

Lecture Notes 14

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0.1 Continuous distribution

0.1.1 Uniform distribution

In continuous regime, we define a uniform random variable on a close interval $[a, b]$, where $a < b$, and denote it as $\text{Uni}(a, b)$. If $X \sim \text{Uni}(a, b)$, then

- $f_X(x|a, b) = \frac{1}{b-a}$
- $\mathbb{E}[X|a, b] = \frac{a+b}{2}$
- $\text{Var}[X|a, b] = \frac{(b-a)^2}{12}$

0.1.2 Exponential family

Here, we define three highly related continuous random variables: exponential, Weibull, and gamma. We first write down their distribution respectively, then introduce their relationship and properties.

Exponential: Exponential random variable captures a single interleaving time of a Poisson process with frequency $1/\beta$. If $X \sim \text{exponential}(\beta)$

$$f_X(x|\beta) = \frac{1}{\beta} e^{-x/\beta} \mathbf{1}_{(0, \infty)}(x)$$

Weibull: If $Y = X^{1/\gamma}$, where $X \sim \text{exponential}(\beta)$ and $\gamma > 0$, we say Y has a Weibull(β, γ) distribution.

$$f_Y(y|\beta, \gamma) = \frac{\gamma}{\beta} y^{\gamma-1} e^{-y^\gamma/\beta} \mathbf{1}_{(0, \infty)}(y)$$

In other words, Weibull random variable is a power transformed version of exponential random variable. And as $\gamma = 1$, the Weibull degenerates to exponential.

Gamma: Intuitively, gamma distribution captures the total interleaving time up to more than $\alpha + 1$ appearances. We use two parameters α, β to define a gamma random variable and denote it as $\text{Gamma}(\alpha, \beta)$. If $X \sim \text{Gamma}(\alpha, \beta)$, then

$$f_X(x|\alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha) \beta^\alpha} \mathbf{1}_{(0, \infty)}$$

As we mentioned earlier, these three distributions are highly related to Poisson distribution in the sense that they describe the waiting time of a counting process given the number of desired observations while Poisson distribution captures the number of appearances given the amount of

observing time. The two aspects are just two side of a coin, and we can use the following equation to relate them all together: Let $\{N(t)\}$ be a counting process and T be the corresponding waiting time for a single event to happen. We have

$$\{T > t\} = \{N(t) = 0\}$$

If we write down the probability of each side and do some computation, we can derive a relationship between exponential distribution and Poisson distribution.

0.1.3 Exponential family to Poisson

Exponential: Let $F_X(x)$ be the distribution function of a random variable $X \sim \text{exponential}(\beta)$ and t_1 be the interleaving time of next arrival. Exponential distribution captures the probability of interleaving time less than or equal to t . We can interpreted it as at least one arrival happened up to time t . So it follows:

$$F_X(x) = P(t_1 \leq x) = 1 - P(t_1 > x) = 1 - P(N(t) = 0) = \frac{(\lambda x)^0 e^{-\lambda x}}{0!}$$

We have

$$f_X(x|\beta) = \frac{1}{\beta} e^{-x/\beta} \mathbf{1}_{(0,\infty)}(x)$$

Gamma: Let $F_X(x)$ be the distribution function of a random variable $X \sim \text{gamma}(\alpha, \beta)$. The distribution captures the probability that there are at least α arrival up to time t . We can interpreted it as the probability a poisson process with intensity $\frac{1}{\beta}$ up to time t with at least α arrivals.

If we have $Y \sim \text{Poisson}(x/\beta)$ then,

$$P(X \leq x|\alpha, \beta) = P(Y \geq \alpha|\frac{x}{\beta})$$

Property 1 let $X \sim \text{Gamma}(\alpha, \beta)$

1. $E[X|\alpha, \beta] = \alpha\beta$
2. $\text{Var}[X|\alpha, \beta] = \alpha\beta^2$
3. $M_X(t) = (\frac{1}{1-\beta t})^\alpha, t < \frac{1}{\beta}$
4. $f_X(x|\alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} \mathbf{1}_{(0,\infty)}$

Inverse Gamma: Let $X \sim \text{Gamma}(\alpha, \beta)$ then $Y = \frac{1}{X}$ follows Inverse Gamma distribution (α, β) . Its moments can be expressed as:

$$E[Y^n] = \frac{T(\alpha - n)\beta^{\alpha-n}}{T(\alpha)\beta^\alpha}$$

Chi-square: Let X be a random variable follows Chi-square distribution with k degrees of freedom.

$$X \stackrel{d}{=} \text{Gamma}(\frac{k}{2}, 2)$$

0.1.4 Normal distribution

Normal: If $X \sim \text{Normal}(\mu, \sigma^2)$ then X has density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathbb{1}_{x \in \mathcal{R}}(x)$$

Log-normal: If $\ln X \sim N(\mu, \sigma^2)$ then $X \sim \text{Lognormal}(\mu, \sigma) \mathbb{1}_{x \in \mathcal{R}^+}(x)$

$$f_X(x) = \frac{1}{\sqrt{2\pi x}\sigma} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

Cauchy: Cauchy is a symmetric distribution with more heavy tails than normal distribution, if $X \sim \text{Cauchy}(0,1)$ then:

$$f_X(x) = \frac{1}{\pi(1+x^2)} \mathbb{1}_{x \in \mathcal{R}}(x)$$

Logistic: Logistic has tails between Cauchy and Normal, if $X \sim \text{Logistic}(0,1)$ then:

$$F_X(x) = \frac{e^x}{1+e^x} \mathbb{1}_{x \in \mathcal{R}}(x)$$

Property 2 If $X \sim N(\mu, \sigma^2)$ and $Z = \frac{x-\mu}{\sigma} \sim (0,1)$ then the m.g.f of X is:

$$\begin{aligned} M_X(t) &= E[e^{t(\mu+\sigma Z)}] = e^{\mu t} M_Z(\sigma t) \\ &= e^{\mu t} \int_{-\infty}^{\infty} e^{\sigma z t} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma t)^2}{2}} e^{\frac{\sigma^2 t^2}{2}} dz \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}} \end{aligned}$$

Theorem 1 Let $X \sim N(\mu, \sigma^2)$ and $g(x)$ be differentiable function satisfying $E[|g'(x)|] < \infty$ then we have

$$E[g(x)(x - \mu)] = \sigma^2 E[g'(x)]$$

Proof:

$$\begin{aligned} \sigma^2 E[g'(x)] &= \sigma^2 \int_{-\infty}^{\infty} g'(x) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} g'(x) dx \int_{-\infty}^x -\frac{z-\mu}{\sigma^2} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \\ &= \int_{-\infty}^{\infty} g'(x) dx \int_{-\infty}^x -(z-\mu) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \\ &= \int_{-\infty}^{\infty} -(z-\mu) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \int_z^{\infty} g'(x) dx \\ &= \int_{-\infty}^{\infty} (z-\mu) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \lim_{b \rightarrow \infty} (g(z) - g(b)) dz \\ &= E[g(x)(x - \mu)] \end{aligned}$$

