18.03 Uniqueness of Laplace Transform

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In order to use the Laplace transform we need the inverse transform. To do this we should make sure there is such an inverse.

One way to do this is to write a formula for the inverse. This can be done, but it requires either some really fiddly real analysis or some relatively straight-forward complex analyis.

A simple way to show that the inverse exists is to prove the following theorem. As usual, we restrict attention to functions of exponential type. To be precise, we assume $f(t) < M\mathrm{e}^{at}$ for a fixed a and M.

We also only consider continuous functions. The generalization to piecewise continuous functions is easy, as long as we allow the functions to differ at a discrete set of points.

Theorem: Suppose f and g are continuous functions. If $\mathcal{L}f(s) = \mathcal{L}g(s)$ then $f(t) = g(t)$. (Technically we only look at s with $\mathrm{Re}(s) > a$.)

Proof: By linearity it is enough to show that if $\mathcal{L}f = 0$ then $f = 0$.

We prove this as follows.

Lemma: If $h(u)$ is continuous on [0,1] and $\displaystyle\int_0^1 h(u)u^n\,du = 0$ for $n = 0, 1, 2, \ldots$ then $h(u) = 0$.

Proof: Any continuous function on [0,1] is the uniform limit of polynomials. (This means for any small number ϵ we can find a polynomial P_ϵ such that $|h(u) - P_\epsilon(u)| < \epsilon$ for all u.) The hypothesis implies $\displaystyle\int_0^1 h(u)P_\epsilon(u)\,du = 0$. Taking the limit as $\epsilon \to 0$ this implies $\displaystyle\int_0^1 h(u)\cdot h(u)\,du = 0$. Since $h(u)^2 \geq 0$ this implies $h(u) = 0$. QED

Next we assume $\mathcal{L}f(s) = 0$ for all s with $\mathrm{Re}(s) > a$. This implies

$$\int_0^\infty f(t)\mathrm{e}^{-st}\,dt = 0$$

whenever $\mathrm{Re}(s) > a$. Fix s_0 positive with $s_0 > a$ real. Make the change of variables $u = \mathrm{e}^{-t}$. Then at $s = s_0 + n + 1$, $n = 0, 1, 2, \ldots$, we get

$$0 = \mathcal{L}f(s) = \int_0^\infty f(t)\mathrm{e}^{-nt}\mathrm{e}^{-s_0 t}\mathrm{e}^{-t}\,dt = \int_0^1 u^n(u^{s_0}f(-\ln u))\,du$$

By the lemma, $u^{s_0}f(-\ln u) = 0 \;\Rightarrow\; f(t) = 0$. QED.

(There is no problem at $u = 0$ because the assumption on exponential type implies $\displaystyle\lim_{u\to 0} u^{s_0}f(-\ln u) = \lim_{t\to\infty} \mathrm{e}^{-s_0 t}f(t) = 0$.)