Chordal graphs: a linear testing algorithm

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14 June 2017

Motivations

- Many problems are in general **NP**-hard for graphs. For example find Hamiltonian cycles or maximal cliques.
- On particular classes of graphs though these problems can become low-degree polynomial.
- Chordal graphs are one of those classes for which many **NP**-hard problems become low-degree polynomial (e.g. maximal cliques).
- At this point become important that testing chordality is itself low-degree polynomial.

The algorithm

Preliminary definitions

Definition (Undirected graph)

A *graph* G is a pair (V, E) where V is a finite set and E is a set of $2-$subsets of V. Elements of V are called *vertices* while elements of E are called *edges*. $|V| = n$, $|E| = m$.

Definitions (Path, Cycle, Chord)

- A *path* π is a sequence of distinct vertices $v_1, v_2, \cdots v_i, \cdots, v_k$ where $\{v_i, v_{i+1}\} \in E$ with $1 \leq i < k$.
- A *cycle* of lenght $k + 1$ is a closed path, i.e. a path for which $\{v_1, v_k\} \in E$.
- A *chord* is an edge connecting two nonconsecutive vertices of a cycle.

Definition (Chordal graph)

A graph is chordal if every cycle of lenght at least four has a chord.

Orderings and fill-ins

Definition (Ordering)

An *ordering* is a bijection $\alpha : V \to \{1, 2, \cdots, n\}$. $v <_\alpha w$ iff $V(v) < V(w)$.

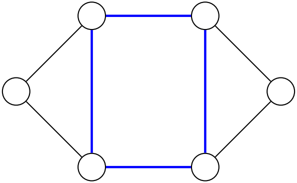
Definition (Fill-in)

A *fill-in* induced by an ordering α is a set of edges $F(\alpha) \notin E$ such that there exists a path containing only u, v and vertices ordered after both u and v. $F(\alpha)$ is *zero fill-in* if $F(\alpha) = \varnothing$ and α is a zero fill-in ordering.

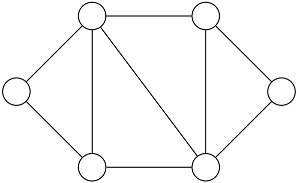
Definition (Elimination graph)

The *elimination graph* of G w.r.t. the ordering α is $G(\alpha) = (V, E \cup F(\alpha))$.

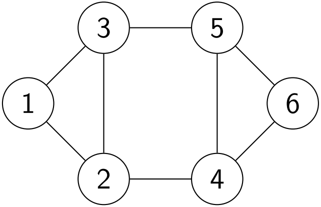
Example



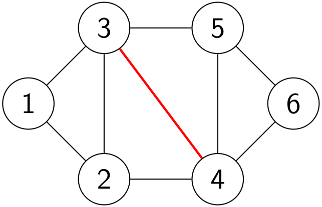
not chordal



chordal



ordered



elimination graph

Testing chordality

Theorem

A graph is chordal if and only if it has a zero fill-in ordering.

We want to compute an ordering α that is zero fill-in if and only if G is chordal. In this way we can compute $F(\alpha)$. G is chordal if and only if $F(\alpha) = \varnothing$.

Definition (Maximum cardinality search (MCS))

It's an ordering algorithm in which at each step i (from 1 to n) the vertex selected and numberd with i among the unnumbered ones is that adjacent to the largest number of previously numberd vertices, breaking ties arbitrarily.

Theorem

An ordering generated by MCS is zero fill-in if the graph is chordal.

MCS - complexity analysis

Theorem

Complexity of the algorithm that computes the MCS ordering is $\mathcal{O}(n+m)$.

Proof.

The first two for loops are $\mathcal{O}(n)$. The third is also $\mathcal{O}(n)$ but there are inner loops. However in the first one every edge is scanned at most twice, while the second one is executed at most n times yielding a total cost of the outer loop of $\mathcal{O}(n+m)$. □

```
for i = 0 to n − 1 do
    set[i] := ∅
end for
for all v in V do
    size[v] := 0; set[0] := set[0] ∪ {v}
end for
j := 0
for i = 1 to n do
    v := delete any node from set[j]
    α[v] := i; α^{-1}[i] := v; size[v] := −1
    for all (u, v) ∈ E and size[u] ≥ 0 do
        delete u from set[size[u]]
        size[u] := size[u] + 1
        set[size[u]] := set[size[u]] ∪ {u}
    end for
    j := j + 1
    while j ≥ 0 and set[j] = ∅ do
        j := j − 1
    end while
end for
```

Computing the fill-in

Definition

The *follower* of a vertex v, $f(v)$ is the vertex w of largest number (w.r.t to α) both adjacent to v in $G(\alpha)$ and such that $w <_\alpha v$. For $i \geq 0$, $f^0(v) = v$ and $f^{i+1}(v) = f(f^i(v))$.

Theorem

If $x, w \in V$ with $w <_\alpha x$, then $(x, w) \in E \cup F(\alpha)$ if and only if there is a vertex v such that $(v, w) \in E$ and $f^i(v) = x$ for some $i \geq 0$.

With this theorem we can compute for any vertex w, the set $A(w) = \{x|(x, w) \in E \cup F(\alpha), w <_\alpha x\}$ and also all vertices x such that $f(x) = w$.

Computing the fill-in - complexity analysis

Theorem

Complexity of the algorithm that computes the fill-in of a graph G is $\mathcal{O}(n + m')$ where $m' = |E \cup F(\alpha)|$.

Proof.

The outer loop is executed n times. The inner one is scans each vertex of the elimination graph at most twice yielding a total cost of the outer loop of $\mathcal{O}(n + m')$. □

for $i = n$ **to** 1 **do**
    $w := \alpha^{-1}[i]$
    $f[w] := w$
    $index[w] := i$
    for all $v \in V$ s.t. $(v, w) \in E$ and $\alpha[v] > i$
        $x := v$
        while $index[x] > i$ **do**
            $index[x] := i$
            add (x, w) to $E \cup F(\alpha)$
            $x := f[x]$
        end while
        if $f[x] = x$ **then**
            $f[x] := w$
        end if
    end for
end for

Testing cordality - complexity analysis

Theorem

Complexity of the algorithm that recognizes cordality of a graph G is $\mathcal{O}(n+m)$.

Proof.

The outer loop is executed n times. The inner one is scans each vertex of the G at most twice yielding a total cost of the outer loop of $\mathcal{O}(n+m)$. □

```
for i = n to 1 do
    w := α^{-1}[i]
    f[w] := w
    index[w] := i
    for all v ∈ V s.t. (v, w) ∈ E and α[v] > i
        x := v
        while index[x] > i do
            index[x] := i
            if (x, w) ∉ E and x ≠ w then
                return false
            end if
            x := f[x]
        end while
        if f[x] = x then
            f[x] := w
        end if
    end for
end for
return true
```

Implementation

Language and libraries

- **JavaSE 8** was used to implement the algorithm.
- Graph data structures were provided by **JUNG** library and graphs were stored in **PajekNet** format.
- **Apache Maven** was used to handle the project.
- **Eclipse** was used as IDE.
- The code is released under **GPL 3** license.
- **Oracle Java Mission Control** with **Flight Recorder** was used to profile the application.

Data structures

Two data structures were defined to implement the algorithms.

Vertex

- `alpha:int`
- `size:int`
- `index:int`
- `follower:Vertex`

ChordalAlgorithms

- `order:List<Vertex>`
- `graph:UndirectedGraph<Vertex,Integer>`
- `maximumCardinalitySearch():void`
- `isChordal():boolean`

Testing environment

- **JVM**: Oracle 1.8.0_111-b14
- **OS**: Ubuntu 15.10
- **Linux kernel**: 4.2.0-42-generic
- **libc**: glibc 2.21
- **System architecture**: x86_64
- **CPU**: Intel Core i7-2600K CPU @ 3.40GHz
- **Memory**: 8 GB

Datasets

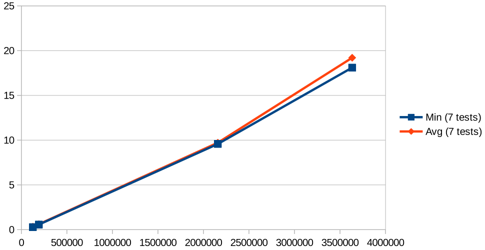
- I chose to use **real networks** data to test the algorithms[1].
- All graphs from datasets were obviously non chordal.
- For every datasets, I computed the elimination graph, so to have a chordal graph and I exported it.
- To measure performances I used only chordal graphs so that the complexity is $\Theta(n+m)$.

[1] `http://vlado.fmf.uni-lj.si/pub/networks/data/`; M. Boguña, R. Pastor-Satorras, A. Diaz-Guilera and A. Arenas, Physical Review E, vol. 70, 056122 (2004)

Profiling

Total time is the actual time spent by the application to execute the two methods of the algorithm, `maximumCardinalitySearch()` and `isChordal()`. The data structure `ChordalAlgorithms` was already filled in with the graph read from file. Seven tests were executed for each dataset.

Dataset	n+m	Total average time	Total minimum time
YeastChordal	126547	328 ms	282 ms
PGPChordal	192058	608 ms	569 ms
MiniDaysAllChordal	2157699	9s 572 ms	9 s 711 ms
DaysAllChordal	3633865	19s 209 ms	18 s 103 ms



Parallel versions

Parallel versions of the algorithm

Recognizing chordal graphs is in the complexity class **NC**, as shown for the first time by Edenbrandt (1986), using a different characterization of chordal graphs. After this seminal paper other results were achieved in optimizing the algorithm.

Authors	**Complexity**	**Number of processors**	**Cost model**
Edenbrandt	$\mathcal{O}(\log n)$	$\mathcal{O}(n^5)$	PRAM CREW
Chandrasekharan et al.	$\mathcal{O}(\log n)$	$\mathcal{O}(n^4)$	PRAM CRCW
Klein	$\mathcal{O}(\log^2 n)$	$\mathcal{O}(n+m)$	PRAM CRCW
Klein	$\mathcal{O}(\log^2 n)$	$\mathcal{O}(\frac{n+m}{\log n})$	PRAM CRCW (randomized)

Bibliografia

N. Chandrasekharan and S. S. Iyengar.
NC algorithms for recognizing chordal graphs and k trees.
IEEE Transactions on Computers, 37(10):1178–1183, October 1988.

Anders Edenbrandt.
Chordal graph recognition is in NC.
Information Processing Letters, 24(4):239–241, March 1987.

Jean-François Huard.
Chordal Graphs: Their Testing and Their Role Chordal Graphs: Their Testing and Their Role.

P. Klein.
Efficient Parallel Algorithms for Chordal Graphs.
SIAM J. Comput., 25(4):797–827, August 1996.

R. Tarjan and M. Yannakakis.
Simple Linear-Time Algorithms to Test Chordality of Graphs, Test Acyclicity of Hypergraphs, and Selectively Reduce Acyclic Hypergraphs.
SIAM J. Comput., 13(3):566–579, August 1984.