Markov chains - Examples

1 Finite state space

1.1 Gambler's ruin

1.1.1 Simple case

Model At each step, flip a coin. The bank gives you 1$ if you get heads, but you lose 1$ if you get tails. You win if you reach an amount of 5$, and obviously lose if you do not have any more money.

State space $S =$

Transition matrix

$$P = \begin{pmatrix} \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

Graph

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

1.1.2 General case

Model At each step, flip a coin. The bank gives you 1\$ if you get heads, which happens with probability $p \in (0,1)$, but you lose 1\$ if you get tails (which happens with probability $1-p$). You win if you reach an amount of N coins, and lose if you do not have any more money.

State space $S =$

Transition matrix

$$P = \begin{pmatrix} \\ \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

Graph

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

1.2 Social mobility (Durrett, p. 3)

Model Let X_n to be the social class of a family in the n-th generation, which can be lower- (L), middle- (M) or upper-class (U). A simple model is that the status changes at each generation according to a Markov chain with transition matrix P below.

State space $S = \{L, M, U\}$.

Transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} L & M & U \end{array} \\ \begin{array}{c} L \\ M \\ U \end{array} & \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix} \end{array}$$

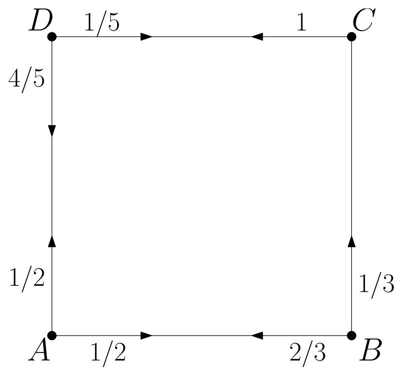
Graph

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

1.3 Random walk on a graph

1.3.1 Random walk on a square

Model



State space $S =$

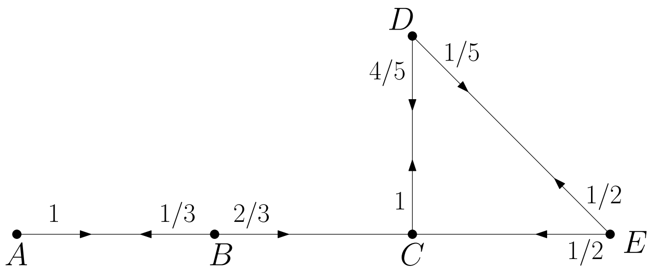
Transition matrix

$$P = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

1.3.2 On another graph

Model



State space $S =$

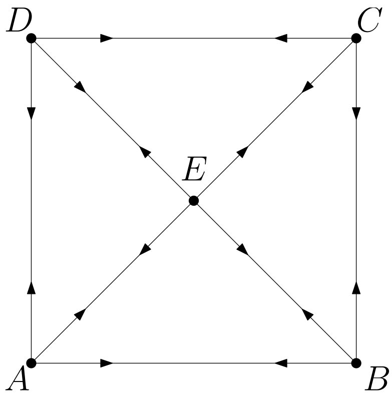
Transition matrix

$$P = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

1.3.3 Simple random walk

Model In the following graph, from some vertex, jump to a neighbor with equal probability.



State space $S =$

Transition matrix

$$P = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

1.4 Ehrenfest model

Model Consider two urns, containing a total number of N balls. At each step, pick a ball uniformly at random among all the possible balls, and move it to the other urn. The number X_n of urns in the left urn is a Markov chain called Ehrenfest chain.

State space $S =$

Transition matrix

$$P = \begin{pmatrix} \\ \\ \\ \\ \\ \end{pmatrix}$$

Graph

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

2 Infinite state space

2.1 Random Walk

2.1.1 Asymmetric simple random walk in 1D

Model Consider $(X_i)_{i \geq 1}$ a sequence of i.i.d. random variables with values in $\mathbb{Z}$, with $\mathbb{P}(X_i = +1) = p$, $\mathbb{P}(X_i = -1) = 1 - p$, $p \neq 1/2$. The Markov chain defined by

$$S_n = X_1 + \cdots + X_n, \quad n \geq 1$$

is called an asymmetric simple random walk.

State space $S =$

Transition probabilities

Graph

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

2.1.2 Symmetric simple random walk in 1D

Model It is the same, with $p = 1/2$.

State space $S =$

Transition probabilities

Graph

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

2.2 Symmetric random walk in d dimensions

Model Consider (S_n) a Markov chain on $\mathbb{Z}^d$, where, at each step, the chain jumps to one of the $2d$ neighbors with probability $1/(2d)$. It is called a symmetric random walk on $\mathbb{Z}^d$.

State space $S =$

Transition probabilities and properties

Graph

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

2.3 Gambling strategies

2.3.1 A very bad strategy

Model Play a game, where at each step, you bet all your money. You win 1$ with probability p, and lose (everything) with probability $1-p$. You lose when you have nothing remaining.

State space $S =$

Transition matrix

$$P = \begin{pmatrix} \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

Graph

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	

2.3.2 A very bad strategy, but a nice dealer

Model Play a game, where at each step, you bet all your money. You win 1$ with probability p, and lose (everything) with probability $1 - p$. However, when you do not have any money left, the dealer still allows you to play.

State space $S =$

Transition matrix and properties

$$P = \begin{pmatrix} \\ \\ \\ \\ \\ \end{pmatrix}$$

Graph

Irreducible?	
Irreducible classes	
Absorbing states	
Recurrent states	
Transient states	
Period	