

Lecture Notes 6

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0.1 Density function

Theorem 1 $P(X = x) = F(x) - F(x^-)$, where $F(x^-) = \lim_{y \uparrow x} F(y)$.

Proof: Since $P(X = x) = F(X \leq x) - F(X < x)$ and notice that $y \downarrow x$ then $\{X \leq y\} \downarrow \{X \leq x\}$ and $y \uparrow x$ then $\{X \leq y\} \uparrow \{X < x\}$, we have

$$P(X = x) = F(x) - F(x^-)$$

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When X is a discrete random variable, $F(x)$ is a step function which jumps at possible outcome.

Theorem 2 If $F(x)$ is differentiable, there exists a unique probability density function $f(x)$ such that $F(c) = \int_{-\infty}^c f(x)dx \forall c \in \mathcal{R}$.

Proof: The result is quite trivial, just apply the Fundamental Theorem of Calculus, we get

$$F'(x) = f(x)$$

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The question arises when considering the physical meaning of $f(x)$, is $f(x)$ a probability measure? Go back to the definition of differentiation, we have

$$F'(x) = \lim_{\Delta \rightarrow 0} \frac{F(x + \frac{\Delta}{2}) - F(x - \frac{\Delta}{2})}{\Delta} = \frac{P((x - \frac{\Delta}{2}, x + \frac{\Delta}{2}))}{\Delta} = \frac{Probability}{Interval}$$

$f(x)$ is not usually probability measure, it is of intensity/density sense.