Statistical Inference I	**Prof. Chin-Tsang Chiang**
Lecture Notes 11	
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0.1 Convergence of m.g.f

Example: Consider $f_{X_n}(x) = \binom{n}{x} P_n^x (1-P_n)^{n-x} \mathbb{1}_{\{0,1,2\ldots n\}}(x)$, the corresponding m.g.f $M_{X_n}(t) = (P_n e^t + (1-P_n))^n$. As $n \to \infty$, $nP_n \to \lambda$, we have $P_n = \frac{\lambda}{n}(1 + O(1))$ and

$$M_{X_n}(t) \to M_X(t) = e^{\lambda(e^t - 1)}$$

which is the m.g.f of Poisson distribution.

Proof: Let $y = (P_n e^t + (1-P_n))^n$ we have $\ln y = \frac{\ln(P_n e^t + (1-P_n))}{\frac{1}{n}} = \frac{\ln(\frac{\lambda}{n} e^t (1+O(1)) + (1 - \frac{\lambda}{n}(1+O(1)))}{\frac{1}{n}}$ applying $L'h\hat{o}pital's$ rule we get

$$\lim_{n\to\infty} \ln y = \lambda(e^t - 1)$$

so

$$M_{X_n}(t) \to M_X(t) = e^{\lambda(e^t-1)} \; as \; n \to \infty \; nP_n \to \lambda$$

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Intuition (From the basic view)

The example construct a relationship between Poisson distribution and Binomial distribution. And it is quite reasonable from the definition of Poisson distribution.
Recall the definition of Poisson:

- It's a **counting process**. That is, $N(t)$ that counts the number of appearances before time t.
- (**Boundary condition**) $N(0) = 0$
- (**Stationary**) $\forall t_1 < t_2$, $N(t_2) - N(t_1) \sim N(t_2 - t_1)$
- (**Independence**) $\forall t_1 < t_2 < t_3 < t_4$, $N(t_4) - N(t_3) \sim N(t_2) - N(t_1)$
- (**Fixed frequency**) $\lim_{\Delta\to 0^+} \frac{Pr[N(\Delta) - N(0) = 1]}{\Delta} = \lambda$, and $\lim_{\Delta\to 0^+} \frac{Pr[N(\Delta)-N(0)>1]}{\Delta} = 0$
- (**Density function**) $f_\lambda(t,k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \mathbf{1}_{\{k=0,1,2,\ldots\}}$

Since $\lim_{\Delta\to 0^+} \frac{Pr[N(\Delta)=1}{\Delta} = \lambda$ simply let $\Delta\lambda$ be the success probability of P_n.

0.2 Basic property of characteristic function

Property 1 (relation to moment) *Let X be a random variable. If $E[|X^n|] < \infty$, then $\frac{d^n}{(dt)^n}\phi_X(t)$ exists for all t and*

$$\frac{d^n}{(dt)^n}\phi_X(t) = E[e^{itX}(iX)^n]$$

so the lower moments are

$$E[X^n] = (-i)^n \frac{d^n}{(dt)^n}\phi_X(0)$$

Property 2 (Basic) *Let X and Y be random variables.*

1. $\phi_X(0) = 1$ *and* $|\phi_X(t)| \leq 1\ \forall t$.
2. $\phi_{-X}(t) = \overline{\phi_X(t)}$ *where bar denotes complex conjugation.*
3. $\phi_{aX+b}(t) = e^{itb}\phi_X(at)$.
4. *If X and Y are independent,* $\phi_{X+Y}(t) = \phi_X(t) \times \phi_Y(t)$.