} \int_0^\infty \frac{\sin \alpha x}{x} dx &= \int_0^\infty \int_0^\infty \sin \alpha x e^{-ux} du dx \\ (\text{by Fubini's thm}) &= \int_0^\infty \int_0^\infty \sin \alpha x e^{-ux} dx du \\ &= \dots \text{some change of integrals} \dots \\ &= \frac{\pi}{2} \text{sign}(\alpha) \end{aligned}$$

2. Consider

$$\begin{aligned}
\frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt &= \frac{1}{2\pi} \int_{-T}^T \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} e^{itx} dF_X(x) dt \\
&= \frac{1}{2\pi} \int_{-T}^T \int_{-\infty}^{\infty} \frac{\cos t(x-a) + i \sin t(x-a)}{it} dF_X(x) dt \\
&= \frac{1}{2\pi} \int_{-T}^T \int_{-\infty}^{\infty} \frac{\cos t(x-b) + i \sin t(x-b)}{it} dF_X(x) dt \\
(\because \text{symmetry}) &= \frac{1}{\pi} \int_0^T \int_{-\infty}^{\infty} \frac{\sin t(x-a) - \sin t(x-b)}{t} dF_X(x) dt \\
(\text{by Fubini's thm}) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x)
\end{aligned}$$

3. By dominant convergence theorem,

$$\begin{aligned}
\frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt &\xrightarrow{T \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&= \frac{1}{\pi} \int_{x \in (-\infty, a)} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&\quad + \frac{1}{\pi} \int_{x=a} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&\quad + \frac{1}{\pi} \int_{x \in (a, b)} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&\quad + \frac{1}{\pi} \int_{x=b} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
&\quad + \frac{1}{\pi} \int_{x \in (b, \infty)} \int_0^T \frac{\sin t(x-a) - \sin t(x-b)}{t} dt dF_X(x) \\
(\text{by the tool in 1.}) &= 0 + \frac{1}{2} P(X = a) + P(a < X < b) + \frac{1}{2} P(X = b) + 0
\end{aligned}$$

4. Suppose X is continuous and $\int_{\mathcal{R}} |\phi_X(t)| dt < \infty$,

$$\begin{aligned}
\int_a^b f(x) dx &= \frac{1}{2\pi} \int_a^b \int_{\mathcal{R}} e^{-itx} \phi_X(t) dt dx \\
&= \frac{1}{2\pi} \int_{\mathcal{R}} \left(\int_a^b e^{-itx} dx \right) \phi_X(t) dt \\
&= \frac{1}{2\pi} \int_{\mathcal{R}} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt \\
&= \frac{1}{2} P(X = a) + P(a < X < b) + \frac{1}{2} P(X = b) \\
&= P(a < X < b) \quad (\text{further set } b=x, a \downarrow -\infty)
\end{aligned}$$

■

Corollary 1 For probability measures μ_X and μ_Y on $\mathcal{B}(\mathcal{R})$, the equality $\phi_{\mu_X} = \phi_{\mu_Y}$ implies that $\mu_X = \mu_Y$.

Proof: From inversion theorem, we have $\mu_X((a, b)) = \mu_Y((a, b)) \forall a, b \in C$, where C is the set of all $z \in \mathcal{R}$ such that $\mu_X(\{z\}) = \mu_Y(\{z\}) = 0$. Since C^c is at most countable. The family of $\{(a, b) : a, b \in C\}$ of intervals is a π -system generating $\mathcal{B}(\mathcal{R})$. μ_X and μ_Y agrees on a π -system also agrees on the σ -algebra generated by it. ■

0.2 Convergence

Theorem 1 Let $\{X_n\}$ be a sequence of random variables with characteristic functions $\phi_{X_n}(t)$. Suppose that

- $\lim_{n \rightarrow \infty} \phi_{X_n}(t) = \phi_X(t)$ for t in a neighborhood of 0. (pairwise convergence)
- $\phi_X(t)$ is a characteristic function of some random variable X .

Then,

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x), \forall x \text{ such that } F_X(x) \text{ is continuous (weakly convergence)}$$

Proof: Let a and b be continuous points of $F_X(x)$ and $F_{X_n}(x)$ for $n \geq N_0$ for some $n \in \mathcal{N}$

$$\begin{aligned} F_X(b) - F_X(a) &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \lim_{n \rightarrow \infty} \phi_{X_n}(t) dt \\ &= \lim_{n \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi_{X_n}(t) dt \text{ (since } |\phi_{X_n}(t)| \leq 1, \text{ dominated)} \\ &= \lim_{n \rightarrow \infty} (F_{X_n}(b) - F_{X_n}(a)) \end{aligned}$$

By setting $b=x$, $a \downarrow -\infty$ one obtains $\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x)$. ■

Remark: We don't have to worry about the discrete points since they must converge to the right value.

Intuition (convergence of characteristic function)

Theorem 1 tells us that if r.v.s converge in characteristic functions, then r.v.s also converge in distribution.

Remark: Tips for calculating MGF: consider the MGF of binomial distributed random variable X such that $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \mathbf{1}_{0,1,\dots,n}$. We have

$$\begin{aligned}
M_{X_n}(t) &= \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x} e^{tx} \\
&= \sum_{x=0}^n \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\
&= \sum_{x=0}^n \binom{n}{x} \frac{(pe^t)^x (1-p)^{n-x}}{[(pe^t) + (1-p)]^n} [(pe^t) + (1-p)]^n \\
(\text{set } p' &= \frac{pe^t}{pe^t + (1-p)}) = \sum_{x=0}^n \binom{n}{x} p'^x (1-p')^{n-x} [(pe^t) + (1-p)]^n \\
&= [(pe^t) + (1-p)]^n \sum_{x=0}^n \binom{n}{x} p'^x (1-p')^{n-x} \\
&= [(pe^t) + (1-p)]^n
\end{aligned}$$