We have all the properties that we need to calculate the ruin probabilities now. We will start from solving integral equation to derive the famous Pollaczeck-Khinchine formula and give a more general result to the example 5.13.

0.1 Integral equation of ultimate ruin

Since the Poisson process is a renewal process and since ruin cannot occur before the first claim arrival T_1, then the the survival probability $\bar{\varphi}(u)$ conditioning on no claim in $(0,T_1)$ satisfies following relation:

$$\begin{aligned}\bar{\varphi}(u) =& E[\bar{\varphi}(u+\beta T_1 - U_1)] \\ &= \int_0^\infty \lambda e^{-\lambda s} \int_0^{u+\beta s} \bar{\varphi}(u+\beta s - z) dF_U(z) ds \\ &= \frac{\lambda}{\beta} e^{\lambda \frac{u}{\beta}} \int_u^\infty e^{-\lambda \frac{x}{\beta}} \int_0^x \bar{\varphi}(x-z) dF_U(z) dx\end{aligned}$$

and since $\bar{\varphi}(u)$ is differentiable [I] we have

$$\bar{\varphi}'(u) = \frac{\lambda}{\beta}\bar{\varphi}(u) - \frac{\lambda}{\beta}\int_0^u \bar{\varphi}(u-z) dF_U(z)$$

Theorem 1. *The ruin function satisfies*

$$\beta\varphi(u) = \lambda(\int_u^\infty \bar{F}_U(x)dx + \int_0^u \varphi(u-x)\bar{F}_U(x)dx)$$

and

$$\varphi(0) = \frac{\lambda u}{\beta}, \varphi(\infty) = 0$$

If $F_U(x)$ are exponentially distributed with mean μ

$$\varphi(u) = \frac{1}{1+\theta} e^{-\frac{\theta u}{\mu(1+\theta)}}$$

[I]The discussion of differentiability can be found in both Renewal Risk Processes with Stochastic Returns on Investments - A Unified Approach and Analysis of the Ruin Probabilities section 2.2 and Stochastic Processes for Insurance and Finance p.163

Proof. By integrating (0,u] leads to

$$\begin{aligned}\frac{\beta}{\lambda}(\bar{\varphi}(u)-\bar{\varphi}(0)) &= \frac{1}{\lambda}\int_0^u \beta\bar{\varphi}'(x)dx \\ &= \int_0^u \bar{\varphi}(x)dx - \int_0^u dx\int_0^x \bar{\varphi}(x-y)dF_U(y) \\ &= \int_0^u \bar{\varphi}(x)dx - \int_0^u dF_U(y)\int_y^u \bar{\varphi}(x-y)dx \\ &= \int_0^u \bar{\varphi}(x)dx - \int_0^u dF_U(y)\int_0^{u-y} \bar{\varphi}(x)dx \\ &= \int_0^u \bar{\varphi}(x)dx - \int_0^u dx\int_0^{u-x} \bar{\varphi}(x)dF_U(y) \\ &= \int_0^u \bar{\varphi}(x)(1-F_U(u-x))dx \\ &= \int_0^u \bar{\varphi}(u-x)\bar{F}_U(x)dx\end{aligned}$$

Now letting u $\to \infty$,we have

$$\beta(\bar{\varphi}(\infty)-\bar{\varphi}(0)) = \lambda\lim_{u\to\infty}\int_0^u \bar{\varphi}(u-x)\bar{F}_U(u-x)dx$$

From net profit condition, we know $\lim_{n\to\infty} W_n = -\infty$ and $F_U(\infty) = 0$ so M can only take on finite positive number, we have

$$\bar{\varphi}(\infty) = 1$$

Then by applying *Dominated convergence theorem* to the right-hand side we get,

$$\beta(1-\bar{\varphi}(0)) = \lambda\int_0^\infty 1\cdot\bar{F}_U(u-x))dx = \lambda\mu^{\mathrm{II}}$$

Thus,

$$\bar{\varphi}(0) = 1-\frac{\lambda\mu}{\beta}$$

By changing $\bar{\varphi}(u)$ to $1-\bar{\varphi}(u) = \varphi(u)$,

$$\begin{aligned}\beta\varphi(u) &= \beta\varphi(0) - \lambda\int_0^u (1-\varphi(u-x))\bar{F}_U(x)dx \\ &= \lambda\mu - \lambda\int_0^u \bar{F}_U(x)dx + \lambda\int_0^u \varphi(u-x)\bar{F}_U(x)dx \\ &= \lambda(\int_u^\infty \bar{F}_U(x)dx + \int_0^u \varphi(u-x)\bar{F}_U(x)dx\end{aligned}$$

[II] Essential for Stochastic Process P.220 $E[X] = \int_0^\infty P(X>t)dt$

If $F_U(x)$ are exponentially distributed , $\bar{\varphi}(u)$ will satisfies this ODE

$$\bar{\varphi}''(u) + \frac{1}{\mu}\frac{\theta}{1+\theta}\bar{\varphi}'(u) = 0$$

and the initial conditions

$$\bar{\varphi}(\infty) = 1 \quad and \quad \bar{\varphi}(0) = 1 - \frac{\lambda\mu}{\beta} = \frac{\theta}{1+\theta}$$

gives the solution

$$\varphi(u) = 1 - \bar{\varphi}(u) = \frac{1}{1+\theta}e^{-\frac{\theta u}{\mu(1+\theta)}}$$

$\square$

0.2 Pollaczeck-Khinchine formula

In this section, we will use Laplace transform to show that $\bar{\varphi}(u)$ is actually compound geometric distributed to give the general n-fold solution to it .

Theorem 2. *Pollaczeck-Khinchine formula*

$$\varphi(u) = (1 - \frac{\lambda\mu}{\beta})\sum_{n=1}^{\infty}(\frac{\lambda\mu}{\beta})^n(1 - (F_U^I)^{*n}(u))$$

with F_U^I *is the intergrating tail distribution related to* F_U *denoted by,*

$$F_U^I(z) = \frac{1}{\mu}\int_0^z(1 - F_U(x))dx$$

and density

$$f_U^I(z) = \frac{1}{\mu}\bar{F}_U(z)$$

Proof. Taking Laplace transform of $\varphi(u) = \frac{\lambda}{\beta}(\int_u^\infty \bar{F}_U(x)dx + \int_0^u \varphi(u-x)\bar{F}_U(x)dx)$ we get,

$$\begin{aligned}
\hat{L}_\varphi(s) &= \int_0^\infty \varphi(u)e^{-su}du \\
&= \frac{\lambda}{\beta}\int_0^\infty [\int_u^\infty \bar{F}_U(x)dx + \int_0^u \varphi(u-x)\bar{F}_U(x)dx]e^{-su}du \\
&= \frac{\lambda}{\beta}\int_0^\infty (\mu - \int_0^u \bar{F}_U(x)dx)e^{-su}du + \frac{\lambda}{\beta}\int_0^\infty (\int_0^u \varphi(u-x)\mu f_U^I(x)dx)e^{-su}du \\
&= \frac{\lambda\mu}{\beta}\int_0^\infty (1 - F_U^I(u))e^{-su}du + \frac{\lambda\mu}{\beta}\hat{L}_\varphi(s)\hat{L}_{f_U^I}(s)^{\mathrm{III}} \\
&= \frac{\lambda\mu}{\beta}\hat{L}_{1-F_U^I}(s) + \frac{\lambda\mu}{\beta}\hat{L}_\varphi(s)\hat{L}_{f_U^I}(s) \\
&= \frac{\lambda\mu}{\beta}\frac{1-\hat{L}_{f_U^I}(s)}{s} + \frac{\lambda\mu}{\beta}\hat{L}_\varphi(s)\hat{L}_{f_U^I}(s)^{\mathrm{IV}}
\end{aligned}$$

Thus, by rearranging the equation

$$\begin{aligned}
\hat{L}_\varphi(s) &= \frac{1}{s}\frac{\lambda\mu}{\beta}\frac{1-\hat{L}_{f_U^I}(s)}{1-\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s)} \\
&= \frac{1}{s}\frac{\lambda\mu}{\beta}(\frac{1-\hat{L}_{f_U^I}(s)}{1-\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s)} - 1 + 1) \\
&= \frac{1}{s}\frac{\lambda\mu}{\beta}(\frac{\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s) - \hat{L}_{f_U^I}(s)}{1-\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s)} + 1) \\
&= \frac{1}{s}\frac{\lambda\mu}{\beta}(1 - \frac{(1-\frac{\lambda\mu}{\beta})\hat{L}_{f_U^I}(s)}{1-\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s)})
\end{aligned}$$

[III] An Introduction to Probability Theory and its Applications Volume 2 p.434 , f(x),g(x) and u(x) their convolutions $u(x) = \int_0^x g(x-y)f(y)dy$, their Laplace transform satisfies $\hat{L}_u(s) = \hat{L}_f(s)\hat{L}_g(s)$ if all exists.

[IV] An Introduction to Probability Theory and its Applications Volume 2 p.435 2.7, F(x) and f(x) be cumulative and density function of a random variable respectively, then $\hat{L}_{1-F}(s) = \frac{1-\hat{L}_f(s)}{s}$

Within the parentheses, it is actually the Laplace transform of compound geometric distribution G with density g [V] characterizing as $(1-\frac{\lambda\mu}{\beta}, F_U^I)$

$$\begin{aligned}\hat{L}_\varphi(s) &= \frac{\lambda\mu}{\beta}\frac{1-\hat{L}_g(s)}{s} \\ &= \frac{\lambda\mu}{\beta}\hat{L}_{\bar{G}}(s) \\ &= \int_0^\infty e^{-su}\frac{\lambda\mu}{\beta}\bar{G}(u)du\end{aligned}$$

And since the Laplace transform is unique[VI], it implies that $\varphi(u)$ has the same distribution as $\frac{\lambda\mu}{\beta}\bar{G}(u)$

$$\begin{aligned}\varphi(u) &= \frac{\lambda\mu}{\beta}\bar{G}(u) \\ &= \frac{\lambda\mu}{\beta}(1-\sum_{n=1}^{\infty}(1-\frac{\lambda\mu}{\beta})(\frac{\lambda\mu}{\beta})^{n-1}(F_U^I)^{*n}(u))^{\text{VII}} \\ &= \frac{\lambda\mu}{\beta}(\sum_{n=1}^{\infty}(1-\frac{\lambda\mu}{\beta})(\frac{\lambda\mu}{\beta})^{n-1}-\sum_{n=1}^{\infty}(1-\frac{\lambda\mu}{\beta})(\frac{\lambda\mu}{\beta})^{n-1}(F_U^I)^{*n}(u)) \\ &= (1-\frac{\lambda\mu}{\beta})\sum_{n=1}^{\infty}(\frac{\lambda\mu}{\beta})^n(1-(F_U^I)^{*n}(u))\end{aligned}$$

Since the strong connection between ruin theory and queing theory, the equation is actually equivalent to the well-known waiting time distribution Pollaczeck-Khinchine formula and thus has the same name. □

0.3 Martingale Approximation

The explicit expression of Pollaczeck-Khinchine formula is sometimes to hard to compute. Thus, we use the same martingale technique as section 1.1 to give the exponential bound of ultimate ruin.

Theorem 3. *Lundberg inequality*

$$\varphi(u) \le e^{-Lu}$$

[V]Theorem 3, second corollary, and Laplace-Stieltjes transform is actually Laplace transform but focus on cumulative function not density function

[VI]An Introduction to Probability Theory and its Applications Volume 2 p.430, Distinct probability distributions has distinct Laplace transforms

[VI]Applying theorem 2 and setting $p_0 = 0$

where L is called the Lundberg exponent, the positive solution of $\lambda(\hat{m}_U(s) - 1) - \beta s = 0$.

Proof. We first construct a martingale for the *risk reserve process* R(t), s,t>0

$$\begin{aligned}
E[e^{-sR(t)}] &= E[e^{-s(u+\beta t - X(t))}] = E[e^{s(X(t))}]e^{-s(u+\beta t)} \\
&= e^{-s(u+\beta t)}E[e^{s(U_1+U_2+\ldots+U_{N(t)})}] \\
&= e^{-s(u+\beta t)}\sum_{k=0}^{\infty}(E[e^{s(U_1+U_2+\ldots+U_k})]P(N(t)=k)) \\
&= e^{-s(u+\beta t)}\sum_{k=0}^{\infty}(E[e^{sU}]^k\frac{(\lambda t)^k}{k!}e^{-\lambda t}) \\
&= e^{-s(u+\beta t)}e^{-\lambda t}\sum_{k=0}^{\infty}(\hat{m}_U(s)^k\frac{(\lambda t)^k}{k!}) \\
&= e^{-s(u+\beta t)}e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\hat{m}_U(s)\lambda t)^k}{k!} \\
&= e^{-s(u+\beta t)}e^{-\lambda t}e^{\hat{m}_U(s)\lambda t} \\
&= e^{-su+(\lambda(\hat{m}_U(s)-1)-\beta s)t} \\
&= e^{-su+g(s)t}
\end{aligned}$$

Recall the definition of ruin,

$$\tau(u) = inf\{t \geq 0 : S(t) > u\}$$

Obviously $\tau(u)$ is a $\mathcal{F}_t^S$ stopping time. Put

$$M_t = \frac{e^{-r(R(t))}}{e^{g(r)t}}$$

For $0 \leq s \leq t$,we have

$$\begin{aligned}
E[M_t|\mathcal{F}_s^S] &= E[\frac{e^{-r(u+\beta t - X(t))}}{e^{g(r)t}}|\mathcal{F}_s^S] \\
&= E[\frac{e^{-r(u+\beta s - X(s)))}}{e^{g(r)s}}\frac{e^{-r(\beta t - X(t) - \beta s + X(s))}}{e^{g(r)(t-s)}}|\mathcal{F}_s^S] \\
&= M_s \cdot E[\frac{e^{-r(\beta t - X(t) - \beta s + X(s))}}{e^{g(r)(t-s)}}|\mathcal{F}_s^S] \\
&= M_s
\end{aligned}$$

So M_t is a martingale so we can apply the same method of section 1.1 to calculate the exponential bound of ultimate ruin. Further let L be the positive solution of g(s)=0, we know $M'(t) = e^{-LR(t)}$ is still a martingale.

$$\begin{aligned}
E[M'_0] &= E[M'_{\tau(u)\wedge t}] \\
&= E[M'_{\tau(u)}; \tau(u) \leq t] + E[M'_t; \tau(u) > t] \\
&\geq E[e^{-LR(\tau(u))} | \tau(u) \leq t] \times P(\tau(u) \leq t) \\
&\geq P(\tau(u) \leq t) \qquad\qquad since \quad R(\tau(u)) \leq 0
\end{aligned}$$

$$P(\tau(u) < \infty) = \lim_{t\to\infty} P(\tau(u) \leq t) \leq E[M_0] = e^{-Lu}$$

□

Although the martingale technique makes the approximation very easy, one must have to aware that if the claim size distribution is heavy-tailed, $\hat{m}_U(s)$ does not exist for s>0 and the martingale technique can not be used.