On the Interaction Between Risk and Queueing Theories*)

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Introduction

It is now well-known that there exists a strong interaction between risk and queueing theories. However, its deep significance and its contribution to each theory are not sufficiently understood. We give here precise relations showing the contributions and the restrictions of this interaction. A general conclusion will be that for the "classical models", many results of queueing theory may be translated into risk theory for the asymptotic case (i.e. computation of ruin probability in a infinite time) but the inverse is true for the transient study.
We also give some examples of some models which can be used in both theories and we show their interest for the numerical point of view.

§ 1. The GI/G/1 queueing model

The interaction between risk and queueing theories was pointed out by Prabhu (1961), Seal (1972) and more recently by Jewell (1980). Though this interaction is known, it is not really studied and a fortiori used. Also, let us begin with a short description of the GI/G/1 queueing model (using Kendall's notations) that is a system with one server and FIFO (first in, first out) as discipline rule. It can be entirely specified by two renewal processes defined on a suitable probability space $(\Omega, \mathfrak{A}, \mathbb{P})$:

$$(A_n, n \geqq 1), \quad (B_n, n \geqq 1) \tag{1.1}$$

the first one being the interarrival process and the other one the service times process characterized respectively by the d. f. $A(\cdot)$ and $B(\cdot)$ such that $A(0) < 1$ and $B(0) < 1$.
More precisely, assuming that a customer – called number 0 – just arrives at $t = 0$ and is immediately served, let B_n the service time of the $(n-1)$ customer $(n \geqq 1)$ and A_n the interarrival time between the $(n-1)$th and nth customers $(n \geqq 1)$. For the facility, we introduce random variables A_0 and B_0 a. s. equal to 0.
We shall represent by $(N_t, t \geqq 0)$ the counting process related to the renewal process (A_n).
We associate to each queueing model the four following stochastic processes

(i) The process $(W_n, n \geqq 0)$ where W_n represents the *waiting time* of the nth customer, that is the time he waits in the system before beginning his service.
(ii) The process $(W(t), t \geqq 0)$ where $W(t)$ represents the *waiting time of the last customer entered in the system before or at time* t; if follows that:

$$W(t) = W_{N(t)}. \tag{1.2}$$

(iii) The process $(\eta(t), t \geqq 0)$ where $\eta(t)$ is the *virtual waiting time at* t, i.e. the time that a customer would have to wait if he arrived at time t. By opposition, $W(t)$ is

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called the *actual waiting time at* t and W_n *the actual waiting time of the* nth *customer.* If

$$T_0 = 0, \quad T_n = \sum_{i=1}^{n} A_i, \quad n \geqq 1 \tag{1.3}$$

we have the following relations between the three types of waiting times:

$$W_n \;\;\; = \eta(T_n - 0) \tag{1.4}$$

$$W(t) = \eta(T_{N(t)} - 0) \tag{1.5}$$

$$\eta(t) \;\; = \sup\{0, W_{N(t)} + B_{N(t)} - (t - T_{N(t)})\} \tag{1.6}$$

$\eta(t)$ also represents the *service load* of the server at time t, i.e. the time required to serve all customers present in the system at t. A typical trajectory of this process is given in Fig. 1.

(iv) The process $(Q(t), t \geqq 0)$ where $Q(t)$ is the total number of customers in the system at time t.
If $\tilde{Q}(t)$ is the total number of customers actually waiting at t to be served, we clearly have:

$$\tilde{Q}(t) = \begin{cases} -1 + Q(t) & \text{if} \quad Q(t) > 0, \\ \quad 0 & \text{if} \quad Q(t) = 0. \end{cases} \tag{1.7}$$

(v) Let us suppose now that $W_0 = u$ so that the initial customer has to wait a time u before beginning his service. The *busy period* $T(u)$ is defined as the number of steps the server remains continuously busy. We have

$$T(u) = \inf\{n : W_n = 0\}$$

(vi) The traffic intensity φ is defined as the ratio β/α with $\alpha = \mathbb{E}(A_n)$ and $\beta = \mathbb{E}(B_n)$ supposed to be finite.

Remarks

1. The *particular models* frequently used in queueing problems are:
   a) The M/G/1 model for which the d.f. $A(\cdot)$ is exponential negative of parameter λ;
   b) the G/M/1 model for which the d.f. $B(\cdot)$ is exponential negative of parameter μ;
   c) the M/M/1 model for which $A(\cdot)$ and $B(\cdot)$ are both exponential respectively of parameter λ and μ. In this case

$$\varphi = \frac{\lambda}{\mu}. \tag{1.8}$$

2. *Stationary version of the* GI/G/1 *model*
   If the time origin does not coincide with the first arrival time, this last random variable A_0 has a d.f. in general different from the d.f. $A(\cdot)$ so that the renewal process defined by the random sequence $(A_n, n \geqq 0)$ is a so-called general or delayed renewal process (see Feller (1968)). This process is stationary iff the d.f. of A_0 is

$$\begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^x (1 - A(y)\,dy, & x \geqq 0; \\ \quad 0 & x < 0. \end{cases} \tag{1.9}$$

In this case, we shall speak of the stationary version of the GI/G/1 model.

§ 2. Connection with the classical risk theory

We only consider in this paper what we call the classical risk models i.e. the E. S. Andersen (1957) presentation of risk model and its most important particular cases. With the same notations as those given in the preceding paragraph we have the following interpretations:

(i) $(A_n, n \geqq 1)$ is the process of successive interarrivals of claims,
(ii) $(B_n, n \geqq 1)$ is the process of successive claim amounts.

We suppose that we start just after a claim arrival and that the initial fortune or reserve of the insurance company after payment of the initial claim is u ($u \geqq 0$). We also suppose that

(iii) the incomes of the company have a constant rate $c\,(c > 0)$.

Remark

Without loss of generality, we may suppose $c = 1$.
Indeed, if $c \neq 1$, we can introduce the sequence $(\tilde{A}_n)$ such that

$$\tilde{A}_n = c\,A_n$$

so that $\tilde{F}$, the d.f. of $\tilde{A}_n$ is given by:

$$\tilde{F}(x) = F\left(\frac{x}{c}\right)$$

and if

$$\tilde{\alpha}_n = \mathbb{E}(\tilde{A}_n)$$

we have

$$\tilde{\alpha}_n = c\,\alpha_n\,.$$

Then the sequences $((\tilde{A}_n), (B_n))$ define a risk model equivalent to the model treated and for which the condition $c = 1$ is satisfied.
This model is called a *positive capital risk*. For a *negative capital risk model* we have:

(i)′ as (i) above
(ii)′ $(B_n, n \geqq 1)$ is the process of successive incomes of the company
(iii)′ the outcomes of the company have a constant rate $c\,(c < 0)$ (without loss of generality, we may suppose that $c = 1$).

When in the sequel we will speak of the risk model, we always mean the positive capital risk version.
We associate to the risk model the following stochastic processes:

(i) $(X_n, n \geqq 0)$, where $X_n = B_n - c\,A_n,\ n \geqq 0$ (2.1)

(ii) $(S_n, n \geqq 0)$, where $S_n = \sum_1^n X_k,\quad n \geqq 0,\quad S_0 = 0$ (2.2)

(iii) $(T_n, n \geqq 0)$, $T_n = \sum_{i=1}^n A_i,\quad n > 0,\quad T_0 = 0$ (2.3)

(iv) $(M_n, n \geqq 0)$, $M_n = \sup\{S_0, S_1, \ldots, S_n\}$ (2.4)

(v) $(N(t), t \geqq 0)$, where $N(t)$ is the total number of claims in $(0, t]$

(vi) $(X(t), t > 0)$, where $X(t)$ represents the total amount of claims paid on $(0, t]$; clearly

$$X(t) = \begin{cases} 0, & t < A_1, \\ \sum_{k=1}^{N(t)} B_k, & t \geqq A_1, \end{cases} \tag{2.5}$$

(vii) $(Z(t), t > 0)$, where $Z(t)$ represents the "fortune" of the insurer at time t, that is

$$Z(t) = u + c\,t - X(t)\,. \tag{2.6}$$

Remarks

1. The particular queueing models are of course valid in the risk version; in particular the M/G/1 case is the Lundberg-Cramer risk model.
2. The stationary version defined in § 1, is of course still valid and was introduced by Janssen (1969) and also studied by Thorin (1975).

The two main problems in risk theory are

1°) *the transient problem:* How to find the value of $\psi(u, t)$, the *non-ruin probability on* $(0, t]$, defined by:

$$\psi(u, t) = \mathbb{P}[Z(t') \geqq 0, t' \in (0, t) \,|\, Z(0) = u], \quad t > 0, \quad u \geqq 0 \tag{2.7}$$

2°) *the asymptotical problem:* How to find the value of $\psi(u)$, *the non-ruin probability on* $(0, \infty)$, defined by

$$\psi(u) = \mathbb{P}[Z(t') \geqq 0, t' \in (0, \infty) \,|\, Z(0) = u], \quad u \geqq 0\,. \tag{2.8}$$

As almost all the trajectories of the process $(Z(t), t \geqq 0)$ are caglad – i.e. they are left continued and have existing right limits at all points – this process is separable and consequently:

$$\psi(u) = \lim_{t \to \infty} \psi(u, t) \tag{2.9}$$

a result, first proved by Cramer (1955) for the Lundberg-Cramer model and generalized after by Brans (1966, 1967).

§ 3. Duality risk theory – queueing theory

The connection between trajectories of the virtual waiting time process and those of the $Z(t)$-process in risk theory with $c = 1$ is intuitively clear if we reverse for example the time in Fig. 1 on $[0, T]$. We then get the Figure 2, that is a trajectory of $Z(\cdot)$ on $[0, T]$. Of course, this procedure is involutive. Then, we obtain trajectories having the same structure from the geometric point of view. The basic problem is to show that this "reversal time" technique leads to an equivalence between the two models from the probabilistic point of view. This is easily done here and this equivalence still remains for more general models (see Janssen (1977, 1978)).
From these results, it follows that:

$$\mathbb{P}[W_n \leqq x] = \mathbb{P}[M_n \leqq x]\,, \tag{3.1}$$

$$\mathbb{P}[W_{N(t)} \leqq x] = \mathbb{P}[M_{N(t)} \leqq x]\,. \tag{3.2}$$

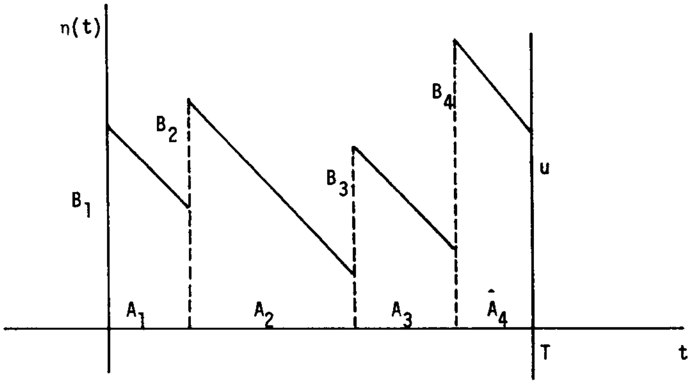

Fig. 1. A trajectory of $\eta(\cdot)$ on [0, T] ...

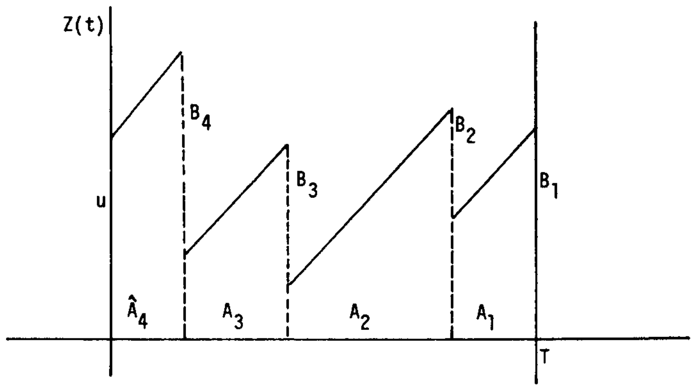

Fig. 2. ... and its reversed time version as a trajectory of $Z(\cdot)$ on [0, T].

As the second member of (3.1) is the probability of non-ruin after the nth claim – noted $\psi_n(x)$ – and the one of (3.2) the probability of non-ruin on (0, T], we get

$$\mathbb{P}[W_n \leqq x] = \psi_n(x)\,, \tag{3.3}$$

$$\mathbb{P}[W_{N(t)} \leqq x] = \psi(x, t)\,. \tag{3.4}$$

The equality (3.3) – already quoted by Feller (1966) in the framework of random walk theory – is trivial from the fact that the relation

$$W_0 = 0$$

$$W_n = \sup\{0, W_{n-1} + X_n\}$$

given

$$W_n = \sup\{X_n, X_n + X_{n-1}, \dots, X_n + \dots + X_1, 0\}$$

and from the independence of the variables $(X_n, n > 0)$.
Let us also remark that the discrete time process $(W_n, n \geqq 0)$ is easier to study than $(M_n, n \geqq 0)$ because it is a markovian process.

§ 4. Consequences for the transient problem

For positive u, t, ξ, let us define the following probability:

$$\psi(u, t, \xi) = \mathbb{P}[T_{N(t)} \leqq t - \xi, W_{N(t)} \leqq u] \; (= \mathbb{P}[T_{N(t)} \leqq t - \xi, M_{N(t)} \leqq u]) . \tag{4.1}$$

Then, it can be shown (Janssen (1978)) that :

$$\psi(u, t, \xi) = \int_0^{t-\xi} [1 - A(t - \tau)] \, G(d\tau, u) \tag{4.2}$$

where

$$G(t, x) = \sum_{n=0}^{\infty} R^{[n]}(t, x) \tag{4.3}$$

with

$$R(t, x) = \int_0^t B(a + x) \, A(dx), \quad t \geqq 0, \quad x \in \mathbb{R} \text{ (that is } R(t, x) = \mathbb{P}[A_n \leqq t, X_n \leqq x]) \tag{4.4}$$

and

$$R^{[0]}(t, x) = R^+(t, x) \; (= U_0(t) \, U_0(x) \, R(t, x)) \tag{4.5}$$

$$R^{[n]}(t, x) = (R \otimes R^{[n-1]})^+(t, x), \quad n \geqq 1 \tag{4.6}$$

the symbol $\otimes$ representing the two-dimensional convolution product.
From (4.3), it can be seen that the function $(G(\dots)$ is solution of the integral equation:

$$G(t, x) = U_0(t) \, U_0(x) \left[1 + \int_0^t \int_0^{\infty} R(t - \bar{t}, x - \bar{x}) \, G(d\bar{t}, d\bar{x}) \right] \tag{4.7}$$

It follows a theoretical "explicit" expression of $\psi(\dots)$:

$$\psi(u, t) = \int_0^t [1 - A(t - \tau)] \, G(d\tau, u) \tag{4.8}$$

and consequently, from (3.2), also for the distribution of the waiting time $W_{N(t)}$,

§ 5. Consequence for the asymptotical behaviour

From the preceding section, we have

$$\psi(t, x) = \mathbb{P}[W_{N(t)} \leqq x] \tag{5.1}$$

and

$$\lim_t \psi(t, x) = \lim_t \mathbb{P}[W_{N(t)} \leqq x] \tag{5.2}$$

As the limit of the second member is also the $\lim_n P[W_n \leqq x]$, we get

$$\psi(x) = \lim_n \mathbb{P}[W_n \leqq x] \tag{5.3}$$

Now, the question is: what can we find in the abundant queueing literature as results related to the second member of relation (5.3)?
It is a fact that almost all the results concern the limit distribution of the virtual waiting time distribution. However if we set

$$\psi^*(x) = \lim_t \mathbb{P}[\eta(t) \leqq x] \tag{5.4}$$

there exists simple relations between $\psi^*(\cdot)$ and $\psi(\cdot)$ (see Takacs (1963) and more recently Harrison and Lemoine (1976)), of course in the case $\varphi < 1$:

$$\psi^*(x) = (1 - \varphi) + \varphi\, \psi * \hat{B}(x) \tag{5.5}$$

$$\psi^*(x) = U_0(x) \cdot \psi * E(x) \tag{5.6}$$

with

$$\hat{B}(x) = \frac{1}{\beta} \int_0^x (1 - B(t))\, dt \tag{5.7}$$

$$E(x) = \int_{\mathbb{R}} B(x + y)\, d\, \hat{A}(y)\,. \tag{5.8}$$

It follows that the knowledge of ψ gives directly ψ^* by an integration and that inversely, the knowledge of ψ^* gives ψ up to a simple integral equation. However, in the M/G/1 model we get:

$$\hat{A}(\cdot) = A(\cdot) \tag{5.9}$$

so that from (5.8)

$$E(x) = F(x)$$

with

$$F(x) = \mathbb{P}(X_n \leqq x) \tag{5.10}$$

and (5.6) becomes

$$\psi^*(x) = U_0(x) \cdot \psi * F(x) \tag{5.11}$$

so that

$$\psi^*(x) = \psi(x) \tag{5.12}$$

by the well-known Lindley's equation (Lindley (1952)):

$$\psi(x) = U_0(x)\, \psi * F(x) \tag{5.13}$$

It follows that for the Cramer risk model, each result on the virtual waiting time can be transposed on the non-ruin probability $\psi(x)$ but in general that is not the case since for the GI/G/1 model $\psi(\cdot) \not\equiv \psi^*(\cdot)$.

The following figure summarizes the interaction between risk and queueing theories in the asymptotical case:

Model	Queueing Theory	Risk Theory
M/G/1	dis. virtual waiting t dis. actual waiting t	$\equiv$ non-ruin probab. $\psi(\cdot)$
GI/G/1	dis. virtual waiting t $\Uparrow \; \# \; \Downarrow$ dis. actual waiting t	$\equiv$ non-ruin probab. $\psi(\cdot)$

Note

$\Rightarrow$: means, given ψ, we get ψ^* by an integration,

$\leftarrow\cdots$: means, given ψ^*, we get ψ up to the resolution of a integral equation (in fact, we have the expression of the Laplace transform of ψ by (5.5) or (5.6).

§ 6. Particular cases

This section gives some applications of the preceding results. For the simplicity, we use queueing notations.

6.1 Asymptotical results

1. The M/M/1 model

If λ and μ are the parameters respectively of the d.f. of the interarrival times and the service times, it is well-known that if $\varphi = \frac{\lambda}{\mu} < 1$, then

$$\psi(x) = 1 - \varphi\, e^{-\mu x(1-\varphi)}, \quad x \geqq 0 \tag{6.1}$$

a classical result both in queueing and risk theories.

2. The M/G/1 model

Let λ be the parameter of the negative exponential distribution of the interarrival times; the famous Pollaczeck-Khinchine formula gives $\hat{\psi}$, the Laplace transform of ψ in term of $\hat{B}$, the one of the d.f. B:

$$\hat{\psi}(s) = \frac{(1-\varphi)\, s}{\lambda\, \hat{B}(s) - \lambda + s} \tag{6.2}$$

so that (Janssen (1969)), for $x \geqq 0$:

$$\psi(x) = (1-\varphi) \sum_{n=0}^{\infty} \varphi^n\, \hat{B}^{(n)}(x) \tag{6.3}$$

where $\hat{B}^{(n)}$ is the n-fold convoluated of $\hat{B}$ defined by

$$\hat{B}(x) = \begin{cases} 0\,, & x < 0 \\ \frac{1}{\beta} \int_0^x (1 - B(v))\, dv\,, & x \geqq 0 \end{cases} \tag{6.4}$$

3. The $E_k | E_{k'} | 1$ model ($k', k \in \mathbb{N}_0$)

Let us recall briefly that the density function on $\mathbb{R}^+$ is of Erlang type of parameter k if it has the form:

$$f(x) = \frac{(\mu\, k)^k}{(k-1)!}\, x^{k-1}\, e^{-k\mu x} \tag{6.5}$$

(with $\mu > 0$).

In this case, the mean is $\frac{1}{\mu}$ and the variance $\frac{1}{k\,\mu^2}$. In fact, the random X having such a distribution is characterized as a sum of k i.i.d. random variables having a negative exponential distribution of parameter $\frac{1}{k\,\mu}$.

Prabhu (1965) shows that for this model, we have:

$$\psi^*(x) = 1 - \sum_{r=1}^{k'} A_r\, e^{-\theta r x} (x \geqq 0) \tag{6.6}$$

where

$$A_r = \left(1 - \theta_r \frac{\beta}{k'}\right)^{k'} \prod_{\substack{p=1 \\ p \neq r}}^{k'} \left(1 - \frac{\theta_r}{\theta_p}\right)^{-1}, \qquad r = 1, \ldots, k' \tag{6.7}$$

$\theta_1, \ldots, \theta_p$ being the roots with positive real parts of:

$$\left(\frac{k'}{\beta} - \theta\right)^{k'} = \left(\frac{k'}{\beta}\right)^{k'} \left(\frac{k}{k + \alpha\theta}\right)^{k}. \tag{6.8}$$

For the particular model $M/E_{k'}/1$, – for which $\psi \equiv \psi^*$ –, it follows from the preceding results (Prabhu (1965)) that

$$\psi(0) = 1 - \varphi \tag{6.9}$$

with $\varphi = \lambda\beta$ (λ being the unit rate of arrivals) and

$$\frac{d\psi}{dx}(x) = (1 - \varphi) \sum_{\nu=1}^{k'} \bar{A}_\nu m_\nu k' \xi \, e^{-k'\xi(1-m_\nu)x} \tag{6.10}$$

with

$$\bar{A}_\nu = \prod_{i \neq \nu} \left(1 - \frac{m_i}{m_\nu}\right)^{-1}, \quad 1 \leqq \nu \leqq k' \tag{6.11}$$

$m_i = \dfrac{1}{Z_i}$, $|m_i| < 1$, $i = 1, \ldots, k'$ and $Z_1, \ldots, Z_{k'}$ are roots of the equation:

$$Z^{k'} + Z^{k'-1} + \ldots + Z = \frac{\varphi}{k'}. \tag{6.12}$$

4. The M/D/1 model

For this model, the d.f. A is negative exponential of parameter λ and the service times are deterministic, i.e. $B_n = \beta$ for all n. This model has already been considered by Erlang (1909) who showed that

$$\psi(x) = (1 - \varphi) \sum_{r=0}^{m} e^{\lambda(x - r\beta)} \frac{[-\lambda(x - r\,b)]^r}{r!} \tag{6.13}$$

if $x = n + b\,t$, $n \in \mathbb{N}$, $0 \leqq t < \beta$.

5. The $D/E_{k'}/1$ model

To obtain the limit distribution of the actual waiting time, Lindley (1952) showed that:

$$\psi(x) = 1 + \sum_{i=1}^{k'} c_i \, e^{\lambda_i x} \tag{6.14}$$

when c_i, λ_i, $i = 1, \ldots, k'$ are complex numbers with $\mathrm{Re}(\lambda_i) < 0$ uniquely determined by the relations:

$$\left(\frac{k'\mu}{k'\mu + \lambda_i}\right)^{k'} = e^{-\lambda_i a} \quad i = 1, \ldots, k' \tag{6.15}$$

$$\sum_{i=0}^{k'} \frac{c_i}{(k'\mu + \lambda_i)^{s+1}} = 0 \qquad s = 0, \ldots, k'-1 \tag{6.16}$$

with

$$\lambda_0 = 0, \quad c_0 = 1, \quad \mu = \frac{1}{\beta}. \tag{6.17}$$

6. The M/PH/1 model

The family of phase type distributions have been introduced by Neuts (1975) by the following manner: let Q be the infinitesimal generator of a (m + 1)-state continuous parameter Markov chain ($m \geqq 1$) of the form:

$$Q = \begin{pmatrix} T & T^0 \\ O & 0 \end{pmatrix} \tag{6.18}$$

where: T is a $m \times m$ matrix with $T_{ii} < 0$, $i = 1, \ldots, m$

$O = (0, \ldots, 0)$

T^0 a m-dimensional vector

$Te + T^0 = O'$ with $e' = (1, \ldots, 1)$ *).

It can be shown that the state (m + 1) is absorbing and that the first m states are transient iff the matrix T is inversible. In this case, it follows from the theory of Markov chains (see for example Chung (1960)) that the event "absorption into the state (m + 1)" is a.s. certain. Let (α, α_{m+1}) be an initial distribution – with $\alpha = (\alpha_1, \ldots, \alpha_m)$ – for the considered Markov chain. Then, Neuts (1975) proves that the d.f. H of time till absorption is given by:

$$H(x) = 1 - \alpha \exp T x e, \quad x \geqq 0 . \tag{6.19}$$

By definition, H is a d.f. of phase type with representation (α, T). As particular cases, let us mention:

a) the generalized Erlang d.f.

Such a d.f. corresponds to the convolution of m independent negative exponential distributions of parameters $\mu_1, \ldots, \mu_m$ so that the Laplace transform of the derivative h of it is given by

$$\tilde{h}(s) = \frac{\mu_1}{\mu_1 + s} \cdot \ldots \cdot \frac{\mu_m}{\mu_m + s} . \tag{6.20}$$

This corresponds to a phase type distribution for which:

$$\alpha_1 = 1, \quad \alpha_i = 0, \quad 2 \leqq i \leqq m + 1$$

$$T = \begin{pmatrix} -\mu_1 & \mu_1 & & & 0 \\ 0 & -\mu_2 & \mu_2 & 0 \ldots & 0 \\ \cdot & & \cdot & & \\ \cdot & & & & \\ \cdot & & & \cdot & \\ 0 & & & \cdot & \mu_{m-1} \\ & & & \cdot & -\mu_m \end{pmatrix} \tag{6.21}$$

b) the hyperexponential d.f.

Here

$$H(x) = \sum_{i=1}^{m} \alpha_i (1 - e^{-\mu_i x}) \tag{6.22}$$

$$\alpha_i > 0, \quad \mu_i > 0, \quad i = 1, \ldots, m, \quad \sum_{j=1}^{m} \alpha_j = 1 .$$

This is a phase type d.f. for which:

$$\alpha = (\alpha_1, \ldots, \alpha_m), \quad \alpha_{m+1} = 0$$

T is the diagonal matrix with $(-\mu_1, \ldots, -\mu_m)$ as diagonal.

*) The symbol ′ denotes the transposed vector or matrix.

Now if we consider a queueing system with a negative exponential distribution of parameter λ for the interarrival times and with a phase type distribution of representation (α, T) for the service times, we have the model $M|PH|1$ treated by Neuts (1977). Of course when $\varphi < 1$, he showed that the d.f. ψ of the limiting waiting time is also of phase type of representation (γ, L) where:

$$\gamma = \varphi \pi, \quad \gamma_{m+1} = 1 - \varphi \tag{6.23}$$

$$L = T + \varphi T^0 \pi, \quad L^0 = (1 - \varphi) T^0 \tag{6.24}$$

with

$$\pi = \begin{pmatrix} \pi_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & \pi_m \end{pmatrix} \text{ such that } \quad \pi(T + T^0 A) = 0, \quad \pi e = 1 , \tag{6.25}$$

where

$$A_{ij} = (\delta_{ij} \alpha_j) \text{ i; } \quad j = 1, \ldots, m . \tag{6.26}$$

This result leads to approximation results such that

$$1 - \psi(x) \sim e^{\eta x}$$

(where η is the largest negative real eigenvalue of $S = T + \varphi T^0 \pi$, and $K = \varphi(\pi u)(ve)$ where u and v are respectively right and left eigenvectors of S corresponding to η such that their scalar product is 1 and also to efficient algorithms of computation.
Let us also mention that there exist analogous results for the GI/PH/1 model both for actual and virtual waiting times (Neuts (1978)).

6.2. Transient results

The preceding subsection gives asymptotical results coming from queueing theory which may be very useful for risk theory. If we look for transient results, we see that despite the intensive literature on queueing models such results are very rare – (with some exceptions, as for example Ledermann & Reuter (1954)) – and in this case, the literature on risk theory is more interesting. As transient results for the M/G/1 have just been treated by Delfosse & Janssen (1982), we will here give only two examples: the M/M/1 and the M/D/1 models.

1. The M/M/1 model

Without loss of generality, we set $\lambda = \mu = 1$. In this case, the well-known Benès-Prabhu-Seal formula (see for example Seal (1978)) gives:

$$\psi(0, t) = e^{-t} + \int_0^{ct} \left(1 - \frac{y}{ct}\right) t \frac{e^{-t-y}}{\sqrt{ty}} I_1(2\sqrt{ty})\, dy \tag{6.27}$$

$$\psi(u, t) = e^{-t} + \int_0^{u+ct} \frac{t\, e^{-t-y}}{\sqrt{ty}} I_1(2\sqrt{ty})\, dy$$

$$- c \int_0^t \psi(0, t-r)\, r \frac{e^{-u-cr-r}}{\sqrt{r(u+cr)}} I_1(2\sqrt{r(u+cr)})\, dr \tag{6.28}$$

where $I_1(\cdot)$ is the first modified Bessel function.

It is possible to compute directly $\psi(u, t)$ by the formula

$$\psi(u, t) = e^{-t} + \int_0^{u+ct} f_t(x)\, dx + c \int_0^t e^{(z-t)(1-c)} f_z(u + c\, t)\, dz \tag{6.29}$$

where

$$f_t(x) = t\, e^{-t-x} \frac{1}{\sqrt{t\, x}} I_1(2\sqrt{t\, x}), \quad x \geqq 0 .$$

These relations lead to good numerical results for every value of λ and μ (see Delfosse & Janssen (1982)).

2. The M/D/1 model

Starting from the same Prabhu-Benès-Seal formula, it can be shown that, for $\lambda = 1$ and $\beta = h$:

$$\psi(0, t) = \sum_{n=0}^{\left[\frac{c\, t}{h}\right]} e^{-t} \frac{t^n}{n!}\left(1 - \frac{n\, h}{c\, t}\right) \tag{6.30}$$

$$\psi(u, t) = \sum_{n=0}^{m} e^{-t} \frac{t^j}{j!} - \sum_{n=k}^{m} e^{-(n\, h-u)/c} \frac{[(n\, h - u)/c]^n}{n!} \cdot \psi\left(0, t - \frac{(n\, h - u)}{c}\right) \tag{6.31}$$

where

$$m = \left[\frac{u + c\, t}{h}\right], \quad k = \left[\frac{u}{h}\right] + 1 .$$

§ 7. Conclusion

For the asymptotical behaviour, the passage from queueing theory to risk theory is the most interesting way to obtain some explicit expressions of $\psi(u)$ on computational algorithms.

Nevertheless, the passage risk → queueing can sometimes give useful informations to get the virtual waiting distribution in other models than the M/G/1. For the transient behaviour, the passage risk theory to queueing theory seems to be the only interesting way to get transient results for the waiting time distribution.

From the conceptual point of view, we must mention that the literature on queueing gives a lot of non-classical models which may be useful to obtain risk models taking account of new concepts such as environment, economical situation, ... and so giving a more realistic approach of the problem.

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Zusammenfassung

In diesem Artikel werden in präziser Weise der Beitrag und die Grenzen der Wechselbeziehung zwischen Risiko- und Warteschlangentheorie aufgezeigt. Daraus ergibt sich, daß viele Ergebnisse der Warteschlangentheorie auf die Risikotheorie für den asymptotischen Fall übertragen werden können, während die Umkehrung für transiente Untersuchungen gültig ist (sowohl bzgl. der Wartezeitverteilungen als auch der Ruinwahrscheinlichkeiten).
Es werden einige Beispiele angegeben, die vom numerischen Standpunkt aus das Interesse an dieser Wechselbeziehung zeigen.

Summary

This paper gives precise relations showing the contribution and the restriction of the interaction between risk and queueing theories from which it follows that many results of queueing theory may be translated into risk theory for the asymptotic case but the inverse is true for the transient study (concerning both waiting time distributions and ruin probabilities).
Some examples are given showing the interest of this interaction from the numerical point of view.