

1 Random Variables

Definition 1 (image of set function) Let $X : \Omega \rightarrow \mathcal{X} \subset \mathbb{R}$ be a set function. Then the preimage of $S \subset \mathbb{R}$ over X is

$$X^{-1} := \{w \in \Omega : X(w) \in S\}$$

Property 2 (Properties of set function) Let $X : \Omega \rightarrow \mathcal{X} \subset \mathbb{R}$ be a set function, then the following properties hold. (Note that here X is not necessary a random variable)

1. (Close under complementation): $X^{-1}(S^C) = (X^{-1}(S))^C$
2. (Close under union): $X^{-1}(\bigcup_{\alpha \in \Gamma} S_\alpha) = \bigcup_{\alpha \in \Gamma} X^{-1}(S_\alpha)$, where $\{S_\alpha\}_{\alpha \in \Gamma}$
3. (Close under intersection): $X^{-1}(\bigcap_{\alpha \in \Gamma} S_\alpha) = \bigcap_{\alpha \in \Gamma} X^{-1}(S_\alpha)$, where $\{S_\alpha\}_{\alpha \in \Gamma}$

By definition, a random variable should satisfy the condition that the preimage of every Borel set should in the event space \mathcal{A} . However, it's difficult to check since it's hard to enumerate all Borel set and claim the results. Thus, we would like to find a relaxed but necessary condition for a set function to be a random variable. And the following theorem does so.

Theorem 3 (necessary and sufficient condition for random variable) Let $X : \Omega \rightarrow \mathcal{X} \subset \mathbb{R}$ be a set function, then

$$X \text{ is a random variable} \Leftrightarrow \forall x \in \mathbb{R}, \{w : X(w) \leq x\} \in \mathcal{A}$$

The proof is left in Appendix A

2 From Set Function to Value Function

Note that random variable is a function that helps us map the event set \mathcal{A} onto the Borel set. The importance here is that now we can do the equivalent operation on **real number** instead of arbitrary σ -algebra. This not only provides us a general and uniform way to play but also gives us the opportunity to operate on an **ordered** set.

Soon, we might wonder can we play with an even more general function: value function instead of just a set function onto real number? And by the construction of random variable, there are two direct value functions that play an important role in probability theory.

Theorem 4 (value functions) Let X be a random variable w.r.t. (Ω, \mathcal{A}, P) , then we can define

- The measure function of X is $\mu_X : \mathcal{B} \rightarrow \mathbf{R}^+$ such that

$$\forall B \in \mathcal{B}, \mu_X(B) := P\{w : X(w) \in B\}$$

- The cumulative distribution of X is $F_X : \mathbf{R} \rightarrow \mathbf{R}^+$ such that

$$\forall x \in \mathbf{R}, F_X(x) := P\{w : X(w) \leq x\}$$

Here, we also have a necessary and sufficient condition for cumulative function. Intuitively, when a function F satisfies the following conditions, then there's a random variable with its unique cumulative distribution being F .

Theorem 5 (necessary and sufficient condition of cumulative distribution) F is a cumulative distribution iff

- (upper and lower bound) $\lim_{x \rightarrow -\infty} F(x) = 0, \lim_{x \rightarrow \infty} F(x) = 1$
- (non-decreasing) $\forall x \leq y, F(x) \leq F(y)$
- (right continuous) $\lim_{x \rightarrow x_0^+} F(x) = F(x_0)$

The proof is left in Appendix B.

Intuition (set function and value function)

Careful with the difference of

- (random variable): $X : \Omega \rightarrow S \subset \mathbf{R}$
- (measure function): $\mu_X : \mathcal{B} \rightarrow \mathbf{R}^+$
- (cumulative distribution): $F_X : \mathbf{R} \rightarrow \mathbf{R}^+$

A Proof of the necessary and sufficient condition of random variable

Recall that Theorem 3 provides a necessary and sufficient condition for a set function to be a random variable. Here, we prove the correctness of the theorem.

(\Rightarrow) First, we can see that $\{w : X(w) \leq x\} = X^{-1}[(-\infty, x)]$. Thus, it's sufficient to show that $(-\infty, x)$ is in Borel set $\forall x$. And this is simple since $(-\infty, x) = \bigcup_{n \in \mathbf{N}} [-n, x] \in \mathcal{B}$.

(\Leftarrow) This direction can be proved in two steps:

1. Show that $\forall a < b, X^{-1}([a, b]) \in \mathcal{A}$.
2. Show that $\xi = \{S : \exists w \in \Omega, X(w) = S\}$ is a σ -algebra.

With these two results, we can see that the image of X contains \mathcal{B} . Thus, X is a random variable. The following show the correctness of these two:

1. $\forall a < b$, consider

$$\begin{aligned}
X^{-1}([a, b]) &= X^{-1}((-\infty, b] \setminus (-\infty, a]) = X^{-1}((-\infty, b] \cap (-\infty, a])^C) \\
&= X^{-1}((-\infty, b]) \cap X^{-1}((-\infty, a])^C \\
&= X^{-1}((-\infty, b]) \cap X^{-1}((-\infty, a])^C \in \mathcal{A}
\end{aligned}$$

2. Check the three axioms of σ -algebra:

- (a) Clearly, $\emptyset \in \xi$.
- (b) $\forall S \in \xi$, $X^{-1}(X^C) = X^{-1}(S)^C \in \mathcal{A}$ by Property 2. Thus $S^C \in \xi$.
- (c) $\forall \{S_\alpha\}_{\alpha \in \Gamma} \in \xi$, $X^{-1}(\bigcup_{\alpha \in \Gamma} S_\alpha) = \bigcup_{\alpha \in \Gamma} X^{-1}(S_\alpha) \in \mathcal{A}$. Thus $\bigcup_{\alpha \in \Gamma} S_\alpha \in \xi$.

B Proof of the necessary and sufficient condition of cumulative function