

0.1 Logistic distribution

The cumulative distribution function of logistic distribution is

$$F_X(x) = \frac{e^x}{1 + e^x} \mathbf{1}_{(-\infty, \infty)}(x)$$

The importance of logistic distribution is that it lies between Cauchy distribution and Gaussian distribution. That is, the decaying rate of logistic distribution is somewhere between $O(2^{-n})$ and $O(2^{-n^2})$.

Another important application of logistic distribution is *logistic regression*. Here, we sketch the formulation of logistic regression:

- Binary response: $Y \in \{0, 1\}$
- Explanatory variables: Z_1, \dots, Z_p
- Odds ratio: $\frac{P(Y=1|Z_1, \dots, Z_p)}{1-P(Y=1|Z_1, \dots, Z_p)}$
- Logistic transformation:

$$\ln \frac{P(Y = 1|Z_1, \dots, Z_p)}{1 - P(Y = 1|Z_1, \dots, Z_p)} = \beta_0 + \beta_1 Z_1 + \dots + \beta_p Z_p$$

- Positive probability:

$$P(Y = 1|Z_1, \dots, Z_p) = \frac{e^{\beta_0 + \beta_1 Z_1 + \dots + \beta_p Z_p}}{1 + e^{\beta_0 + \beta_1 Z_1 + \dots + \beta_p Z_p}}$$

To sum up, logistic regression is a special case of generalized linear model and aims to predict the probability of certain outcome. **Generalized linear model:**

$$P(Y|Z_1, \dots, Z_p) = F_0(\beta_0 + \beta_1 Z_1 + \dots + \beta_p Z_p)$$

, where F_0 is a cumulative distribution function.

0.2 Beta distribution

Beta distribution is a distribution that describes the battle between 0 and 1. We can create various of distribution in the interval $[0,1]$ with beta distribution. With this abundance property, beta distribution can be used as a prior function in Bayesian analysis. Formally speaking, the density function of beta distribution is

$$f_X(x|\alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\text{Beta}(\alpha, \beta)} \mathbf{1}_{[0,1]}(x)$$

, where $\alpha, \beta > 0$ and $\text{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Now, let's see some basic properties of beta distribution:

- $\mathbb{E}[X|\alpha, \beta] = \frac{\alpha}{\alpha+\beta}$
- $\text{Var}[X|\alpha, \beta] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
- Shape:
 - $\alpha > 1, \beta = 1$, increasing.
 - $\alpha = 1, \beta > 1$, decreasing.
 - $\alpha < 1, \beta < 1$, U shape.
 - $\alpha > 1, \beta > 1$, unimodal.
 - $\alpha = \beta$, symmetric.
 - $\alpha = \beta = 1$, uniform.
- The relation between beta and binomial: $X \sim \text{Beta}(\alpha, \beta)$, $Y \sim \text{Binomial}(n, p)$

$$P(X \geq p|x+1, n-x) = P(Y \leq x|n, p)$$

0.3 Double exponential distribution (Laplace distribution)

The density function of Laplace distribution is

$$f_X(x|\mu, \sigma^2) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma} \mathbf{1}_{(-\infty, \infty)}$$

The following is some basic properties:

- $\mathbb{E}[X|\mu, \sigma^2] = \mu$
- $\text{Var}[X|\mu, \sigma^2] = \sigma^2$

Note that here σ is not variance.

0.4 Log-normal distribution

Let X be a log-normal distribution with parameter (μ, σ^2) , then

$$Y = \ln X \sim N(\mu, \sigma^2)$$

The following is some basic properties:

- $\mathbb{E}[X|\mu, \sigma^2] \neq \mu$
- median = μ
- pdf: $f_X(x|\mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \frac{1}{x} e^{-(\ln x - \mu)^2 / 2\sigma^2} \mathbf{1}_{(0, \infty)}(x)$
- Have moments but no mgf.
- $\mathbb{E}[X|\mu, \sigma^2] = e^{\mu + \sigma^2/2}$
- $Var[X|\mu, \sigma^2] = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$

0.5 Cauchy distribution

The density function of Cauchy distribution is

$$f_X(x|\mu, \sigma) = \frac{1}{\pi\sigma(1 + (\frac{x-\mu}{\sigma})^2)} \mathbf{1}_{(-\infty, \infty)}(x)$$

The following is some basic properties

- $\mathbb{E}[|X| | \mu, \sigma^2] = \infty$
- median = μ