Chapter 2

POISSON PROCESSES

2.1 Introduction

A Poisson process is a simple and widely used stochastic process for modeling the times at which arrivals enter a system. It is in many ways the continuous-time version of the Bernoulli process that was described briefly in Subsection 1.3.5.

Recall that the Bernoulli process is defined by a sequence of IID binary rv's $Y_1, Y_2 \dots$, with PMF $p_Y(1) = q$ specifying the probability of an arrival in each time slot $i > 0$. There is an associated counting process $\{N(t);\, t \geq 0\}$ giving the number of arrivals up to and including time slot t. The PMF for $N(t)$, for integer $t > 0$, is the binomial $p_{N(t)}(n) = \binom{t}{n} q^n (1-q)^{t-n}$. There is also a sequence $S_1, S_2, \dots$ of integer arrival times (epochs), where the rv S_i is the epoch of the ith arrival. Finally there is an associated sequence of interarrival times, $X_1, X_2, \dots$, which are IID with the geometric PMF, $p_{X_i}(x) = q(1-q)^{x-1}$ for positive integer x. It is intuitively clear that the Bernoulli process is fully specified by specifying that the interarrival intervals are IID with the geometric PMF.

For the Poisson process, arrivals may occur at any time, and the probability of an arrival at any particular instant is 0. This means that there is no very clean way of describing a Poisson process in terms of the probability of an arrival at any given instant. It is more convenient to define a Poisson process in terms of the sequence of interarrival times, $X_1, X_2, \dots$, which are defined to be IID. Before doing this, we describe arrival processes in a little more detail.

2.1.1 Arrival processes

An *arrival process* is a sequence of increasing rv's , $0 < S_1 < S_2 < \cdots$, where $S_i < S_{i+1}$ means that $S_{i+1} - S_i$ is a positive rv, *i.e.*, a rv X such that $\Pr\{X \leq 0\} = 0$. These random variables are called arrival epochs (the word *time* is somewhat overused in this subject) and represent the times at which some repeating phenomenon occurs. Note that the process starts at time 0 and that multiple arrivals can't occur simultaneously (the phenomenon of bulk arrivals can be easily handled by the simple extension of associating a positive integer rv to each arrival). We will often specify arrival processes in a way that allows an arrival at time 0 or simultaneous arrivals as events of zero probability, but such zero probability events can usually be ignored. In order to fully specify the process by the sequence $S_1, S_2, \ldots$ of rv's, it is necessary to specify the joint distribution of the subsequences $S_1, \ldots, S_n$ for all $n > 1$.

Although we refer to these processes as arrival processes, they could equally well model departures from a system, or any other sequence of incidents. Although it is quite common, especially in the simulation field, to refer to incidents or arrivals as events, we shall avoid that here. The nth arrival epoch S_n is a rv and $\{S_n \le t\}$, for example, is an event. This would make it confusing to also refer to the nth arrival itself as an event.

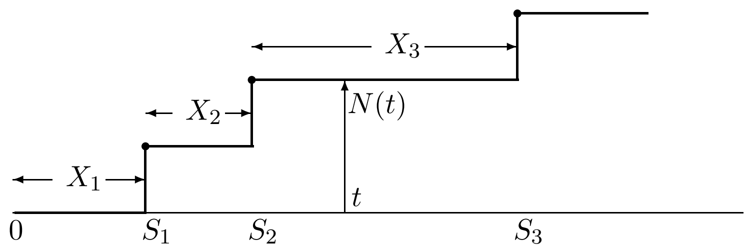


Figure 2.1: An arrival process and its arrival epochs $\{S_1, S_2, \ldots\}$, its interarrival intervals $\{X_1, X_2, \ldots\}$, and its counting process $\{N(t);\, t \ge 0\}$

As illustrated in Figure 2.1, any arrival process can also be specified by two other stochastic processes. The first is the sequence of interarrival times, $X_1, X_2, \ldots$,. These are positive rv's defined in terms of the arrival epochs by $X_1 = S_1$ and $X_i = S_i - S_{i-1}$ for $i > 1$. Similarly, given the X_i, the arrival epochs S_i are specified as

$$S_n = \sum\nolimits_{i=1}^{n} X_i. \tag{2.1}$$

Thus the joint distribution of $X_1, \ldots, X_n$ for all $n > 1$ is sufficient (in principle) to specify the arrival process. Since the interarrival times are IID, it is usually much easier to specify the joint distribution of the X_i than of the S_i.

The second alternative to specify an arrival process is the counting process $N(t)$, where for each $t > 0$, the rv $N(t)$ is the number of arrivals up to and including time t.

The counting process $\{N(t);\, t > 0\}$, illustrated in Figure 2.1, is an uncountably infinite family of rv's $\{N(t);\, t \ge 0\}$ where $N(t)$, for each $t > 0$, is the number of arrivals in the interval $(0, t]$. Whether the end points are included in these intervals is sometimes important, and we use parentheses to represent intervals without end points and square brackets to represent inclusion of the end point. Thus (a, b) denotes the interval $\{t : a < t < b\}$, and $(a, b]$ denotes $\{t : a < t \le b\}$. The *counting rv's* $N(t)$ for each $t > 0$ are then defined as the number of arrivals in the interval $(0, t]$. $N(0)$ is defined to be 0 with probability 1, which means, as before, that we are considering only arrivals at strictly positive times.

The counting process $\{N(t), t \ge 0\}$ for any arrival process has the properties that $N(\tau) \ge N(t)$ for all $\tau \ge t > 0$ (i.e., $N(\tau) - N(t)$ is a non-negative random variable).

For any given integer $n \geq 1$ and time $t \geq 0$, the nth arrival epoch, S_n, and the counting random variable, $N(t)$, are related by

$$\{S_n \leq t\} = \{N(t) \geq n\}. \tag{2.2}$$

To see this, note that $\{S_n \leq t\}$ is the event that the nth arrival occurs by time t. This event implies that $N(t)$, the number of arrivals by time t, must be at least n; i.e., it implies the event $\{N(t) \geq n\}$. Similarly, $\{N(t) \geq n\}$ implies $\{S_n \leq t\}$, yielding the equality in (2.2). This equation is essentially obvious from Figure 2.1, but is one of those peculiar obvious things that is often difficult to see. One should be sure to understand it, since it is fundamental in going back and forth between arrival epochs and counting rv's. In principle, (2.2) specifies the joint distributions of $\{S_i; i > 0\}$ and $\{N(t);\, t > 0\}$ in terms of each other, and we will see many examples of this in what follows.

In summary, then, an arrival process can be specified by the joint distributions of the arrival epochs, the interarrival intervals, or the counting rv's. In principle, specifying any one of these specifies the others also.[1]

2.2 Definition and properties of the Poisson process

The Poisson process is an example of an arrival process, and the interarrival times provide the most convenient description since the interarrival times are defined to be IID. Processes with IID interarrival times are particularly important and form the topic of Chapter 3.

Definition 2.1. *A renewal process is an arrival process for which the sequence of interarrival times is a sequence of IID rv's.*

Definition 2.2. *A Poisson process is a renewal process in which the interarrival intervals have an exponential distribution function; i.e., for some parameter λ, each X_i has the density $f_X(x) = \lambda \exp(-\lambda x)$ for $x \geq 0$*[2].

The parameter λ is called the rate of the process. We shall see later that for any interval of size t, λt is the expected number of arrivals in that interval. Thus λ is called the arrival rate of the process.

2.2.1 Memoryless property

What makes the Poisson process unique among renewal processes is the memoryless property of the exponential distribution.

[1]By definition, a stochastic process is a collection of rv's, so one might ask whether an arrival process (as a stochastic process) is 'really' the arrival epoch process $0 \leq S_1 \leq S_2 \leq \cdots$ or the interarrival process $X_1, X_2, \dots$ or the counting process $\{N(t);\, t \geq 0\}$. The arrival time process comes to grips with the actual arrivals, the interarrival process is often the simplest, and the counting process 'looks' most like a stochastic process in time since $N(t)$ is a rv for each $t \geq 0$. It seems preferable, since the descriptions are so clearly equivalent, to view arrival processes in terms of whichever description is more convenient at the time.

[2]With this density, $\Pr\{X_i{=}0\} = 0$, so that we can regard X_i as a positive random variable. Since events of probability zero can be ignored, the density $\lambda \exp(-\lambda x)$ for $x \geq 0$ and zero for $x < 0$ is effectively the same as the density $\lambda \exp(-\lambda x)$ for $x > 0$ and zero for $x \leq 0$.

Definition 2.3. Memoryless random variables: *A non-negative non-deterministic rv X possesses the memoryless property if, for every $x \geq 0$ and $t \geq 0$,*

$$\Pr\{X > t+x\} = \Pr\{X > x\}\Pr\{X > t\}\,. \tag{2.3}$$

Note that (2.3) is a statement about the complementary distribution function of X. There is no intimation that the *event* $\{X > t+x\}$ in the equation has any relation to the events $\{X > t\}$ or $\{X > x\}$. For the case $x = t = 0$, (2.3) says that $\Pr\{X > 0\} = [\Pr\{X > 0\}]^2$, so, since X is non-deterministic, $\Pr\{X > 0\} = 1$. In a similar way, by looking at cases where $x = t$, it can be seen that $\Pr\{X > t\} > 0$ for all $t \geq 0$. Thus (2.3) can be rewritten as

$$\Pr\{X > t+x \mid X > t\} = \Pr\{X > x\}\,. \tag{2.4}$$

If X is interpreted as the waiting time until some given arrival, then (2.4) states that, given that the arrival has not occured by time t, the distribution of the remaining waiting time (given by x on the left side of (2.4)) is the same as the original waiting time distribution (given on the right side of (2.4)), *i.e.*, the remaining waiting time has no memory of previous waiting.

Example 2.2.1. If X is the waiting time for a bus to arrive, and X is memoryless, then after you wait 15 minutes, you are no better off than you were originally. On the other hand, if the bus is known to arrive regularly every 16 minutes, then you know that it will arrive within a minute, so X is not memoryless. The opposite situation is also possible. If the bus frequently breaks down, then a 15 minute wait can indicate that the remaining wait is probably very long, so again X is not memoryless. We study these non-memoryless situations when we study renewal processes in the next chapter.

For an exponential rv X of rate λ, $\Pr\{X > x\} = e^{-\lambda x}$, so (2.3) is satisfied and X is memoryless. Conversely, it turns out that an arbitrary non-negative non-deterministic rv X is memoryless only if it is exponential. To see this, let $h(x) = \ln[\Pr\{X > x\}]$ and observe that since $\Pr\{X > x\}$ is nonincreasing in x, $h(x)$ is also. In addition, (2.3) says that $h(t+x) = h(x) + h(t)$ for all $x, t \geq 0$. These two statements (see Exercise 2.6) imply that $h(x)$ must be linear in x, and $\Pr\{X > x\}$ must be exponential in x.

Although the exponential distribution is the only memoryless distribution, it is interesting to note that if we restrict the definition of memoryless to integer times, then the geometric distribution is memoryless, so the Bernoulli process in this respect seems like a discrete-time version of the Poisson process.

We now use the memoryless property of the exponential rv to find the distribution of the first arrival in a Poisson process after some given time $t > 0$. We not only find this distribution, but also show that this first arrival after t is independent of all arrivals up to and including t. Note that t is an arbitrarily selected constant here; it is not a random variable. Let Z be the duration of the interval from t until the first arrival after t. First we find $\Pr\{Z > z \mid N(t) = 0\}$.

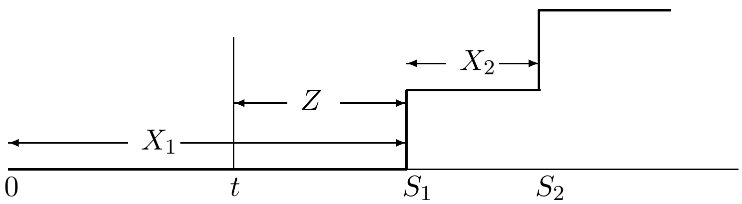


Figure 2.2: For some fixed t, consider the event $N(t) = 0$. Conditional on this event, Z is the interval from t to S_1; i.e., $Z = X_1 - t$.

As illustrated in Figure 2.2, for $\{N(t) = 0\}$, the first arrival after t is the first arrival of the process. Stating this more precisely, the following events are identical:[3]

$$\{Z > z\} \bigcap \{N(t) = 0\} = \{X_1 > z + t\} \bigcap \{N(t) = 0\}.$$

The conditional probabilities are then

$$\begin{aligned}
\Pr\{Z > z \mid N(t){=}0\} &= \Pr\{X_1 > z + t \mid N(t){=}0\} \\
&= \Pr\{X_1 > z + t \mid X_1 > t\} && (2.5) \\
&= \Pr\{X_1 > z\} = e^{-\lambda z}. && (2.6)
\end{aligned}$$

In (2.5), we used the fact that $\{N(t) = 0\} = \{X_1 > t\}$, which is clear from Figure 2.1. In (2.6) we used the memoryless condition in (2.4) and the fact that X_1 is exponential.

Next consider the condition that there are n arrivals in $(0, t]$ and the nth occurs at epoch $S_n = \tau \leq t$. The argument here is basically the same as that with $N(t) = 0$, with a few extra details (see Figure 2.3).

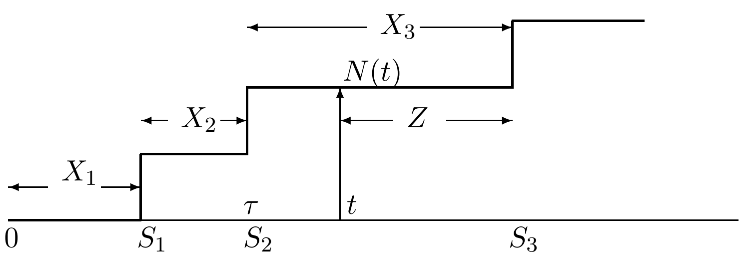


Figure 2.3: Given $N(t) = 2$, and $S_2 = \tau$, X_3 is equal to $Z + (t - \tau)$. Also, the event $\{N(t){=}2,\ S_2{=}\tau\}$ is the same as the event $\{S_2{=}\tau,\ X_3{>}t{-}\tau\}$.

Conditional on $N(t) = n$ and $S_n = \tau$, the first arrival after t is the first arrival after the arrival at S_n, *i.e.*, $Z = z$ corresponds to $X_{n+1} = z{+}t{-}\tau$. Stating this precisly, the following events are identical:

$$\{Z > z\} \bigcap \{N(t) = n\} \bigcap \{S_n = \tau\} = \{X_{n+1} > z{+}t{-}\tau\} \bigcap \{N(t) = n\} \bigcap \{S_n = \tau\}.$$

[3]It helps intuition to sometimes think of one event A as conditional on another event B. More precisely, A given B is the set of sample points in B that are also in A, which is simply $A \bigcap B$.

Note that $S_n = \tau$ is an event of zero probability, but S_n is a sum of n IID random variables with densities, and thus has a density itself, so that other events can be conditioned on it.

$$
\begin{aligned}
\Pr\{Z > z \mid N(t){=}n, S_n{=}\tau\} &= \Pr\{X_{n+1} > z{+}t{-}\tau \mid N(t){=}n, S_n{=}\tau\} && (2.7)\\
&= \Pr\{X_{n+1} > z{+}t{-}\tau \mid X_{n+1}{>}t{-}\tau, S_n{=}\tau\} && (2.8)\\
&= \Pr\{X_{n+1} > z{+}t{-}\tau \mid X_{n+1}{>}t{-}\tau\} && (2.9)\\
&= \Pr\{X_{n+1} > z\} = e^{-\lambda z}. && (2.10)
\end{aligned}
$$

In (2.8), we have used the fact that, given $S_n = \tau$, the event $N(t) = n$ is the same as $X_{n+1} > t - \tau$ (see Figure 2.3). In (2.9) we used the fact that X_{n+1} is independent of S_n. In (2.10) we used the memoryless condition in (2.4) and the fact that X_{n+1} is exponential.

The same argument applies if, in (2.7), we condition not only on S_n but also on $S_1, \dots, S_{n-1}$. Since this is equivalent to conditioning on $N(\tau)$ for all τ in $(0, t]$, we have

$$\Pr\{Z > z \mid \{N(\tau), 0 < \tau \le t\}\} = \exp(-\lambda z). \tag{2.11}$$

The following theorem states this in words.

Theorem 2.1. *For a Poisson process of rate λ, and any given time $t > 0$, the interval from t until the first arrival after t is a nonnegative rv Z with the distribution function $1 - \exp[-\lambda z]$ for $z \ge 0$. This rv is independent of all arrival epochs before time t and independent of $N(\tau)$ for all $\tau \le t$.*

The length of our derivation of (2.11) somewhat hides its conceptual simplicity. Z, conditional on the time τ of the last arrival before t, is simply the remaining time until the next arrival, which, by the memoryless property, is independent of $\tau \le t$, and hence also independent of everything before t.

Next consider subsequent interarrival intervals after a given time t. For $m \ge 2$, let Z_m be the interarrival interval from the $m-1$st arrival epoch after t to the mth arrival epoch after t. Given $N(t) = n$, we see that $Z_m = X_{m+n}$, and therefore $Z_2, Z_3, \dots,$ are IID exponentially distributed random variables, conditional on $N(t) = n$ (see Exercise 2.8). Let Z in (2.11) become Z_1 here. Since Z_1 is independent of $Z_2, Z_3, \dots$ and independent of $N(t)$, we see that $Z_1, Z_2, \dots$ are unconditionally IID and also independent of $N(t)$. It should also be clear that $Z_1, Z_2, \dots$ are independent of $\{N(\tau); 0 < \tau \le t\}$.

The above argument shows that the portion of a Poisson process starting at some time $t > 0$ is a probabilistic replica of the process starting at 0; that is, the time until the first arrival after t is an exponentially distributed rv with parameter λ, and all subsequent arrivals are independent of this first arrival and of each other and all have the same exponential distribution.

Definition 2.4. *A counting process $\{N(t);\, t \ge 0\}$ has the stationary increment property if for every $t' > t > 0$, $N(t') - N(t)$ has the same distribution function as $N(t' - t)$.*

Let us define $\widetilde{N}(t, t') = N(t') - N(t)$ as the number of arrivals in the interval $(t, t']$ for any given $t' \ge t$. We have just shown that for a Poisson process, the rv $\widetilde{N}(t, t')$ has the same distribution as $N(t'-t)$, which means that a Poisson process has the stationary increment property. Thus, the distribution of the number of arrivals in an interval depends on the size of the interval but not on its starting point.

Definition 2.5. *A counting process $\{N(t);\, t \geq 0\}$ has the independent increment property if, for every integer $k > 0$, and every k-tuple of times $0 < t_1 < t_2 < \cdots < t_k$, the k-tuple of rv's $N(t_1)$, $\widetilde{N}(t_1,t_2), \dots, \widetilde{N}(t_{k-1},t_k)$ of rv's are statistically independent.*

For the Poisson process, Theorem 2.1 says that for any t, the time Z_1 until the next arrival after t is independent of $N(\tau)$ for all $\tau \leq t$. Letting $t_1 < t_2 < \cdots t_{k-1} < t$, this means that Z_1 is independent of $N(t_1), \widetilde{N}(t_1,t_2), \dots, \widetilde{N}(t_{k-1},t)$. We have also seen that the subsequent interarrival times after Z_1, and thus $\widetilde{N}(t,t')$ are independent of $N(t_1), \widetilde{N}(t_1,t_2), \dots, \widetilde{N}(t_{k-1},t)$. Renaming t as t_k and t' as t_{k+1}, we see that $\widetilde{N}(t_k,t_{k+1})$ is independent of $N(t_1), \widetilde{N}(t_1,t_2), \dots, \widetilde{N}(t_{k-1},t_k)$. Since this is true for all k, the Poisson process has the independent increment property. In summary, we have proved the following:

Theorem 2.2. *Poisson processes have both the stationary increment and independent increment properties.*

Note that if we look only at integer times, then the Bernoulli process also has the stationary and independent increment properties.

2.2.2 Probability density of S_n and $S_1, \dots S_n$

Recall from (2.1) that, for a Poisson process, S_n is the sum of n IID rv's, each with the density function $f(x) = \lambda \exp(-\lambda x)$, $x \geq 0$. Also recall that the density of the sum of two independent rv's can be found by convolving their densities, and thus the density of S_2 can be found by convolving $f(x)$ with itself, S_3 by convolving the density of S_2 with $f(x)$, and so forth. The result, for $t \geq 0$, is called the *Erlang density*,[4]

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} \exp(-\lambda t)}{(n-1)!}. \tag{2.12}$$

We can understand this density (and other related matters) much better by reviewing the above mechanical derivation more carefully. The joint density for two continuous independent rv's X_1 and X_2 is given by $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$. Letting $S_2 = X_1 + X_2$ and substituting $S_2 - X_1$ for X_2, we get the following joint density for X_1 and the sum S_2,

$$f_{X_1 S_2}(x_1 s_2) = f_{X_1}(x_1) f_{X_2}(s_2 - x_1).$$

The marginal density for S_2 then results from integrating x_1 out from the joint density, and this, of course, is the familiar convolution integration. For IID exponential rv's X_1, X_2, the joint density of X_1, S_2 takes the following interesting form:

$$f_{X_1 S_2}(x_1 s_2) = \lambda^2 \exp(-\lambda x_1) \exp(-\lambda(s_2 - x_1)) \;=\; \lambda^2 \exp(-\lambda s_2) \quad \text{for } 0 \leq x_1 \leq s_2. \tag{2.13}$$

[4]Another (somewhat poorly chosen and rarely used) name for the Erlang density is the *gamma density.*

This says that the joint density does not contain x_1, except for the constraint $0 \le x_1 \le s_2$. Thus, for fixed s_2, the joint density, and thus the conditional density of X_1 given $S_2 = s_2$ is uniform over $0 \le x_1 \le s_2$. The integration over x_1 in the convolution equation is then simply multiplication by the interval size s_2, yielding the marginal distribution $f_{S_2}(s_2) = \lambda^2 s_2 \exp(-\lambda s_2)$, in agreement with (2.12) for $n = 2$.

This same curious behavior exhibits itself for the sum of an arbitrary number n of IID exponential rv's. That is, $f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \lambda^n \exp(-\lambda x_1 - \lambda x_2 - \cdots - \lambda x_n)$. Letting $S_n = X_1 + \cdots + X_n$ and substituting $S_n - X_1 - \cdots - X_{n-1}$ for X_n, this becomes

$$f_{X_1\cdots X_{n-1}S_n}(x_1,\dots,x_{n-1},s_n) = \lambda^n \exp(-\lambda s_n).$$

since each x_i cancels out above. This equation is valid over the region where each $x_i \ge 0$ and $s_n - x_1 - \cdots - x_{n-1} \ge 0$. The density is 0 elsewhere.

The constraint region becomes more clear here if we replace the interarrival intervals $X_1,\dots,X_{n-1}$ with the arrival epochs $S_1,\dots,S_{n-1}$ where $S_1 = X_1$ and $S_i = X_i + S_{i-1}$ for $2 \le i \le n-1$. The joint density then becomes[5]

$$f_{S_1\cdots S_n}(s_1,\dots,s_n) = \lambda^n \exp(-\lambda s_n) \qquad \text{for } 0 \le s_1 \le s_2 \cdots \le s_n. \tag{2.14}$$

The interpretation here is the same as with S_2. The joint density does not contain any arrival time other than s_n, except for the ordering constraint $0 \le s_1 \le s_2 \le \cdots \le s_n$, and thus this joint density is the same for all choices of arrival times satisfying the ordering constraint. Mechanically integrating this over s_1, then s_2, etc. we get the Erlang formula (2.12). The Erlang density then is the joint density in (2.14) times the volume $s_n^{n-1}/(n-1)!$ of the region of $s_1,\dots,s_{n-1}$ satisfing $0 < s_1 < \cdots < s_n$. This will be discussed further later.

2.2.3 The PMF for $N(t)$

The Poisson counting process, $\{N(t);\, t > 0\}$ consists of a discrete rv $N(t)$ for each $t > 0$. In this section, we show that the PMF for this rv is the well-known Poisson PMF, as stated in the following theorem. We give two proofs for the theorem, each providing its own type of understanding and each showing the close relationship between $\{N(t) = n\}$ and $\{S_n = t\}$.

Theorem 2.3. *For a Poisson process of rate λ, and for any $t > 0$, the PMF for $N(t)$ (*i.e.,* the number of arrivals in $(0,t]$) is given by the Poisson PMF,*

$$p_{N(t)}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}. \tag{2.15}$$

Proof 1: This proof, for given n and t, is based on two ways of calculating the probability $\Pr\{t < S_{n+1} \le t + \delta\}$ for some vanishingly small δ. The first way is based on the already

[5] The random vector $\boldsymbol{S} = (S_1,\dots,S_n)$ is then related to the interarrival intervals $\boldsymbol{X} = (X_1,\dots,X_n)$ by a linear transformation, say $\boldsymbol{S} = \mathsf{A}\boldsymbol{X}$. In general, the joint density of $\boldsymbol{S}$ at $\boldsymbol{s} = \mathsf{A}\boldsymbol{x}$ is $f_{\boldsymbol{S}}(\boldsymbol{s}) = f_{\boldsymbol{X}}(\boldsymbol{x})/|\det \mathsf{A}|$. This is because the transformation A carries a cube δ on a side into a parallelepiped of volume $\delta^n|\det \mathsf{A}|$. In the case here, A is upper triangular with 1's on the diagonal, so $\det \mathsf{A} = 1$.

known density of S_{n+1} and gives

$$\Pr\{t < S_{n+1} \le t+\delta\} = \int_t^{t+\delta} f_{S_n}(\tau)\,d\tau = f_{S_n}(t)\,(\delta + o(\delta)).$$

The term $o(\delta)$ is used to describe a function of δ that goes to 0 faster than δ as $\delta \to 0$. More precisely, a function $g(\delta)$ is said to be of order $o(\delta)$ if $\lim_{\delta\to 0} \frac{g(\delta)}{\delta} = 0$. Thus $\Pr\{t < S_n \le t+\delta\} = f_{S_n}(t)(\delta + o(\delta))$ is simply a consequence of the fact that S_n has a continuous probability density in the interval $[t, t+\delta]$.

The second way is that $\{t < S_{n+1} \le t+\delta\}$ occurs if exactly n arrivals arrive in the interval $(0,t]$ and one arrival occurs in $(t, t+\delta]$. Because of the independent increment property, this is an event of probability $p_{N(t)}(n)(\lambda\delta + o(\delta))$. It is also possible to have fewer than n arrivals in $(0,t]$ and more than one in $(t, t+\delta]$, but this has probability $o(\delta)$. Thus

$$p_{N(t)}(n)(\lambda\delta + o(\delta)) + o(\delta) = f_{S_{n+1}}(t)(\delta + o(\delta)).$$

Dividing by λ and taking the limit $\delta \to 0$, we get

$$\lambda p_{N(t)}(n) = f_{S_{n+1}}(t).$$

Using the density for f_{S_n} given in (2.12), we get (2.15). □

Proof 2: The approach here is to use the fundamental relation that $\{N(t \ge n\} = \{S_n \le t\}$. Taking the probabilities of these events,

$$\sum_{i=n}^{\infty} p_{N(t)}(i) = \int_0^t f_{S_n}(\tau)\,d\tau \qquad \text{for all } n \ge 1 \text{ and } t > 0.$$

The term on the right above is the distribution function of S_n for each $n \ge 1$ and the term on the left is the complementary distribution function of $N(t)$ for each $t > 0$. Thus this equation (for all $n \ge 1, t > 0$), uniquely specifies the PMF of $N(t)$ for each $t > 0$. The theorem will then be proven by showing that

$$\sum\nolimits_{i=n}^{\infty} \frac{(\lambda t)^i \exp(-\lambda t)}{i!} = \int_0^t f_{S_n}(\tau)\,d\tau. \tag{2.16}$$

If we take the derivative with respect to t of each side of (2.16), we find that almost magically each term except the first on the left cancels out, leaving us with

$$\frac{\lambda^n t^{n-1} \exp(-\lambda t)}{(n-1)!} = f_{S_n}(t).$$

Thus the derivative with respect to t of each side of (2.16) is equal to the derivative of the other for all $n \ge 1$ and $t > 0$. The two sides of (2.16) are also equal in the limit $t \to 0$, so it follows that (2.16) is satisfied everywhere, completing the proof. □

2.2.4 Alternate definitions of Poisson processes

Definition 2 of a Poisson process: *A Poisson counting process* $\{N(t);\, t \geq 0\}$ *is a counting process that satisfies (2.15) (*i.e., *has the Poisson PMF) and has the independent and stationary increment properties.*

We have seen that the properties in Definition 2 are satisfied starting with Definition 1 (using IID exponential interarrival times), so Definition 1 implies Definition 2. Exercise 2.4 shows that IID exponential interarrival times are implied by Definition 2, so the two definitions are equivalent.

It may be somewhat surprising at first to realize that a counting process that has the Poisson PMF at each t is not necessarily a Poisson process, and that the independent and stationary increment properties are also necessary. One way to see this is to recall that the Poisson PMF for all t in a counting process is equivalent to the Erlang density for the successive arrival epochs. Specifying the probability density for $S_1, S_2, \dots$, as Erlang specifies the *marginal* densities of $S_1, S_2, \dots$,, but need not specify the *joint* densities of these rv's. Figure 2.4 illustrates this in terms of the joint density of S_1, S_2, given as

$$f_{S_1S_2}(s_1s_2) = \lambda^2 \exp(-\lambda s_2) \qquad \text{for } 0 \leq s_1 \leq s_2$$

and 0 elsewhere. The figure illustrates how the joint density can be changed without changing the marginals.

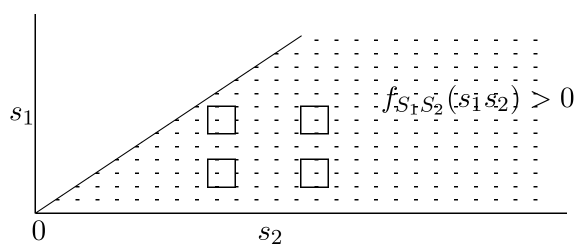


Figure 2.4: The joint density of S_1, S_2 is nonzero in the region shown. It can be changed, while holding the marginals constant, by reducing the joint density by ε in the upper left and lower right squares above and increasing it by ε in the upper right and lower left squares.

There is a similar effect with the Bernoulli process in that a discrete counting process for which the number of arrivals from 0 to t, for each integer t is a binomial rv, but the process is not Bernoulli. This is explored in Exercise 2.5.

The next definition of a Poisson process is based on its incremental properties. Consider the number of arrivals in some very small interval $(t, t+\delta]$. Since $\widetilde{N}(t, t+\delta)$ has the same distribution as $N(\delta)$, we can use (2.15) to get

$$\begin{aligned}
\Pr\left\{\widetilde{N}(t,t+\delta)=0\right\} &= e^{-\lambda\delta} \approx 1-\lambda\delta+o(\delta) \\
\Pr\left\{\widetilde{N}(t,t+\delta)=1\right\} &= \lambda e^{-\lambda\delta} \approx \lambda\delta+o(\delta) \\
\Pr\left\{\widetilde{N}(t,t+\delta)\geq 2\right\} &\approx o(\delta).
\end{aligned} \tag{2.17}$$

Definition 3 of a Poisson process: *A Poisson counting process is a counting process that satisfies (2.17) and has the stationary and independent increment properties.*

We have seen that Definition 1 implies Definition 3. The essence of the argument the other way is that for any interarrival interval X, $F_X(x+\delta)-F_X(x)$ is the probability of an arrival in an appropriate infinitesimal interval of width δ, which by (2.17) is $\lambda\delta+o(\delta)$. Turning this into a differential equation (see Exercise 2.7), we get the desired exponential interarrival intervals. Definition 3 has an intuitive appeal, since it is based on the idea of independent arrivals during arbitrary disjoint intervals. It has the disadvantage that one must do a considerable amount of work to be sure that these conditions are mutually consistent, and probably the easiest way is to start with Definition 1 and derive these properties. Showing that there is a unique process that satisfies the conditions of Definition 3 is even harder, but is not necessary at this point, since all we need is the use of these properties. Section 2.2.5 will illustrate better how to use this definition (or more precisely, how to use (2.17)).

What (2.17) accomplishes, beyond the assumption of independent and stationary increments, in Definition 3 is the prevention of bulk arrivals. For example, consider a counting process in which arrivals always occur in pairs, and the intervals between successive pairs are IID and exponentially distributed with parameter λ (see Figure 2.5). For this process, $\Pr\left\{\widetilde{N}(t,t+\delta)=1\right\} = 0$, and $\Pr\left\{\widetilde{N}(t,t+\delta)=2\right\} = \lambda\delta+o(\delta)$, thus violating (2.17). This process has stationary and independent increments, however, since the process formed by viewing a pair of arrivals as a single incident is a Poisson process.

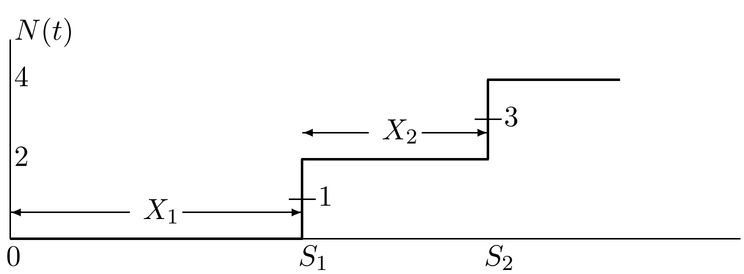


Figure 2.5: A counting process modeling bulk arrivals. X_1 is the time until the first pair of arrivals and X_2 is the interval between the first and second pair of arrivals.

2.2.5 The Poisson process as a limit of shrinking Bernoulli processes

The intuition of Definition 3 can be achieved in a much less abstract way by starting with the Bernoulli process, which has the properties of Definition 3 in a discrete-time sense. We then go to an appropriate limit of a sequence of these processes, and find that this sequence of Bernoulli processes converges in various ways to the Poisson process.

Recall that a Bernoulli process is an IID sequence, $Y_1, Y_2, \dots$, of binary random variables for which $p_Y(1) = q$ and $p_Y(0) = 1 - q$. We can visualize $Y_i = 1$ as an *arrival* at time i and $Y_i = 0$ as no arrival, but we can also 'shrink' the time scale of the process so that for some integer $j > 0$, Y_i is an arrival or no arrival at time $i2^{-j}$. We consider a sequence indexed by j of such shrinking Bernoulli processes, and in order to keep the arrival rate constant, we let $q = \lambda 2^{-j}$ for the jth process. Thus for each unit increase in j, the Bernoulli process shrinks by replacing each slot with two slots, each with half the previous arrival probability. The expected number of arrivals per time unit is then λ, matching the Poisson process that we are approximating.

If we look at this jth process relative to Definition 3 of a Poisson process, we see that for these regularly spaced increments of size $\delta = 2^{-j}$, the probability of one arrival in an increment is $\lambda\delta$ and that of no arrival is $1 - \lambda\delta$, and thus (2.17) is satisfied, and in fact the $o(\delta)$ terms are exactly zero. For arbitrary sized increments, it is clear that disjoint increments have independent arrivals. The increments are not quite stationary, since, for example, an increment of size 2^{-j-1} might contain a time that is a multiple of 2^{-j} or might not, depending on its placement. However, for any fixed increment of size δ, the number of multiples of 2^{-j} (*i.e.*, the number of possible arrival points) is either $\lfloor \delta 2^j \rfloor$ or $1 + \lfloor \delta 2^j \rfloor$. Thus in the limit $j \to \infty$, the increments are both stationary and independent.

For each j, the jth Bernoulli process has an associated Bernoulli counting process $N_j(t) = \sum_{i=1}^{\lfloor t2^j \rfloor} Y_i$. This is the number of arrivals up to time t and is a discrete rv with the binomial PMF. That is, $p_{N_j(t)}(n) = \binom{\lfloor t2^j \rfloor}{n} q^n (1-q)^{\lfloor t2^j \rfloor - n}$ where $q = \lambda 2^{-j}$. We now show that this PMF approaches the Poisson PMF as j increases

Theorem 2.4. *Consider the sequence of shrinking Bernoulli processes with arrival probability $\lambda 2^{-j}$ and time-slot size 2^{-j}. Then for every fixed time $t > 0$ and fixed number of arrivals n, the counting PMF $p_{N_j(t)}(n)$ approaches the Poisson PMF (of the same λ) with increasing j,* i.e.,

$$\lim_{j \to \infty} p_{N_j(t)}(n) = p_{N(t)}(n). \tag{2.18}$$

Proof: We first rewrite the binomial PMF, for $\lfloor t2^j \rfloor$ variables with $q = \lambda 2^{-j}$ as

$$
\begin{aligned}
\lim_{j\to\infty} p_{N_j(t)}(n) &= \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp[\lfloor t2^j \rfloor (\ln(1-\lambda 2^{-j})] \\
&= \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp(-\lambda t) \qquad (2.19) \\
&= \lim_{j\to\infty} \frac{\lfloor t2^j \rfloor \cdot \lfloor t2^j{-}1 \rfloor \cdots \lfloor t2^j{-}n{+}1 \rfloor}{n!} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp(-\lambda t) \qquad (2.20) \\
&= \frac{(\lambda t)^n \exp(-\lambda t)}{n!}. \qquad (2.21)
\end{aligned}
$$

We used $\ln(1-\lambda 2^{-j}) = -\lambda 2^{-j} + o(2^{-j})$ in (2.19) and expanded the combinatorial term in (2.20). In (2.21), we recognized that $\lim_{j\to\infty} \lfloor t2^j - i \rfloor \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right) = \lambda t$ for $0 \le i \le n-1$. $\square$

Since the binomial PMF (scaled as above) has the Poisson PMF as a limit for each n, the distribution function of $N_j(t)$ also converges to the Poisson distribution function for each t. In other words, for each $t > 0$, the counting random variables $N_j(t)$ of the Bernoulli processes converge in distribution to $N(t)$ of the Poisson process. By Definition 2, then,[6] this limiting process is a Poisson process.

With the same scaling, the distribution function of the geometric distribution converges to the exponential distribution function,

$$\lim_{j\to\infty} (1-\lambda 2^{-j})^{\lfloor t2^j - 1 \rfloor} = \exp(-\lambda t).$$

Note that no matter how large j is, the corresponding shrunken Bernoulli process can have arrivals only at the discrete times that are multiples of 2^{-j}. Thus the interarrival times and the arrival epochs are discrete rv's and in no way approach the densities of the Poisson process. The distribution functions of these rv's, however quickly approach the distribution functions of the corresponding Poisson rv's. This is a good illustration of why it is sensible to focus on distribution functions rather than PDF's or PMF's.

2.3 Combining and splitting Poisson processes

Suppose that $\{N_1(t), t \ge 0\}$ and $\{N_2(t), t \ge 0\}$ are independent Poisson counting processes[7] of rates λ_1 and λ_2 respectively. We want to look at the sum process where $N(t) = N_1(t) +$

[6]Note that the counting process is one way of defining the Poisson process, but this requires the joint distributions of the counting variables, not just the marginal distributions. This is why Definition 2 requires not only the Poisson distribution for each $N(t)$ (*i.e.*, each marginal distribution), but also the stationary and independent increment properties. Exercise 2.5 gives an example for how the binomial distribution can be satisfied without satisfying the discrete version of the independent increment property.

[7]Two processes $\{N_1(t); t \ge 0\}$ and $\{N_2(t); t \ge 0\}$ are said to be independent if for all positive integers k and all sets of times $t_1, \dots, t_k$, the random variables $N_1(t_1), \dots, N_1(t_k)$ are independent of $N_2(t_1), \dots, N_2(t_k)$. Here it is enough to extend the independent increment property to independence between increments over the two processes; equivalently, one can require the interarrival intervals for one process to be independent of the interarrivals for the other process.

$N_2(t)$ for all $t \geq 0$. In other words, $\{N(t), t \geq 0\}$ is the process consisting of all arrivals to both process 1 and process 2. We shall show that $\{N(t), t \geq 0\}$ is a Poisson counting process of rate $\lambda = \lambda_1 + \lambda_2$. We show this in three different ways, first using Definition 3 of a Poisson process (since that is most natural for this problem), then using Definition 2, and finally Definition 1. We then draw some conclusions about the way in which each approach is helpful. Since $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are independent and both possess the stationary and independent increment properties, it follows from the definitions that $\{N(t);\, t \geq 0\}$ also possesses the stationary and independent increment properties. Using the approximations in (2.17) for the individual processes, we see that

$$\begin{aligned}\Pr\left\{\widetilde{N}(t,t+\delta)=0\right\} &= \Pr\left\{\widetilde{N}_1(t,t+\delta)=0\right\}\Pr\left\{\widetilde{N}_2(t,t+\delta)=0\right\}\\ &= (1-\lambda_1\delta)(1-\lambda_2\delta) \approx 1-\lambda\delta.\end{aligned}$$

where $\lambda_1\lambda_2\delta^2$ has been dropped. In the same way, $\Pr\left\{\widetilde{N}(t,\, t{+}\delta)=1\right\}$ is approximated by $\lambda\delta$ and $\Pr\left\{\widetilde{N}(t,t+\delta)\geq 2\right\}$ is approximated by 0, both with errors proportional to δ^2. It follows that $\{N(t), t \geq 0\}$ is a Poisson process.

In the second approach, we have $N(t) = N_1(t) + N_2(t)$. Since $N(t)$, for any given t, is the sum of two independent Poisson rv's , it is also a Poisson rv with mean $\lambda t = \lambda_1 t + \lambda_2 t$. If the reader is not aware that the sum of two independent Poisson rv's is Poisson, it can be derived by discrete convolution of the two PMF's (see Exercise 1.18). More elegantly, one can observe that we have already implicitly shown this fact. That is, if we break an interval I into disjoint subintervals, I_1 and I_2, the number of arrivals in I (which is Poisson) is the sum of the number of arrivals in I_1 and in I_2 (which are independent Poisson). Finally, since $N(t)$ is Poisson for each t, and since the stationary and independent increment properties are satisfied, $\{N(t);\, t \geq 0\}$ is a Poisson process.

In the third approach, X_1, the first interarrival interval for the sum process, is the minimum of X_{11}, the first interarrival interval for the first process, and X_{21}, the first interarrival interval for the second process. Thus $X_1 > t$ if and only if both X_{11} and X_{21} exceed t, so

$$\Pr\{X_1 > t\} = \Pr\{X_{11} > t\}\Pr\{X_{21} > t\} = \exp(-\lambda_1 t - \lambda_2 t) = \exp(-\lambda t).$$

Using the memoryless property, each subsequent interarrival interval can be analyzed in the same way.

The first approach above was the most intuitive for this problem, but it required constant care about the order of magnitude of the terms being neglected. The second approach was the simplest analytically (after recognizing that sums of independent Poisson rv's are Poisson), and required no approximations. The third approach was very simple in retrospect, but not very natural for this problem. If we add many independent Poisson processes together, it is clear, by adding them one at a time, that the sum process is again Poisson. What is more interesting is that when many independent counting processes (not necessarily Poisson) are added together, the sum process often tends to be approximately Poisson if the individual processes have small rates compared to the sum. To obtain some crude intuition about why this might be expected, note that the interarrival intervals for each process (assuming no bulk arrivals) will tend to be large relative to the mean interarrival interval for the sum process. Thus arrivals that are close together in time will typically come from different processes. The number of arrivals in an interval large relative to the combined mean interarrival interval, but small relative to the individual interarrival intervals, will be the sum of the number of arrivals from the different processes; each of these is 0 with large probability and 1 with small probability, so the sum will be approximately Poisson.

2.3.1 Subdividing a Poisson process

Next we look at how to break $\{N(t), t \geq 0\}$, a Poisson counting process of rate λ, into two processes, $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$. Suppose that each arrival in $\{N(t), t \geq 0\}$ is sent to the first process with probability p and to the second process with probability $1-p$ (see Figure 2.6). Each arrival is switched independently of each other arrival and independently of the arrival epochs. We shall show that the resulting processes are each Poisson, with rates $\lambda_1 = \lambda p$ and $\lambda_2 = \lambda(1-p)$ respectively, and that furthermore the two processes are independent. Note that, conditional on the original process, the two new processes are not independent; in fact one completely determines the other. Thus this independence might be a little surprising.

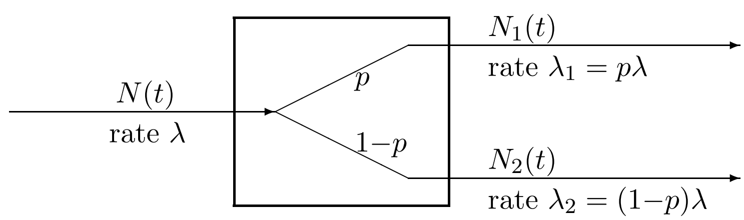


Figure 2.6: Each arrival is independently sent to process 1 with probability p and to process 2 otherwise.

First consider a small increment $(t, t+\delta]$. The original process has an arrival in this incremental interval with probability $\lambda\delta$ (ignoring δ^2 terms as usual), and thus process 1 has an arrival with probability $\lambda\delta p$ and process 2 with probability $\lambda\delta(1-p)$. Because of the independent increment property of the original process and the independence of the division of each arrival between the two processes, the new processes each have the independent increment property, and from above have the stationary increment property. Thus each process is Poisson. Note now that we cannot verify that the two processes are independent from this small increment model. We would have to show that the number of arrivals for process 1 and 2 are independent over $(t, t+\delta]$. Unfortunately, leaving out the terms of order δ^2, there is at most one arrival to the original process and no possibility of an arrival to each new process in $(t, t+\delta]$. If it is impossible for both processes to have an arrival in the same interval, they cannot be independent. It is possible, of course, for each process to have an arrival in the same interval, but this is a term of order δ^2. Thus, without paying attention to the terms of order δ^2, it is impossible to demonstrate that the processes are independent.

To demonstrate that process 1 and 2 are independent, we first calculate the joint PMF for $N_1(t)$, $N_2(t)$ for arbitrary t. Conditioning on a given number of arrivals $N(t)$ for the original process, we have

$$\Pr\{N_1(t)=m, N_2(t)=k \mid N(t)=m+k\} = \frac{(m+k)!}{m!k!}p^m(1-p)^k. \qquad (2.22)$$

Equation (2.22) is simply the binomial distribution, since, given $m+k$ arrivals to the original process, each independently goes to process 1 with probability p. Since the event $\{N_1(t) = m,\ N_2(t) = k\}$ is a subset of the conditioning event above,

$$\Pr\{N_1(t)=m, N_2(t)=k \mid N(t)=m+k\} = \frac{\Pr\{N_1(t)=m, N_2(t)=k\}}{\Pr\{N(t)=m+k\}}.$$

Combining this with (2.22), we have

$$\Pr\{N_1(t)=m, N_2(t)=k\} = \frac{(m+k!)}{m!k!}p^m(1-p)^k\frac{(\lambda t)^{m+k}e^{-\lambda t}}{(m+k)!}. \qquad (2.23)$$

Rearranging terms, we get

$$\Pr\{N_1(t)=m, N_2(t)=k\} = \frac{(p\lambda t)^m e^{-\lambda pt}}{m!}\ \frac{[(1-p)\lambda t]^k e^{-\lambda(1-p)t}}{k!}. \qquad (2.24)$$

This shows that $N_1(t)$ and $N_2(t)$ are independent. To show that the processes are independent, we must show that for any $k > 1$ and any set of times $0 \le t_1 \le t_2 \le \cdots \le t_k$, the sets $\{N_1(t_i) 1 \le i \le k\}$ and $\{N_2(t_j);\ 1 \le j \le k\}$ are independent of each other. It is equivalent to show that the sets $\{\widetilde{N}_1(t_{i-1}, t_i); 1 \le i \le k\}$ and $\{\widetilde{N}_2(t_{j-1}, t_j); 1 \le j \le k\}$ (where t_0 is 0) are independent. The argument above shows this independence for $i = j$, and for $i \ne j$, the independence follows from the independent increment property of $\{N(t);\ t \ge 0\}$.

2.3.2 Examples using independent Poisson processes

We have observed that if the arrivals of a Poisson process are split into two new arrival processes, each arrival of the original process independently going into the first of the new processes with some fixed probability p, then the new processes are Poisson processes and are independent. The most useful consequence of this is that any two independent Poisson processes can be viewed as being generated from a single process in this way. Thus, if one process has rate λ_1 and the other has rate λ_2, they can be viewed as coming from a process of rate $\lambda_1 + \lambda_2$. Each arrival to the combined process then goes to the first process with probability $p = \lambda_1/(\lambda_1 + \lambda_2)$ and to the second process with probability $1 - p$.

The above point of view is very useful for finding probabilities such as $\Pr\{S_{1k} < S_{2j}\}$ where S_{1k} is the epoch of the kth arrival to the first process and S_{2j} is the epoch of the jth arrival to the second process. The problem can be rephrased in terms of a combined process to ask: out of the first $k+j-1$ arrivals to the combined process, what is the probability that k or more of them are switched to the first process? (Note that if k or more of the first $k+j-1$ go to the first process, at most $j-1$ go to the second, so the kth arrival to the first precedes the jth arrival to the second; similarly if fewer than k of the first $k+j-1$ go to the first process, then the jth arrival to the second process precedes the kth arrival to the first). Since each of these first $k+j-1$ arrivals are switched independently with the same probability p, the answer is

$$\Pr\{S_{1k} < S_{2j}\} = \sum_{i=k}^{k+j-1} \frac{(k+j-1)!}{i!(k+j-1-i)!} p^i (1-p)^{k+j-1-i}. \tag{2.25}$$

As an example of this, suppose a queueing system has arrivals according to a Poisson process (process 1) of rate λ. There is a single server who serves arriving customers in order with a service time distribution $F(y) = 1-\exp[-\mu y]$. Thus during periods when the server is busy, customers leave the system according to a Poisson process (process 2) of rate μ. Thus, if j or more customers are waiting at a given time, then (2.25) gives the probability that the kth subsequent arrival comes before the jth departure.

2.4 Non-homogeneous Poisson processes

The Poisson process, as we defined it, is characterized by a constant arrival rate λ. It is often useful to consider a more general type of process in which the arrival rate varies as a function of time. A *non-homogeneous Poisson process* with time varying arrival rate $\lambda(t)$ is defined[8] as a counting process $\{N(t);\, t \geq 0\}$ which has the independent increment property and, for all $t \geq 0, \delta > 0$, also satisfies:

$$\begin{aligned}
\Pr\Big\{\widetilde{N}(t, t+\delta) = 0\Big\} &= 1 - \delta\lambda(t) + o(\delta) \\
\Pr\Big\{\widetilde{N}(t, t+\delta) = 1\Big\} &= \delta\lambda(t) + o(\delta) \\
\Pr\Big\{\widetilde{N}(t, t+\delta) \geq 2\Big\} &= o(\delta).
\end{aligned} \tag{2.26}$$

where $\widetilde{N}(t, t+\delta) = N(t+\delta) - N(t)$. The non-homogeneous Poisson process does not have the stationary increment property.

One common application occurs in optical communication where a non-homogeneous Poisson process is often used to model the stream of photons from an optical modulator; the modulation is accomplished by varying the photon intensity $\lambda(t)$. We shall see another application shortly in the next example. Sometimes a Poisson process, as we defined it earlier, is called a homogeneous Poisson process.

We can use a "shrinking Bernoulli process" again to approximate a non-homogeneous Poisson process. To see how to do this, assume that $\lambda(t)$ is bounded away from zero. We partition the time axis into increments whose lengths δ vary inversely with $\lambda(t)$, thus holding the probability of an arrival in an increment at some fixed value $q = \delta\lambda(t)$. Thus,

[8] We assume that $\lambda(t)$ is right continuous, i.e., that for each t, $\lambda(t)$ is the limit of $\lambda(t+\varepsilon)$ as ε approaches 0 from above. This allows $\lambda(t)$ to contain discontinuities, as illustrated in Figure 2.7, but follows the convention that the value of the function at the discontinuity is the limiting value from the right. This convention is required in (2.26) to talk about the distribution of arrivals just to the right of time t.

temporarily ignoring the variation of $\lambda(t)$ within an increment,

$$\begin{aligned}
\Pr\left\{\widetilde{N}\left(t, t+\frac{q}{\lambda(t)}\right)=0\right\} &= 1-q+o(q) \\
\Pr\left\{\widetilde{N}\left(t, t+\frac{q}{\lambda(t)}\right)=1\right\} &= q+o(q) \\
\Pr\left\{\widetilde{N}\left(t, t+\frac{q}{\lambda(t)}\right)\geq 2\right\} &= o(\varepsilon).
\end{aligned} \tag{2.27}$$

Ths partition is defined more precisely by defining $m(t)$ as

$$m(t)=\int_0^t \lambda(\tau)d\tau. \tag{2.28}$$

Then the ith increment ends at that t for which $m(t)=qi$.

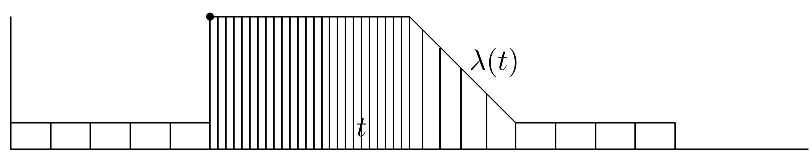


Figure 2.7: Partitioning the time axis into increments each with an expected number of arrivals equal to q. Each rectangle above has the same area, which ensures that the ith partition ends where $m(t)=qi$.

As before, let $\{Y_i;\, i\geq 1\}$ be a sequence of IID binary rv's with $\Pr\{Y_i=1\}=q$ and $\Pr\{Y_i=0\}=1-q$. Consider the counting process $\{N(t);\, t\geq 0\}$ in which Y_i, for each $i\geq 1$, denotes the number of arrivals in the interval $(t_{i-1}, t_i]$, where t_i satisfies $m(t_i)=iq$. Thus, $N(t_i)=Y_1+Y_2+\cdots+Y_i$. If q is decreased as 2^{-j} each increment is successively split into a pair of increments. Thus by the same argument as in (2.21),

$$\Pr\{N(t)=n\}=\frac{[1+o(q)][m(t)]^n\exp[-m(t)]}{n!}. \tag{2.29}$$

Similarly, for any interval $(t,\tau]$, taking $\widetilde{m}(t,\tau)=\int_t^\tau \lambda(u)du$, and taking $t=t_k$, $\tau=t_i$ for some k, i, we get

$$\Pr\left\{\widetilde{N}(t,\tau)=n\right\}=\frac{[1+o(q)][\widetilde{m}(t,\tau)]^n\exp[-\widetilde{m}(t,\tau)]}{n!}. \tag{2.30}$$

Going to the limit $q\to 0$, the counting process $\{N(t);\, t\geq 0\}$ above approaches the non-homogeneous Poisson process under consideration, and we have the following theorem:

Theorem 2.5. *For a non-homogeneous Poisson process with right-continuous arrival rate $\lambda(t)$ bounded away from zero, the distribution of $\widetilde{N}(t,\tau)$, the number of arrivals in $(t,\tau]$, satisfies*

$$\Pr\left\{\widetilde{N}(t,\tau)=n\right\}=\frac{[\widetilde{m}(t,\tau)]^n\exp[-\widetilde{m}(t,\tau)]}{n!} \qquad \text{where } \widetilde{m}(t,\tau)=\int_t^\tau \lambda(u)\,du. \tag{2.31}$$

Hence, one can view a non-homogeneous Poisson process as a (homogeneous) Poisson process over a non-linear time scale. That is, let $\{N^*(s); s \geq 0\}$ be a (homogeneous) Poisson process with rate 1. The non-homogeneous Poisson process is then given by $N(t) = N^*(m(t))$ for each t.

Example 2.4.1 (THE M/G/∞ Queue). Queueing theorists use a standard notation of characters separated by slashes to describe common types of queueing systems. The first character describes the arrival process to the queue. M stands for memoryless and means a Poisson arrival process; D stands for deterministic and means that the interarrival interval is fixed and non-random; G stands for general interarrival distribution. We assume that the interarrival intervals are IID (thus making the arrival process a renewal process), but many authors use GI to explicitly indicate IID interarrivals. The second character describes the service process. The same letters are used, with M indicating the exponential service time distribution. The third character gives the number of servers. It is assumed, when this notation is used, that the service times are IID, independent of the arrival times, and independent of the which server is used.

With this notation, M/G/∞ indicates a queue with Poisson arrivals, a general service distribution, and an infinite number of servers. Similarly, the example at the end of Section 2.3 considered an M/M/1 queue. Since the M/G/∞ queue has an infinite number of servers, no arriving customers are ever queued. Each arrival immediately starts to be served by some server, and the service time Y_i of customer i is IID over i with some distribution function $G(y)$; the service time is the interval from start to completion of service and is also independent of arrival epochs. We would like to find the distribution function of the number of customers being served at a given epoch τ.

Let $\{N(t);\, t \geq 0\}$ be the Poisson counting process of customer arrivals. Consider the arrival times of those customers that are still in service at some fixed time τ. In some arbitrarily small interval $(t, t+\delta]$, the probability of an arrival is $\delta\lambda + o(\delta)$ and the probability of 2 or more arrivals is negligible (i.e., $o(\delta)$). The probability that an arrival occurred in $(t, t+\delta]$ and that that customer is still being served at time $\tau > t$ is then $\delta\lambda[1 - G(\tau - t)] + o(\delta)$. Consider a counting process $\{N_1(t);\ 0{\leq}t{\leq}\tau\}$ where $N_1(t)$ is the number of arrivals between 0 and t that are still in service at τ. This counting process has the independent increment property. To see this, note that the overall arrivals in $\{N(t);\, t \geq 0\}$ have the independent increment property; also the arrivals in $\{N(t);\, t \geq 0\}$ have independent service times, and thus are independently in or not in $\{N_1(t); 0 \leq t < \tau\}$. It follows that $\{N_1(t); 0 \leq t < \tau\}$ is a non-homogeneous Poisson process with rate $\lambda[1 - G(\tau - t)]$ at time $t \leq \tau$. The expected number of arrivals still in service at time τ is then

$$m(\tau) = \lambda \int_{t=0}^{\tau} [1 - G(\tau - t)]\, dt = \lambda \int_{t=0}^{\tau} [1 - G(t)]\, dt. \tag{2.32}$$

and the PMF of the number in service at time τ is given by

$$\Pr\{N_1(\tau) = n\} = \frac{m(\tau)^n \exp(-m(\tau))}{n!}. \tag{2.33}$$

Note that as $\tau \to \infty$, the integral in (2.32) approaches the mean of the service time distribution (i.e., it is the integral of the complementary distribution function, $1 - G(t)$, of the

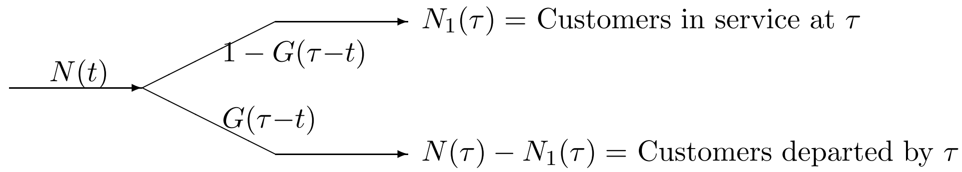


Figure 2.8: Poisson arrivals $\{N(t);\, t \geq 0\}$ can be considered to be split in a non-homogeneous way. An arrival at t is split with probability $1 - G(\tau - t)$ into a process of customers still in service at τ.

service time). This means that in steady-state (as $\tau \to \infty$), the distribution of the number in service at τ depends on the service time distribution only through its mean. This example can be used to model situations such as the number of phone calls taking place at a given epoch. This requires arrivals of new calls to be modeled as a Poisson process and the holding time of each call to be modeled as a random variable independent of other holding times and of call arrival times. Finally, as shown in Figure 2.8, we can regard $\{N_1(t);\ 0{\le}t \leq \tau\}$ as a splitting of the arrival process $\{N(t);\ t{\ge}0\}$. By the same type of argument as in Section 2.3, the number of customers who have completed service by time τ is independent of the number still in service.

2.5 Conditional arrival densities and order statistics

A diverse range of problems involving Poisson processes are best tackled by conditioning on a given number n of arrivals in the interval $(0, t]$, i.e., on the event $N(t) = n$. Because of the incremental view of the Poisson process as independent and stationary arrivals in each incremental interval of the time axis, we would guess that the arrivals should have some sort of uniform distribution given $N(t) = n$. More precisely, the following theorem shows that the joint density of $\boldsymbol{S}^{(n)} = (S_1, S_2, \dots, S_n)$ given $N(t) = n$ is uniform over the region $0 < S_1 < S_2 < \cdots < S_n < t$.

Theorem 2.6. *Let $f_{\boldsymbol{S}^{(n)}|N(t)}(\boldsymbol{s}^{(n)} \mid n)$ be the joint density of $\boldsymbol{S}^{(n)}$ conditional on $N(t) = n$. This density is constant over the region $0 < s_1 < \cdots < s_n < t$ and has the value*

$$f_{\boldsymbol{S}^{(n)}|N(t)}(\boldsymbol{s}^{(n)} \mid n) \;=\; \frac{n!}{t^n}. \tag{2.34}$$

Two proofs are given, each illustrative of useful techniques.

Proof 1: Recall that the joint density of the first $n + 1$ arrivals $\boldsymbol{S}^{n+1} = (S_1 \dots, S_n, S_{n+1}$ with no conditioning is given in (2.14). We first use Bayes law to calculate the joint density of $\boldsymbol{S}^{n+1}$ conditional on $N(t) = n$.

$$f_{\boldsymbol{S}^{(n+1)}|N(t)}(\boldsymbol{s}^{(n+1)} \mid n)\, p_{N(t)}(n) = p_{N(t)|\boldsymbol{S}^{(n+1)}}(n|\boldsymbol{s}^{(n+1)}) f_{\boldsymbol{S}^{(n+1)}}(\boldsymbol{s}^{(n+1)}).$$

Note that $N(t) = n$ if and only if $S_n \leq t$ and $S_{n+1} > t$. Thus $p_{N(t)|\boldsymbol{S}^{(n+1)}}(n|\boldsymbol{s}^{(n+1)})$ is 1 if $S_n \leq t$ and $S_{n+1} > t$ and is 0 otherwise. Restricting attention to the case $N(t) = n$, $S_n \leq t$

and $S_{n+1} > t$,

$$\begin{aligned}
f_{\boldsymbol{S}^{(n+1)}|N(t)}(\boldsymbol{s}^{(n+1)} \mid n) &= \frac{f_{\boldsymbol{S}^{(n+1)}}(\boldsymbol{s}^{(n+1)})}{p_{N(t)}(n)} \\
&= \frac{\lambda^{n+1}\exp(-\lambda s_{n+1})}{(\lambda t)^n \exp(-\lambda t)\,/n!} \\
&= \frac{n!\lambda\exp(-\lambda(s_{n+1}-t)}{t^n}. \qquad (2.35)
\end{aligned}$$

This is a useful expression, but we are interested in $\boldsymbol{S}^{(n)}$ rather than $\boldsymbol{S}^{(n+1)}$. Thus we break up the left side of (2.35) as follows:

$$f_{\boldsymbol{S}^{(n+1)}|N(t)}(\boldsymbol{s}^{(n+1)} \mid n) = f_{\boldsymbol{S}^{(n)}|N(t)}(\boldsymbol{s}^{(n)} \mid n)\; f_{S_{n+1}|\boldsymbol{S}^{(n)}N(t)}(s_{n+1}|\boldsymbol{s}^{(n)}, n).$$

Conditional on $N(t) = n$, S_{n+1} is the first arrival epoch after t, which by the memoryless property is independent of $\boldsymbol{S}^n$. Thus that final term is simply $\lambda\exp(-\lambda(s_{n+1} - t))$ for $s_{n+1} > t$. Substituting this into (2.35), the result is (2.34). $\square$

Proof 2: This alternative proof derives (2.34) by looking at arrivals in very small increments of size δ (see Figure 2.9). For a given t and a given set of n times, $0 < s_1 < \cdots, < s_n < t$, we calculate the probability that there is a single arrival in each of the intervals $(s_i, s_i + \delta], 1 \leq i \leq n$ and no other arrivals in the interval $(0, t]$. Letting $A(\delta)$ be this event,

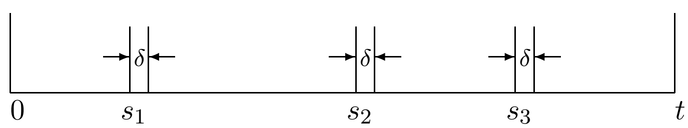


Figure 2.9: Intervals for arrival density.

$$\Pr\{A(\delta)\} = p_{N(s_1)}(0)\; p_{\widetilde{N}(s_1,s_1+\delta)}(1)\; p_{\widetilde{N}(s_1+\delta,s_2)}(0)\; p_{\widetilde{N}(s_2,s_2+\delta)}(1)\cdots p_{\widetilde{N}(s_n+\delta,t)}(0).$$

The sum of the lengths of the above intervals is t, so if we represent $p_{\widetilde{N}(s_i,s_i+\delta)}(1)$ as $\lambda\delta\exp(-\lambda\delta) + o(\delta)$ for each i, then

$$\Pr\{A(\delta)\} = (\lambda\delta)^n \exp(-\lambda t) + \delta^{n-1}o(\delta).$$

The event $A(\delta)$ can be characterized as the event that, first, $N(t) = n$ and, second, that the n arrivals occur in $(s_i, s_i{+}\delta]$ for $1 \leq i \leq n$. Thus we conclude that

$$f_{\boldsymbol{S}^{(n)}|N(t)}(\boldsymbol{s}^{(n)}) = \lim_{\delta\to 0} \frac{\Pr\{A(\delta)\}}{\delta^n p_{N(t)}(n)},$$

which simplifies to (2.34). $\square$

The joint density of the interarrival intervals, $\boldsymbol{X}^{(n)} = X_1 \ldots, X_n$ given $N(t) = n$ can be found directly from Theorem 2.6 simply by making the linear transformation $X_1 = S_1$

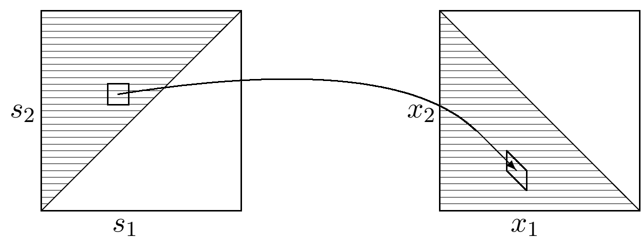


Figure 2.10: Mapping from arrival epochs to interarrival times. Note that incremental cubes in the arrival space map into parallelepipeds of the same volume in the interarrival space.

and $X_i = S_i - S_{i-1}$ for $2 \le i \le n$. The density is unchanged, but the constraint region transforms into $\sum_{i=1}^{n} X_i < t$ with $X_i > 0$ for $1 \le i \le n$ (see Figure 2.10).

$$f_{\boldsymbol{X}^{(n)}|N(t)}(\boldsymbol{x}^{(n)} \mid n) \;=\; \frac{n!}{t^n} \qquad \text{for } \boldsymbol{X}^{(n)} > 0,\ \sum\nolimits_{i=1}^{n} X_i < t. \tag{2.36}$$

It is also instructive to compare the joint distribution of $\boldsymbol{S}^{(n)}$ conditional on $N(t) = n$ with the joint distribution of n IID uniformly distributed random variables, $\boldsymbol{U}^{(n)} = U_1, \ldots, U_n$ on $(0, t]$. For any point $\boldsymbol{U}^{(n)} = \boldsymbol{u}^{(n)}$, this joint density is

$$f_{\boldsymbol{U}^{(n)}}(\boldsymbol{u}^{(n)}) = 1/t^n \text{ for } 0 < u_i \le t,\ 1 \le i \le n.$$

Both $f_{\boldsymbol{S}^{(n)}}$ and $f_{\boldsymbol{U}^{(n)}}$ are uniform over the volume of n-space where they are non-zero, but as illustrated in Figure 2.11 for $n = 2$, the volume for the latter is $n!$ times larger than the volume for the former. To explain this more fully, we can define a set of random variables $S_1, \ldots, S_n$, not as arrival epochs in a Poisson process, but rather as the order statistics function of the IID uniform variables $U_1, \ldots, U_n$; that is

$$S_1 = \min(U_1, \ldots, U_n);\ S_2 = 2^{\text{nd}} \text{ smallest } (U_1, \ldots, U_n);\ \text{etc.}$$

The n-cube is partitioned into $n!$ regions, one where $u_1 < u_2 < \cdots < u_n$. For each permutation $\pi(i)$ of the integers 1 to n, there is another region[9] where $u_{\pi(1)} < u_{\pi(2)} < \cdots < u_{\pi(n)}$. By symmetry, each of these regions has the same volume, which then must be $1/n!$ of the volume t^n of the n-cube.

Each of these $n!$ regions map into the same region of ordered values. Thus these order statistics have the same joint probability density function as the arrival epochs $S_1, \ldots, S_n$ conditional on $N(t) = n$. Thus anything we know (or can discover) about order statistics is valid for arrival epochs given $N(t) = n$ and vice versa.[10]

[9] As usual, we are ignoring those points where $u_i = u_j$ for some i, j, since the set of such points has 0 probability.

[10] There is certainly also the intuitive notion, given n arrivals in $(0, t]$, and given the stationary and independent increment properties of the Poisson process, that those n arrivals can be viewed as uniformly distributed. It does not seem worth the trouble, however, to make this precise, since there is no natural way to associate each arrival with one of the uniform rv's.

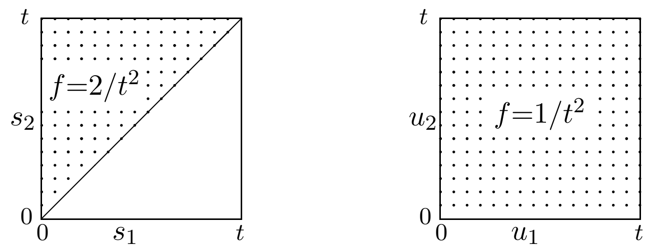


Figure 2.11: Density for the order statistics of an IID 2 dimensional uniform distribution. Note that the square over which $f_{\boldsymbol{U}^{(2)}}$ is non-zero contains one triangle where $u_2 > u_1$ and another of equal size where $u_1 > u_2$. Each of these maps, by a permutation mapping, into the single triangle where $s_2 > s_1$.

Next we want to find the marginal distribution functions of the individual S_i conditional on $N(t) = n$. Starting with S_1, and viewing it as the minimum of the IID uniformly distributed variables $U_1, \dots, U_n$, we recognize that $S_1 > \tau$ if and only if $U_i > \tau$ for all i, $1 \leq i \leq n$. Thus,

$$\Pr\{S_1 > \tau \mid N(t){=}n\} = \left[\frac{t-\tau}{t}\right]^n \qquad \text{for } 0 < \tau \leq t. \tag{2.37}$$

Since this is the complement of the distribution function of S_1, conditional on $N(t) = n$, we can integrate it to get the conditional mean of S_1,

$$\mathsf{E}\left[S_1 \mid N(t){=}n\right] = \frac{t}{n+1}. \tag{2.38}$$

We come back later to the distribution functions of $S_2, \dots, S_n$, and first look at the marginal distributions of the interarrival intervals. Recall from (2.36) that

$$f_{\boldsymbol{X}^{(n)}|N(t)}(\boldsymbol{x}^{(n)} \mid n) \;=\; \frac{n!}{t^n} \qquad \text{for } \boldsymbol{X}^{(n)} > 0,\; \sum\nolimits_{i=1}^n X_i < t. \tag{2.39}$$

The joint density is the same for all points in the constraint region, and the constraint does not distinguish between X_1 to X_n. Thus they must all have the same marginal distribution, and more generally the marginal distribution of any subset of the X_i can depend only on the size of the subset. We have found the distribution of S_1, which is the same as X_1, and thus

$$\Pr\{X_i > \tau \mid N(t){=}n\} \;=\; \left[\frac{t-\tau}{t}\right]^n \qquad \text{for } 1 \leq i \leq n \text{ and } 0 < \tau \leq t. \tag{2.40}$$

$$\mathsf{E}\left[X_i \mid N(t){=}n\right] \;=\; \frac{t}{n+1} \qquad \text{for } 1 \leq i \leq n. \tag{2.41}$$

Next define $X^*_{n+1} = t - S_n$ to be the interval from the largest of the IID variables to t, the right end of the interval. Using (2.39)

$$f_{\boldsymbol{X}^{(n)}|N(t)}(\boldsymbol{x}^{(n)} \mid n) \;=\; \frac{n!}{t^n} \qquad \text{for } \boldsymbol{X}^{(n)} > 0,\; X^*_{n+1} > 0, \sum\nolimits_{i=1}^n X_i + X^*_{n+1} = t.$$

The constraints above are symmetric in $X_1, \dots, X_n, X^*_{n+1}$, and the density of $X_1, \dots, X_n$ within the constrainty region is uniform. This density can be replaced by a density over any other n rv's out of $X_1, \dots, X_n, X^*_{n+1}$ by a linear transformation with unit determinant. Thus X^*_{n+1} has the same marginal distribution as each of the X_i. This gives us a partial check on our work, since the interval $(0, t]$ is divided into $n+1$ intervals of sizes $X_1, X_2, \dots, X_n, X^*_{n+1}$, and each of these has a mean size $t/(n+1)$. We also see that the joint distribution function of any proper subset of $X_1, X_2, \dots X_n, X^*_{n+1}$ is independent of the order of the variables.

Next consider the distribution function of X_{i+1} for $(i < n)$, conditional both on $N(t) = n$ and $S_i = s_i$ (or conditional on any given values for $X_1, \dots, X_i$ summing to s_i). We see that X_{i+1} is just the wait until the first arrival in the interval $(s_i, t]$, given that this interval contains $n - i$ arrivals. From the same argument as used in (2.37), we have

$$\Pr\{X_{i+1} > \tau \mid N(t){=}n,\, S_i{=}s_i\} = \left[\frac{t - s_i - \tau}{t - s_i}\right]^{n-i}. \tag{2.42}$$

Since S_{i+1} is $X_{i+1} + S_i$, this immediately gives us the conditional distribution of S_{i+1}

$$\Pr\{S_{i+1} > s_{i+1} \mid N(t) = n, S_i = s_i\} = \left[\frac{t - s_{i+1}}{t - s_i}\right]^{n-i}. \tag{2.43}$$

We note that this is independent of $S_1, \dots, S_{i-1}$. As a check, one can find the conditional densities from (2.43) and multiply them all together to get back to (2.34) (see Exercise 2.25).

W can also find the distribution of each S_i conditioned on $N(t) = n$ but unconditioned on $S_1, S_2, \dots, S_{i-1}$. The density for this is calculated by looking at n uniformly distributed rv's in $(0, t]$. The probability that one of these lies in the interval $(x, x+dt]$ is $(n\,dt)/t$. Out of the remaining $n - 1$, the probability that $i - 1$ lie in the interval $(0, x]$ is given by the binomial distribution with probability of success x/t. Thus the desired density is

$$\begin{aligned} f_{S_i}(x \mid N(t){=}n)\ dt &= \frac{x^{i-1}(t-x)^{n-i}(n-1)!}{t^{n-1}(n-i)!(i-1)!} \quad \frac{n\,dt}{t} \\ f_{S_i}(x \mid N(t){=}n) &= \frac{x^{i-1}(t-x)^{n-i}n!}{t^n(n-i)!(i-1)!}. \end{aligned} \tag{2.44}$$

2.6 Summary

We started the chapter with three equivalent definitions of a Poisson process—first as a renewal process with exponentially distributed inter-renewal intervals, second as a stationary and independent increment counting process with Poisson distributed arrivals in each interval, and third essentially as a limit of shrinking Bernoulli processes. We saw that each definition provided its own insights into the properties of the process. We emphasized the importance of the memoryless property of the exponential distribution, both as a useful tool in problem solving and as an underlying reason why the Poisson process is so simple.

We next showed that the sum of independent Poisson processes is again a Poisson process. We also showed that if the arrivals in a Poisson process are independently routed to different locations with some fixed probability assignment, then the arrivals at each of these locations form independent Poisson processes. This ability to view independent Poisson processes either independently or as a splitting of a combined process is a powerful technique for finding almost trivial solutions to many problems.

It was next shown that a non-homogeneous Poisson process could be viewed as a (homogeneous) Poisson process on a non-linear time scale. This allows all the properties of (homogeneous) Poisson properties to be applied directly to the non-homogeneous case. The simplest and most useful result from this is (2.31), showing that the number of arrivals in any interval has a Poisson PMF. This result was used to show that the number of customers in service at any given time τ in an M/G/∞ queue has a Poisson PMF with a mean approaching λ times the expected service time in the limit as $\tau \to \infty$.

Finally we looked at the distribution of arrivals conditional on n arrivals in the interval $(0, t]$. It was found that these arrivals had the same joint distribution as the order statistics of n uniform IID rv's in $(0, t]$. By using symmetry and going back and forth between the uniform variables and the Poisson process arrivals, we found the distribution of the interarrival times, the arrival epochs, and various conditional distributions.

2.7 Exercises

Exercise 2.1. a) Find the Erlang density $f_{S_n}(t)$ by convolving $f_X(x) = \lambda \exp(-\lambda x)$ with itself n times.

b) Find the moment generating function of X (or find the Laplace transform of $f_X(x)$), and use this to find the moment generating function (or Laplace transform) of $S_n = X_1 + X_2 + \cdots + X_n$. Invert your result to find $f_{S_n}(t)$.

c) Find the Erlang density by starting with (2.14) and then calculating the marginal density for S_n.

Exercise 2.2. a) Find the mean, variance, and moment generating function of $N(t)$, as given by (2.15).

b) Show by discrete convolution that the sum of two independent Poisson rv's is again Poisson.

c) Show by using the properties of the Poisson process that the sum of two independent Poisson rv's must be Poisson.

Exercise 2.3. The purpose of this exercise is to give an alternate derivation of the Poisson distribution for $N(t)$, the number of arrivals in a Poisson process up to time t; let λ be the rate of the process.

a) Find the conditional probability $\Pr\{N(t) = n \mid S_n = \tau\}$ for all $\tau \leq t$.

b) Using the Erlang density for S_n, use (a) to find $\Pr\{N(t) = n\}$.

Exercise 2.4. Assume that a counting process $\{N(t); t\geq 0\}$ has the independent and stationary increment properties and satisfies (2.15) (for all $t > 0$). Let X_1 be the epoch of the first arrival and X_n be the interarrival time between the $n-1^{\text{st}}$ and the nth arrival.

a) Show that $\Pr\{X_1 > x\} = e^{-\lambda x}$.

b) Let S_{n-1} be the epoch of the $n-1^{\text{st}}$ arrival. Show that $\Pr\{X_n > x \mid S_{n-1} = \tau\} = e^{-\lambda x}$.

c) For each $n > 1$, show that $\Pr\{X_n > x\} = e^{-\lambda x}$ and that X_n is independent of S_{n-1}.

d) Argue that X_n is independent of $X_1, X_2, \dots X_{n-1}$.

Exercise 2.5. The point of this exercise is to show that the sequence of PMF's for the counting process of a Bernoulli process does not specify the process. In other words, knowing that $N(t)$ satisfies the binomial distribution for all t does not mean that the process is Bernoulli. This helps us understand why the second definition of a Poisson process requires stationary and independent increments as well as the Poisson distribution for $N(t)$.

a) For a sequence of binary rv's $Y_1, Y_2, Y_3, \dots,$ in which each rv is 0 or 1 with equal probability, find a joint distribution for Y_1, Y_2, Y_3 that satisfies the binomial distribution, $p_{N(t)}(k) = \binom{t}{k}2^{-k}$ for $t = 1, 2, 3$ and $0 \leq k \leq$, but for which Y_1, Y_2, Y_3 are not independent.

Your solution should contain four 3-tuples with probability 1/8 each, two 3-tuples with probability 1/4 each, and two 3-tuples with probability 0. Note that by making the subsequent arrivals IID and equiprobable, you have an example where $N(t)$ is binomial for all t but the process is not Bernoulli. Hint: Use the binomial for $t = 3$ to find two 3-tuples that must have probability 1/8. Combine this with the binomial for $t = 2$ to find two other 3-tuples with probability 1/8. Finally look at the constraints imposed by the binomial distribution on the remaining four 3-tuples.

b) Generalize part a) to the case where Y_1, Y_2, Y_3 satisfy $\Pr\{Y_i = 1\} = q$ and $\Pr\{Y_i = 0\} = 1 - q$. Assume $q < 1/2$ and find a joint distribution on Y_1, Y_2, Y_3 that satisfies the binomial distribution, but for which the 3-tuple $(0, 1, 1)$ has zero probability.

c) More generally yet, view a joint PMF on binary t-tuples as a non-negative vector in a 2^t dimensional vector space. Each binomial probability $p_{N(\tau)}(k) = \binom{\tau}{k}q^k(1-q)^{\tau-k}$ constitutes a linear constraint on this vector. For each τ, show that one of these constraints may be replaced by the constraint that the components of the vector sum to 1.

d) Using part c), show that at most $(t + 1)t/2 + 1$ of the binomial constraints are linearly independent. Note that this means that the linear space of vectors satisfying these binomial constraints has dimension at least $2^t - (t + 1)t/2 - 1$. This linear space has dimension 1 for $t = 3$ explaining the results in parts a) and b). It has a rapidly increasing dimensional for $t > 3$, suggesting that the binomial constraints are relatively ineffectual for constraining the joint PMF of a joint distribution. More work is required for the case of $t > 3$ because of all the inequality constraints, but it turns out that this large dimensionality remains.

Exercise 2.6. Let $h(x)$ be a positive function of a real variable that satisfies $h(x + t) = h(x) + h(t)$ and let $h(1) = c$.

a) Show that for integer $k > 0$, $h(k) = kc$.

b) Show that for integer $j > 0$, $h(1/j) = c/j$.

c) Show that for all integer k, j, $h(k/j) = ck/j$.

d) The above parts show that $h(x)$ is linear in positive *rational* numbers. For very picky mathematicians, this does not guarantee that $h(x)$ is linear in positive *real* numbers. Show that if $h(x)$ is also monotonic in x, then $h(x)$ is linear in $x > 0$.

Exercise 2.7. Assume that a counting process $\{N(t); t{\geq}0\}$ has the independent and stationary increment properties and, for all $t > 0$, satisfies

$$\begin{aligned}
\Pr\left\{\widetilde{N}(t, t + \delta) = 0\right\} &= 1 - \lambda\delta + o(\delta) \\
\Pr\left\{\widetilde{N}(t, t + \delta) = 1\right\} &= \lambda\delta + o(\delta) \\
\Pr\left\{\widetilde{N}(t, t + \delta) > 1\right\} &= o(\delta).
\end{aligned}$$

a) Let $F_0(\tau) = \Pr\{N(\tau) = 0\}$ and show that $dF_0(\tau)/d\tau = -\lambda F_0(\tau)$.

b) Show that X_1, the time of the first arrival, is exponential with parameter λ.

c) Let $F_n(\tau) = \Pr\left\{\widetilde{N}(t, t + \tau) = 0 \mid S_{n-1} = t\right\}$ and show that $dF_n(\tau)/d\tau = -\lambda F_n(\tau)$.

d) Argue that X_n is exponential with parameter λ and independent of earlier arrival times.

Exercise 2.8. Let $t > 0$ be an arbitrary time, let Z_1 be the duration of the interval from t until the next arrival after t. Let Z_m, for each $m > 1$, be the interarrival time from the epoch of the $m - 1^{\text{st}}$ arrival after t until the mth arrival.

a) Given that $N(t) = n$, explain why $Z_m = X_{m+n}$ for $m > 1$ and $Z_1 = X_{n+1} - t + S_n$.

b) Conditional on $N(t) = n$ and $S_n = \tau$, show that $Z_1, Z_2, \ldots$ are IID.

c) Show that $Z_1, Z_2, \ldots$ are IID.

Exercise 2.9. Consider a "shrinking Bernoulli" approximation $N_\delta(m\delta) = Y_1 + \cdots + Y_m$ to a Poisson process as described in Subsection 2.2.5.

a) Show that

$$\Pr\{N_\delta(m\delta) = n\} = \binom{m}{n} (\lambda\delta)^n (1 - \lambda\delta)^{m-n}.$$

b) Let $t = m\delta$, and let t be fixed for the remainder of the exercise. Explain why

$$\lim_{\delta \to 0} \Pr\{N_\delta(t) = n\} = \lim_{m \to \infty} \binom{m}{n} \left(\frac{\lambda t}{m}\right)^n \left(1 - \frac{\lambda t}{m}\right)^{m-n}.$$

where the limit on the left is taken over values of δ that divide t.

c) Derive the following two equalities:

$$\lim_{m\to\infty} \binom{m}{n} \frac{1}{m^n} = \frac{1}{n!}; \qquad \text{and} \qquad \lim_{m\to\infty} \left(1 - \frac{\lambda t}{m}\right)^{m-n} = e^{-\lambda t}.$$

d) Conclude from this that for every t and every n, $\lim_{\delta\to 0} \Pr\{N_\delta(t){=}n\} = \Pr\{N(t){=}n\}$ where $\{N(t);\, t \geq 0\}$ is a Poisson process of rate λ.

Exercise 2.10. Let $\{N(t);\, t \geq 0\}$ be a Poisson process of rate λ.

a) Find the joint probability mass function (PMF) of $N(t)$, $N(t+s)$ for $s > 0$.

b) Find $\mathsf{E}\left[N(t) \cdot N(t+s)\right]$ for $s > 0$.

c) Find $\mathsf{E}\left[\widetilde{N}(t_1, t_3) \cdot \widetilde{N}(t_2, t_4)\right]$ where $\widetilde{N}(t, \tau)$ is the number of arrivals in $(t, \tau]$ and $t_1 < t_2 < t_3 < t_4$.

Exercise 2.11. An experiment is independently performed N times where N is a Poisson rv of mean λ. Let $\{a_1, a_2, \dots, a_K\}$ be the set of sample points of the experiment and let p_k, $1 \leq k \leq K$, denote the probability of a_k.

a) Let N_i denote the number of experiments performed for which the output is a_i. Find the PMF for N_i $(1 \leq i \leq K)$. (Hint: no calculation is necessary.)

b) Find the PMF for $N_1 + N_2$.

c) Find the conditional PMF for N_1 given that $N = n$.

d) Find the conditional PMF for $N_1 + N_2$ given that $N = n$.

e) Find the conditional PMF for N given that $N_1 = n_1$.

Exercise 2.12. Starting from time 0, northbound buses arrive at 77 Mass. Avenue according to a Poisson process of rate λ. Passengers arrive according to an independent Poisson process of rate μ. When a bus arrives, all waiting customers instantly enter the bus and subsequent customers wait for the next bus.

a) Find the PMF for the number of customers entering a bus (more specifically, for any given m, find the PMF for the number of customers entering the mth bus).

b) Find the PMF for the number of customers entering the mth bus given that the interarrival interval between bus $m-1$ and bus m is x.

c) Given that a bus arrives at time 10:30 PM, find the PMF for the number of customers entering the next bus.

d) Given that a bus arrives at 10:30 PM and no bus arrives between 10:30 and 11, find the PMF for the number of customers on the next bus.

e) Find the PMF for the number of customers waiting at some given time, say 2:30 PM (assume that the processes started infinitely far in the past). Hint: think of what happens moving backward in time from 2:30 PM.

f) Find the PMF for the number of customers getting on the next bus to arrive after 2:30. (Hint: this is different from part (a); look carefully at part e).

g) Given that I arrive to wait for a bus at 2:30 PM, find the PMF for the number of customers getting on the next bus.

Exercise 2.13. a) Show that the arrival epochs of a Poisson process satisfy

$$f_{\boldsymbol{S}^{(n)}|S_{n+1}}(\boldsymbol{s}^{(n)}|s_{n+1}) = n!/s_{n+1}^n.$$

Hint: This is easy if you use only the results of Section 2.2.2.

b) Contrast this with the result of Theorem 2.6

Exercise 2.14. Equation (2.44) gives $f_{S_i}(x \mid N(t){=}n)$, which is the density of random variable S_i conditional on $N(t) = n$ for $n \geq i$. Multiply this expression by $\Pr\{N(t) = n\}$ and sum over n to find $f_{S_i}(x)$; verify that your answer is indeed the Erlang density.

Exercise 2.15. Consider generalizing the bulk arrival process in Figure 2.5. Assume that the epochs at which arrivals occur form a Poisson process $\{N(t);\, t \geq 0\}$ of rate λ. At each arrival epoch, S_n, the number of arrivals, Z_n, satisfies $\Pr\{Z_n{=}1\} = p$, $\Pr\{Z_n{=}2)\} = 1-p$. The variables Z_n are IID.

a) Let $\{N_1(t); t \geq 0\}$ be the counting process of the epochs at which single arrivals occur. Find the PMF of $N_1(t)$ as a function of t. Similarly, let $\{N_2(t); t \geq 0\}$ be the counting process of the epochs at which double arrivals occur. Find the PMF of $N_2(t)$ as a function of t.

b) Let $\{N_B(t); t \geq 0\}$ be the counting process of the total number of arrivals. Give an expression for the PMF of $N_B(t)$ as a function of t.

Exercise 2.16. a) For a Poisson counting process of rate λ, find the joint probability density of $S_1, S_2, \dots, S_{n-1}$ conditional on $S_n = t$.

b) Find $\Pr\{X_1 > \tau \mid S_n{=}t\}$.

c) Find $\Pr\{X_i > \tau \mid S_n{=}t\}$ for $1 \leq i \leq n$.

d) Find the density $f_{S_i}(x|S_n{=}t)$ for $1 \leq i \leq n-1$.

e) Give an explanation for the striking similarity between the condition $N(t) = n-1$ and the condition $S_n = t$.

Exercise 2.17. a) For a Poisson process of rate λ, find $\Pr\{N(t)=n \mid S_1=\tau\}$ for $t > \tau$ and $n \geq 1$.

b) Using this, find $f_{S_1}(\tau \mid N(t)=n)$

c) Check your answer against (2.37).

Exercise 2.18. Consider a counting process in which the rate is a rv Λ with probability density $f_\Lambda(\lambda) = \alpha e^{-\alpha\lambda}$ for $\lambda > 0$. Conditional on a given sample value λ for the rate, the counting process is a Poisson process of rate λ (i.e., nature first chooses a sample value λ and then generates a sample function of a Poisson process of that rate λ).

a) What is $\Pr\{N(t)=n \mid \Lambda=\lambda\}$, where $N(t)$ is the number of arrivals in the interval $(0, t]$ for some given $t > 0$?

b) Show that $\Pr\{N(t)=n\}$, the unconditional PMF for $N(t)$, is given by

$$\Pr\{N(t)=n\} = \frac{\alpha t^n}{(t+\alpha)^{n+1}}.$$

c) Find $f_\Lambda(\lambda \mid N(t)=n)$, the density of λ conditional on $N(t)=n$.

d) Find $\mathsf{E}[\Lambda \mid N(t)=n]$ and interpret your result for very small t with $n = 0$ and for very large t with n large.

e) Find $\mathsf{E}[\Lambda \mid N(t)=n, S_1, S_2, \ldots, S_n]$. (Hint: consider the distribution of $S_1, \ldots, S_n$ conditional on $N(t)$ and Λ). Find $\mathsf{E}[\Lambda \mid N(t)=n, N(\tau)=m]$ for some $\tau < t$.

Exercise 2.19. a) Use Equation (2.44) to find $\mathsf{E}[S_i \mid N(t)=n]$. Hint: When you integrate $x f_{S_i}(x \mid N(t)=n)$, compare this integral with $f_{S_{i+1}}(x \mid N(t)=n+1)$ and use the fact that the latter expression is a probability density.

b) Find the second moment and the variance of S_i conditional on $N(t)=n$. Hint: Extend the previous hint.

c) Assume that n is odd, and consider $i = (n+1)/2$. What is the relationship between S_i, conditional on $N(t)=n$, and the sample median of n IID uniform random variables.

d) Give a weak law of large numbers for the above median.

Exercise 2.20. Suppose cars enter a one-way infinite length, infinite lane highway at a Poisson rate λ. The ith car to enter chooses a velocity V_i and travels at this velocity. Assume that the V_i's are independent positive rv's having a common distribution F. Derive the distribution of the number of cars that are located in an interval $(0, a)$ at time t.

Exercise 2.21. Consider an M/G/∞ queue, i.e., a queue with Poisson arrivals of rate λ in which each arrival i, independent of other arrivals, remains in the system for a time X_i, where $\{X_i; i \geq 1\}$ is a set of IID rv's with some given distribution function $F(x)$.

You may assume that the number of arrivals in any interval $(t, t+\varepsilon)$ that are still in the system at some later time $\tau \geq t+\varepsilon$ is statistically independent of the number of arrivals in that same interval $(t, t+\varepsilon)$ that have departed from the system by time τ.

a) Let $N(\tau)$ be the number of customers in the system at time τ. Find the mean, $m(\tau)$, of $N(\tau)$ and find $\Pr\{N(\tau) = n\}$.

b) Let $D(\tau)$ be the number of customers that have departed from the system by time τ. Find the mean, $\mathsf{E}[D(\tau)]$, and find $\Pr\{D(\tau) = d\}$.

c) Find $\Pr\{N(\tau) = n, D(\tau) = d\}$.

d) Let $A(\tau)$ be the total number of arrivals up to time τ. Find $\Pr\{N(\tau) = n \mid A(\tau) = a\}$.

e) Find $\Pr\{D(\tau+\varepsilon) - D(\tau) = d\}$.

Exercise 2.22. The voters in a given town arrive at the place of voting according to a Poisson process of rate $\lambda = 100$ voters per hour. The voters independently vote for candidate A and candidate B each with probability 1/2. Assume that the voting starts at time 0 and continues indefinitely.

a) Conditional on 1000 voters arriving during the first 10 hours of voting, find the probability that candidate A receives n of those votes.

b) Again conditional on 1000 voters during the first 10 hours, find the probability that candidate A receives n votes in the first 4 hours of voting.

c) Let T be the epoch of the arrival of the first voter voting for candidate A. Find the density of T.

d) Find the PMF of the number of voters for candidate B who arrive before the first voter for A.

e) Define the nth voter as a *reversal* if the nth voter votes for a different candidate than the $n-1^{st}$. For example, in the sequence of votes $AABAABB$, the third, fourth, and sixth voters are reversals; the third and sixth are A to B reversals and the fourth is a B to A reversal. Let $N(t)$ be the number of reversals up to time t (t in hours). Is $\{N(t);\, t \geq 0\}$ a Poisson process? Explain.

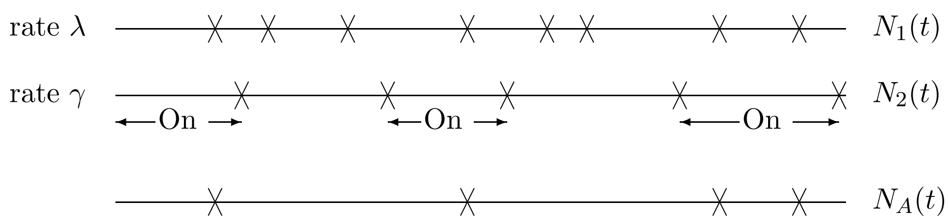
f) Find the expected time (in hours) between reversals.

g) Find the probability density of the time between reversals.

h) Find the density of the time from one A to B reversal to the next A to B reversal.

Exercise 2.23. Let $\{N_1(t); t \geq 0\}$ be a Poisson counting process of rate λ. Assume that the arrivals from this process are switched on and off by arrivals from a second independent Poisson process $\{N_2(t); t \geq 0\}$ of rate γ.

Let $\{N_A(t); t{\geq}0\}$ be the switched process; that is $N_A(t)$ includes the arrivals from $\{N_1(t); t \geq 0\}$ during periods when $N_2(t)$ is even and excludes the arrivals from $\{N_1(t); t \geq 0\}$ while $N_2(t)$ is odd.



a) Find the PMF for the number of arrivals of the first process, $\{N_1(t); t \geq 0\}$, during the nth period when the switch is on.

b) Given that the first arrival for the second process occurs at epoch τ, find the conditional PMF for the number of arrivals of the first process up to τ.

c) Given that the number of arrivals of the first process, up to the first arrival for the second process, is n, find the density for the epoch of the first arrival from the second process.

d) Find the density of the interarrival time for $\{N_A(t); t \geq 0\}$. Note: This part is quite messy and is done most easily via Laplace transforms.

Exercise 2.24. Let us model the chess tournament between Fisher and Spassky as a stochastic process. Let X_i, for $i \geq 1$, be the duration of the ith game and assume that $\{X_i;\ i{\geq}1\}$ is a set of IID exponentially distributed rv's each with density $f(x) = \lambda e^{-\lambda x}$. Suppose that each game (independently of all other games, and independently of the length of the games) is won by Fisher with probability p, by Spassky with probability q, and is a draw with probability $1-p-q$. The first player to win n games is defined to be the winner, but we consider the match up to the point of winning as being embedded in an unending sequence of games.

a) Find the distribution of time, from the beginning of the match, until the completion of the first game that is won (i.e., that is not a draw). Characterize the process of the number $\{N(t);\ t \geq 0\}$ of games won up to and including time t. Characterize the process of the number $\{N_F(t); t \geq 0\}$ of games won by Fisher and the number $\{N_S(t); t \geq 0\}$ won by Spassky.

b) For the remainder of the problem, assume that the probability of a draw is zero; i.e., that $p+q = 1$. How many of the first $2n-1$ games must be won by Fisher in order to win the match?

c) What is the probability that Fisher wins the match? Your answer should not involve any integrals. Hint: consider the unending sequence of games and use part b.

d) Let T be the epoch at which the match is completed (i.e., either Fisher or Spassky wins). Find the distribution function of T.

e) Find the probability that Fisher wins and that T lies in the interval $(t, t+\delta)$ for arbitrarily small δ.

Exercise 2.25. Using (2.43), find the conditional density of S_{i+1}, conditional on $N(t) = n$ and $S_i = s_i$, and use this to find the joint density of $S_1, \dots, S_n$ conditional on $N(t) = n$. Verify that your answer agrees with (2.34).

Exercise 2.26. A two-dimensional Poisson process is a process of randomly occurring special points in the plane such that (i) for any region of area A the number of special points in that region has a Poisson distribution with mean λA, and (ii) the number of special points in nonoverlapping regions is independent. For such a process consider an arbitrary location in the plane and let X denote its distance from its nearest special point (where distance is measured in the usual Euclidean manner). Show that

a) $\Pr\{X > t\} = \exp(-\lambda \pi t^2)$

b) $\mathsf{E}[X] = 1/(2\sqrt{\lambda})$.

Exercise 2.27. This problem is intended to show that one can analyze the long term behavior of queueing problems by using just notions of means and variances, but that such analysis is awkward, justifying understanding the strong law of large numbers. Consider an M/G/1 queue. The arrival process is Poisson with $\lambda = 1$. The expected service time, $\mathsf{E}[Y]$, is 1/2 and the variance of the service time is given to be 1.

a) Consider S_n, the time of the nth arrival, for $n = 10^{12}$. With high probability, S_n will lie within 3 standard derivations of its mean. Find and compare this mean and the 3σ range.

b) Let V_n be the total amount of time during which the server is busy with these n arrivals (i.e., the sum of 10^{12} service times). Find the mean and 3σ range of V_n.

c) Find the mean and 3σ range of I_n, the total amount of time the server is idle up until S_n (take I_n as $S_n - V_n$, thus ignoring any service time after S_n).

d) An idle period starts when the server completes a service and there are no waiting arrivals; it ends on the next arrival. Find the mean and variance of an idle period. Are successive idle periods IID?

e) Combine (c) and (d) to estimate the total number of idle periods up to time S_n. Use this to estimate the total number of busy periods.

f) Combine (e) and (b) to estimate the expected length of a busy period.

Exercise 2.28. The purpose of this problem is to illustrate that for an arrival process with independent but not identically distributed interarrival intervals, $X_1, X_2, \dots$,, the number of arrivals $N(t)$ in the interval $(0, t]$ can be a defective rv. In other words, the 'counting process' is not a stochastic process according to our definitions. This illustrates that it is necessary to prove that the counting rv's for a renewal process are actually rv's .

a) Let the distribution function of the ith interarrival interval for an arrival process be $F_{X_i}(x_i) = 1 - \exp(-\alpha^i x_i)$ for some fixed $\alpha \in (0, 1)$. Let $S_n = X_1 + \cdots X_n$ and show that

$$\mathsf{E}[S_n] = \frac{1 - \alpha^{n-1}}{1 - \alpha}.$$

b) Sketch a 'reasonable' sample function for $N(t)$.

c) Find $\sigma^2_{S_n}$.

d) Use the Chebyshev inequality on $\Pr\{S_n \geq t\}$ to find an upper bound on $\Pr\{N(t) \leq n\}$ that is smaller than 1 for all n and for large enough t. Use this to show that $N(t)$ is defective for large enough t.