Statistical Inference I	**Prof. Chin-Tsang Chiang**
Lecture Notes 6	
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1 Random Variables

Definition 1 (image of set function) *Let $X : \Omega \to \mathcal{X} \subset \mathbb{R}$ be a set function. Then the preimage of $S \subset \mathbb{R}$ over X is*

$$X^{-1} := \{w \in \Omega : X(w) \in S\}$$

Property 2 (Properties of set function) *Let $X : \Omega \to \mathcal{X} \subset \mathbb{R}$ be a set function, then the following properties hold. (Note that here X is not necessary a random variable)*

1. *(Close under complementation): $X^{-1}(S^C) = (X^{-1}(S))^C$*
2. *(Close under union): $X^{-1}(\bigcup_{\alpha\in\Gamma} S_\alpha) = \bigcup_{\alpha\in\Gamma} X^{-1}(S_i)$, where $\{S_\alpha\}_{\alpha\in\Gamma}$*
3. *(Close under intersection): $X^{-1}(\bigcap_{\alpha\in\Gamma} S_\alpha) = \bigcap_{\alpha\in\Gamma} X^{-1}(S_i)$, where $\{S_\alpha\}_{\alpha\in\Gamma}$*

By definition, a random variable should satisfy the condition that the preimage of every Borel set should in the event space $\mathcal{A}$. However, it's difficult to check since it's hard to enumerate all Borel set and claim the results. Thus, we would like to find a relaxed but necessary condition for a set function to be a random variable. And the following theorem does so.

Theorem 3 (necessary and sufficient condition for random variable) *Let $X : \Omega \to \mathcal{X} \subset \mathbb{R}$ be a set function, then*

$$X \text{ is a random variable} \Leftrightarrow \forall x \in \mathbb{R},\ \{w : X(w) \le x\} \in \mathcal{A}$$

The proof is left in Appendix A

2 From Set Function to Value Function

Note that random variable is a function that helps us map the event set $\mathcal{A}$ onto the Borel set. The importance here is that know we can do the equivalent operation on **real number** instead of arbitrary σ-algebra. This not only provides us a general and uniform way to play but also gives us the opportunity to operate on an **ordered** set.

Soon, we might wonder can we play with an even more general function: value function instead of just a set function onto real number? And by the construction of random variable, there are two direct value functions that play an important role in probability theory.

Theorem 4 (value functions) *Let X be a random variable w.r.t. $(\Omega, \mathcal{A}, P)$, then we can define*

- *The measure function of X is $\mu_X : \mathcal{B} \to \mathbf{R}^+$ such that*

$$\forall B \in \mathcal{B},\ \mu_X(B) := P\{w : X(w) \in B\}$$

- *The cumulative distribution of X is $F_X : \mathbf{R} \to \mathbf{R}^+$ such that*

$$\forall x \in \mathbf{R},\ F_X(x) := P\{w : X(w) \leq x\}$$

Here, we also have a necessary and sufficient condition for cumulative function. Intuitively, when a function F satisfies the following conditions, then there's a random variable with its unique cumulative distribution being F.

Theorem 5 (necessary and sufficient condition of cumulative distribution) *F is a cumulative distribution iff*

- *(upper and lower bound)* $\lim_{x\to\infty} F(x) = 0,\ \lim_{x\to\infty} F(x) = 1$
- *(non-decreasing)* $\forall x \leq y,\ F(x) \leq F(y)$
- *(right continuous)* $\lim_{x\to x_0^+} = F(x_0)$

The proof is left in Appendix B.

Intuition (set function and value function)

Careful with the difference of

- (**random variable**): $X : \Omega \to S \subset \mathbb{R}$
- (**measure function**): $\mu_X : \mathcal{B} \to \mathbb{R}^+$
- (**cumulative distribution**): $F_X : \mathbb{R} \to \mathbb{R}^+$

A Proof of the necessary and sufficient condition of random variable

Recall that Theorem 3 provides a necessary and sufficient condition for a set function to be a random variable. Here, we prove the correctness of the theorem.

($\Rightarrow$) First, we can see that $\{w : X(w) \leq x\} = X^{-1}[(-\infty, x)]$. Thus, it's sufficient to show that $(-\infty, x)$ is in Borel set $\forall x$. And this is simple since $(-\infty, x) = \bigcup_{n\in\mathbb{N}}[-n, x] \in \mathcal{B}$.

($\Leftarrow$) This direction can be proved in two steps:

1. Show that $\forall a < b,\ X^{-1}([a, b]) \in \mathcal{A}$.
2. Show that $\xi = \{S : \exists w \in \Omega,\ X(w) = S\}$ is a σ-algebra.

With these two results, we can see that the image of X contains $\mathcal{B}$. Thus, X is a random variable. The following show the correctness of these two:

1. $\forall a < b$, consider

$$\begin{aligned} X^{-1}([a,b]) &= X^{-1}((-\infty,b]\backslash(-\infty,a]) = X^{-1}((-\infty,b] \cap (-\infty,a])^C) \\ &= X^{-1}((-\infty,b]) \cap X^{-1}((-\infty,a])^C) \\ &= X^{-1}((-\infty,b]) \cap X^{-1}((-\infty,a]))^C \in \mathcal{A} \end{aligned}$$

2. Check the three axioms of σ-algebra:

   (a) Clearly, $\emptyset \in \xi$.

   (b) $\forall S \in \xi$, $X^{-1}(X^C) = X^{-1}(S)^C \in \mathcal{A}$ by Property 2. Thus $S^C \in \xi$.

   (c) $\forall \{S_\alpha\}_{\alpha\in\Gamma} \in \xi$, $X^{-1}(\bigcup_{\alpha\in\Gamma} S_\alpha) = \bigcup_{\alpha\in\Gamma} X^{-1}(S_\alpha) \in \mathcal{A}$. Thus $\bigcup_{\alpha\in\Gamma} S_\alpha \in \xi$.

B Proof of the necessary and sufficient condition of cumulative function