Chordal graphs: a linear testing algorithm

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Preliminary definitions

Definition (Undirected graph)

A *graph* G is a pair (V, E) where V is a finite set and E is a set of $2-$subsets of V. Elements of V are called *vertices* while elements of E are called *edges*. $|V| = n$, $|E| = m$.

Definitions (Path, Cycle, Chord)

- A *path* π is a sequence of distinct vertices $v_1, v_2, \cdots v_i, \cdots, v_k$ where $\{v_i, v_{i+1}\} \in E$ with $1 \leq i < k$.
- A *cycle* of lenght $k + 1$ is a closed path, i.e. a path for which $\{v_1, v_k\} \in E$.
- A *chord* is an edge connecting two nonconsecutive vertices of a cycle.

Definition (Chordal graph)

A graph is chordal if every cycle of lenght at least four has a chord.

Orderings and fill-ins

Definition (Ordering)

An *ordering* is a bijection $\alpha : V \to \{1, 2, \cdots, n\}$. $v <_\alpha w$ iff $V(v) < V(w)$.

Definition (Fill-in)

A *fill-in* induced by an ordering α is a set of edges $F(\alpha) \notin E$ such that there exists a path containing only u, v and vertices ordered after both u and v. $F(\alpha)$ is *zero fill-in* if $F(\alpha) = \varnothing$ and α is a zero fill-in ordering.

Definition (Elimination graph)

The *elimination graph* of G w.r.t. the ordering α is $G(\alpha) = (V, E \cap F(\alpha))$.

Testing chordality

Theorem

A graph is chordal if and only if it has a zero fill-in ordering.

We want to compute an ordering α that is zero fill-in if and only if G is chordal. In this way we can compute $F(\alpha)$. G is chordal if and only if $F(\alpha) = \varnothing$.

Definition (Maximum cardinality search (MCS))

It's an ordering algorithm in which at each step i (from 1 to n) the vertex selected and numberd with i among the unnumbered ones is that adjacent to the largest number of previously numberd vertices, breaking ties arbitrarily.

Theorem

An ordering generated by MCS is zero fill-in if the graph is chordal.