Ruin Theory:
An Application of Stochastic Process

Stochastic Process: Second Project

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Abstract

The project was originally a brief study of example 5.13 in *Rick Durrett*, Essentials of Stochastic Processes, but the assumption of this problem was somehow weird. I tried to extend the model and found that it is an interesting application of tools we taught in class, which plays a big rule in insurance pricing. *Risk theory*, which studies the ruin probability of an insurer with given initial reserve and some kind of premiums and claims types. And once we get the ruin probability of this insurance, insures can quantify the risk they took to decide whether they issue the insurance or come up with another income type.

In this project, I will first give the formal definition of risk theory and terminology of Cramèr-Lundberg Model. Then, I will compute the ruin probability in infinite time horizon. Next, I will illustrate the connection between queuing theory and ruin theory. Finally, I will extend the model to have more general assumption via stochastic integral, which has a special name called perturbed risk process. And do some simulation of Cramèr-Lundberg Model model.

Keywords: Ruin theory, Queuing theory, Poisson Process, Laplace Transform, Convolution, Stochastic Integral, Pollaczek-Khinchin formula

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1 Ruin Theory

Ruin theory focuses on how to compute the ruin probability, that is to say, the insurer has how much chance to lose all of his initial reserve before certain time point under pre-defined structure of income and outcome flows. We will first study the money flow structure providing by Durrett in example 5.13 and fix it to the well-known Cramèr Lundberg model.

1.1 Original question

Let S_n be the total total assets of the insurer at year n and C be the fixed amount of premiums (money earns from insuree) in every year and Y_i be the amount of claims (money pays to insuree) in ith year. Then, we have

$$S_n = S_{n-1} + C - Y_n$$

which is the insurer's asset at the end of year i. We further let

$$X_n = C - Y_n$$

where $X_1, X_2, ..., X_n$ be the deficit of insurer in every year. We assume $\{X_n\}$ are i.i.d. following $\mathrm{N}(\mu, \sigma^2)$ and S_0 is the initial reserve greater than zero.

We want to show that the probability of event B, which the insurer becomes insolvency at some time n has an exponential bound

$$P(B) \leq e^{-2\mu S_0/\sigma^2}$$

Proof. Let T=inf{n: $S_n \leq 0$} which is the stopping time of the insurer first reaches bankrupt. We find that the event B is actually {T<∞} so we try to construct a martingale to study the behaviour of stopping time.

Theorem 1. *Consider a random walk $\{S_n\}$, $S_n = \sum_{i=1}^{n} X_i$ and $\hat{m}_X(s)$ is finite for some $s \in R$, then $\{M_n\}$ given by $M_n = e^{sS_n}(\hat{m}_X(s))^{-n}$ is an $\{\mathcal{F}_n^X\}$ martingale.*

Proof.

$$\begin{aligned} E[M_{n+1}|\mathcal{F}_n^X] &= \frac{E(e^{sS_n}e^{X_{n+1}}|\mathcal{F}_n^X)}{\hat{m}_X(s)^{n+1}} = \frac{e^{sS_n}E(e^{X_{n+1}}|\mathcal{F}_n^X)}{\hat{m}_X(s)^{n+1}} \\ &= \frac{e^{sS_n}E(e^{X_{n+1}})}{\hat{m}_X(s)^{n+1}} = \frac{e^{sS_n}}{\hat{m}_X(s)^n} = M_n \end{aligned}$$

□

We now know that $M_n = e^{sS_n}(\hat{m}_X(s))^{-n}$ is an $\{\mathcal{F}_n^X\}$ martingale. If we further let $\hat{m}_X(s) = 1$ we will have a martingale $e^{\gamma S_n}$ which looks like the right-hand side of desired result and γ is the solution of equation $\hat{m}_X(s) = 1$. Since m.g.f of normal distribution is

$$\hat{m}_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$$

We can find that γ equals $-2\mu/\sigma^2$ and $e^{-2\mu/\sigma^2 S_n}$ is a martingale so

$$\begin{aligned} E[M_0] &= E[M_{T\wedge n}] = E[e^{-2\mu S_{T\wedge n}/\sigma^2}] \\ &= E[M_T; T \le n] + E[M_n; T > n] \\ &\ge e^{-2\mu S_T/\sigma^2} \times P(T \le n) \\ &\ge P(T \le n) \end{aligned}$$

By *monotone convergence theorem*,

$$P(B) = P(T < \infty) = \lim_{n\to\infty} P(T \le n) \le E[M_0] = e^{-2\mu S_0/\sigma^2}$$

□

The result implies that the failure probability decreases exponentially fast as initial reserve grows linearly which does not violate the intuition that you will have more chance to win if you have more money at first. But the assumption of deficit process $\{X_n\}$ being $\mathrm{N}(\mu, \sigma^2)$ will make the claim process $\{Y_n\}$ becomes $\mathrm{N}(C-\mu, \sigma^2)$ which can take values in R^- which violates the definition of claim. We should change the underlying distribution function of deficit process or come up with a more general way to compute the ruin probability besides setting them normal.

1.2 Risk processes

To give a more general model, we will first rewrite the situation of insurer faced in more explicit way and give definitions of some important terminology.

1.2.1 Definition of the model

To define a risk process, we have to give the time and size at which the claims occurred and the rate that the insurer collects premiums.

Definition. *Risk model*

- random epochs $\sigma_1, \sigma_2, ...$with $0 = \sigma_0 < \sigma_1 < \sigma_2$...at which the claims occur, where the random variables σ_n can be discrete or continuous.
- the corresponding inter-arrival times $\{T_n\}$ and claim sizes $\{U_n\}$ where T_i be the time difference between two claims σ_{i-1} , σ_i and U_i be the amount of money that insure had to pay when the claims occurred at σ_i. U_i is non-negative.
- the initial reserve of the insurer is u $\geq$ 0.
- B(t) is the total amount of the premiums up to time t, which influenced by the risk insurer took.
- We ignore all expenses and other influences.

To study the behaviour of the risk model at time n, we first need to find out the amount of arriving claims in the interval (0,t], and it is a *counting process* {N(t), t$\geq$ 0} given by

$$N(t) = \sum_{k=1}^{\infty} \mathbb{1}(\sigma_k \leq t)$$

so the aggregate money that insurer had to pay in (0,t] forms the *cumulative arrival process* {X(t), t$\geq$ 0}

$$X(t) = \sum_{k=1}^{\infty} U_k \mathbb{1}(\sigma_k \leq t) = \sum_{k=1}^{N(t)} U_k \ \ (\sum_{k=1}^{0} \stackrel{def}{=} 0)$$

After figuring out the behaviour of claims, it is easy to define the *risk reserve process* {R(t), t$\geq$ 0} which represents the money insurer has at time t

$$R(t) = u + B(t) - \sum_{k=1}^{N(t)} U_i$$

while the *claim surplus process* {S(t), t$\geq$ 0} represents the total deficit till time t

$$S(t) = \sum_{k=1}^{N(t)} U_i - B(t)$$

.

Observation. *time of ruin $\tau(u) = inf\{t : R(t) < 0\} = inf\{t : S(t) > u\}$ is the first epoch when the risk reserve process becomes negative or,equivalently, when the claim surplus process cross the level u.*

Finally, we can define the probability that we interested in.

Definition. *Ruin probability*

- *finite-horizon ruin probability* $\varphi(u;x) = P(\tau(u) \leq x)$.
- *infinite-horizon ruin probability or ultimate ruin* $\varphi(u) = \lim_{x\to\infty} \varphi(u;x) = P(\tau(u) < \infty)$.
- *survival probability* $\bar{\varphi}(u) = 1 - \varphi(u)$.

1.2.2 Compound Distribution

The structure of X(t)=$\sum_{k=1}^{N(t)} U_i$ has a special name called *compound process* which is the major cornerstone of ruin theory.

Definition. *Compound distributions*
Let N be a non-negative integer-valued random variable and $U_1, U_2, ...$ a sequence of non-negative random variables. Then the random variable

$$X = \begin{cases} \sum_{i=1}^{N} U_i & if N \geq 1, \\ 0 & if N = 0, \end{cases} \tag{1}$$

is called the compound distribution. We further assume $N, U_1, U_2, ...$ are independent and $U_1, U_2, ...$ are identically distributed.

Theorem 2. *The distribution of X is given by*

$$F_X = \sum_{k=0}^{\infty} p_k F_U^{*k},$$

where F_U^{*k} denotes the k-fold convolution of F_U which is the distribution function of U_i and $p_k = P(N = k)$.

Proof. Conditioning on N, we have

$$F_X(x) = P(X \leq x) = \sum_{k=0}^{\infty} P(X \leq x | N = k) p_k$$

since N=0 implies X=0, we have

$$P(X \leq x | N = 0) = 1_{[0,\infty)}(x) = F_U^{*0}(x)$$

and for k$\geq$ 1,we have

$$P(X \leq x | N = k) = P(U_1 + U_2 + ... + U_k \leq x) = F_U^{*k}(x)$$

so that

$$F_X = F_U^{*0} p_0 + \sum_{k=1}^{\infty} p_k F_U^{*k} = \sum_{k=0}^{\infty} p_k F_U^{*k}$$

□

Theorem 3. *For each $s \geq 0$,*

$$\hat{l}_X(s) = \hat{g}_N(\hat{l}_U(s))$$

where $\hat{l}_X(s)$ is the Laplace-Stieltjes transform of X and $\hat{g}_N(s)$ is the generating function of N.

Proof.

$$\begin{aligned} E[e^{-s\sum_{i=1}^{N} U_i}] &= \sum_{k=0}^{\infty} E[e^{-s\sum_{i=1}^{N} U_i} | N = k] P(N = k) \\ &= \sum_{k=0}^{\infty} E[e^{-s\sum_{i=1}^{k} U_i}] P(N = k) \\ &= \sum_{k=0}^{\infty} E[\prod_{i=1}^{k} e^{-sU_i}] P(N = k) \\ &= \sum_{k=0}^{\infty} E[e^{-sU}]^k P(N = k) = \hat{g}_N(E[e^{-sU}]) \end{aligned}$$

□

Corollary. *Assume that the relevant second moments exist. Then*

$$E[X] = E[N]E[U], \;\; Var[X] = Var[N]E[U]^2 + E[N]Var[U]$$

Proof.

$$E[X] = -d/ds \hat{l}_X(s)|_{s=0+} = \hat{g}_N^{(1)}(1) \hat{l}_U^{(1)}(0+)$$

$$\begin{aligned} E[X^2] = -d^2/ds^2 \hat{l}_X(s)|_{s=0+} &= \hat{g}_N^{(1)}(1) \hat{l}_U^{(2)}(0+) + \hat{g}_N^{(2)}(1) (\hat{l}_U^{(1)}(0+))^2 \\ &= E[N][U^2] + E[N(N-1)]E[U]^2 \\ &= E[N^2]E[U]^2 + E[N](E[U^2] - E[U]^2) \\ &= E[N^2]E[U]^2 + E[N]Var[U] \end{aligned}$$

$$Var[X] = E[X^2] - E[X]^2 = Var[N]E[U]^2 + E[N]Var[U]$$

□

There are especially two cases of compound distributions that we are interested in.

Definition.

- *Compound Poisson distribution* , where N has Poisson distribution determined by λ , characterizing as (λ, F_U).
- *Compound pascal/negative-binomial distribution* where N has negative binomial distribution $NB(\alpha, p)$, characterizing as (α, p, F_U).

Special case $\alpha = 1$ of *compound negative binomial distribution* is called *compound geometric distribution* which plays a big part in calculating the infinite-horizon ruin probability.

Corollary. *if X follows compound geometric distribution with* $s \geq 0$

$$\hat{l}_X(s) = \frac{p \cdot \hat{l}_U(s)}{1 - (1-p) \cdot \hat{l}_U(s)}$$

Proof. We can directly use the result of *Theorem 3.*

$$\begin{aligned}
\hat{l}_X(s) = \hat{g}_N(\hat{l}_U(s)) &= \sum_{k=0}^{\infty} \hat{l}_U(s)^k P(N = k) \\
&= 0 + \sum_{k=1}^{\infty} \hat{l}_U(s)^k p(1-p)^{k-1} \\
&= p \cdot \hat{l}_U(s) \sum_{k=1}^{\infty} \hat{l}_U(s)^{k-1} (1-p)^{k-1} \\
&= \frac{p \cdot \hat{l}_U(s)}{1 - (1-p) \cdot \hat{l}_U(s)}
\end{aligned}$$

□

1.2.3 Cramèr-Lundberg Model

The most well-known risk model is called the *Cramèr-Lundberg Model* or classical risk model, which takes additional assumption besides risk model mentioned above for calculating the ruin probabilities. In section 2 and 3, we will study the time of ruin under this model.

Definition. *Additional Assumption*

- Premiums B(t)=βt is just a linear function of time t with a positive constant β which is called *gross risk premium rate*.
- F_U has mean μ and variance σ^2, $F_U(0) = 0$.
- The inter-arrival time $\{T_n\}$ are i.i.d. exponentially distributed with mean=λ^{-1}.

Observation. *The third assumption is equivalent to a Homogenious Poisson Process with intensity λ and then counting process N(t) follows Poisson(λt) so the cumulative claim process X(t) is a compound Poisson process. Thus the Cramèr-Lundberg Model is also called the compound Poisson model.*

Remark. *The compound Poisson process is just like compound Poisson distribution but now N changes to N(t) and have distribution Poisson(λt).*

1.3 Limiting behaviour

In infinite-horizon ruin probabilities case, we can modify the claim surplus process to a random walk to study the limiting behaviour of it since ruin can occur only at claim times.

$$\begin{aligned}\varphi(u) &= P(u + \beta\sigma_n - X(\sigma_n) \le 0, n \ge 1)\\ &= P(u + \sum_{k=1}^{n}(\beta T_k - U_k) \le 0, n \ge 1)\\ &= P(sup_{n\ge1}\sum_{k=1}^{n}(U_k - \beta T_k) \ge u)\end{aligned}$$

With: $S'_k = U_k - \beta T_k, W_n = \sum_{k=1}^{n} S'_k$ and

$$M = sup_{n\ge1}\sum_{k=1}^{n} S'_k = sup_{n\ge1}W_n$$

$$\varphi(u) = P(M > u)$$

Lemma 1.

$$\begin{aligned}\varphi(u) &= 1 \qquad if\ E[S'_i] \ge 0,\\ \varphi(u) &< 1 \qquad if\ E[S'_i] < 0,\end{aligned}$$

Proof. For $E[S_i']$>0, from strong law of large number we have

$$W_n/n \overset{a.s.}{\to} E[S_i'] \quad as \quad n \to \infty$$

Then it follows that

$$\lim_{n\to\infty} W_n = \infty$$

So $M = sup_{n>=1}W_n$ can not smaller than any finite positive number u so we have $\varphi(u) = P(M > u) = 1$, and for $E[S_i']$<0 we have W_n analogously

$$\lim_{n\to\infty} W_n = -\infty$$

$M = sup_{n>=1}W_n$ can not always greater than any finite positive number u so we have $\varphi(u) = P(M > u) < 1$.
For the $E[S_i']$=0, recalled that this random walk W_n is recurrent at origin and has positive probability to reach u+c from origin (c>0)[I]. We have

$$P(W_n = 0 \quad i.o.) = P(W_n = u + c \quad i.o.) = 1$$

So $\varphi(u) = P(M > u) = 1$ □

Observation. *It is only meaningful to compute ruin probabilities under* $E[S_i'] < 0$ *since not only does it be smaller than one but also suggest premium is greater that claim in that epoch.We have*

$$E[S_i'] = E[U_i - \beta T_i] = \mu - \beta\lambda^{-1} < 0$$

$$\beta > \lambda\mu$$

Define the safety loading coefficient θ *by*

$$\theta = \frac{\beta}{\lambda\mu} - 1$$

If $\theta > 0, E[S_i'] = -\mu\theta < 0$ *we say it satisfies the net profit condition (NPC)*

2 Ultimate Ruin

We have all the properties that we need to calculate the ruin probabilities now. We will start from solving integral equation to derive the famous Pollaczeck-Khinchine formula and give a more general result to the example 5.13.

[I]More details and proofs about recurrence can be found in Multi-Parameter Processes: An Introduction to Random Fields, Davar Khoshnevisan, chapter3

2.1 Integral equation of ultimate ruin

Since the Poisson process is a renewal process and since ruin cannot occur before the first claim arrival T_1, then the the survival probability $\bar{\varphi}(u)$ conditioning on no claim in $(0,T_1)$ satisfies following relation:

$$\begin{aligned}\bar{\varphi}(u) =& E[\bar{\varphi}(u+\beta T_1 - U_1)] \\ =& \int_0^\infty \lambda e^{-\lambda s} \int_0^{u+\beta s} \bar{\varphi}(u+\beta s - z) dF_U(z) ds \\ =& \frac{\lambda}{\beta} e^{\lambda \frac{u}{\beta}} \int_u^\infty e^{-\lambda \frac{x}{\beta}} \int_0^x \bar{\varphi}(x-z) dF_U(z) dx\end{aligned}$$

and since $\bar{\varphi}(u)$ is differentiable [II] we have

$$\bar{\varphi}'(u) = \frac{\lambda}{\beta}\bar{\varphi}(u) - \frac{\lambda}{\beta}\int_0^u \bar{\varphi}(u-z) dF_U(z)$$

Theorem 4. *The ruin function satisfies*

$$\beta\varphi(u) = \lambda(\int_u^\infty \bar{F}_U(x)dx + \int_0^u \varphi(u-x)\bar{F}_U(x)dx)$$

and

$$\varphi(0) = \frac{\lambda u}{\beta}, \varphi(\infty) = 0$$

If $F_U(x)$ are exponentially distributed with mean μ

$$\varphi(u) = \frac{1}{1+\theta} e^{-\frac{\theta u}{\mu(1+\theta)}}$$

[II]The discussion of differentiability can be found in both Renewal Risk Processes with Stochastic Returns on Investments - A Unified Approach and Analysis of the Ruin Probabilities section 2.2 and Stochastic Processes for Insurance and Finance p.163

Proof. By integrating (0,u] leads to

$$\begin{aligned}
\frac{\beta}{\lambda}(\bar{\varphi}(u)-\bar{\varphi}(0)) &= \frac{1}{\lambda}\int_0^u \beta\bar{\varphi}'(x)dx \\
&= \int_0^u \bar{\varphi}(x)dx - \int_0^u dx \int_0^x \bar{\varphi}(x-y)dF_U(y) \\
&= \int_0^u \bar{\varphi}(x)dx - \int_0^u dF_U(y) \int_y^u \bar{\varphi}(x-y)dx \\
&= \int_0^u \bar{\varphi}(x)dx - \int_0^u dF_U(y) \int_0^{u-y} \bar{\varphi}(x)dx \\
&= \int_0^u \bar{\varphi}(x)dx - \int_0^u dx \int_0^{u-x} \bar{\varphi}(x)dF_U(y) \\
&= \int_0^u \bar{\varphi}(x)(1-F_U(u-x))dx \\
&= \int_0^u \bar{\varphi}(u-x)\bar{F}_U(x)dx
\end{aligned}$$

Now letting u $\rightarrow\infty$,we have

$$\beta(\bar{\varphi}(\infty)-\bar{\varphi}(0)) = \lambda \lim_{u\rightarrow\infty}\int_0^u \bar{\varphi}(u-x)\bar{F}_U(u-x)dx$$

From net profit condition, we know $\lim_{n\rightarrow\infty} W_n = -\infty$ and $F_U(\infty) = 0$ so M can only take on finite positive number, we have

$$\bar{\varphi}(\infty) = 1$$

Then by applying *Dominated convergence theorem* to the right-hand side we get,

$$\beta(1-\bar{\varphi}(0)) = \lambda\int_0^\infty 1\cdot\bar{F}_U(u-x))dx = \lambda\mu^{\mathrm{III}}$$

Thus,

$$\bar{\varphi}(0) = 1 - \frac{\lambda\mu}{\beta}$$

By changing $\bar{\varphi}(u)$ to $1-\bar{\varphi}(u) = \varphi(u)$,

$$\begin{aligned}
\beta\varphi(u) &= \beta\varphi(0) - \lambda\int_0^u (1-\varphi(u-x))\bar{F}_U(x)dx \\
&= \lambda\mu - \lambda\int_0^u \bar{F}_U(x)dx + \lambda\int_0^u \varphi(u-x)\bar{F}_U(x)dx \\
&= \lambda(\int_u^\infty \bar{F}_U(x)dx + \int_0^u \varphi(u-x)\bar{F}_U(x)dx
\end{aligned}$$

[III] Essential for Stochastic Process P.220 $E[X] = \int_0^\infty P(X>t)dt$

If $F_U(x)$ are exponentially distributed , $\bar{\varphi}(u)$ will satisfies this ODE

$$\bar{\varphi}''(u) + \frac{1}{\mu}\frac{\theta}{1+\theta}\bar{\varphi}'(u) = 0$$

and the initial conditions

$$\bar{\varphi}(\infty) = 1 \quad and \quad \bar{\varphi}(0) = 1 - \frac{\lambda\mu}{\beta} = \frac{\theta}{1+\theta}$$

gives the solution

$$\varphi(u) = 1 - \bar{\varphi}(u) = \frac{1}{1+\theta}e^{-\frac{\theta u}{\mu(1+\theta)}}$$

□

2.2 Pollaczeck-Khinchine formula

In this section, we will use Laplace transform to show that $\bar{\varphi}(u)$ is actually compound geometric distributed to give the general n-fold solution to it .

Theorem 5. *Pollaczeck-Khinchine formula*

$$\varphi(u) = (1 - \frac{\lambda\mu}{\beta})\sum_{n=1}^{\infty}(\frac{\lambda\mu}{\beta})^n(1 - (F_U^I)^{*n}(u))$$

with F_U^I *is the intergrating tail distribution related to* F_U *denoted by,*

$$F_U^I(z) = \frac{1}{\mu}\int_0^z (1 - F_U(x))dx$$

and density

$$f_U^I(z) = \frac{1}{\mu}\bar{F}_U(z)$$

Proof. Taking Laplace transform of $\varphi(u) = \frac{\lambda}{\beta}(\int_u^\infty \bar{F}_U(x)dx + \int_0^u \varphi(u-x)\bar{F}_U(x)dx)$ we get,

$$\begin{aligned}
\hat{L}_\varphi(s) &= \int_0^\infty \varphi(u)e^{-su}du \\
&= \frac{\lambda}{\beta}\int_0^\infty [\int_u^\infty \bar{F}_U(x)dx + \int_0^u \varphi(u-x)\bar{F}_U(x)dx]e^{-su}du \\
&= \frac{\lambda}{\beta}\int_0^\infty (\mu - \int_0^u \bar{F}_U(x)dx)e^{-su}du + \frac{\lambda}{\beta}\int_0^\infty (\int_0^u \varphi(u-x)\mu f_U^I(x)dx)e^{-su}du \\
&= \frac{\lambda\mu}{\beta}\int_0^\infty (1 - F_U^I(u))e^{-su}du + \frac{\lambda\mu}{\beta}\hat{L}_\varphi(s)\hat{L}_{f_U^I}(s)^{\mathrm{IV}} \\
&= \frac{\lambda\mu}{\beta}\hat{L}_{1-F_U^I}(s) + \frac{\lambda\mu}{\beta}\hat{L}_\varphi(s)\hat{L}_{f_U^I}(s) \\
&= \frac{\lambda\mu}{\beta}\frac{1-\hat{L}_{f_U^I}(s)}{s} + \frac{\lambda\mu}{\beta}\hat{L}_\varphi(s)\hat{L}_{f_U^I}(s)^{\mathrm{V}}
\end{aligned}$$

Thus, by rearranging the equation

$$\begin{aligned}
\hat{L}_\varphi(s) &= \frac{1}{s}\frac{\lambda\mu}{\beta}\frac{1-\hat{L}_{f_U^I}(s)}{1-\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s)} \\
&= \frac{1}{s}\frac{\lambda\mu}{\beta}(\frac{1-\hat{L}_{f_U^I}(s)}{1-\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s)} - 1 + 1) \\
&= \frac{1}{s}\frac{\lambda\mu}{\beta}(\frac{\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s) - \hat{L}_{f_U^I}(s)}{1-\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s)} + 1) \\
&= \frac{1}{s}\frac{\lambda\mu}{\beta}(1 - \frac{(1-\frac{\lambda\mu}{\beta})\hat{L}_{f_U^I}(s)}{1-\frac{\lambda\mu}{\beta}\hat{L}_{f_U^I}(s)})
\end{aligned}$$

[IV] An Introduction to Probability Theory and its Applications Volume 2 p.434 , f(x),g(x) and u(x) their convolutions $u(x) = \int_0^x g(x-y)f(y)dy$, their Laplace transform satisfies $\hat{L}_u(s) = \hat{L}_f(s)\hat{L}_g(s)$ if all exists.

[V] An Introduction to Probability Theory and its Applications Volume 2 p.435 2.7, F(x) and f(x) be cumulative and density function of a random variable respectively, then $\hat{L}_{1-F}(s) = \frac{1-\hat{L}_f(s)}{s}$

Within the parentheses, it is actually the Laplace transform of compound geometric distribution G with density g [VI] characterizing as $(1 - \frac{\lambda\mu}{\beta}, F_U^I)$

$$\begin{aligned}\hat{L}_\varphi(s) &= \frac{\lambda\mu}{\beta}\frac{1 - \hat{L}_g(s)}{s} \\ &= \frac{\lambda\mu}{\beta}\hat{L}_{\bar{G}}(s) \\ &= \int_0^\infty e^{-su}\frac{\lambda\mu}{\beta}\bar{G}(u)du\end{aligned}$$

And since the Laplace transform is unique[VII], it implies that $\varphi(u)$ has the same distribution as $\frac{\lambda\mu}{\beta}\bar{G}(u)$

$$\begin{aligned}\varphi(u) &= \frac{\lambda\mu}{\beta}\bar{G}(u) \\ &= \frac{\lambda\mu}{\beta}(1 - \sum_{n=1}^\infty (1 - \frac{\lambda\mu}{\beta})(\frac{\lambda\mu}{\beta})^{n-1}(F_U^I)^{*n}(u))^{\text{VIII}} \\ &= \frac{\lambda\mu}{\beta}(\sum_{n=1}^\infty (1 - \frac{\lambda\mu}{\beta})(\frac{\lambda\mu}{\beta})^{n-1} - \sum_{n=1}^\infty (1 - \frac{\lambda\mu}{\beta})(\frac{\lambda\mu}{\beta})^{n-1}(F_U^I)^{*n}(u)) \\ &= (1 - \frac{\lambda\mu}{\beta})\sum_{n=1}^\infty (\frac{\lambda\mu}{\beta})^n(1 - (F_U^I)^{*n}(u))\end{aligned}$$

Since the strong connection between ruin theory and queing theory, the equation is actually equivalent to the well-known waiting time distribution Pollaczeck-Khinchine formula and thus has the same name. □

2.3 Martingale Approximation

The explicit expression of Pollaczeck-Khinchine formula is sometimes to hard to compute. Thus, we use the same martingale technique as section 1.1 to give the exponential bound of ultimate ruin.

Theorem 6. *Lundberg inequality*

$$\varphi(u) \le e^{-Lu}$$

[VI]Theorem 3, second corollary, and Laplace-Stieltjes transform is actually Laplace transform but focus on cumulative function not density function

[VII]An Introduction to Probability Theory and its Applications Volume 2 p.430, Distinct probability distributions has distinct Laplace transforms

[VII]Applying theorem 2 and setting $p_0 = 0$

where L is called the Lundberg exponent, the positive solution of $\lambda(\hat{m}_U(s) - 1) - \beta s = 0$.

Proof. We first construct a martingale for the *risk reserve process* R(t), s,t>0

$$
\begin{aligned}
E[e^{-sR(t)}] &= E[e^{-s(u+\beta t-X(t))}] = E[e^{s(X(t))}]e^{-s(u+\beta t)} \\
&= e^{-s(u+\beta t)}E[e^{s(U_1+U_2+\ldots+U_{N(t)})}] \\
&= e^{-s(u+\beta t)}\sum_{k=0}^{\infty}(E[e^{s(U_1+U_2+\ldots+U_k})]P(N(t)=k)) \\
&= e^{-s(u+\beta t)}\sum_{k=0}^{\infty}(E[e^{sU}]^k\frac{(\lambda t)^k}{k!}e^{-\lambda t}) \\
&= e^{-s(u+\beta t)}e^{-\lambda t}\sum_{k=0}^{\infty}(\hat{m}_U(s)^k\frac{(\lambda t)^k}{k!}) \\
&= e^{-s(u+\beta t)}e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\hat{m}_U(s)\lambda t)^k}{k!} \\
&= e^{-s(u+\beta t)}e^{-\lambda t}e^{\hat{m}_U(s)\lambda t} \\
&= e^{-su+(\lambda(\hat{m}_U(s)-1)-\beta s)t} \\
&= e^{-su+g(s)t}
\end{aligned}
$$

Recall the definition of ruin,

$$\tau(u) = inf\{t \geq 0 : S(t) > u\}$$

Obviously $\tau(u)$ is a $\mathcal{F}_t^S$ stopping time. Put

$$M_t = \frac{e^{-r(R(t))}}{e^{g(r)t}}$$

For $0 \leq s \leq t$,we have

$$
\begin{aligned}
E[M_t|\mathcal{F}_s^S] &= E[\frac{e^{-r(u+\beta t-X(t))}}{e^{g(r)t}}|\mathcal{F}_s^S] \\
&= E[\frac{e^{-r(u+\beta s-X(s)))}}{e^{g(r)s}}\frac{e^{-r(\beta t-X(t)-\beta s+X(s))}}{e^{g(r)(t-s)}}|\mathcal{F}_s^S] \\
&= M_s \cdot E[\frac{e^{-r(\beta t-X(t)-\beta s+X(s))}}{e^{g(r)(t-s)}}|\mathcal{F}_s^S] \\
&= M_s
\end{aligned}
$$

So M_t is a martingale so we can apply the same method of section 1.1 to calculate the exponential bound of ultimate ruin. Further let L be the positive solution of g(s)=0, we know $M'(t) = e^{-LR(t)}$ is still a martingale.

$$\begin{aligned} E[M'_0] &= E[M'_{\tau(u)\wedge t}] \\ &= E[M'_{\tau(u)}; \tau(u) \leq t] + E[M'_t; \tau(u) > t] \\ &\geq E[e^{-LR(\tau(u))} | \tau(u) \leq t] \times P(\tau(u) \leq t) \\ &\geq P(\tau(u) \leq t) \qquad\qquad since \quad R(\tau(u)) \leq 0 \end{aligned}$$

$$P(\tau(u) < \infty) = \lim_{t\to\infty} P(\tau(u) \leq t) \leq E[M_0] = e^{-Lu}$$

□

Although the martingale technique makes the approximation very easy, one must have to aware that if the claim size distribution is heavy-tailed, $\hat{m}_U(s)$ does not exist for s>0 and the martingale technique can not be used.

3 Connection Between Queuing Theory

3.1 A Dual Queueing Model

Mathematical modelling is used to capture the most intrinsic fact of a real problem into a stochastic framework. For this reason, mathematicians sometimes find out that they can use same model to describe different facts which seem to have less in common. Mathematical models are rather universal. A concrete example is that we can use the ruin theory to analyse a single server with exponential inter-arrival times quest and arbitrary service time distribution. In this section, we will first point out where they are equivalent and use the behaviour of the server to derive the ruin probability reversely.

3.1.1 Queuing model and waiting time

The server mentioned above in reality is like a clinic with one doctor. Patients come to the clinic follow some random patterns and they have to wait for previous patients to finish their diagnosis first. The doctor spends different times on patients. The most interesting problem is that each patient has to spend how much time in the waiting room. This kind of problem has a special name in mathematics which is called the queuing theory. We will first give the explicit model of this clinic.

Definition. *Queuing model*

- The diagnosis time of different patients are independent and both follow the same distribution F_U. U_i stands for the diagnosis time of i-th patient.
- The inter-arrival time between i and i+1 patients are βT_i. $\{T_i\}$ are i.i.d exponentially distributed with parameter λ.
- Doctor never rests except there are no patients waiting.
- Patients are treated in order of arrival. Once the patients come, they have to stay in the waiting room till all the previous patients over. The waiting room has infinite space.
- L_i is the just time that $i+1$th patient has to stay in the waiting room and $L_0 = 0$.

It can be easily figure out that the waiting time of patients follow the recurrence relation.

$$L_n = (L_{n-1} + U_n - \beta T_n)_+$$

And the rounding to zero part is because the inter-arrival time may be very big to make L_i be negative i.e. the doctor finishes all his/her patients before new patient arrives, but L_i is the waiting time has minimum 0 so we round out the negative part and this part is actually the rest period of the doctor between two arrival.

3.1.2 Elegant relation

Lemma 2. *Suppose $Y_i = U_i - \beta T_i$,*

$$L_n = max(0, Y_n, Y_{n-1} + Y_n, ..., Y_n + ...Y_2 + Y_1)$$

Proof. The intuitive solution is just $\sum_{i=1}^{n} Y_i$ but it is wrong since the waiting time can not be negative and Y_i can be negative so L_n can not simply be that form. Now we consider the situation $L_i = 0$ i.e. the i+1 patient comes and finds that he needs not to wait for anybody and assume all the patients i+2,i+3,...n has to wait. Then the rounding to zero part in the recurrence relation above is not applied. And L_n is just $Y_{i+1} + Y_{i+2} ... + Y_n$. In reality, i can only be one of 0,1,2...n so L_n is one of

$$(0, Y_n, Y_{n-1} + Y_n, ..., Y_n + ...Y_2 + Y_1)$$

To figure out which one is L_n, we first assume the solution is $Y_{i+1} + Y_{i+2}... + Y_n$ as previous assumption. The solution is true when $L_i = 0$ and all the patients i+2,i+3,...n has to wait. That is to say, the doctor does not have the time to rest so $\sum_{k=i+1}^{j<n} Y_k \geq 0$ thus $L_n = Y_{i+1} + Y_{i+2}... + Y_n = \sum_{k=i+1}^{j<n} Y_k + \sum_{k=j+1}^{n} Y_k$ implies

$$(Y_{i+2} + Y_{i+3}... + Y_n, Y_{i+3}... + Y_n, ..., Y_n) = \sum_{k=j+1}^{n} Y_k, j = i...n-1 \leq L_n$$

Now we trace back all the waiting time of previous patients. Somebody may have zero waiting time or everybody has to wait (except the first patient). If every body has to wait then $\sum_{k=1}^{j<i} Y_k \geq 0$ but $\sum_{k=1}^{i} Y_k \leq 0$ since $L_i = 0$ so $\sum_{k=j}^{i} Y_k \leq 0$ thus L_n implies

$$(Y_1 + Y_2... + Y_n, Y_2... + Y_n, ..., Y_{i-1} + Y_n) = L_n + \sum_{k=j}^{i} Y_k \leq L_n$$

If some previous patients have zero waiting time, let $w = argsup\{L_p = 0, p < i\}$ we find $L_w = 0$ which means the w+1 patient has zero waiting time and all patient between w+1 and i+1 has to wait. It is just like the situation of previous paragraph if we see w+1 as 1. Thus,

$$(Y_{w+1} + Y_{w+2}... + Y_n, Y_{w+2}... + Y_n, ..., Y_{i-1} + Y_n) \leq L_n$$

Further consider $w_{-1} = argsup\{L_p = 0, p < w < i\}$, we can fold the time line making w and i coincide and w_{-1} is just like w in previous paragraph. This trick can be applied to both w_{-1}, w_{-2}..... so

$$(Y_1 + Y_2... + Y_n, Y_2... + Y_n, ..., Y_{i-1} + Y_n) \leq L_n$$

Since i is not fixed and for any i, L_n will the the max of the possible values. It follows that

$$L_n = max\{0, Y_n, Y_{n-1} + Y_n, ..., Y_n + ...Y_2 + Y_1\}$$

□

Lemma 3. *for all n=1,2...*

$$(Y_n, Y_{n-1} + Y_n, ..., Y_n + ...Y_2 + Y_1) \stackrel{d}{=} (Y_1, Y_1 + Y_2, ..., Y_1 + ...Y_{n-1} + Y_n)$$

Proof. Since both $\{U_i, i \geq 1\}$ and $\{T_i, i \geq 1\}$ are i.i.d themselves and β is just a constant, we have $\{Y_i, i \geq 1\}$ are i.i.d.. From the definition of joint distribution, we have

$$P(Y_1 \in A_1, Y_1 + Y_2 \in A_2, Y_1 + Y_2 + Y_3 \in A_3, ..., Y_1 + ...Y_{n-1} + Y_n \in A_n)$$
$$= P(Y_1 \in A_1)P(Y_1 + Y_2 \in A_2|Y_1 \in A_1)P(Y_1 + Y_2 + Y_3 \in A_3|Y_1 + Y_2 \in A_2, Y_1 \in A_1)...$$
$$= P(Y_n \in A_1)P(Y_n + Y_{n-1} \in A_2|Y_n \in A_1)P(Y_n + Y_{n-1} + Y_{n-2} \in A_3|Y_n + Y_{n-1} \in A_2, Y_n \in A_1)...$$
$$= P(Y_n \in A_1, Y_n + Y_{n-1} \in A_2, Y_n + Y_{n-1} + Y_{n-2} \in A_3, ..., Y_n + ...Y_2 + Y_1 \in A_n)$$

□

Theorem 7. *Recall definition of W_n and M in section 1.3. Then,*

$$P(L_n > u) = P(\sup_{1 \leq i \leq n} W_i > u)$$

$$\lim_{n \to \infty} P(L_n > u) = P(\sup_{i \geq 1} W_i > u) = P(M > u) = \varphi(u)$$

Proof.

$$\begin{aligned} P(L_n > u) &= P(max\{0, Y_n, Y_{n-1} + Y_n, ..., Y_n + ...Y_2 + Y_1\} > u) \\ &= P(max\{0, Y_1, Y_1 + Y_2, ..., Y_1 + ...Y_{n-1} + Y_n\} > u) \\ &= P(max\{S'_1, S'_1 + S'_2, ..., S'_1 + ...S'_{n-1} + S'_n\} > u) \\ &= P(\sup_{1 \leq i \leq n} W_i > u) \end{aligned}$$

Since $max\{0, Y_1, Y_1 + Y_2, ..., Y_1 + ...Y_{n-1} + Y_n\} \leq max\{0, Y_1, Y_1 + Y_2, ..., Y_1 + ...Y_{n-1} + Y_n + Y_{n+1}\}$ so $P(L_n > u) \leq P(L_{n+1} > u) \leq 1$. Thus, *monotone convergence theorem* yields the second equation of the theorem.

This clinic is somehow a insurance company! The waiting time of the transfinite patient bigger than u is just the probability of ultimate ruin. Of course, the doctor is free from karoshi. This clinic model can be modify to a M/G/1 queue, which has been studied extensively. □

3.2 M/G/1 Queue

Queuing theory is developed to study the behaviour of waiting lines. In this theory, we construct a mathematical model so as to predict the queue length(number of customers in system) and waiting time of the system in interest. A queueing node can be classified using *Kendall's notation* in the form A/S/C where A describes the time of inter-arrivals to the queue, S the size of job (service time) and C the number of servers at the node.

3.2.1 Basic definition and property

Definition. *M/G/1 queue*

$$\begin{cases} M: \textit{Markov or memoryless, arrivals occur as a Poisson process with rate } \lambda \\ G: \textit{General service time distribution with E[S]=u}^{-1} \\ 1: \textit{Single server} \end{cases}$$

To analyse the average time spent in this M/G/1 queue by a customer(arriver) and the average number of customer in this system, we have to introduce three useful property of M/G/1 system which are *Little's Law, Level Crossing Low and PASTA*. We assume the queue length is bounded and the system will reach a equilibrium, and mainly focus on the behaviour of the system in equilibrium.

Let A(t) be the number of arrivals to the system in [0,t] and D(t) be the departure. Then, N(t), the number of customers in the system (both in service and waiting) can be expressed as

$$N(t) = A(t) - D(t) + N(0)$$

where N(0) is the number of customers at time 0. We often assume system is initially empty i.e. N(0)=0 and operates in a way such that N(t)=o(A(t)) thus

$$\lim_{t \to \infty} \frac{N(t)}{A(t)} = 0$$

And T_i be the sojourn time (total amount of time will spent in the server) of the ith customer. We further define the following average:

$$\begin{cases} \bar{N}(t) = \frac{1}{t} \int_0^t N(x)dx & \text{The time average of the number of customers until t} \\ \bar{T}(t) = \frac{1}{A(t)} \sum_{i=1}^{A(t)} T_i & \text{The customer average of response time until t} \\ \bar{A}(t) = \frac{A(t)}{t} & \text{Average arrival rate of customers by time t} \end{cases}$$

Since we assume the system will reach equilibrium and thus following limits exist

$$\begin{cases} \bar{N}(t) \overset{a.s.}{\to} N \quad as \quad t \to \infty \\ \bar{T}(t) \overset{a.s.}{\to} T \quad as \quad t \to \infty \\ \bar{A}(t) \overset{a.s.}{\to} \lambda \quad as \quad t \to \infty \end{cases}$$

where N,T,λ are the queue length, sojourn time and arrival rate of the system in equilibrium respectively. Our final purpose is to study the behaviour of random variable N and T so as to derive how much chain will the system in some state.

Property. *Little's Law*

$$N = \lambda T$$

Proof. Consider if whenever a customer in the system, he or she has to pay \$1/unit of time. Then we can view whole amount of money till time t via the queue length (system view) and sojourn time (customer view) at he same time. And E(t) be the unrealized sojourn time of customers that arrived before t.

$$Money(t) = \int_0^t N(t)dt = \sum_{i=1}^{A(t)} T_i - E(t)$$

Consider the average,

$$\begin{aligned}\bar{N}(t) = \frac{1}{t}\int_0^t N(t)dt &= \frac{1}{t}\frac{A(t)}{A(t)}\sum_{i=1}^{A(t)} T_i - \frac{1}{t}E(t)\\ &= \bar{A}(t)\bar{T}(t) - \frac{E(t)}{t}\end{aligned}$$

Now let $t \to \infty$, since we assume queue length is finite and customer will not always stays in the queue.Thus $\frac{E(t)}{t} \to 0$ and then

$$N = \lambda T$$

Little's law was used as a folk theorem in queuing theorem before and a more rigorous proof can be found here.[VIII] □

Besides the relation between queue length and sojourn time, we are also interested in what kind of state of system saw by arrived customer and departure customer. Let a_i and d_i be the fraction of arriving/departing customers see i customers in the system(not including itself) in equilibrium.

Property. *Level Crossing Law*

$$a_i = d_i \quad i = 0, 1, ...$$

Proof. Let $A_i(t)$ be the number of arriving customers in [0,t] that find the system with i customers and $D_i(t)$ respectively.We have,

$$\begin{cases} \frac{A_i(t)}{A(t)} \overset{a.s.}{\to} a_i \quad as \quad t \to \infty \quad i = 0, 1, ... \\ \frac{D_i(t)}{D(t)} \overset{a.s.}{\to} d_i \quad as \quad t \to \infty \quad i = 0, 1, ... \end{cases}$$

[VIII] *Ward Whitt* , A review of L=λW and extensions

Consider an arriving customer who see i customers in the system, this arrival will cause the system transit from i to i+1. We donate this transition as upward cross the level i,and i+1 to i the downward cross caused i by the departing customer. Since arrivals and departures occur one at a time, we can view the number of customers in the system as a stair. When there is an arrival, we move up a step and move down when there is a departure. Then a stairstep cannot be crossed upward two times before a downward cross and vice versa. It follows that

$$|A_i(t) - D_i(t)| \leq 1$$

By simple algebra,

$$\begin{aligned}\frac{A_i(t)}{A(t)} - \frac{D_i(t)}{D(t)} &= \frac{A_i(t) - D_i(t)}{A(t)} + \frac{D_i(t)}{D(t)}\frac{D(t) - A(t)}{A(t)} \\ &= \frac{A_i(t) - D_i(t)}{A(t)} - \frac{D_i(t)}{D(t)}\frac{N(t)}{A(t)}\end{aligned}$$

Then let $t \to \infty$, since A(t)and D(t) becomes ∞ and N(t)=o(A(t)), the right-hand side converges to zero so we conclude $a_i = d_i$ for all i. □

We now know that seeing from arrival and departure are stochastically the same. But such result only consider the system state when the customer arrives or departs, how about the system state in whole time line? Does the fraction of arriving customer see the system in state i equals the fraction of time the system is in state i ? The equality is not always true for all kinds of queues. But for queues with Poisson arrivals, it is true so we can use the view of arriving/departing customer to determine the time average of system behaviour. This property of M/G/1 queue is called PASTA.

Property. *Poisson Arrival See Time Average*

$$Z(t) \to V(\infty) \quad as \quad t \to \infty \iff V(t) \quad converges \quad to \quad V(\infty)$$

$$where \begin{cases} U(t) = \mathbb{1}_{\{N(t)\in B\}} & V(t) = \frac{1}{t}\int_0^t U(x)dx \\ Y(t) = \int_0^t U(x)dA(x) & Z(t) = Y(t)/A(t) \end{cases}$$

Proof. The rigorous proof can be found here.[IX] The meaning of this property is that the probability of an arrival find the system in state B is like

[IX] *Ronald W. Wolff* , Poisson Arrivals See Time Averages the probability that the system is in state B at a random time when reaches equilibrium. Intuitively, this is because the memoryless property of exponential distribution. The new arrival is regardless of the past. $P(N(t) \in B|$ a arrival in (t,t+h],h$\to$ 0$) = P($an arrival find $N(t) \in B) = P(N(t) \in B)$. $\square$

Observation. *Fraction of time system is busy*

$$P(N > 0) = \rho = \frac{\lambda}{\mu} = \lambda E[s]$$

Since the average service time is μ^{-1}, the system can service μ customer every time unit in average. We assume the system will reach equilibrium so $\lambda < \mu$ otherwise we will accumulate more and more customers in queue.The system can not be stable. Consider $A_n = a_1 + a_2.. + a_n$ be the time of nth arrival and s_i be the service time of ith customer and $Z(t)$ be the non-realized service time of customers which are already in system at time t. By strong law of large number we have,

$$\frac{\frac{\sum_i^n s_i - Z(A_n)}{n}}{\frac{A_n}{n}} \to \frac{\mu^{-1} + 0}{\lambda^{-1}}$$

since Z(An) is finite because of the queue length and service time are finite.

3.2.2 Pollaczek-Khinchine formula revisited

Now we have all tools to derive the distribution of queue length N, sojourn time T and waiting time W (sojourn time minus service time) in equilibrium. The formula is known as the Pollaczek-Khinchin formula. We will fist start from deriving the distribution of queue length.

Lemma 4. *The probability generating function of N is given by*

$$\hat{G}_N(z) = \hat{L}_S(\lambda(1-z))\frac{(1-\rho)(1-z)}{\hat{L}_S(\lambda(1-z)) - z}$$

Proof. Let V_i be the number of arrivals to the system during the ith service time interval. Since inter-arrivals are exponential distributed, and thus memoryless. V_i are independent from each other. Let N_i^d denote the number of customers in the system saw by the departing customer i. We have,

$$N_i^d = \begin{cases} N_{i-1}^d - 1 + V_i, & N_{i-1}^d \geq 1 \\ V_i, & N_{i-1}^d = 0 \end{cases}$$

When customer $i-1$ leaves, the system has queue length N_{i-1}^d and enters the service time of customer i. When the service time finishes, the queue length saw by i decreases one (itself) from N_{i-1}^d and gains the arrival in i's service time which is V_i. If $N_{i-1}^d = 0$, the queue is empty at first then adds one (arrival of i) and gains the arrival in i's service time which is V_i and finally decreases one when i leaves. We can simplify it as

$$N_i^d = (N_{i-1}^d - 1)^+ + V_i, \quad i = 1, 2...$$

As $i \to \infty$ the random variable N_i^d converges to the N^d which is the customers in the system saw by a departing customer when the system reaches equilibrium. Further let N^a be the number of customers saw by a arriving customer in equilibrium. Then from the level crossing law and PASTA we have,

$$N^d = N^a = N$$

So,

$$N = (N-1)^+ + V$$

And since V is independent of queue length,we have

$$\hat{G}_N(z) = \hat{G}_{(N-1)^+}(z)\hat{G}_V(z)$$

Now focusing on $\hat{G}_{(N-1)^+}(z)$,

$$\begin{aligned}
\hat{G}_{(N-1)^+}(z) &= \sum_{i=0}^{\infty} P((N-1)^+ = i)z^i \\
&= z^0 P((N-1)^+ = 0) + \sum_{i=1}^{\infty} z^i P((N-1)^+ = i) \\
&= z^0(P(N=0) + P(N=1)) + \frac{1}{z}\sum_{i=2}^{\infty} z^i P(N=i) \\
&= P(N=0) + \frac{1}{z}(\sum_{i=2}^{\infty} z^i P(N=i) + z^1 P(N=1)) \\
&= 1 - \rho + \frac{1}{z}(\hat{G}_N(z) - (1-\rho)) \\
&= \frac{\hat{G}_N(z) - (1-\rho)(1-z)}{z}
\end{aligned}$$

And for V, given that a service interval has length x, the distribution is Poisson with parameter λx and the probability that service time in (x,x+dx) is given by the density function of S, which is $f_S(x)$,

$$\begin{aligned}
\hat{G}_V(z) &= \sum_{i=0}^{\infty}(\int_0^{\infty} \frac{(\lambda x)^i}{i!} e^{-\lambda x} f_S(x)dx)z^i \\
&= \int_0^{\infty} e^{-\lambda x}(\sum_{i=0}^{\infty} \frac{(\lambda x z)^i}{i!}) f_S(x)dx \\
&= \int_0^{\infty} e^{-\lambda x(1-z)} f_S(x)dx \\
&= \hat{L}_S(\lambda(1-z))
\end{aligned}$$

Finally we have,

$$\begin{aligned}
\hat{G}_N(z) &= \hat{G}_{(N-1)^+}(z)\hat{G}_V(z) \\
&= \frac{\hat{G}_N(z) - (1-\rho)(1-z)}{z}\hat{L}_S(\lambda(1-z)) \\
&= \hat{L}_S(\lambda(1-z))\frac{(1-\rho)(1-z)}{\hat{L}_S(\lambda(1-z)) - z}
\end{aligned}$$

□

Lemma 5. *The Laplace transform of T and W are,*

$$\hat{L}_T(s) = \hat{L}_S(s)\frac{s(1-\rho)}{s-\lambda+\lambda\hat{L}_S(s)}$$

$$\hat{L}_W(s) = \frac{s(1-\rho)}{s-\lambda+\lambda\hat{L}_S(s)}$$

Proof. Consider a random customer J that is tagged on its arrival to the queue. Then the customers in the queue at J's departure are customers arrive the system during J's sojourn time. And this number is just N because of the level crossing law and PASTA. Now we can apply the method which we use to derive V on N so we have,

$$\hat{G}_N(z) = \hat{L}_T(\lambda(1-z))$$

and setting $s = \lambda(1-z)$ leads to

$$\hat{L}_T(s) = \hat{L}_S(s)\frac{s(1-\rho)}{s-\lambda+\lambda\hat{L}_S(s)}$$

and since the waiting time of a customer is independent of its service time

$$\hat{L}_W(s) = \frac{s(1-\rho)}{s-\lambda+\lambda\hat{L}_S(s)}$$

□

3.3 Ruin and Queuing Theory

In previous section, we have already shown the one to one correspondence of the clinic queuing model with ultimate ruin of our risk model and the distribution of M/G/1 queue length. Now it's time for us to show why the formula of ultimate ruin probability is called as Pollaczek-Khinchine formula. To do this, we have to first modify our clinic queue to M/G/1 queue.

Recall our clinic queue and M/G/1 queue model.

$$\mathit{Clinic}\ \begin{cases}\text{Arrival rate: The inter-arrival time } \beta T_i. \\ \text{Service time:} E[U] = \mu \\ \text{Same paremeters as risk model}\end{cases} \qquad M/G/1\ \begin{cases}\text{Arrival rate: Poisson process with parameter } \lambda \\ \text{Service time:} E[s] = k^-1 \\ \text{Busy rate:} \frac{\lambda}{k}\end{cases}$$

These two models are not the same generally since exponential distribution times a constant factor is not an exponential distribution. However, if β is set to one, the patients will come as Poisson with paremeter λ. And we can always set any β to do this trick by simply modifying our risk model.

Consider a new claim surplus process

$$\begin{aligned} S^*(t) &= \frac{S(t)}{\beta} \\ &= 1 \cdot t + \sum_{i=1}^{N(t)} \frac{U_i}{\beta} = 1 \cdot t + \sum_{i=1}^{N(t)} U_i' \end{aligned}$$

Then

$$\varphi(u) = P(\sup_{t \geq 0} S(t) > u) = P(\sup_{t \geq 0} S^*(t) > \frac{u}{\beta}) = \varphi'(\frac{u}{\beta})$$

From figure 1, it is like a new insurer with initial reserve $\frac{u}{\beta}$, *gross premium rate =1* and claim size being β times smaller than original model. We can construct a clinic model with this new risk model and link the clinic model to the M/G/1 queue.[X] So we have

$$\begin{aligned} \varphi(u) &= \varphi'(\frac{u}{\beta}) \\ &= P(\text{Patients waiting time} > \frac{u}{\beta}) \\ &= 1 - P(\text{Waing time} \leq \frac{u}{\beta} \text{ in a M/G/1 queue}) \end{aligned}$$

[X]NPC implies $\beta > \lambda\mu$ so $\rho = \lambda E[s] = \lambda \frac{\mu}{\beta} < 1$, the system can reach equilibrium

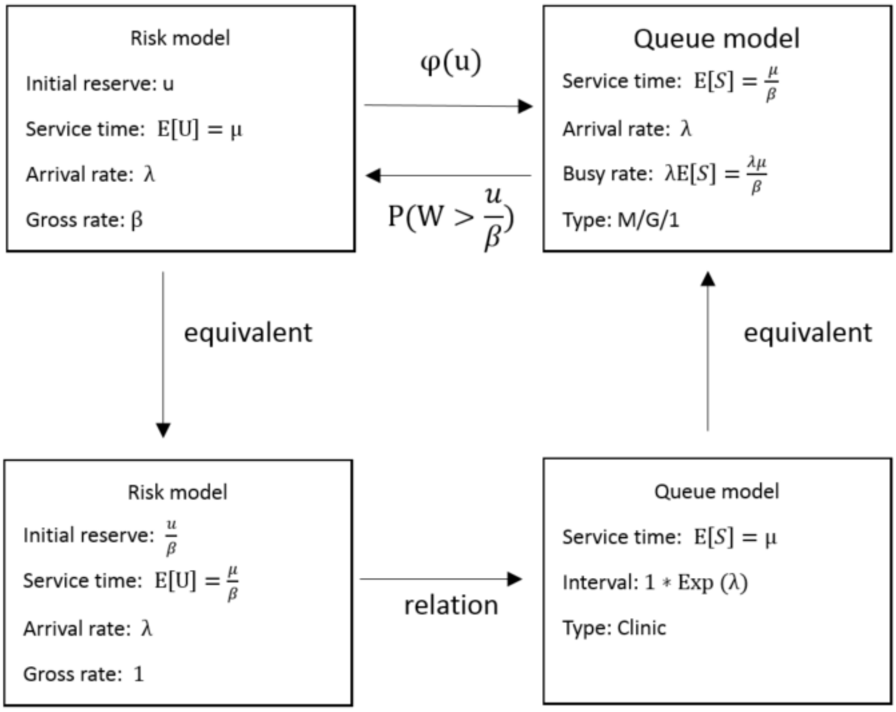

Figure 1: Relation between models

We can use the Pollaczek-Khinchine formula for waiting time distribution to calculate the ruin probability.!

Theorem 8. *With a random variable R, W is actually a compound distribution*

$$\hat{L}_W(s) = \frac{1-\rho}{1-\rho\hat{L}_R(s)}$$

$$W = \begin{cases} \sum_{i=1}^{N} R_i & if N \geq 1, \\ 0 & if N = 0, \end{cases}$$

Where R has density $\frac{\bar{F}_S(t)}{E[s]}$ *and* $P(N = i) = (1-\rho)(\rho)^i$

Proof. We can rewrite the Laplace transform,

$$\begin{aligned}
\hat{L}_W(s) = \frac{s(1-\rho)}{s-\lambda+\lambda\hat{L}_S(s)} &= \frac{s(1-\rho)E[s]}{sE[s]-\lambda E[s]+\lambda\hat{L}_S(s)E[s]} \\
&= \frac{s(1-\rho)E[s]}{sE[s]-\rho+\rho\hat{L}_S(s)} \\
&= \frac{(1-\rho)}{1-\rho(\frac{1-\hat{L}_S(s)}{sE[s]})}
\end{aligned}$$

From footnote VI, the Laplace transform of R equals

$$\hat{L}_R(s) = \frac{1-\hat{L}_S(s)}{sE[s]}$$

Also,

$$\begin{aligned}
\hat{L}_W(s) &= \frac{1-\rho}{1-\rho\hat{L}_R(s)} \\
&= \sum_{i=0}^{\infty}(1-\rho)(\rho)^i(\hat{L}_R(s))^i \\
&= \hat{G}_N(\hat{L}_R(s))
\end{aligned}$$

By theorem 3, we know W is compound geometric distributed.

Remark. *R is actually the the integral tail distribution we had already used in section 2.2. And,*

$$\begin{aligned}
F_R(z) = F_S^I(z) &= \frac{1}{E[s]}\int_0^z(1-F_S(x))dx \\
&= \frac{1}{\frac{\mu}{\beta}}\int_0^z(1-F_U(\beta x))dx \\
&= \frac{1}{\mu}\int_0^{\beta z}(1-F_U(t))dt = F_U^I(\beta z)
\end{aligned}$$

Applying theorem 2, we have

$$\begin{aligned}
\varphi(u) &= 1 - F_W(\frac{u}{\beta}) \\
&= 1 - \sum_{i=0}^{\infty}(1-\rho)(\rho)^i (F_S^I)^{*i}(\frac{u}{\beta}) \\
&= 1 - (1-\rho) - \sum_{i=1}^{\infty}(1-\rho)(\rho)^i (F_S^I)^{*i}(\frac{u}{\beta}) \\
&= \sum_{i=1}^{\infty}(1-\rho)(\rho)^i - \sum_{i=1}^{\infty}(1-\rho)(\rho)^i (F_S^I)^{*i}(\frac{u}{\beta}) \\
&= \sum_{i=1}^{\infty}(1-\rho)(\rho)^i (1 - (F_S^I)^{*i}(\frac{u}{\beta})) \\
&= (1 - \frac{\lambda\mu}{\beta}) \sum_{n=1}^{\infty}(\frac{\lambda\mu}{\beta})^n (1 - (F_U^I)^{*n}(u))
\end{aligned}$$

□

This is just as the answer using Integral equation! And this is why it is also called the Pollaczek-Khinchine formula. Survival probability of ruin is just like waiting time distribution in M/G/1 queue. It sounds that they have nothing in common initially, but they are actually the same. How amazing math could be!

4 Perturbed Risk Process

5 Simulation

6 Acknowledgement

Thanks everyone.

7 Appendix

You can find the simulation code on github.

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