Statistical Inference I **Prof. Chin-Tsang Chiang**

Lecture Notes 6

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0.1 Identical distribution

Definition 1 (i.d.d.) *Let X and Y be random variables defined on the probability space* $(\Omega_1, \mathcal{A}_1, P_1)$ *and* $(\Omega_2, \mathcal{A}_2, P_2)$ *respectively. Then X and Y are said to be identically distributed if and only if*

$$P_1(X \in B) = P_2(Y \in B)\ \forall B \in \mathcal{B}$$

Remark 1 X and Y are said to be identically distributed with notation $X \stackrel{d}{=} Y$ noticed that it does not mean X=Y.

Property 1 *X and Y are identically distributed* $\Leftrightarrow F_X(t) = F_Y(t)\ \forall t$ *where* $F_1(t)$ *and* $F_2(t)$ *are the corresponding distribution functions of X and Y.*

Proof:
$\Rightarrow$ The equality follows straightly from i.d.d. and is true for all t since $(-\infty, t] \in \mathcal{B}$.

$$F_X(t) = P(\{\omega : X(\omega) \leq t\}) = P_1(X^{-1}((-\infty, t])) = P_2(Y^{-1}((-\infty, t])) = F_Y(t)$$

$\Leftarrow$ Let $S = \{(a, b] : P(X \in (a, b]) = P_2(Y \in (a, b])\}\ \forall a, b \in \mathcal{R}$ and $\xi = \{B : P_1(X \in B) = P_2(Y \in B)\ \forall B \in \mathcal{B}\}$. We want to show $s = \xi$ to extend agreed on intervals to agreed on all sets. This is true because $B = \sigma(S)$. ■

0.2 Density and mass function

Definition 2 (Continuous random variable) *A random variable X is continuous if* $F_X(x)$ *is a continuous function and discrete if* $F_X(x)$ *is a step function of X.*

Definition 3 (p.m.f) *The probability mass function of a discrete random variable X is defined as*

$$f_X(x) = P(\{w : X(w) = x\}) = P(X = x)\ \forall X$$

Definition 4 (p.d.f) *The probability density function of a continuous random variable X is a function* $f_X(x)$ *satisfying*

$$F_X(x) = \int_{-\infty}^{x} f_X(u) du$$

And $f_X(x) = \frac{dF_X(x)}{dx}$ *almost everywhere.*

Property 2 *A function* $f_X(x)$ *is a p.d.f(p.m.f) of a random variable X iff* *(a)* $f_X(x) \geq 0$ *for all* x.

(b) $\int_{-\infty}^{\infty} f(x)dx = 1$ *or* $\sum_X f(x)dx = 1$.

Proof:
⇒: $F_X(x)$ is a non-decreasing function so $f_X(x) \geq 0\ \forall x$ and $\lim_{x\to\infty} F_X(x) = 1 = \int_{-\infty}^{\infty} f_X(t)dt$.
⇐: Define $F_X(x) = \int_{-\infty}^{x} f_X(u)du$ and the property of distribution can be verified with (a)(b). ■

Intuition (Changed to p.d.f)

In fact, every non-negative function with a finite positive integral(sum) can be turned into a p.d.f/p.m.f. If h(x) is an non-negative function that is positive on a set A and

$$\int_{x\in A} h(x)dx = K < \infty$$

with positive integral then the function $f_X(x) = h(x)/K$ is a p.d.f of a random variable X taking values in A.

Density functions are not always exist for continuous random variable. But if the distribution function is absolutely continuous, density function exists.

Remark 2 [Absolutely Continuous] A real-valued function f(x) is absolutely continuous on [a,b] if $\forall \epsilon > 0\ \exists\ \delta$ s.t. non-overlapping intervals $(Y_i, X_i) \in [a, b]$ for all

$$\sum_i |Y_i - X_i| < \delta$$

implies

$$\sum_i |f(Y_i) - f(X_i)| < \epsilon$$

Theorem 1 $P(X = x) = F(x) - F(x^-)$, *where* $F(x^-) = \lim_{y\uparrow x} F(y)$.

Proof: Since $P(X = x) = F(X \leq x) - F(X < x)$ and notice that $y \downarrow x$ then $\{X \leq y\} \downarrow \{X \leq x\}$ and $y \uparrow x$ then $\{X \leq y\} \uparrow \{X < x\}$,we have

$$P(X = x) = F(x) - F(x^-)$$

■

The question arises when considering the physical meaning of $f(x)$, is $f(x)$ a probability measure? Go back to the definition of differentiation, we have

$$F'(x) = \lim_{\triangle\to 0} \frac{F(x + \frac{\triangle}{2}) - F(x - \frac{\triangle}{2})}{\triangle} = \frac{P((x - \frac{\triangle}{2}, x + \frac{\triangle}{2}))}{\triangle} = \frac{Probability}{Interval}$$

$f(x)$ is not usually probability measure, it is of intensity/density sense.

0.3 Quantative Description of Poisson Random Variable

Q: Please **quantitatively** describe the Poisson random variable.

- It's a **counting process**. That is, $N(t)$ that counts the number of appearances before time t.
- (**Boundary condition**) $N(0) = 0$
- (**Stationary**) $\forall t_1 < t_2$, $N(t_2) - N(t_1) \sim N(t_2 - t_1)$
- (**Independence**) $\forall t_1 < t_2 < t_3 < t_4$, $N(t_4) - N(t_3) \sim N(t_2) - N(t_1)$
- (**Fixed frequency**) $\lim_{\Delta \to 0^+} \frac{Pr[N(\Delta)-N(0)=1]}{\Delta} = \lambda$, and $\lim_{\Delta \to 0^+} \frac{Pr[N(\Delta)-N(0)>1]}{\Delta} = 0$
- (**Density function**) $f_\lambda(t, k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \mathbf{1}_{\{k=0,1,2,\ldots\}}$