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SCHOOL OF EDUCATION, CULTURE AND COMMUNICATION
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MÄLARDALEN UNIVERSITY SWEDEN

Master Thesis in Mathematics with Specialization in Financial Engineering

Mispricing
Due to Nelson-Siegel-Svensson model

Supervisors:
Senior Lecturer Jan Röman
jan.roman@mdh.se

Professor Anatoliy Malyarenco
anatoliy.malyarenko@mdh.se

Author:
Neda Kazemie
kazemieneda@gmail.com

Examiner:
Senior Lecturer Linus Carlsson
linus.carlsson@mdh.se

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Abstract

Most trading software in the market uses linear interpolation in the bootstrap procedure to create a zero coupon interest rate curve. Forward interest rate curves created by interpolation method have a zigzag form that is inconsistent with arbitrage condition. Furthermore in the real world we are not able to detect the spot rates for all periods because the number of traded instruments is limited.

Nelson and Siegel in 1987 proposed a model to estimate the interest rates. They demonstrated that their model was flexible to capture the various possible shapes of the yield curve and estimates the term structure of interest rates over time.

Many Central banks are using the extended Nelson-Siegel model (sometimes called the Nelson-Siegel-Svensson model). This curve is parameterized with exponential functions, is smooth in both zero rate and forward rate and overcome those barrier (data mining problems and zigzag shape of forward interest rate curve), that we face by using linear interpolation to build the yield curve. The objective of this thesis is to study the Nelson and Siegel model and its extended version by Svensson, furthermore, study the adequacy and forecast performance of Nelson-Siegel-Svensson model in building the term structure of interest rate comparing with the linear interpolation method.

Contents

Chapter 1

Introduction

1.1. Background

Term structures of interest rates, also known as yield curves are a measure of the markets expectation of future interest rates, given the current market conditions. When the yield curves are known, they are used to calculate present values of expected future cash-flows from financial contracts. To estimate an accurate term structure of interest rates is therefore one of the most important and challenging subjects in financial area. Without the yield curves we are not able to value financial instrument such as cash-flow instruments, equities, commodities and derivatives. Derivatives are financial contracts where the values are derived from expected future values of some underlying instrument.

Due to the importance of this issue, many researchers have developed methods and models to construct and forecast yield curves from prices of traded assets[1]. Most trading software in the market uses linear interpolation in a bootstrap procedure to create the term structure of interest rate. Such yield curves used for discounting goes under the name zero-coupon curves (or just zero-curves) or discount curves. Yield-curves representing the interest rate between two dates in the future are called term structure of forward rates. A large number of yield curves exists since they can be derived from different markets like the market of government bonds and the inter-bank market.

The forward rates can be calculated, via no arbitrage conditions via the zero rates. Here the market practice has been to use linear interpolation on zero rates. Forward interest rate curves created by a linear interpolation method have a zigzag form that is inconsistent with no arbitrage condition; moreover, illiquidity and missing data points are two main obstacles in bootstrapping methods. Therefore, other methods have been used to overcome these drawbacks.

One of the satisfactory models that provides smooth forward interest rate curve is the Nelson and Siegel model. This model was proposed by Nelson and Siegel in 1987 to estimate the interest rates. They demonstrated that their model is flexible to capture the various possible shapes of the yield curve and can estimate the term structure of interest rates over time. The model is parameterized with exponential functions and provides smooth yield and forward rate curves. This model was extended by Svensson in 1994 when he added a term allowing an extra bump in the yield curve.

According to Bank for International Settlements (BIS) 2005, central banks in Belgium, Finland, France, Germany, Italy, Norway, Spain and Switzerland have used the Nelson-Siegel model or one of its modifications as a method to estimate their term structure of interest rates.

[1] See Nelson, Charles R. and Andrew F.Siegel.1987, Fabozzi, F. J., L. Martellini, and P. Priaulet. 2005, Diebold Francis X. and Canlin Li. 2006, Hagan, Patrick S. and Graeme West. 2006.

1.2. Literature review

The original model of Nelson-Siegel and its extended versions have been the subject of many academic researches and studies since the first introduction in 1987. These researches provide evidence that the model is capable of successfully and accurately building the term structure of interest rate and outperform the competitive models, especially for longer forecast horizons, see Diebold and Li (2006), or Fabozzi, Martellini, and Priaulet (2005). Dullmann and Uhrig-Homburg (2000) use the Nelson-Siegel model to build yield curves of Deutsche Mark-denominated bonds and calculate the risk structure of interest rates.

According to Hodges and Parekh (2006) and Barrett, Gosnell and Heuson (1995) the Nelson and Siegel model is used by Fixed-income portfolio managers to hedge their portfolios. Martellini and Meyfredi (2007) estimate the value-at-risk for fixed-income portfolios by using Nelson-Siegel model.

1.3. Structure

1.3.1. Purpose

The focus of this thesis is on the extended version of Nelson-Siegel model (sometimes called the Nelson-Siegel-Svensson model). I use the Nelson-Siegel-Svensson model to build the term structure of interest rate and apply the term structure that is built by this model to study the mispricing[2] in different financial instruments. I also use linear interpolation to build the term structure of interest rate, and I compare these two methods to see if prices, when using the extended Nelson-Siegel model, are close to the prices generated by interpolation method and market prices. The market instruments used in this thesis consists of Deposits, FRA (Forward Rate Agreements) and Interest Rate Swaps (IRS).

My study confirms the ability of the Nelson-Seigel-Svensson model to capture the various shapes of spot curves. This model provides smooth term structures in both zero rates and forward rates, therefore, provides better estimation to value instruments that have cash-flows on other days than the market data.

Prices obtained by the Nelson-Seigel-Svensson model are close to the market prices and prices generated by the interpolation method. Moreover, this framework avoids data mining problems which we face in bootstrapping and linear interpolation methods and provides accurate estimation for missing data.

[2] The mispricing is defined as the difference between real market prices and prices given by using the Nelson-Siegel-Svensson curve in discounting and/or to generate the cash-flows in the instrument contracts.

1.3.2. Problem specification and discussion

Forward interest rate curves created by the interpolation method have a zigzag form that is inconsistent with arbitrage condition. Moreover, illiquidity and missing data points are two main obstacles in bootstrapping methods. I will examine Nelson-Siegel-Svensson model to see if the term structure and forward rate curve built by this model are accurate and smooth, compared with the curves that are built by the linear interpolation method. If it is the case, we can conclude that using the Nelson-Siegel-Svensson model is a better approach in building the term structure of the interest rate, since this model is parsimonious[3], avoids data mining problems and provides smooth and accurate term structure of interest rate and forward curve.

1.3.3. Target Audience

Target audience for this thesis are students of financial mathematics and financial Engineering on advanced level. Good knowledge in mathematics and finance is assumed.

The remainder of this thesis is organized as follows:
Chapter 2 reviews some definitions that are used in this thesis.
Chapter 3 introduces Nelson-Siegel model in its original form and extended version by Svensson.
Chapter 4 is devoted to the methodology I have used to construct the term structure of interest rate and in both pre- and post-crisis market practices for valuing and pricing interest rate swaps.
Chapter 5 presents the data and empirical analysis and
Chapter 6 summarizes main points and concludes.

[3] A model with few parameters

Chapter 2

Terminology

This Chapter presents some introduction into basic concepts which are mainly addressed in this thesis.[4]

2.1. Interest Rates

An interest rate is a rate a borrower pays to a lender for the use of money, it is usually expressed as annual percentage of the loan. Some of the different interest rates at the market which are used in this thesis are explained in the following Sections.

2.1.1. Spot Rate

The spot rate or short rate is defined as the theoretical profit given by a zero coupon bond. We use this rate when we calculate the amount we will get at time t_1(in the future) if we invest *X* today. (I.e. at time t_0)

$$X_{t_1} = (1 + r(t_0, t_1))^{t_1} X_{t_0}$$

$$PV(X_{t_1}) = \frac{1}{(1 + r(t_0, t_1))^{t_1}} X_{t_1} .$$

where $PV(X_t)$ is the present value of X_t and $r(t_0, t_1)$ is the spot rate.
In most cases, t_0 is the current time ($t_0 = 0$) and is sometimes dropped from notional convenience.

2.1.2. Discount function

The discount function $p(t_0, t)$ describes the present value at time t_0 of a unit cash flow at time t. This is a fundamental function.

The remaining variable *t* refers to the time between t_0 and *t*. The discount function is used as the base for all other interest rates. These functions also represent the value of a zero-coupon bond at time t_0 with maturity *t*.
At maturity a zero-coupon bond pays 1 cash unit ($, EUR, SEK etc.). Therefor $p(t, t) = 1$.

One can prove the relationship between the spot rate and discount function as

$$p(t) \equiv p(0, t) = \frac{1}{(1 + r(t))^t}$$

[4] The information provided in this Chapter is mainly taken from John C.Hull, Options, Futures, and other Derivatives, 7th ed., Prentice Hall India, 2009

Definition: *Simple spot rate* for period $[S, T]$ is defined by

$$r_{simp}(S,T) = -\frac{p(S,T) - p(S,S)}{(T-S)\,p(S,T)} = -\frac{p(S,T) - 1}{(T-S)p(S,T)}$$

Definition: *Continuously compounded spot rate* for period $[S, T]$ is defined by

$$r_{comp}(S,T) = -\frac{\ln\big(p(S,T)\big) - \ln\big(p(S,S)\big)}{(T-S)} = -\frac{\ln\big(p(S,T)\big)}{(T-S)}$$

2.1.3. Discount curve

The plot of the discount factor against time is known as the discount curve. The discount curve is monotonically decreasing and it never reaches zero since all cash flows no matter how far in the future they are paid are always worth more than nothing.

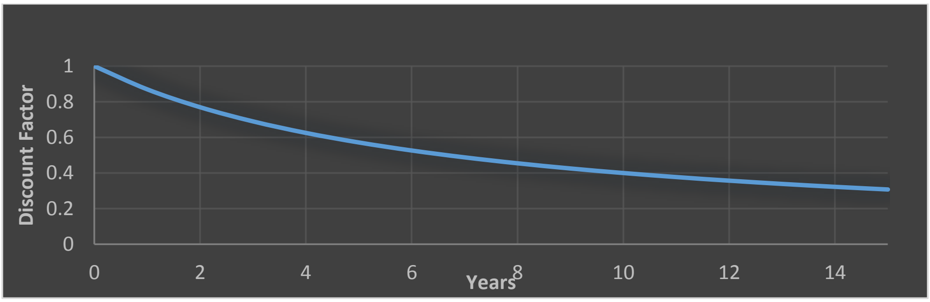


Figure 2.1: Discount Curve when rate =15%

2.1.4. Forward rate

The concept of forward rate is closely related to a forward contract[5]. To define the today price for buying a zero-coupon bond at time t_1 with maturity t_2, $t_1 < t_2$, we should set the future spot rate *r* for the period $[t_1, t_2]$. This future spot rate is called forward rate.

The relationship between forward rate and spot rate is given by

$$(1 + r(t_1)^{spot})^{t_1}\left(1 + R(t_2 - t_1)^{forward}\right)^{t_2 - t_1} = (1 + r(t_2)^{spot})^{t_2} \Rightarrow$$

$$R(t_2 - t_1)^{forward} = \left(\frac{(1 + r(t_2)^{spot})^{t_2}}{(1 + r(t_1)^{spot})^{t_1}}\right)^{\frac{1}{t_2 - t_1}} - 1$$

Definition: Simple forward rate for the period $[t_1, t_2]$ contracted at time *t*

$$R_{simp}(t, t_1, t_2) = -\frac{p(t, t_2) - p(t, t_1)}{(t_2 - t_1).\, p(t, t_2)}$$

Definition: Continuously compounded forward rate for the period $[t_1, t_2]$ contracted at *t*

$$R_{comp}(t, t_1, t_2) = -\frac{\ln(p(t, t_2) - \ln(p(t, t_1)}{(t_2 - t_1)}$$

Definition: Instantaneous forward rate

The instantaneous forward rate with maturity at *T* starting at t_1, contracted at *t* is defined as

$$R(t, T) = \lim_{t_1 \to T} R(t, t_1, T) = -\frac{\partial \ln(p(t, T))}{\partial T}$$

where $t < t_1 < T$.

[5] A forward contract is an agreement to sell or buy an asset at a certain future time for a certain price. (Jan Hull 7th edition (2009))

2.1.5. Yield to maturity

Yield to maturity (YTM), also called Redemption Yield, is the interest an investor gains by holding an interest paying security until maturity, the investor is supposed to reinvest all coupon payments at the same interest rate during the lifetime of the security.

2.1.6. Repo rate

Sometimes trading activities are funded with a repo or repurchase agreement. This is a contract where an investment dealer who owns securities agrees to sell them to another company now and buy them back later at a slightly higher price. The other company is providing a loan to the investment dealer. The difference between the price at which the securities are sold and the price at which they are repurchased is the interest it earns. The interest rate is referred to as the repo rate.[6] The periods in the repo market are usually:

O/N (Over-Night), borrow and lending funds from today until tomorrow[7].
T/N (Tomorrow-Next), borrow and lending funds from tomorrow until next day.
C/W (Corporate-Week), borrow and lending funds from the day after tomorrow for a week.
S/N (Spot-Next), borrow and lending funds from the day after tomorrow until one business day ahead.

2.2. Term structure of interest rate (Yield curve)

The relationship between yield to maturity and time is called the term structure of interest rates. The visual representation of interest rate against time is known as yield curve.
The yield curve gives an idea about the future direction of the interest rate, therefore, building models which explain the shape of the yield curve has always been a challenge for investors and academics.

[6] Jan Hull 7th edition (2009), p75
[7] If today is Friday then the next business day is Monday.

2.3. Par rate

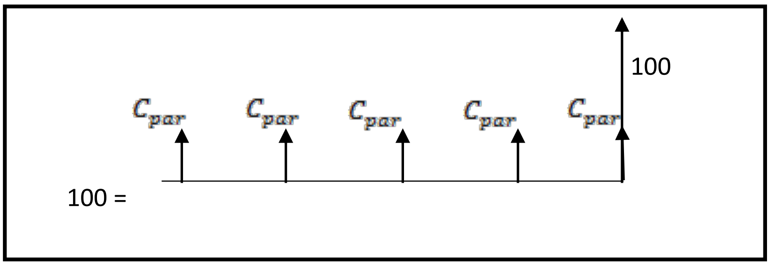
Par rate is the coupon rate that values an instrument to its nominal value.

2.3.1. Par rate for a bond

For simplicity assume that the nominal value of the bond is 100

$$100 = \sum_{i=1}^{n} \big(p(0,t_i)\ C_{par}\big) + (p(0,t_n).100) \qquad \Rightarrow$$

$$C_{par} = \frac{100.(1 - p(0,t_n))}{\sum_{i=1}^{n} p(0,t_i)}.$$

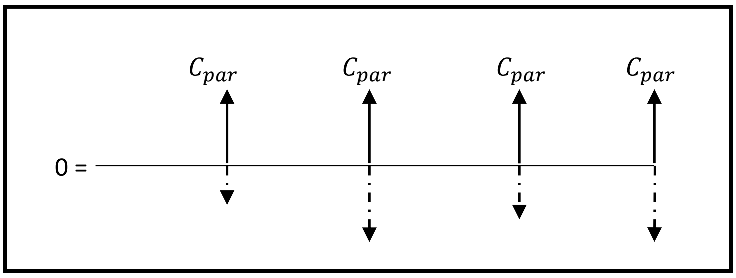


2.3.2. Par rate for a swap

The part rate for a swap is calculated as

$$\sum_{i=1}^{n} \big(p(0,t_i)\ C_{par}\big) = \sum_{i=1}^{n} p(0,t_i)\ R(t_i - t_{i-1})^{forward} \quad \Rightarrow$$

$$C_{par} = \frac{\sum_{i=1}^{n} p(0,t_i) R(t_i - t_{i-1})^{forward}}{\sum_{i=1}^{n} p(0,t_i)}.$$



where dotted lines represent the floating rate at each payment date and straight lines represent the swap's par rate.

2.4. Interbank rate

The Interbank rate is an interest rate decided by a number of investment banks and is defined as an average rate paid for short term loans between banks. Some of the interbank rates are LIBOR-London Interbank Offer Rate, STIBOR- Stockholm Interbank Offer rate, etc.

2.4.1. EURIBOR

Euribor (Euro Interbank Offered Rate) is a benchmark giving an indication of the average rate at which banks lend unsecured funding in the euro interbank market for a given period.

It is the rate at which Euro interbank term deposits are offered by one prime bank to another within the EMU zone (Economic and Monetary Union in Europe). It is produced for one, two and three weeks and for twelve maturities from one to twelve months[8].

2.4.2. EONIA

Eonia (Euro Over-Night Index Average) is the effective overnight reference rate for the Euro. It is computed as a weighted average of all overnight unsecured lending transactions in the interbank market, undertaken in the European Union and European Free Trade Association (EFTA) countries. Eonia is computed with the help of the European Central Bank. The banks contributing to Eonia are the same first class market standing banks as the Panel Banks quoting for Euribor[9].

2.4.3. LIBOR

LIBOR is calculated and published by Thomson Reuters on behalf of the British Bankers' Association (BBA) after 11:00 AM (and generally around 11:45 AM) each day (London time). It is a trimmed average of interbank deposit rates offered by designated contributor banks, for maturities ranging from overnight to one year. LIBOR is calculated for 6 currencies[10]. There are eight, twelve, sixteen or twenty contributor banks on each currency panel, and the reported interest is the mean of the 50% middle values (the interquartile mean). The rates are a benchmark rather than a tradable rate; the actual rate at which banks will lend to one another continues to vary throughout the day. LIBOR is often used as a rate of reference for pound sterling and other currencies.

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[8] http://www.euribor-ebf.eu/euribor-org/about-euribor.html(1/3/14)
[9] http://www.euribor-ebf.eu/euribor-eonia-org/about-eonia.html(1/3/14)
[10] Before May 2013 there were 11 currencies. The following currencies have been removed; NZD, DKK, SEK, AUD and CAD. At the same time the tenors; 2W, 4M, 5M, 7M, 8M, 9M, 10M and 11 M was removed for CHF, EUR, GBP, JPY and USD.

2.5. Instruments

2.5.1. Bonds

Bonds are fixed-income securities, the buyer of the bond pays a price of *P* to the issuer and in returns receives a series of cash flows in the future.

Zero coupon bond (Also called “zero” or “pure discount bond”)

Is a bond that only has a single payment of one unit of currency (also called principal or notional amount), at maturity.

Coupon bond

Is a bond in which the buyer receives a sequence of interest payments (coupons) in specific dates (coupon payment dates) in addition to the principal.

Coupons usually are paid semi-annually or annually.

2.5.2. Swaps

A swap is an agreement between two parties to exchange cash flows during a specific period in the future according to two different indices (typically a floating interest rate-usually a LIBOR rate cash flow against a fixed rate cash flows). Swap agreements are interesting for counterparties due to the comparative advantages they have on borrowing money in capital markets which both parties can benefit (for example when a company wants to transform a floating rate loan into a fixed-rate loan due to the beneficiary it has in paying the fixed-rate interest and vice versa).

Plain vanilla interest-rate swap

Plain vanilla interest-rate swap is an agreement such that one party pays interest at a fixed rate for receiving interest at a floating rate from the other party based on a specified face value. Face value, payment frequency, time to maturity, day count convention, fixed rate and the floating rate must be defined in the contract.
In an interest rate swap, face value is not usually exchanged.

Overnight indexed Swap (OIS)

With an overnight indexed swap, two parties agree that party1 receives a fixed rate from party 2 (the fixed leg of the swap), and party 2, receives a rate equal to the average of overnight rate from party1 (the floating rate of the swap).

OIS is now the market standard for pricing the collateralized agreement since OIS is considered as a good indicator for less risky rate available on the market.

Some of the Overnight indices are:
EONIA (Euro Over-Night Index Average).
SONIA (Sterling Over-Night Index Average).
TONAR (Tokyo Over-Night average rate).
SARON (Swiss Average Rate Overnight).

2.5.3. Forward Rate Agreements (FRA)

FRA is an instrument that two parties agrees to exchange a fixed interest rate (FRA rate) for a future reference rate (usually LIBOR or EURIBOR rate) on a notional amount in a specific time at some future date. Notional amount and the length of the contract are fixed in the contract.

2.5.4. Deposit (Certificate of Deposits)

Deposits are over-the-counter (OTC)[11] contracts that pay a fixed rate of return for a specified period. The interest rate is fixed at the start date and usually is higher than other rates in the market because the money can't be withdrawn before the maturity time unless a penalty is paid for early withdrawal. Deposits usually have relatively short duration.

[11] OTC or Over-the-counter contracts are those contracts which are not traded on an exchange, all terms and conditions are settled and agreed in negotiations between counterparties.

2.6. Bootstrapping interest rate curves

Bootstrapping in finance is a process for extracting zero-coupon rates from a series of market prices of coupon bearing instruments such as bonds and swaps to build the yield curve. Liquidity of instruments is an important aspect in bootstrapping process. In Section 4.2 bootstrapping procedure for different instruments is explained in more detail.

2.7. Interpolation

In addition to required liquidity of instruments, lack of the market data is the other problem for the bootstrapping process. In order to overcome this obstacle interpolation methods are applied to determine the yield rates that are not directly observable from the market data. Interpolation is often used during bootstrapping the yield curve and is in fact a part of the bootstrapping process.

2.8. Day-Count convention

Day-Count convention regulates the way to count the days in an interest payment period and usually is expressed as a fraction in which the numerator represents the number of days in a month and the denominator represents the number of days in a year. Depending on the local calendar, currency or the instrument, each market has its own Day-Count convention.

Some of Day-Count convention alternatives are: 30/360[12], Act/360[13], Act/365, Act/Act and NL/365[14].

[12] Each month is considered to have 30 days and each year 360 days. Exceptions happen when the later date of the period is the last day of February, or if the last date is the 31th and the first day is not 30th or 31th in those cases the month is considered to have its actual number of days.
[13] Act means that the month is considered to have its actual number of days.
[14] NL: the month has its actual number of calendar days, in case of a leap year February is considered to have 28 days instead of 29 days.

Chapter 3

Nelson and Siegel Model

3.1. The Original Model

Charles R. Nelson and Andrew F. Siegel in 1987 presented a simple and parsimonious model of the yield curve. They claimed that their model was flexible enough to capture various possible shapes of the yield curves (monotonic, humped, trough and S shapes).[15]

They introduced the following formula for the instantaneous forward rate[16] at maturity *(m)*

$$R(m) = \beta_0 + \beta_1\, exp\left(\frac{-m}{\tau}\right) + \beta_2\left[\left(\frac{m}{\tau}\right) exp\left(\frac{-m}{\tau}\right)\right] \qquad [1]$$

In Equation [1], $\beta_0, \beta_1, \beta_2$ are constants, $\beta_1\, exp\left(\frac{-m}{\tau}\right)$ is a monotonically decreasing (for positive β_1) and increasing (for negative β_1). The expression $\beta_2\left[\left(\frac{m}{\tau}\right) exp\left(\frac{-m}{\tau}\right)\right]$ generates a hump (for positive β_2) and a trough (for negative β_2). The parameter τ is a time constant which is positive and determines the location of the hump or trough in the yield curve.

By taking the average of the Equation [1] from zero up to the maturity time we get the following expression for the spot rate[17]

$$r(m) = \frac{1}{m}\int_0^m R(s)\, ds$$

$$r(m) = \frac{1}{m}\int_0^m \left[\beta_0 + \beta_1 \exp\left(-\frac{s}{\tau}\right) + \beta_2\left(\frac{s}{\tau}\right). \exp\left(-\frac{s}{\tau}\right)\right] ds$$

Making the following change of variables: $x = \frac{s}{\tau}$, $(ds = \tau dx)$.

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[15] Nelson, Charles R. and Andrew F.Siegel. Parsimonious Modeling of Yield Curves. 1987. The Journal of Business 60, 473-489.

[16] Suppose at date t, one agrees to loan out $1at date T, and get repaid the next day:
1 paid at T, and $1 + f(t,T)\Delta T$ received at $T + \Delta T$
By definition the interest rate to charge is $f(t,T)$=instantaneous forward rate for date T, as seen at date t. (Jan R. M. Röman (2012), Lecture notes in Analytical Finance II)

[17] Lemma: The following holds for $t \le s \le T$:
$p(t,T) = p(t,s)\exp\{-\int_s^T R(t,u)du\} = \exp\{-\int_t^T R(t,u)du\}$
where $p(t,T)$ is the discount function and $f(t,T)$is the instantaneous forward rate at time t. From Section 2.1.2 we have the definition of continuously compounding spot rate:
$r(t,T) = -\frac{\ln[p(t,T)]}{T-t}$
Substituting Equation [2] in the above formula we get:
$r(t,T) = \frac{1}{T-t}\int_t^T R(t,u)du$. (Jan R. M. Röman (2012), Lecture notes in Analytical Finance II)

we get

$$r(m) = \beta_0 + \left(\beta_1 \frac{\tau}{m}\right) \int_0^{\frac{m}{\tau}} e^{-x} dx + \left(\beta_2 \frac{\tau}{m}\right) \int_0^{\frac{m}{\tau}} x e^{-x} dx$$

$$= \beta_0 + \left(\beta_1 \frac{\tau}{m}\right) [-e^{-x}]_0^{\frac{m}{\tau}} + \left(\beta_2 \frac{\tau}{m}\right) \left\{ [-x e^{-x}]_0^{\frac{m}{\tau}} - \int_0^{\frac{m}{\tau}} e^{-x} dx \right\}$$

$$= \beta_0 + \beta_1 \left[\frac{1 - e^{\frac{-m}{\tau}}}{m/\tau} \right] + \beta_2 \left[\frac{1 - e^{\frac{-m}{\tau}}}{m/\tau} - e^{\frac{-m}{\tau}} \right]$$

Finally we get the following formula for spot rates

$$r(m) = \beta_0 + \beta_1 \left(\frac{1 - \exp\left(\frac{-m}{\tau}\right)}{\frac{m}{\tau}} \right) + \beta_2 \left(\frac{1 - \exp\left(\frac{-m}{\tau}\right)}{\frac{m}{\tau}} - \exp\left(\frac{-m}{\tau}\right) \right). \qquad [2]$$

3.2. Deriving the equation for the forward rate by using the Nelson-Siegel model

Let $R(t,T)$ be the forward rate, starting at *t* with maturity *T*. If we divide the interval *[t,T]*, into *n* equal intervals, so that $\Delta t = \frac{T-t}{n}$, we can write the formula for $R(t,T)$, as

$$R(t,T) = R(t, t+\Delta t) \frac{\Delta t}{(T-t)} + R(t+\Delta t, t+2\Delta t) \frac{\Delta t}{(T-t)} + R(t+2\Delta t, t+3\Delta t) \frac{\Delta t}{(T-t)}$$

$$+ \cdots + R(t+(n-1)\Delta t, t+n\Delta t) \frac{\Delta t}{(T-t)}$$

$$= \frac{1}{T-t} \sum_{i=0}^{n-1} R(t+i\Delta t, t+(i+1)\Delta t) \Delta t$$

If we divide the interval *[t,T]* so that $n \to \infty$, then $\Delta t \to 0$ then we can write the above equation as

$$R(t,T) = \frac{1}{T-t} \int_t^T R(s,s)\, ds$$

As we have defined in Chapter 2, *R(s,s)* is the instantaneous forward rate, so by taking the average of Equation [1] from *t* to *T*, we get the formula for the forward rate as

$$R(t,T) = \frac{1}{T-t} \int_t^T \beta_0 + \beta_1\, exp\left(\frac{-s}{\tau}\right) + \beta_2 \left[\left(\frac{s}{\tau}\right) exp\left(\frac{-s}{\tau}\right) \right] ds$$

$$= \beta_0 + \beta_1 \frac{\tau}{T-t}\left[\exp\left(\frac{-t}{\tau}\right) - \exp\left(\frac{-T}{\tau}\right)\right] + \beta_2 \frac{\tau}{T-t}\left[\exp\left(\frac{-t}{\tau}\right) - \exp\left(\frac{-T}{\tau}\right) - \frac{T}{\tau}\exp\left(\frac{-T}{\tau}\right) + \frac{t}{\tau}\exp\left(\frac{-t}{\tau}\right)\right]$$

Nelson-Siegel suggest that for $\tau = 1$, $\beta_0 = 1$ and $\beta_1 = -1$, possible forms of the term structure can be explored and the equation for spot rate depends on a single parameter "a"[18] so that

$$r(m) = 1 - (1-a)\frac{[1-\exp(-m)]}{m} - a\exp(-m)$$

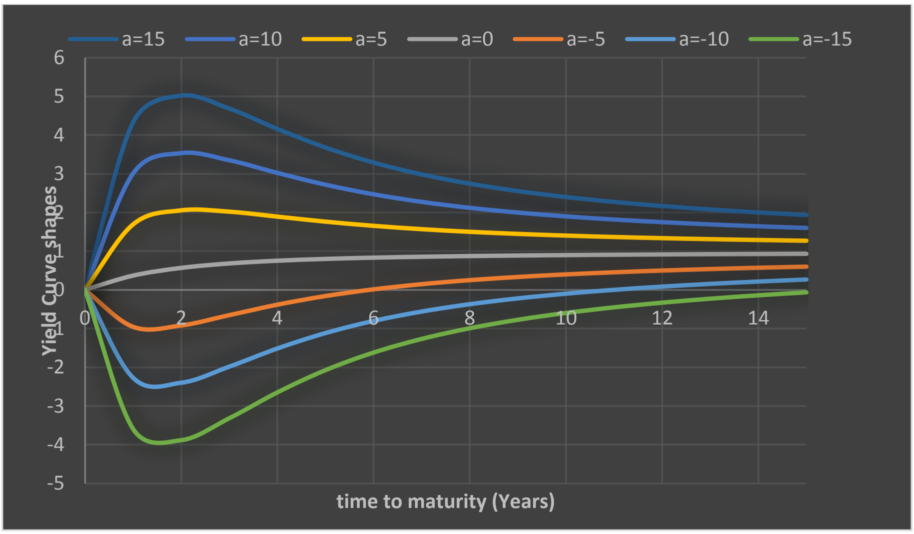

Figure 3.1: Yield curve shapes. Parameter a spans from -15 to 15 with equal increments

We can also interpret the coefficients of the model [1], to observe the shape flexibility of the model.

$$\lim_{m\to\infty} R(m) = \beta_0$$

So β_0 represents the contribution of the long term component of the model.

$$\lim_{m\to 0} R(m) = \beta_0 + \beta_1$$

Therefore $(\beta_0 + \beta_1)$ represents the instantaneous interest rate.

[18] Charles R. Nelson and Andrew F.Siegel. Parsimonious Modeling of Yield Curves. 1987. The Journal of Business 60, 473-489.

Contribution of short-term component is β_1 and β_2 is the contribution of medium-term component. From the Figure of the components of forward rate curve (Figure 3.2) we see that these notations are well-suited. Long-term component (β_0) doesn't decay to zero as m converges to infinity.

From Figure 3.2, we see that $exp\left(\frac{-m}{\tau}\right)$ has a faster decay toward zero than $\left(\frac{m}{\tau}\right) exp\left(\frac{-m}{\tau}\right)$, and therefore is interpreted as the contribution of the short-term component. We also observe that $\left(\frac{m}{\tau}\right) exp\left(\frac{-m}{\tau}\right)$ start at *0,* (therefore is not short-term) and decays to *0* as *m* approaches infinity (therefore is not the long-term component of the model), and represents the contribution of the medium-term component[19].

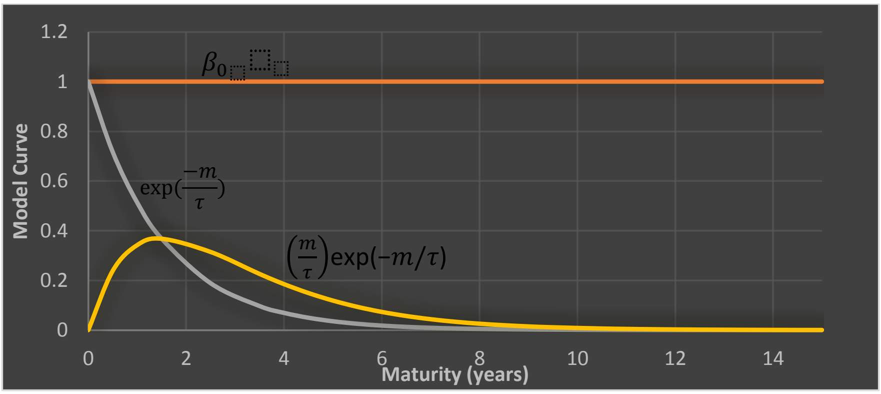

Figure 3.2: Components of the Nelson & Siegel forward curve, fixed parameters are: $\beta_0 = 1$, $\beta_1 = -2$, $\beta_2 = 4.5$, $\tau = 1.5$

[19] Charles R. Nelson and Andrew F.Siegel. Parsimonious Modeling of Yield Curves. 1987. The Journal of Business 60, 473-489.

3.3. The extended version of Svensson

Lars Svensson (1994) extended Nelson and Siegel's function by adding an extra monomial exponential term to improve the fit and flexibility of term structure of interest rate, this extra term provides the capability of a second possible hump or trough for the yield curve.

Instantaneous forward interest rate function presented by Svensson

$$R(m) = \beta_0 + \beta_1 \, exp\left(\frac{-m}{\tau_1}\right) + \beta_2 \left(\frac{m}{\tau_1}\right) exp\left(\frac{-m}{\tau_1}\right) + \beta_3 \left(\frac{m}{\tau_2}\right) \exp\left(\frac{-m}{\tau_2}\right) \quad [3]$$

Parameter τ_2 is a time constant which is positive and determines the location of second possible hump or trough in the yield curve.

The spot rate can be obtained by taking the average of equation [4] from zero up to m

$$r(m) = \beta_0 + \beta_1 \left(\frac{1-\exp(\frac{-m}{\tau_1})}{\frac{m}{\tau_1}}\right) + \beta_2 \left(\frac{1-\exp\left(\frac{-m}{\tau_1}\right)}{\frac{m}{\tau_1}} - \exp(\frac{-m}{\tau_1})\right) + \beta_3 \left(\frac{1-\exp(\frac{-m}{\tau_2})}{(\frac{m}{\tau_2})} - \exp(\frac{-m}{\tau_2})\right) \quad [4]$$

The same discussion as in Section 3.2 holds for the Nelson-Siegel-Svensson model, so the equation for the forward rate can be presented as

$$\begin{aligned} R(t,T) = \beta_0 + \beta_1 \frac{\tau_1}{T-t}&\left[\exp\left(\frac{-t}{\tau_1}\right) - \exp\left(\frac{-T}{\tau_1}\right)\right] \\ + \beta_2 \frac{\tau_1}{T-t}&\left[\exp\left(\frac{-t}{\tau_1}\right) - \exp\left(\frac{-T}{\tau_1}\right) - \frac{T}{\tau_1}\exp\left(\frac{-T}{\tau_1}\right) + \frac{t}{\tau_1}\exp\left(\frac{-t}{\tau_1}\right)\right] \\ + \beta_3 \frac{\tau_2}{T-t}&\left[\exp\left(\frac{-t}{\tau_2}\right) - \exp\left(\frac{-T}{\tau_2}\right) - \frac{T}{\tau_2}\exp\left(\frac{-T}{\tau_2}\right) + \frac{t}{\tau_2}\exp\left(\frac{-t}{\tau_2}\right)\right] \end{aligned}$$

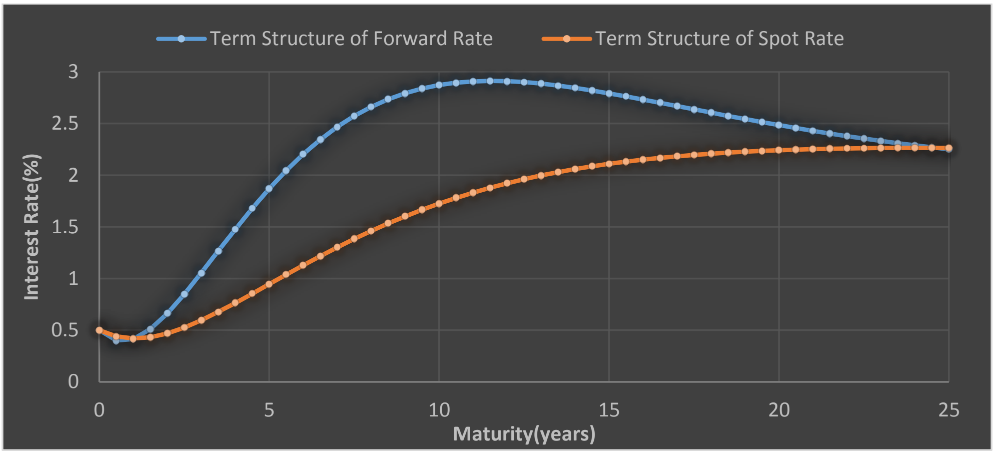

Figure 3.3: Term Structure of Forward & spot Rate by Nelson-Siegel-Svensson model, Fixed parameters are:

$\beta_0 = 2$, $\beta_1 = -1.5$, $\beta_2 = -9$, $\beta_3 = 9$, $\tau_1 = 3.5$, $\tau_2 = 5$.

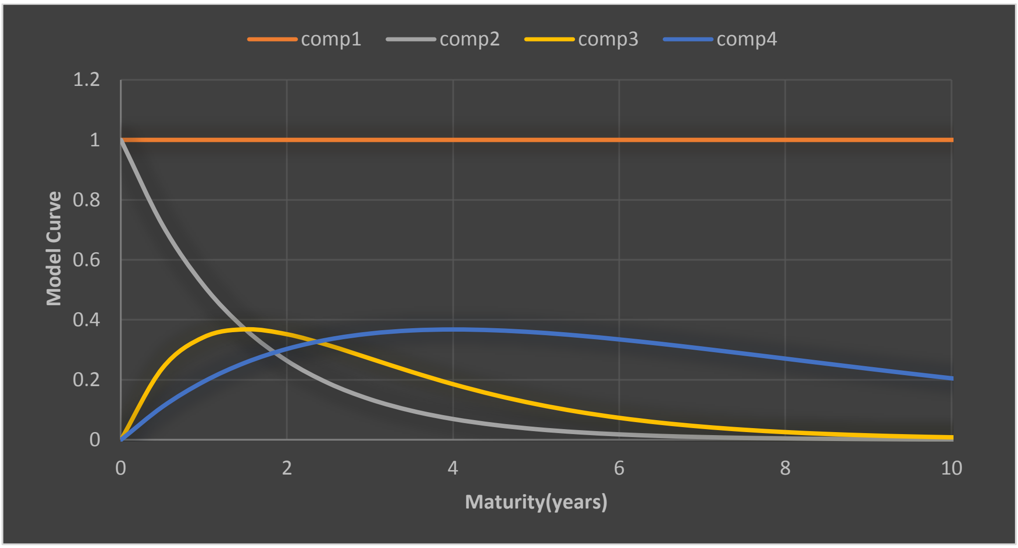

Figure 3.4: Components of the Nelson & Siegel Svensson forward curve

$\beta_0 = 1$, $\beta_1 = -2$, $\beta_2 = 4.5$, $\beta_3 = 9$, $\tau_1 = 1.5$, $\tau_2 = 4$

Comp1=β_0, *Comp2*=$exp\left(\frac{-m}{\tau_1}\right)$, *Comp3*=$\left(\frac{m}{\tau_1}\right) exp\left(\frac{-m}{\tau_1}\right)$, *Comp4*=$\left(\frac{m}{\tau_2}\right) exp\left(\frac{-m}{\tau_2}\right)$

3.4. Fixing the parameters of the model

Suppose we have k spot rates r_{m_i} for different maturities: $m_1, m_2, \dots, m_k$ and let $B = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$ be the vector of parameters of Nelson-Siegel-Svensson model. Our objective is to find the optimal values of the parameters that best fit the given spot rates observed from the market by using the least square method.

$$G = \sum_{i=1}^{k} \left(\beta_0 + \beta_1 \left(\frac{1 - \exp(\frac{-m_i}{\tau_1})}{\frac{m}{\tau_1}} \right) + \beta_2 \left(\frac{1 - \exp\left(\frac{-m_i}{\tau_1}\right)}{\frac{m}{\tau_1}} - \exp(\frac{-m_i}{\tau_1}) \right) + \beta_3 \left(\frac{1 - \exp(\frac{-m_i}{\tau_2})}{(\frac{-m_i}{\tau_2})} - \exp(\frac{-m_i}{\tau_2}) \right) - r(m_i)^M \right)^2$$

Here r^M is the observed rate from the market and r^{NSS} is the estimated rate from Nelson-Siegel-Svensson model. By using the solver function in Excel and adjusting the values of vector *B* we obtain the minimum value for *G*.

By using the solver function in Excel we solve this optimization problem.

Our objective is to minimize the sum of the squared difference between estimated rates taken from Nelson-Siegel-Svensson model, and observed rates from the market.

We proceed by first setting the initial values for parameter vector B, the proper choice for initial values of vector *B* guarantees the convergence toward the minimum of the objective function. Our knowledge of economic interpretation of parameters of the model helps to constrain the parameters to obtain the accurate term structure.

As is mentioned in the above Section, β_0 is interpreted as long-run yield level and therefore, the value of the spot rate with longest maturity in the observed rates from market is an appropriate choice for its initial value. The contribution of the short-term component is β_1, and it determines the slope of the curve. The contribution of the middle-term component is β_2 and degree of curvature is controlled by this factor.

The last expression in model [5] adds a second possible hump to the curve, and parameters τ_1 and τ_2 determine the position of possible humps in the curve and must be positive.

After setting the initial values for the vector *B*, we calculate the spot rates by Nelson-Siegel-Svensson model and we find the minimum value for

$$G = \sum_{i=1}^{k} [r(m_i)^M - r(m_i)^{NSS}]^2$$

Chapter 4

Methodology

4.1. Curve construction of interest rate

Term structure of interest rate or yield curve plays a critical role in valuation of interest rate derivatives. It is also of a great importance to build an accurate discount curve and a smooth forward rate curve to conduct the monetary policy.

An accurate yield curve has a great influence on the performance of the traders, therefore a parsimonious model that construct the entire yield curve with high quality is of the great interest to traders and central banks.

However there is not a unique, single or standard process for constructing the yield curve from a set of observed market rates even when the curve completely reproduce the price of the given instruments. Market practice have been to use linear interpolation in trading software, which leads to a sig-saw-shaped forward curve (see Section 5.3.4), while many central banks use a Nelson-Seigel parameterization.

The challenging question is how to construct an accurate and smooth interest rate curve. The objective of this thesis is to construct the yield curve first by linear interpolation and bootstrapping and then perform the extended version of Nelson and Siegel model to observe and compare the performance ability between the models.
The yield curve is defined as the relationship between interest rates and their time to maturity. The relationship between discount factors and time to maturity is known as the discount curve. However forward rates, discount factors and yield are closely related concepts, once one is derived other ones can be determined.

One approach to construct a yield curve is to evaluate the discount factors and then extract the yield via above formulas. (In general you first calculate the zero rates and then the discount factors. The discount factors are then used as a base to construct all other rates (simple, annual compounding and with different day-count conventions)).

There is no standard model to construct a yield curve, and therefore no standard shape for the term structure of yield curve. While we directly observe some points on the curve from the market data, illiquidity of the instruments and lack of market data are challenges to form the term structure of the interest rate. To overcome these obstacles bootstrapping process and interpolation method must be applied to construct a yield curve.

4.2. Bootstrapping

As it mentioned in Section 2, bootstrapping is a process for extracting zero-coupon rates from a series of market prices of coupon bearing instruments such as bonds or swaps to build the yield curve. This process determines the shape of the yield curve. Banks construct different kind of curves for different kinds of trades. The rate for discounting should reflect the cost of funding for a bank. This funding is close to the interbank rate at which banks can borrow from each other.

However, the bootstrapping procedure is not the same for different instruments. But, the curves (at least when using linear interpolation) should reproduce the market prices.

In the following bootstrapping procedures for cash deposits with maturities O/N, T/N, 1week, 1month, 2 months and 3 months, forward rate agreements (FRA) and swaps with maturity from 1 year up to 30 years are described, since we are dealing with these instruments in this thesis.

Deposit rates are used for constructing the short term of the curve, FRA for short term to medium term and swap rates for constructing the longer term of the curve.

We start by calculating the discount factor and zero rate for short-term of the curve.[20]

$$D_{O/N} = \frac{1}{1 + r_{O/N}\frac{d_{O/N}}{360}}$$

$$Z_{O/N} = -100\frac{\ln(D_{O/N})}{\frac{d_{O/N}}{365}}$$

where $D_{O/N}$ is the discount factor, $r_{O/N}$ is the O/N interest rate, $d_{O/N}$ denotes the length of the contract and $Z_{O/N}$is the zero rate.[21]

We continue by calculating the discount factor and zero rate for T/N

$$D_{T/N} = \frac{D_{O/N}}{1 + r_{T/N}^{Par}\frac{d_{T/N}}{360}}$$

$$Z_{T/N} = -100\frac{\ln(D_{T/N})}{\frac{d_{T/N}}{365}}$$

[20] Day count convention: ACT/360

[21] Zero rates are commonly given as continuous compounding, Act/365.

Discount factor and zero rate for money market instruments

$$D_i = \frac{D_{\frac{T}{N}}}{1 + r_i^{Par}\frac{d_i}{360}} \qquad i = \{1W, 1M, 2M, 3M\}.$$

$$Z_i = -100\frac{\ln(D_i)}{\frac{d_i}{365}} \qquad i = \{1W, 1M, 2M, 3M\}.$$

To calculate the discount factor and zero rate for Forward Rate Agreements we need a rate called stub rate, stub rate should have the maturity date same as the starting date of a forward rate agreement contract, and this rate can be found by linear interpolation.

$$D_{FRA}^{i} = \frac{D_{FRA}^{i-1}}{1 + r_{FRA}^{i}\frac{d_{FRA}^{i}}{360}}$$

where

$$D_{FRA}^{0} = D_{stub}, \qquad r_{FRA}^{0} = r_{stub}$$

and

$$Z_{FRA}^{i}(T) = -100\frac{ln\left(D_{FRA}^{i}\right)}{\frac{d_{FRA}^{i}}{365}}.$$

Here *i* represents the *i*:th FRA contract with maturity at t_i, $t_i < t_{i+1} < t_{i+2\ldots}$

We continue with swaps to get the long-term of the yield curve. The swap par rate is given by

$$r_T^{par} = \frac{D_{T/N} - D_T}{\sum_{t=1}^{T} Y_t D_t},$$

where r_T^{par} is the swap part rate with time of maturity at *T,* and Y_t is the year fraction at time *t* given by

$$Y_t = \frac{360(y_t - y_{t-1}) + 30(m_t - m_{t-1}) + (d_t - d_{t-1})}{360}.$$

where y_t is the year, m_t is the month and d_t the day for the rate.

From this we get the following formula for the discount factor

$$D_T = \frac{D_{T/N} - r_T^{par}\sum_t^{T-1} Y_t D_t}{1 + Y_T r_T^{par}}$$

Finally the zero rate is given by

$$Z_T = -100 \frac{\ln(D_T)}{\frac{d_T}{365}}$$

For the years we don't have the swap rate we use linear interpolation to get the swap rates and from that we calculate the discount factors and zero rates.

4.3. Linear Interpolation[22]

Let *f(x)* be a function, linear interpolation is an approach to approximate a value of f, defined as *p(x)*, by using the values of two known data points, $f(x_1)$ and $f(x_2)$. The linear interpolation function for any $x \in [x_0, x_1]$, is defined as

$$p(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x - x_0)$$

this equation can easily be arranged so that

$$p(x) = \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) + \frac{(x_1 - x)}{(x_1 - x_0)} f(x_0)$$

As it mentioned, illiquidity of the instruments is one of the problem in constructing the yield curve. Interpolation is one approach to complete the missing data.

Bootstrapping and interpolation method are two related procedures, interpolation complete the missing data that are proceed in bootstrapping.

Linear interpolation is popular because of its simplicity, however using the linear interpolation is a not a good approach to construct the yield curve, because of producing a zigzag shaped forward rate curve.

4.3.1. Linear interpolation on discount factor

Let $t \in (t_1, t_n)$, so that, $t \neq t_i$ and $t_i < t < t_{i+1}$, for $i = 1, \ldots, n-1$, If *D(*t_i*)* and *D(*t_{i+1}*)* are given then we can derive *D(t)* by using the linear interpolation

$$\mathrm{D(t)} = \frac{t - t_i}{t_{i+1} - t_i} \mathrm{D}(t_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} \mathrm{D}(t_i)$$

[22] Hagan, Patrick S. and Graeme West. 2006. *Interpolation Methods for Curve Construction*, Applied Mathematical Finance **13**, 89-129.

4.4. Interest Rate Swap valuation (traditional approach)

Swaps are agreements between two parties to exchange a series of cash flows based on variable rates at regular intervals, these rates can be fixed or floating. The payable or receivable cash flows are commonly calculated by multiplying the reference rate or the par rate of the swap by the nominal amount, all the terms must be specified in the contract.

In the plain-vanilla swap one party agrees to pay a series of cash flows on the basis of a fixed rate while the counterpart pays the cash flows on the basis of a floating rate, commonly a LIBOR interest rate is applied on the notional amount stated in the contract. Counterparties usually don't exchange the notional value.

Consider two bonds that one pays a fixed rate coupon and the second one pays a floating rate. The interest rate swaps can be considered as a series of zero coupon bonds, which pay a series of cash flows. To price a swap these cash flows are discounted by a zero-coupon rate corresponding to the date of the payment. The process of swaps valuation is intimately connected to discounting the future cash flows.

Define $p(T)$ as the discount factor for a cash flow at time T, and $0 = \overline{T_0}, \dots, \overline{T_n}$ and $0 = T_0, T_1, \dots, T_m$ respectively the payment dates on the fixed and floating legs, so that $\overline{T_n}=T_m$.

Let $\overline{\Delta_i}= \overline{T_i} - \overline{T_{i-1}}$ and $\Delta_i = T_i - T_{i-1}$ be the length of time interval between two payment dates (day count fraction) on the fixed and floating legs. Then $p(T_i)$, the discount factor is defined by

$$p(T_i) = \frac{p(T_{i-1})}{1 + \Delta_i F_i}.$$

By solving the above equation for F_i, we get the forward rate for the interval $[T_{i-1}, T_i]$

$$F_i = \frac{p(T_{i-1}) - p(T_i)}{p(T_i)\Delta_i}.$$

The cash flow (*CF*) for interval $[T_{i-1}, T_i]$ on the floating leg is determined by

$$\mathrm{CF} = \Delta_i F_i \, p(T_i) = p(T_{i-1}) - p(T_i)\,.$$

The value of the whole floating rate is then determined by

$$V^{flt} = \sum_{i=1}^{m} \{p(T_{i-1}) - p(T_i)\} = 1 - p(T_m) = 1 - p(\overline{T_n})\,.$$

Consequently on the issuance date or any reset date, the value of the floating rate bond is always at par or face value.

The total value of the floating rate bond

$$\sum_{i=1}^{m} \Delta_i F_i \, p(T_i) + p(T_m) = 1 - p(T_m) + p(T_m) = 1 .$$

Values for the fixed leg

Let C be the fixed coupon rate, then the value of the fixed leg is given by:

$$V^{fix} = \sum_{i=1}^{n} C \, \overline{\Delta_i} \, p(\overline{T_i}).$$

At swap's inception there is no advantage to either party, it means that the fair value of the swap is zero and the value of the floating leg and fixed leg must be equal, that is

$$p(\overline{T_n}) + \sum_{i=1}^{n} C \, \overline{\Delta_i} \, p(\overline{T_i}) = 1 \qquad [5]$$

If we solve the above equation for C, we get the price of the swap, which is the interest rate that makes the floating leg have the same value as the fixed leg. This rate is the fixed payment rate of the swap.

$$C = \frac{1 - p(\overline{T_n})}{\sum_{i=1}^{n} \overline{\Delta_i} \, p(\overline{T_i})}.$$

This rate which is known as the par rate makes the swap have the market value of zero. The same technique is used to value the swap during its life time, but over the time, forward rates change and it makes the value of the swap deviate from zero.

If we solve equation [6] for $p(\overline{T_n})$ we get the following formula which is basis for recursive bootstrapping of discount factor

$$p(\overline{T_n}) = \frac{1 - C \, \sum_{i=1}^{n-1} \overline{\Delta_i} \, p(\overline{T_i})}{1 + \overline{\Delta_n} \, C}.$$

Pricing a swap requires some key information. Tenor of the swap, day count conventions, frequency of the payments, maturity of the swap, payment dates, notional value of the swap, swap's rate, floating rate for the current payment and whether the swap is a payer or receiver[23], are among the required information.

[23] A payer swap pays the fixed leg and receives the floating leg. A receiver swap receives the fixed leg and pays the floating leg.

4.5. Financial crisis and its effect on pricing methodology

Sudden increase of basis spreads between similar interest rate instruments with different underlying rate tenors (swaps in particular), is among the consequences of the financial crisis or so called "credit crunch" that began in summer 2007.

Before the credit crunch in 2007, basis spreads was ignorable and therefore market segmentation was not effective, but after the financial crisis, basis spreads between similar instruments with different tenor (for example between EURIBOR and EONIA OIS swaps) significantly diverged from zero and were not negligible anymore. Therefore a new methodology and framework to price and hedge the interest rate derivatives was necessary.

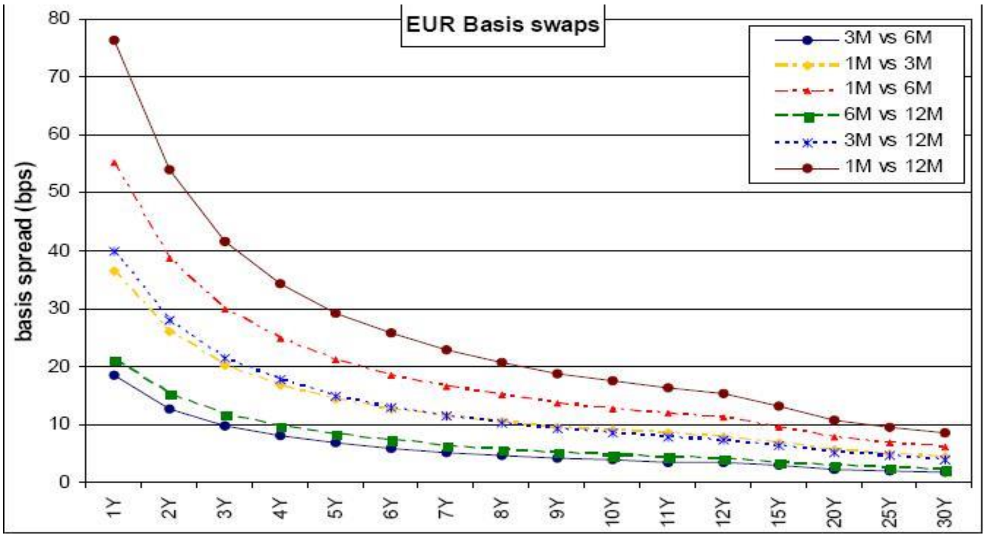

Figure 4.1: Quotations for the six euro basis swap spread curves corresponding to the four Euribor swap curves 1M, 3M, 6M, and 12M. Source: Reuters, February 16, 2009

Prior to the credit crunch a single yield curve was constructed by a selection of liquid market instruments and future cash flows were generated and discounted by the same single curve, but financial crisis implied that one single curve is not sufficient for forward rates with different tenors like 1, 3, 6 and 12 months because of the large basis spread mentioned above thus a multiple "forwarding" curves is in need to account for pricing derivatives in a new framework.

4.5.1. Market practices for pricing interest rate derivatives in multi-curve framework.

Denote the reset days for any swap as: $T_0, T_1, \ldots, T_N$ and define Δ_i as the time interval $T_i - T_{i-1}$. (For the sake of simplicity in calculations, assume that the payment dates for both floating side and fixed side are the same). The holder of a payer swap receives fixed payments at times $T_1, T_2 \ldots T_N$, and pays at the same times floating payments.

For each period $[T_i, T_{i+1}]$ the forward rate $F_{i+1}(T_i)$ is set at time T_i and the cash flow for the floating side $\Delta_{i+1}F_{i+1}(T_i)$ is received at T_{i+1}. For the same period the cash flow for the fixed side $\Delta_{i+1}\, C$, is paid at T_{i+1} where C is the (fixed) swap rate.

The discount factor $p(T_i)$, is defined by

$$p(T_i) = \frac{p(T_{i-1})}{1 + \Delta_i\, F_i}.$$

In a multi-curve framework we might generate the cash-flows with one curve (for example EURIBOR rate swap curve with 3-month tenor, as below) and discount with another (for example a EURIBOR rate swap curve with 6-month tenor, as below). Then, we have to modify the calculation as follows:

The total value of the floating side, V^{flt}

$$V^{flt} = \sum_{i=1}^{N} \Delta_i\, F_{3M}(T_{i-1,} T_i)\, p_{6M}(T_i) = \sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)}\, p_{6M}(T_i)\,.$$

We see that we cannot simplify this as we did when using the same tenors on both curves. The total value at time t for the fixed side, using a 6-month tenor for discounting equals to:

The total value of the fixed side, V^{fix}

$$V^{fix} = \sum_{i=1}^{N} \Delta_i\, C\;\, p_{6M}(T_i) = C \sum_{i=1}^{N} \Delta_i\;\, p_{6M}(T_i)\,.$$

Where C is the swap rate. This is a **par rate** since it makes the price of the swap to be equal zero when entering the swap contract. So the total value of the payer swap *(V)*, is given by

$$V = \sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)}\;\, p_{6M}(T_i) - \; C \sum_{i=1}^{N} \Delta_i\;\, p_{6M}(T_i)$$

$$= \sum_{i=1}^{N} \left(\frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} - C\,\Delta_i \right) p_{6M}(T_i)\,.$$

With different tenors the price of the swap is given by

$$\sum_{i=1}^{N}\left(\frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} - C\,\Delta_i\right)p_{6M}(T_i) = 0$$

$$C = \frac{\sum_{i=1}^{N}\frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)}\quad p_{6M}(T_i)}{\sum_{i=1}^{N}\Delta_i\ \ p_{6M}(T_i)}$$

$$C\ = \frac{1}{T}\sum_{i=1}^{N}\left(\frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)}\right)$$

Or

$$C = \frac{1}{T}\sum_{i=1}^{N}\left(\frac{p_{3M}(T_{i-1})}{p_{3M}(T_i)} - 1\right).$$

Chapter 5

Empirical Analysis

The objective of this Chapter is to assess the adequacy and forecast performance of extended version of Nelson and Siegel model.

5.1. Data and Parameters

All data in this Section are taken from Bloomberg at 2012-11-14 and 2013-08-02. Bloomberg is used by many banks to get market data and to value complex instruments.

In order to construct the yield curves and price the swaps, I have used EONIA (Euro Overnight Index Average) rates and also EURIBOR rates with 3 months and 6 months tenors, containing O/N, T/N and cash deposits from 1 month to 3 months for short-term of the curve and Euro swap rates with maturities from 1 year to 30 years to construct the long-term of the curve, all instruments have stamp date at 2012-11-14. (Deposits, FRA and Interest Rate Swaps, are the most liquid instruments in the market, respectively with the short, middle and long term maturities)

To analyze the forecast performance of extended version of Nelson and Siegel model I have used the same instruments with stamp date at 2013-08-02.

Day count convention: ACT/360
Currency: Euro

5.2. Building the spot rate and forward rate curves

In this Section yield, forward and discount curves for selected data are built, first by using interpolation and bootstrapping methods described in details in Chapter 4 and then by Nelson-Siegel-Svensson method, the model and process of fixing the parameters are explained in details in Chapter 3.

5.2.1. Data

Table 5.1: EURIBOR rates at 2012-11-14, Interval: 3 Months.

Instrument	*Tenor*	*Start date*	*End date*	*Market Rates (%)*
M. Mkt.	*O/N*	*2012-11-14*	*2012-11-15*	*0.01429*
M. Mkt.	*T/N*	*2012-11-14*	*2012-11-16*	*0.01429*
M. Mkt.	*1W*	*2012-11-14*	*2012-11-23*	*0.078*
M. Mkt.	*1M*	*2012-11-14*	*2012-12-17*	*0.108*
M. Mkt.	*2M*	*2012-11-14*	*2013-01-16*	*0.144*
M. Mkt.	*3M*	*2012-11-14*	*2013-02-18*	*0.191*
FRA	*3X6*	*2012-11-14*	*2013-05-16*	*0.16*
FRA	*6X9*	*2012-11-14*	*2013-08-16*	*0.16*
FRA	*9X12*	*2012-11-14*	*2013-11-18*	*0.18*
Swap	*2Y*	*2012-11-14*	*2014-11-17*	*0.234*
Swap	*3Y*	*2012-11-14*	*2015-11-16*	*0.346*
Swap	*4Y*	*2012-11-14*	*2016-11-16*	*0.5085*
Swap	*5Y*	*2012-11-14*	*2017-11-16*	*0.7015*
Swap	*6Y*	*2012-11-14*	*2018-11-16*	*0.9023*
Swap	*7Y*	*2012-11-14*	*2019-11-18*	*1.093*
Swap	*8Y*	*2012-11-14*	*2020-11-16*	*1.261*
Swap	*9Y*	*2012-11-14*	*2021-11-16*	*1.409*
Swap	*10Y*	*2012-11-14*	*2022-11-16*	*1.54*
Swap	*12Y*	*2012-11-14*	*2024-11-18*	*1.766*
Swap	*15Y*	*2012-11-14*	*2027-11-16*	*1.993*
Swap	*20Y*	*2012-11-14*	*2032-11-16*	*2.132*
Swap	*30Y*	*2012-11-14*	*2042-11-17*	*2.188*

Source: Bloomberg, (2012-11-14)

Table 5.2: EONIA rates at 2012-11-14

Instrument	*Tenor*	*Start date*	*End date*	*Market Rates (%)*
M. Mkt.	*O/N*	*2012-11-14*	*2012-11-15*	*0.08*
M. Mkt.	*T/N*	*2012-11-14*	*2012-11-16*	*0.08*
Swap	*1W*	*2012-11-16*	*2012-11-23*	*0.082*
Swap	*1M*	*2012-11-16*	*2012-12-17*	*0.08*
Swap	*2M*	*2012-11-16*	*2013-01-16*	*0.078*
Swap	*3M*	*2012-11-16*	*2013-02-18*	*0.071*
Swap	*4M*	*2012-11-16*	*2013-03-18*	*0.065*
Swap	*5M*	*2012-11-16*	*2013-04-16*	*0.059*
Swap	*6M*	*2012-11-16*	*2013-05-16*	*0.055*
Swap	*7M*	*2012-11-16*	*2013-06-17*	*0.052*
Swap	*8M*	*2012-11-16*	*2013-07-16*	*0.05*
Swap	*9M*	*2012-11-16*	*2013-08-16*	*0.048*
Swap	*10M*	*2012-11-16*	*2013-09-16*	*0.047*
Swap	*11M*	*2012-11-16*	*2013-10-16*	*0.047*
Swap	*1Y*	*2012-11-16*	*2013-11-18*	*0.048*
Swap	*18M*	*2012-11-16*	*2014-05-16*	*0.057*
Swap	*2Y*	*2012-11-16*	*2014-11-17*	*0.085*
Swap	*3Y*	*2012-11-16*	*2015-11-16*	*0.177*
Swap	*4Y*	*2012-11-16*	*2016-11-16*	*0.333*
Swap	*5Y*	*2012-11-16*	*2017-11-16*	*0.526*
Swap	*6Y*	*2012-11-16*	*2018-11-16*	*0.715*
Swap	*7Y*	*2012-11-16*	*2019-11-18*	*0.909*
Swap	*8Y*	*2012-11-16*	*2020-11-16*	*1.076*
Swap	*9Y*	*2012-11-16*	*2021-11-16*	*1.222*
Swap	*10Y*	*2012-11-16*	*2022-11-16*	*1.351*
Swap	*12Y*	*2012-11-16*	*2024-11-18*	*1.578*
Swap	*15Y*	*2012-11-16*	*2027-11-16*	*1.809*
Swap	*20Y*	*2012-11-16*	*2032-11-16*	*1.956*
Swap	*25Y*	*2012-11-16*	*2037-11-16*	*2.00*
Swap	*30Y*	*2012-11-16*	*2042-11-17*	*2.02*

Source: Bloomberg, (2012-11-14)

Table 5.3: EURIBOR rates at 2012-11-14, Interval: 6 Months.

Instrument	*Tenor*	*Start date*	*End date*	*Market Rates (%)*
M. Mkt.	*O/N*	*2012-11-14*	*2012-11-15*	*0.01429*
M. Mkt.	*T/N*	*2012-11-15*	*2012-11-16*	*0.01429*
M. Mkt.	*3 M*	*2012-11-16*	*2013-02-18*	*0.191*
M. Mkt.	*6 M*	*2012-11-16*	*2013-05-16*	*0.358*
Swap	*1 Y*	*2012-11-16*	*2013-11-18*	*0.34*
Swap	*2 Y*	*2012-11-16*	*2014-11-17*	*0.389*
Swap	*3 Y*	*2012-11-16*	*2015-11-16*	*0.495*
Swap	*4 Y*	*2012-11-16*	*2016-11-16*	*0.655*
Swap	*5 Y*	*2012-11-16*	*2017-11-16*	*0.847*
Swap	*6 Y*	*2012-11-16*	*2018-11-16*	*1.043*
Swap	*7 Y*	*2012-11-16*	*2019-11-18*	*1.226*
Swap	*8 Y*	*2012-11-16*	*2020-11-16*	*1.39*
Swap	*9 Y*	*2012-11-16*	*2021-11-16*	*1.533*
Swap	*10 Y*	*2012-11-16*	*2022-11-16*	*1.66*
Swap	*12 Y*	*2012-11-16*	*2024-11-18*	*1.877*
Swap	*15 Y*	*2012-11-16*	*2027-11-16*	*2.093*
Swap	*20 Y*	*2012-11-16*	*2032-11-16*	*2.218*
Swap	*25 Y*	*2012-11-16*	*2037-11-16*	*2.246*
Swap	*30 Y*	*2012-11-16*	*2042-11-17*	*2.255*

Source: Bloomberg, (2012-11-14)

5.3. Fitted Curves

Based on the Euribor rates with various tenors O/N, to 30 years from 2012-11-14 to 2042-22-17 (see Table 5.1, 5.2 and Table 5.3), the yield, forward and discount curves are shown in the following Figures.

5.3.1. Yield Curves

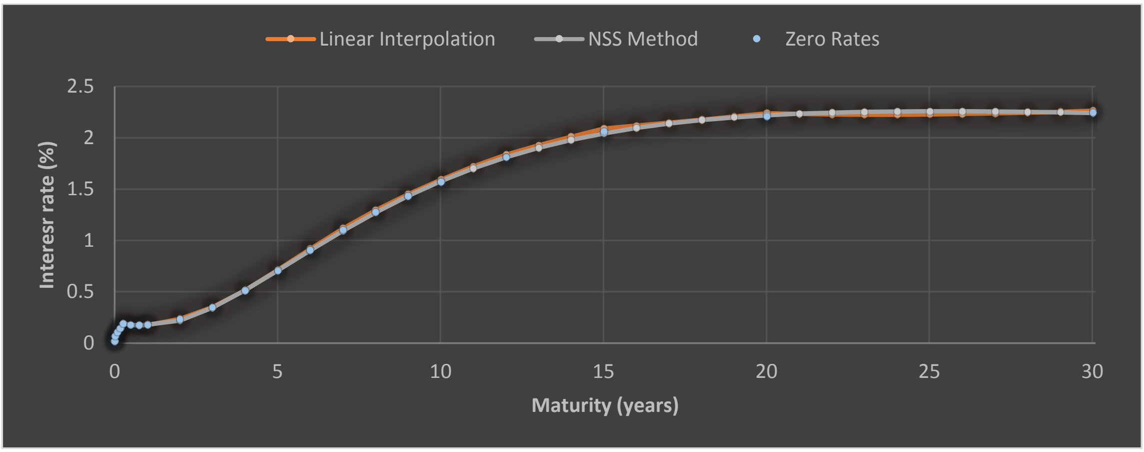

Figure 5.1: Plot of fitted yield curves for 3M Euribor rates, together with actual market rates, (Table 5.1)

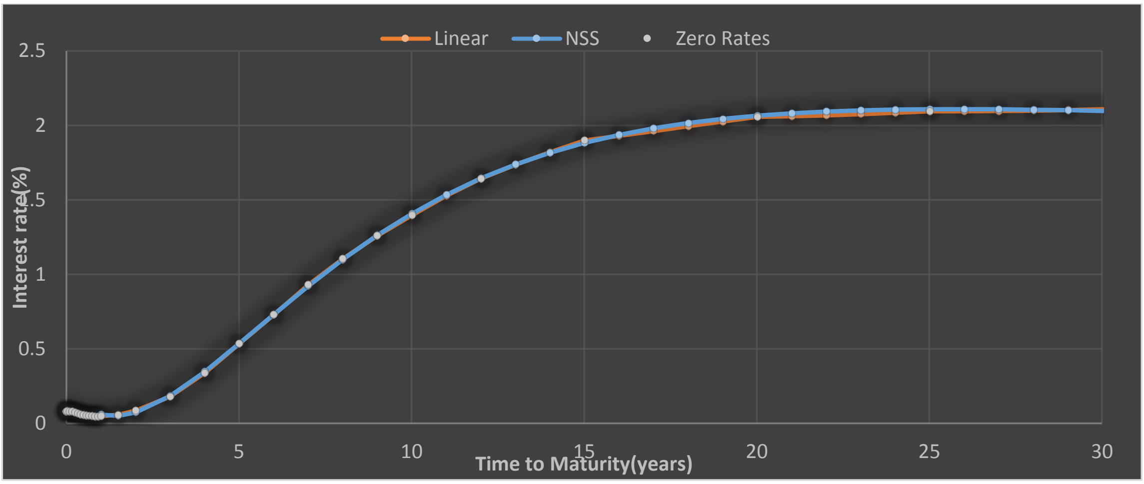

Figure 5.2: plot of fitted yield curves for EONIA rates together with actual market rates, (Table 5.2)

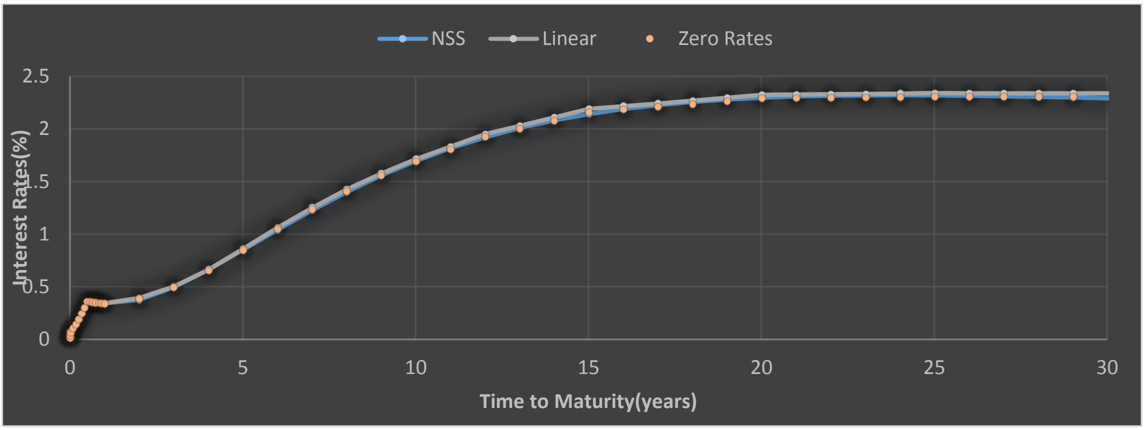

Figure 5.3: plot of fitted yield curves for 6M Euribor rates together with actual market rates, (Table 5.3)

As we see in above Figures time to maturity for all curves span to 30 years, from 2012 to 2042, both NSS and interpolation method generate smooth yield curves, both curves generated by interpolation method and NSS model almost completely overlap.

5.3.2. Forward Curves

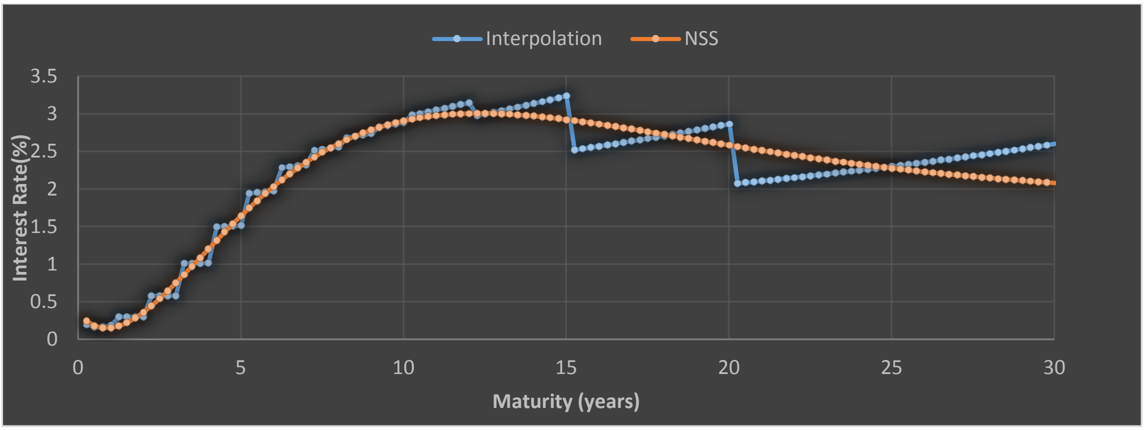

Figure 5.4: 3M Forward Curves for 3M Euribor rates built by linear interpolation and NSS model (Table 5.1)

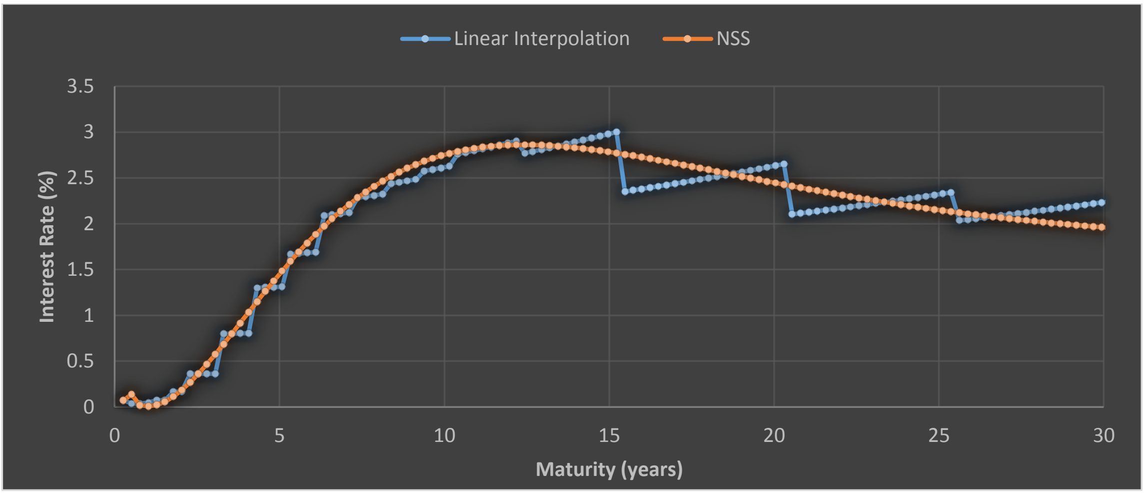

Figure 5.5: 3M Forward Curves for EONIA rates by linear interpolation and NSS model (Table 5.2)

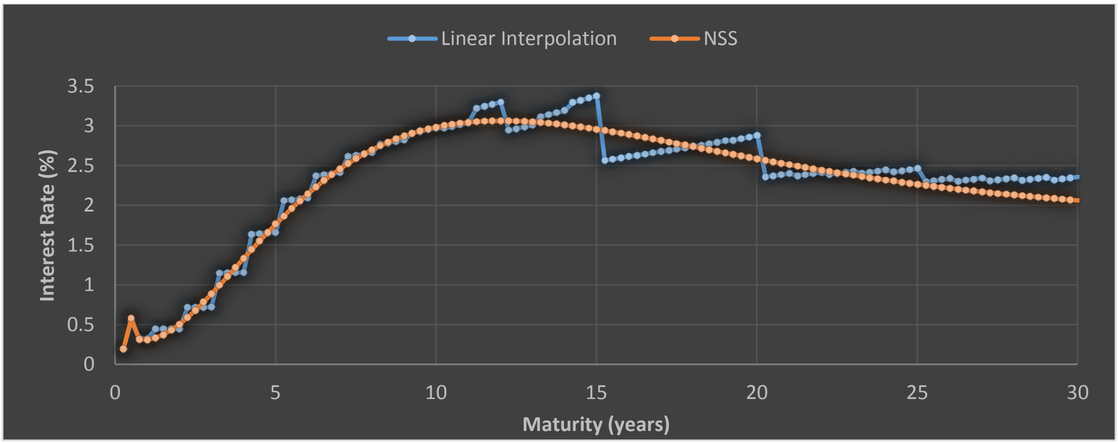

Figure 5.6: 3M Forward Curves for 6M Euribor rates built by linear interpolation and NSS model (Table 5.3)

As depicted in above Figures, forward curve built by NSS model is much smoother than the ones built by interpolation method specially at long tail of the curve, the forward rates generated by NSS model for longer maturities seems to give the average forward rates of that rates provided by interpolation method.

Since for the end tail of the curve we only have the data of 10th, 12th, 15th, 20th and 25th years, interpolation method is applied to the last 20 years , namely starting in 11th year, and that's the starting point where forward curves built by interpolation method start to change from a smooth curve to a zigzag curve.

5.3.3. Discount Curves

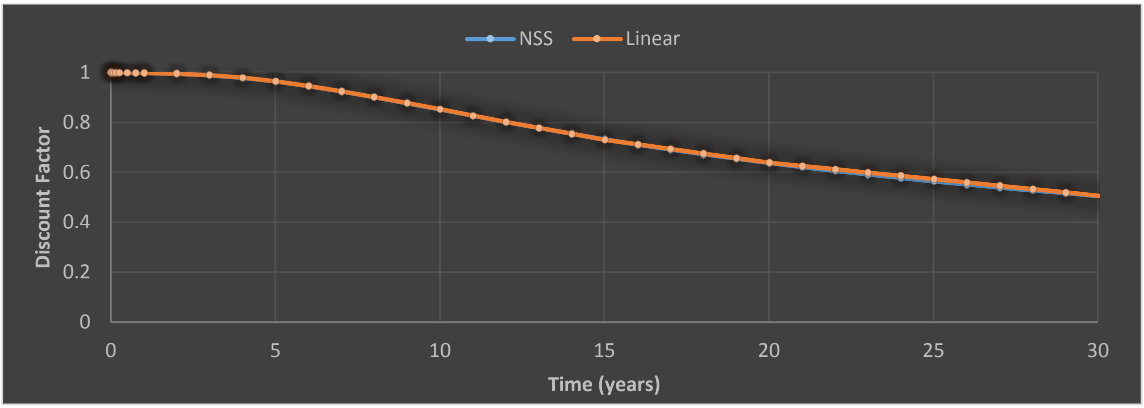

Figure 5.7: Discount Curves for 3M Euribor rates built by linear interpolation and NSS model (Table 5.1)

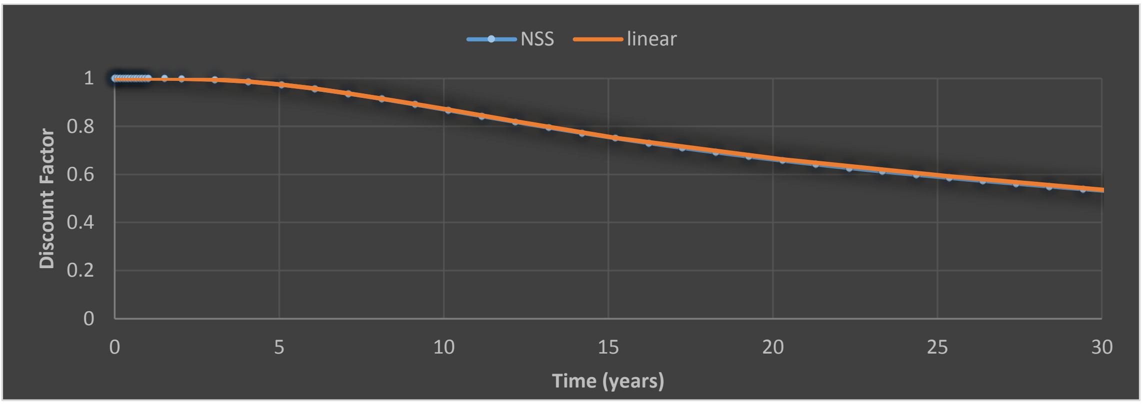

Figure 5.8: Discount Curves for EONIA rates built by linear interpolation and NSS model (Table 5.2)

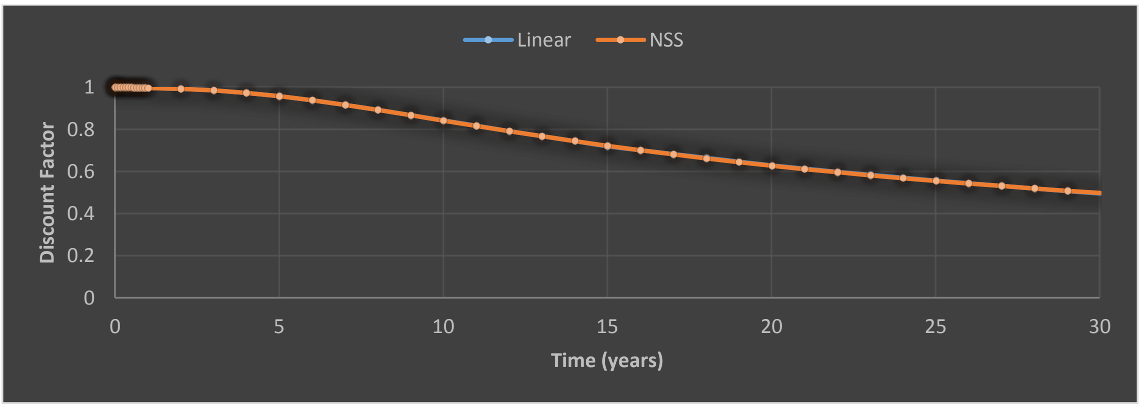

Figure 5.9: Discount Curves for 6M Euribor rates built by linear interpolation and NSS model (Table 5.3)

5.3.4. Instantaneous Forward and Spot Curves by Interpolation

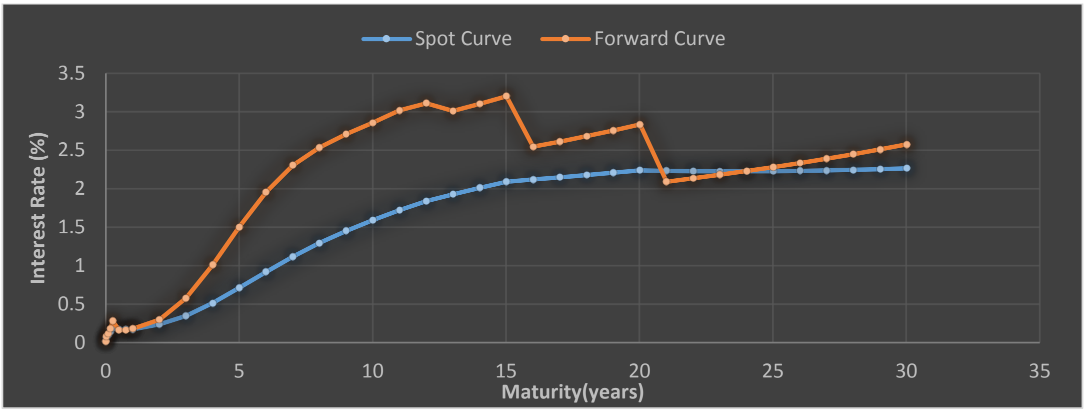

Figure 5.10: Spot and Instantaneous Forward Curves for 3M Euribor rates built by linear interpolation (Table 5.1)

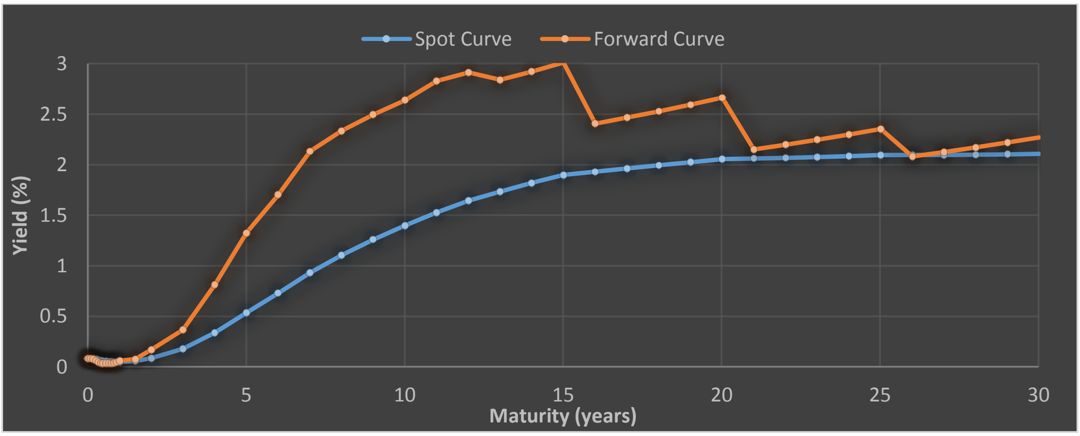

Figure 5.11: Spot and Instantaneous Forward Curves for EONIA rates built by linear interpolation (Table 5.2)

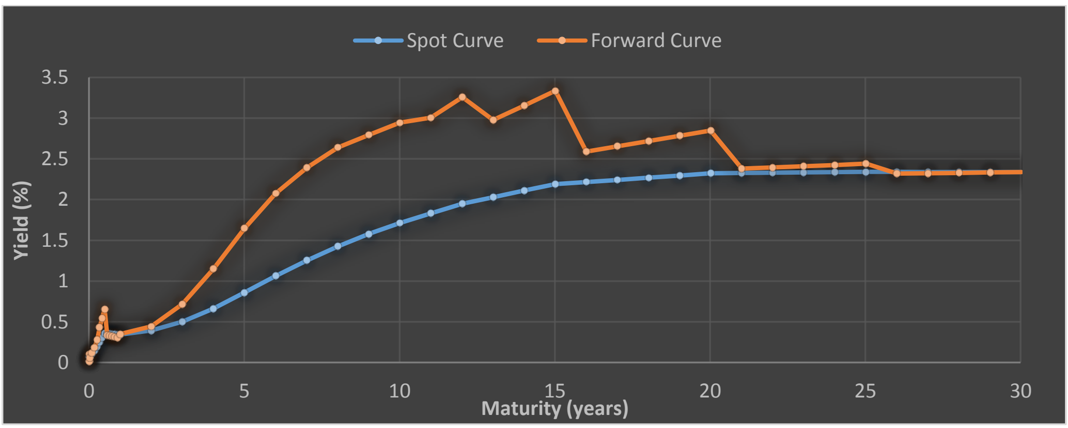

Figure 5.12: Spot and Instantaneous Forward Curves for 6M Euribor rates built by linear interpolation (Table 5.3)

5.3.5. Forward and Spot Curves by Nelson-Siegel-Svensson Model

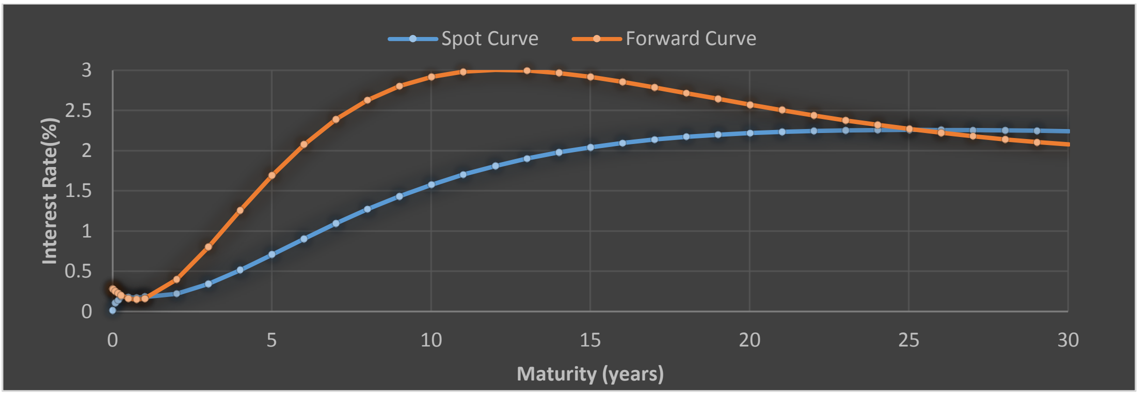

Figure 5.13: Spot and Instantaneous Forward Curves for 3M Euribor rates build by Nelson-Siegel-Svensson Model (Table 5.1)

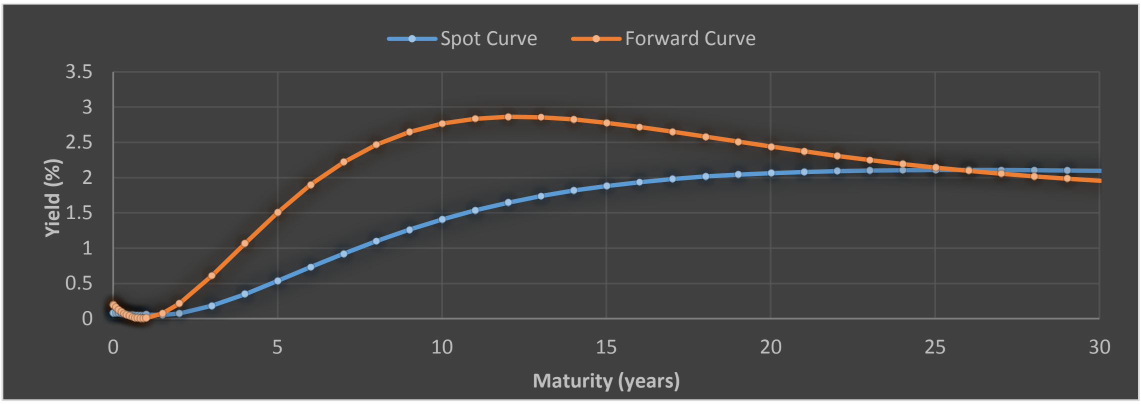

Figure 5.14: Spot and Instantaneous Forward Curves for EONIA rates build by Nelson-Siegel-Svensson Model (Table 5.2)

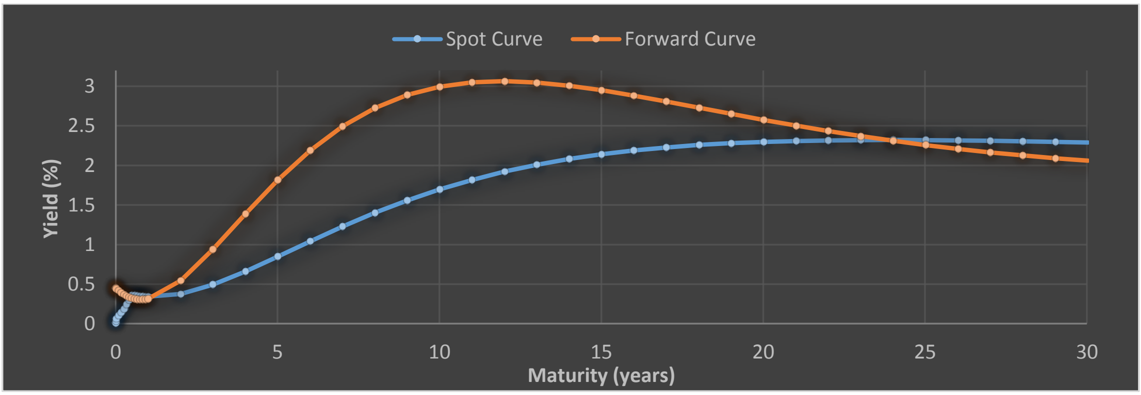

Figure 5.15: Spot and Instantaneous Forward Curves for 6M Euribor rates build by Nelson-Siegel-Svensson Model (Table 5.3)

Table 5.4: Fixed parameters of NSS (3M Euribor rates, 2012-11-14)

Long-run level of Yield curve, $\boldsymbol{\beta_0}$	*1.86*
Short-term component, $\boldsymbol{\beta_1}$	*-1.58*
Medium-term component, $\boldsymbol{\beta_2}$	*-8.20*
Parameter $\boldsymbol{\beta_3}$	*8.36*
Decay parameter, $\boldsymbol{\tau_1}$	*3.57*
Decay parameter, $\boldsymbol{\tau_2}$	*5.73*

Table 5.5: Fixed parameters of NSS (EONIA rates, 2012-11-14)

Long-run level of Yield curve, $\boldsymbol{\beta_0}$	*1.75*
Short-term component, $\boldsymbol{\beta_1}$	*-1.55*
Medium-term component, $\boldsymbol{\beta_2}$	*-8.79*
Parameter $\boldsymbol{\beta_3}$	*8.65*
Decay parameter, $\boldsymbol{\tau_1}$	*3.56*
Decay parameter, $\boldsymbol{\tau_2}$	*5.60*

Table 5.6: Fixed parameters of NSS (6M Euribor rates, 2012-11-14)

Long-run level of Yield curve, $\boldsymbol{\beta_0}$	*1.84*
Short-term component, $\boldsymbol{\beta_1}$	*-1.39*
Medium-term component, $\boldsymbol{\beta_2}$	*-9.04*
Parameter $\boldsymbol{\beta_3}$	*9.46*
Decay parameter, $\boldsymbol{\tau_1}$	*3.63*
Decay parameter, $\boldsymbol{\tau_2}$	*5.58*

Parameters of Nelson-Siegel-Svensson model are fixed to best fit the model with actual market yields, solver function in Excel use Generalized Reduced Gradient (GRG) algorithm, one of the most powerful non-linear optimization algorithms to fix these parameters by minimizing sum of the squared discrepancy between estimated rates generated by NSS model and observed rates in the market.[24]

[24] http://support.microsoft.com/kb/82890(3/1/2014)

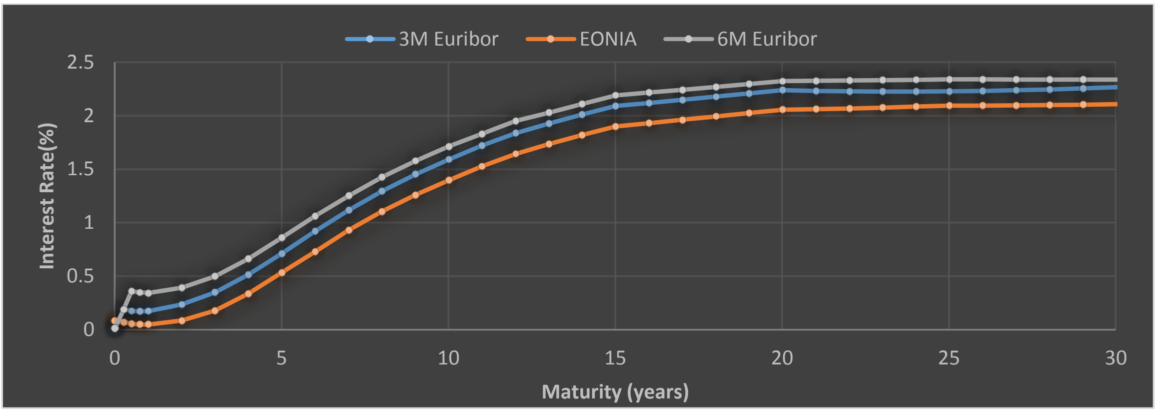

Figure 5.16: Comparison plot of EURIBOR Interest Rate Yield curves with different tenors (Linear Interpolation)

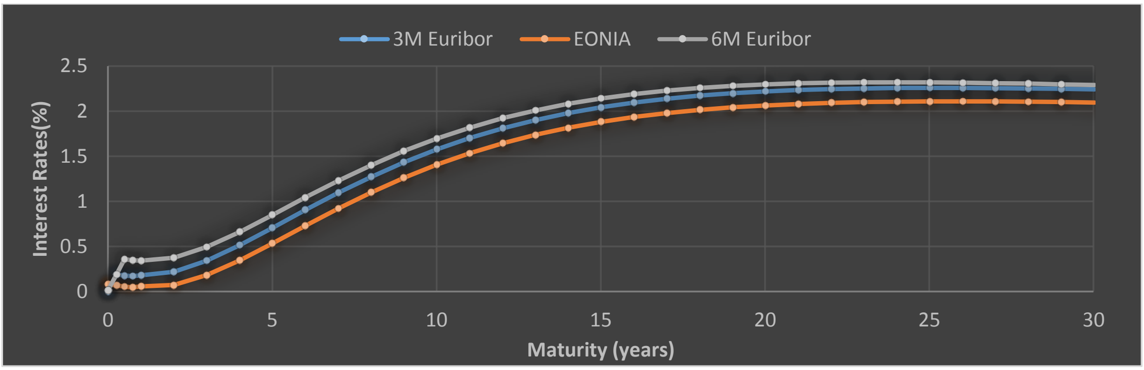

Figure 5.17: Comparison plot of EURIBOR Interest Rate Yield curves with different tenors (NSS Model)

As we see in the above Figures, 3M Euribor Rates are higher than EONIA rates and subsequently the 6M EURIBOR curve lies above the 3M EURIBOR curve. We know from Section 2.4 that Euribor is a benchmark giving an indication of the average rate at which banks lend unsecured funding in the euro interbank market for a given period. We should consider that even if a bank has high investment grade and is able to borrow at Euribor today, it doesn't guarantee that it will remain Euribor rated and be able to borrow at Euribor rate next quarter, this makes a big difference between 6M Euribor and 3M Euribor rate.

Therefore, 6M Euribor rate are higher than a 3M Euribor rates and EONIA rates by its definition satisfy the "credit risk approaching zero" criterion and represent risk-free rates for banks and consequently EONIA curve lies under two other curves.

Now that all spot, forward and discount curves are built we can proceed to price an interest rate swap. The underlying instrument is IRS vs 3M Euribor rates, we use these rates to generate future cash flows, for discounting these cash flows and price the swap, three discount curves obtained by 3M Euribor, 6M Euribor and EONIA rates are used. Using Libor-based rates to discount the cash flows and generate floating rates has been the usual method in pricing the swap. However recently dealers tend to value and price collateralized interest rate swaps by using the OIS curve to mitigate counterparty credit risk.[25]

Linear Interpolation & Bootstrapping

Maturity of Swap	*Discounted by EUR-6M*	*Discounted by EUR-3M*	*Discounted by EONIA*	*Market Price*
5 Years	*0.71*	*0.71*	*0.71*	*0.70*
10 Years	*1.55*	*1.55*	*1.55*	*1.54*
20 Years	*2.13*	*2.14*	*2.14*	*2.13*
30 Years	*2.18*	*2.18*	*2.18*	*2.19*

Nelson-Siegel-Svensson Model

Maturity of Swap	*Discounted by EUR-6M*	*Discounted by EUR-3M*	*Discounted by EONIA*	*Market Price*
5 Years	*0.70*	*0.70*	*0.71*	*0.70*
10 Years	*1.53*	*1.53*	*1.54*	*1.54*
20 Years	*2.11*	*2.12*	*2.13*	*2.13*
30 Years	*2.16*	*2.16*	*2.17*	*2.19*

Table 5.7: Swap rates with maturities of 5, 10, 20 and 30 years for selected data, using Interpolation and NSS Model (see Tables 5.1, 5.2 and 5.3)

As we observe in Table 5.7 fixed rate on the collateralized swap (swap which is priced by using EONIA curve) is slightly higher than other fixed rate with the same maturity, our results confirm the result of studying by Johannes and Sundaresan (2009), they claim that fixed rate on a collateralized swap should be higher than uncollateralized ones and provide evidence for their findings. The reason for higher rates on a collateralized swap is due to the asymmetric impact of collateralization on counterparties, the fixed-rate payer benefits from interest rate volatility while the fixed-rate receiver suffer from such volatility, furthermore posting collateral is costly for counterparties. See Johannes and Sundaresan (2009) for more details.

We should consider that since all cash flows are generated by one unique yield curve (3M Euribor rates) the difference in par rates are due to the different discount curves used for pricing the swaps. 6M Euribor discount factors are lower than 3M Euribor discount factors which leads to higher par rates for the swaps which use 3M Euribor discount curve in pricing process.

The same reasoning applies to swap rates calculated by NSS curves. Forward rates obtained by NSS method are lower than forward rates generated by bootstrapping and linear interpolations, especially at long-tail of the curves, thus fixed rates calculates by NSS model are lower than those which are calculated by bootstrapping and interpolations.

[25] J.Smith, Donald (2012)

5.4. Valuing interest rate swaps

Now that all fixed rates are determined we can proceed by valuing the swap at any time after swap's inception. The same valuation technique which used to price an interest rate swap applies to value the swap during its life. See Section 4.4 for details. The value of the swap is simply the difference between the present value of floating-rate side and fixed-rate side.

To calculate the value of the fixed-side, swap's par rates that we obtained in the previous Section are used, but since over time forward rates change in the market in order to value the floating side, we need Euribor rates in effect for remaining period of swap's life.

In the next Section, 3 months EURIBOR rates, 6 months EURIBOR rates, and EONIA rates at 2013-08-02 are presented, all these rates are taken from Bloomberg at the same date.

The following Tables present the 3M-Euribor, 6M-Euribor and EONIA rates at 2013-08-02.

Table 5.8: EURIBOR rates at 2013-08-02, Interval: 3 Months.

Instrument	*Tenor*	*Start Date*	*End Date*	*Market Rates (%)*
M. Mkt.	*O/N*	*2013-08-02*	*2013-08-05*	*0.0436*
M. Mkt.	*T/N*	*2013-08-02*	*2013-08-06*	*0.0436*
M. Mkt.	*1W*	*2013-08-06*	*2013-08-13*	*0.105*
M. Mkt.	*1M*	*2013-08-06*	*2013-09-06*	*0.129*
M. Mkt.	*2M*	*2013-08-06*	*2013-10-07*	*0.18*
M. Mkt.	*3M*	*2013-08-06*	*2013-11-06*	*0.228*
Fra	*3X6*	*2013-08-06*	*2014-02-06*	*0.27*
Fra	*6X9*	*2013-08-06*	*2014-05-06*	*0.335*
Swap	*1Y*	*2013-08-06*	*2014-08-06*	*0.314*
Swap	*2Y*	*2013-08-06*	*2015-08-06*	*0.443*
Swap	*3Y*	*2013-08-06*	*2016-08-08*	*0.621*
Swap	*4Y*	*2013-08-06*	*2017-08-07*	*0.848*
Swap	*5Y*	*2013-08-06*	*2018-08-06*	*1.077*
Swap	*6Y*	*2013-08-06*	*2019-08-06*	*1.281*
Swap	*7Y*	*2013-08-06*	*2020-08-06*	*1.461*
Swap	*8Y*	*2013-08-06*	*2021-08-06*	*1.622*
Swap	*9Y*	*2013-08-06*	*2022-08-08*	*1.769*
Swap	*10Y*	*2013-08-06*	*2023-08-07*	*1.901*
Swap	*12Y*	*2013-08-06*	*2025-08-06*	*2.113*
Swap	*15Y*	*2013-08-06*	*2028-08-07*	*2.322*
Swap	*20Y*	*2013-08-06*	*2033-08-08*	*2.444*
Swap	*25Y*	*2013-08-06*	*2038-08-06*	*2.465*
Swap	*30Y*	*2013-08-06*	*2043-08-06*	*2.463*

Source: Bloomberg, (2013-08-02)

Table 5.9: EURIBOR rates at 2013-08-02, Interval: 6 Months.

Instrument	*Tenor*	*Start Date*	*End Date*	*Market Rates (%)*
M. Mkt.	*O/N*	*2013-08-02*	*2013-08-05*	*0.0436*
M. Mkt.	*T/N*	*2013-08-02*	*2013-08-06*	*0.0436*
M. Mkt.	*1W*	*2013-08-06*	*2013-08-13*	*0.105*
M. Mkt.	*1M*	*2013-08-06*	*2013-09-06*	*0.129*
M. Mkt.	*2 M*	*2013-08-06*	*2013-10-07*	*0.18*
M. Mkt.	*3 M*	*2013-08-06*	*2013-11-06*	*0.228*
M. Mkt.	*6 M*	*2013-08-06*	*2014-02-06*	*0.341*
Swap	*1Y*	*2013-08-06*	*2014-08-06*	*0.423*
Swap	*2Y*	*2013-08-06*	*2015-08-06*	*0.57*
Swap	*3Y*	*2013-08-06*	*2016-08-08*	*0.751*
Swap	*4Y*	*2013-08-06*	*2017-08-07*	*0.978*
Swap	*5Y*	*2013-08-06*	*2018-08-06*	*1.205*
Swap	*6Y*	*2013-08-06*	*2019-08-06*	*1.405*
Swap	*7Y*	*2013-08-06*	*2020-08-06*	*1.581*
Swap	*8Y*	*2013-08-06*	*2021-08-06*	*1.783*
Swap	*9Y*	*2013-08-06*	*2022-08-08*	*1.881*
Swap	*10Y*	*2013-08-06*	*2023-08-07*	*2.009*
Swap	*12Y*	*2013-08-06*	*2025-08-06*	*2.213*
Swap	*15Y*	*2013-08-06*	*2028-08-07*	*2.411*
Swap	*20Y*	*2013-08-06*	*2033-08-08*	*2.519*
Swap	*25Y*	*2013-08-06*	*2038-08-06*	*2.531*
Swap	*30Y*	*2013-08-06*	*2043-08-06*	*2.522*

Source: Bloomberg, (2013-08-02)

Table 5.10: EONIA rates at 2013-08-02

Instrument	*Tenor*	*Start Date*	*End Date*	*Market Rates (%)*
M. Mkt.	*O/N*	*2013-08-02*	*2013-08-05*	*0.093*
M. Mkt.	*T/N*	*2013-08-02*	*2013-08-06*	*0.093*
M. Mkt.	*1W*	*2013-08-06*	*2013-08-13*	*0.097*
M. Mkt.	*1M*	*2013-08-06*	*2013-09-06*	*0.101*
M. Mkt.	*2M*	*2013-08-06*	*2013-10-07*	*0.105*
M. Mkt.	*3M*	*2013-08-06*	*2013-11-06*	*0.106*
M. Mkt.	*6M*	*2013-08-06*	*2014-02-06*	*0.113*
M. Mkt.	*9M*	*2013-08-06*	*2014-05-06*	*0.13*
Swap	*1Y*	*2013-08-06*	*2014-08-06*	*0.15*
Swap	*2Y*	*2013-08-06*	*2015-08-06*	*0.255*
Swap	*3Y*	*2013-08-06*	*2016-08-08*	*0.42*
Swap	*4Y*	*2013-08-06*	*2017-08-07*	*0.64*
Swap	*5Y*	*2013-08-06*	*2018-08-06*	*0.864*
Swap	*6Y*	*2013-08-06*	*2019-08-06*	*1.065*
Swap	*7Y*	*2013-08-06*	*2020-08-06*	*1.244*
Swap	*8Y*	*2013-08-06*	*2021-08-06*	*1.406*
Swap	*9Y*	*2013-08-06*	*2022-08-08*	*1.555*
Swap	*10Y*	*2013-08-06*	*2023-08-07*	*1.69*
Swap	*12Y*	*2013-08-06*	*2025-08-06*	*1.911*
Swap	*15Y*	*2013-08-06*	*2028-08-07*	*2.135*
Swap	*20Y*	*2013-08-06*	*2033-08-08*	*2.275*
Swap	*25Y*	*2013-08-06*	*2038-08-06*	*2.31*
Swap	*30Y*	*2013-08-06*	*2043-08-06*	*2.318*

Source: Bloomberg, (2013-08-02)

Comparing rates in above Tables with old rates at 2012-11-14 (see Tables 5.1, 5.2, 5.3, 5.6, 5.7 and 5.8) we observe that interest rates have risen. With the above rates we build new yield and discount curves to value our Swaps at 2013-08-02. The whole process is as the same as we did for building curves with old data. Following Figures present the new curves for selected data at 2013-08-02.

5.4.1. Yield Curves

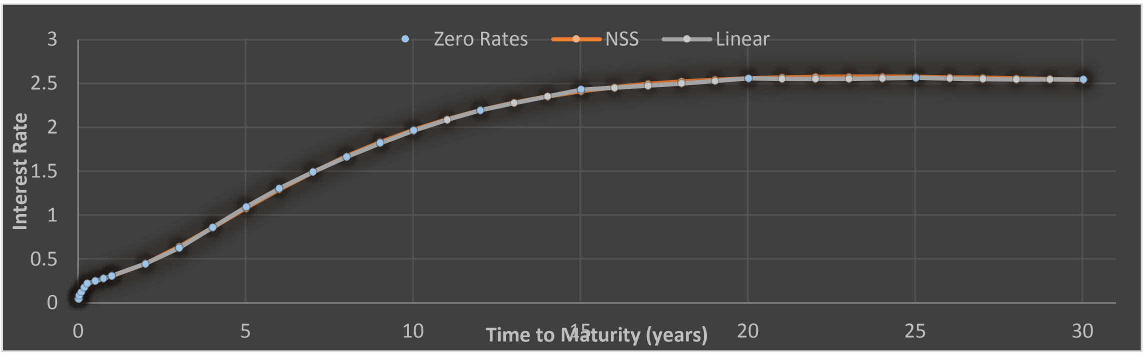

Figure 5.18: Plot of fitted yield curves for 3M Euribor rates, together with actual market rates, (Table 5.6)

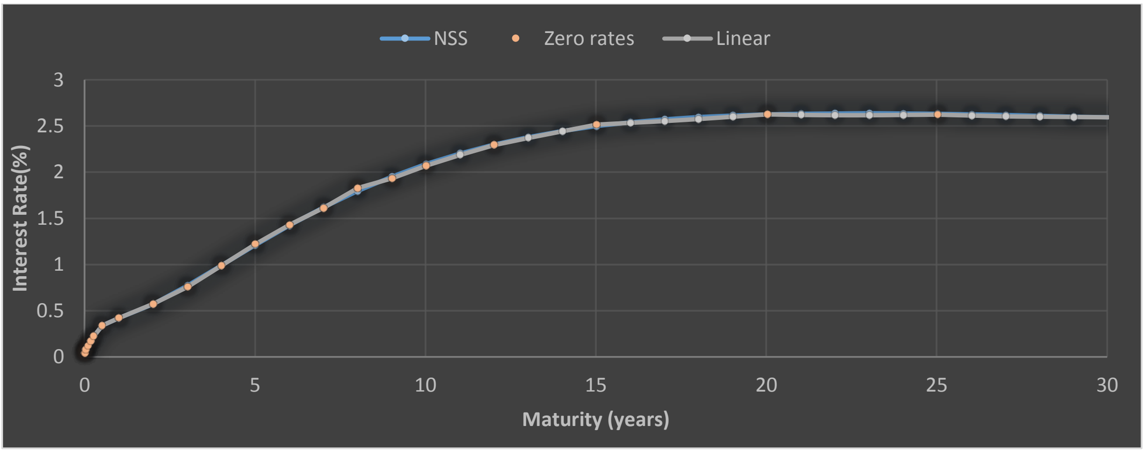

Figure 5.19: Plot of fitted yield curves for 6M Euribor rates, together with actual market rates, (Table 5.7)

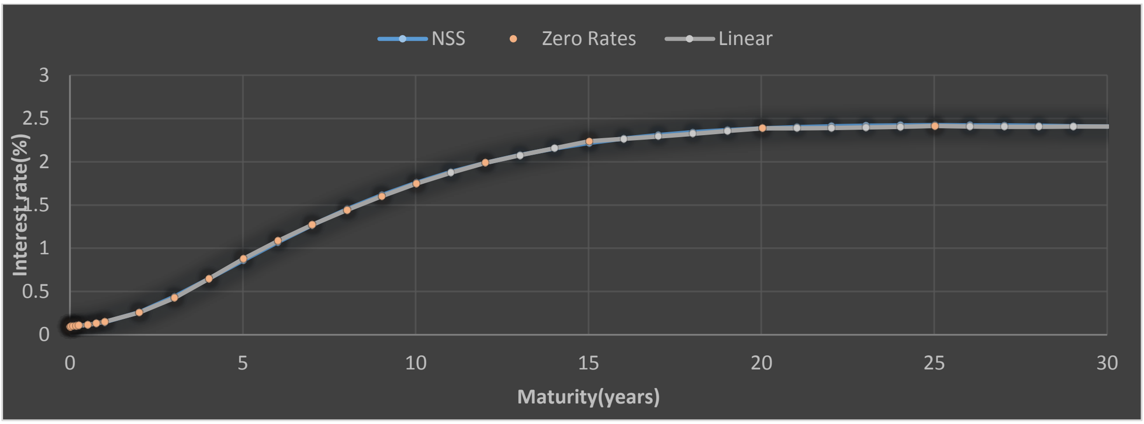

Figure 5.20: Plot of fitted yield curves for EONIA rates, together with actual market rates, (Table 5.8)

5.4.2. Forward Curves

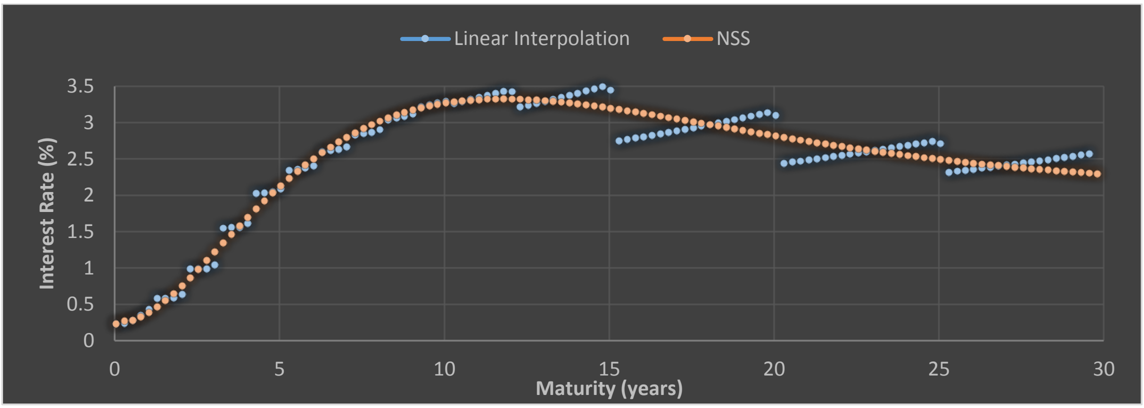

Figure 5.21: 3M Forward Curves for 3M Euribor rates built by linear interpolation and NSS model (Table 5.6)

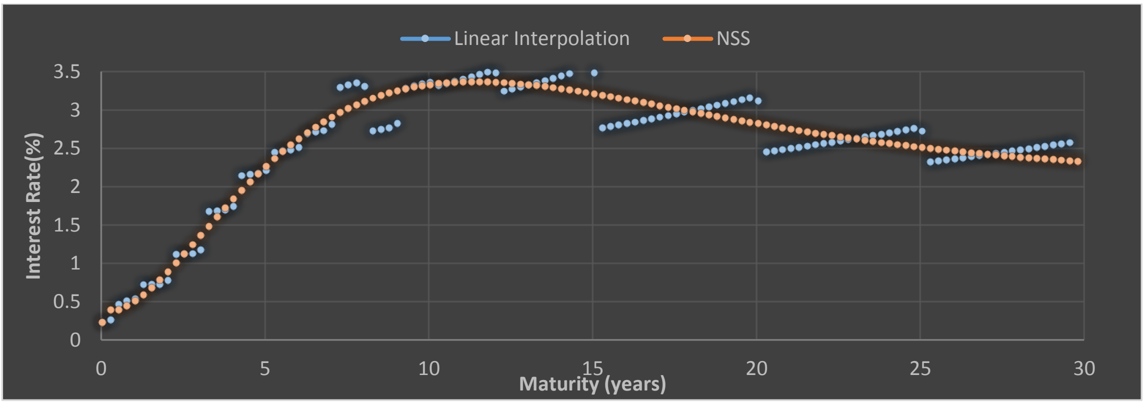

Figure 5.22: 3M Forward Curves for 6M Euribor rates built by linear interpolation and NSS model (Table 5.7)

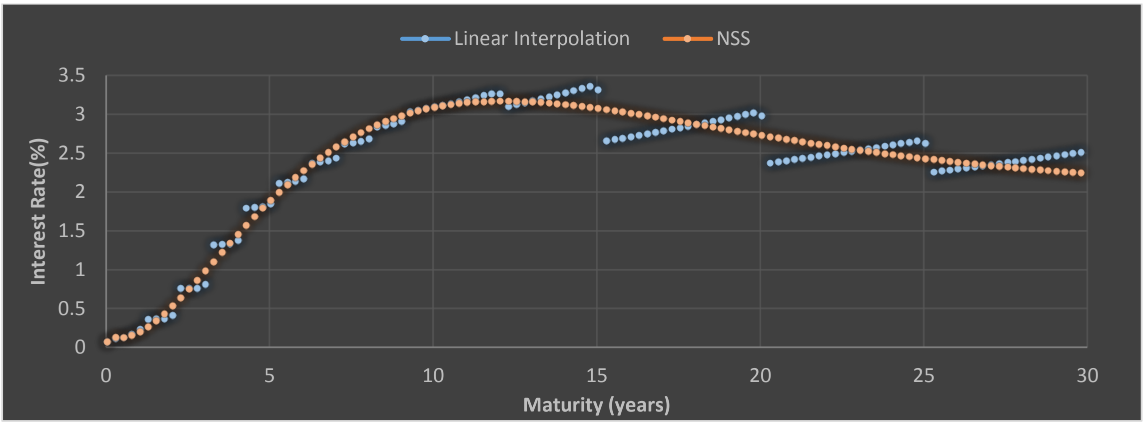

Figure 5.23: 3M Forward Curves for EONIA rates built by linear interpolation and NSS model (Table 5.8)

5.4.3 Discount Curves

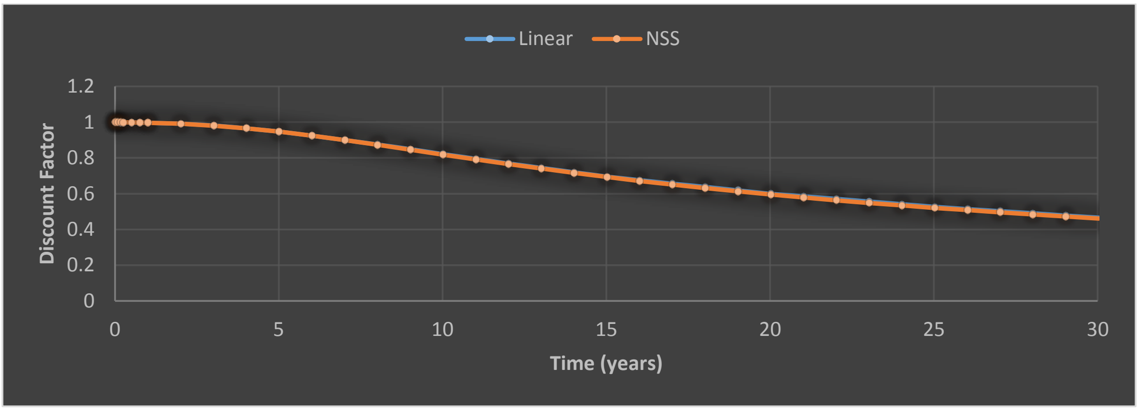

Figure 5.24: Discount Curves for 3M Euribor rates built by linear interpolation and NSS model (Table 5.6)

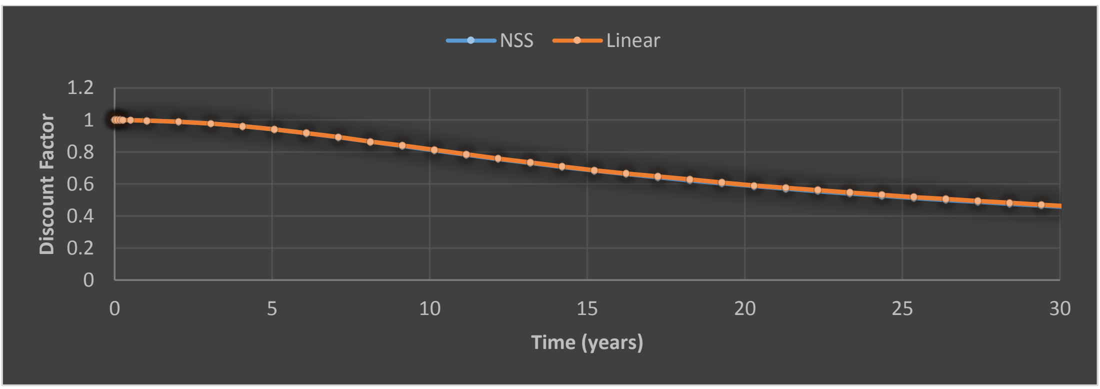

Figure 5.25: Discount Curves for 6M Euribor rates built by linear interpolation and NSS model (Table 5.7)

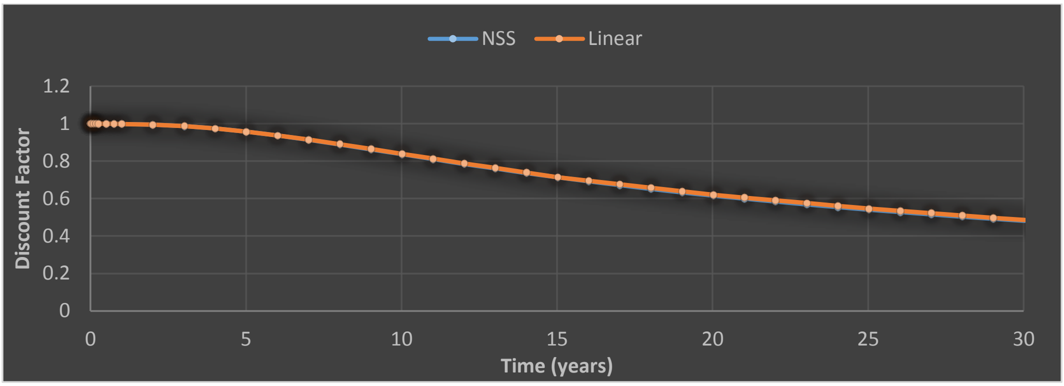

Figure 5.26: Discount Curves for EONIA rates built by linear interpolation and NSS model (Table 5.8)

The following Tables present the value of under studying swaps:
Valuation Date: 2013-08-06
Notional amount: 1,000,000,000 Euro

Maturity of Swap	*Discounted by EUR-6M*	*Discounted by EUR-3M*	*Discounted by EONIA*
5 Years	*-9,410,087*	*-9,433,067*	*-9,481,658*
10 Years	*-23,352,551*	*-23,499,421*	*-23,763,778*
20 Years	*-46,264,191*	*-46,625,684*	*-47,409,913*
30 Years	*-60,462,927*	*-60,959,459*	*-62,239,961*

Table 5.11: Value of the Swaps with maturities of 5, 10, 20 and 30 years for selected data (see Tables 5.1, 5.2, 5.3, 5.6, 5.7 and 5.8), using Linear Interpolation Method.

Maturity of Swap	*Discounted by EUR-6M*	*Discounted by EUR-3M*	*Discounted by EONIA*
5 Years	*-8,999,088*	*-9,019,984*	*-9,070,638*
10 Years	*-25,553,776*	*-25,708,732*	*-26,120,736*
20 Years	*-50,654,829*	*-51,090,601*	*-52,350,453*
30 Years	*-64,482,225*	*-65,113,313*	*-66,897,385*

Table 5.12: Value of the Swaps with maturities of 5, 10, 20 and 30 years for selected data (see Tables 5.1, 5.2, 5.3, 5.6, 5.7 and 5.8), using Nelson-Siegel-Svensson Model

Chapter 6

Conclusions

In this thesis the parsimonious model of Nelson-Siegel-Svensson are applied to the Euro market data. The Nelson and Siegel model, and the extended version of their model by Svensson, are explained in detail. Furthermore, the process of pricing and valuing interest rate swaps and also the effect of financial crisis on pricing methodology are explained.

In Chapter 5 of this thesis the significant basis spread between same instruments with different tenors is depicted in Figures 5.16 and 5.17, which confirms the necessity of a multi-curve framework in pricing and valuing interest rate derivatives after the financial crisis in 2008.

Our study confirms the ability of Nelson-Seigel-Svensson model to capture the various shapes of the spot curves and to estimate the term structure of interest rates over time. Moreover, this model provides smooth term structures in both zero rates and forward rates and therefore provides better estimation to value instruments that have cash flows on other days than the market data. Furthermore, prices given by Nelson-Seigel-Svensson model are very close to market prices and rates generated by interpolation method.

At a short glance the swap rates generated by interpolation and NSS model for maturities from 5 years up to 30 years are either the same or differ up to 0.02 %. Furthermore, swap rates calculated by interpolation differ from quoted swap rates up to 0.02 % and swap rates calculated by NSS model differ from the quoted swap rates up to 0.03%.

Maturity of Swap	*Discounted by EUR-6M*	*Discounted by EUR-3M*	*Discounted by EONIA*	*Market Price*
	Linear Interpolation/NSS	*Linear Interpolation/NSS*	*Linear Interpolation/NSS*	*Linear Interpolation/NSS*
5 Years	*0.71 / 0.70*	*0.71 / 0.70*	*0.71 / 0.71*	*0.70*
10 Years	*1.55 / 1.53*	*1.55 /1.53*	*1.55 / 1.54*	*1.54*
20 Years	*2.13 / 2.11*	*2.14 / 2.12*	*2.14 / 2.13*	*2.13*
30 Years	*2.18 / 2.16*	*2.18 / 2.16*	*2.18 / 2.17*	*2.19*

Table 6.1: Swap rates with maturities of 5, 10, 20 and 30 years for selected data, using Interpolation and NSS Model (see Tables 5.1, 5.2 and 5.3)

The Nelson-Seigel model avoids the data mining problem that we face if we use bootstrapping with linear interpolation. This model provides accurate estimation for missing data. Moreover Nelson-Siegel-Svensson model is a parsimonious model that accurately estimates the entire yield curve therefore the author concludes that this model is a better approach in building the yield and forward curves than the linear interpolation and bootstrapping.

However for shorter maturities (in our case up to one year) the approximation is not satisfactory. One suggestion to overcome this obstacle is to combine the basis spline approximation with Nelson-Seigel approximation to build the short-term of the term structure of interest rate[26]. Moreover fixing the parameters of Nelson-Seigel-Svensson model is challenging. To fix the parameters of this model we need a solver/optimizing software, in this work the Excel solver is used. Getting the optimal values for the parameters is highly related to the initial values that we set. Although the initial values of the vector of parameters can be set by the help of the economic interpretation of these parameters.

[26] Tobler (1999)

Chapter 7

Summary of reflection of objectives in the thesis

This Chapter is devoted to the summarization of reflection of objectives in this thesis.

7.1 Objective 1 - Knowledge and understanding

The author has used a broad range of articles to fulfill this thesis requirements, although not all the resources are cited as the reference. Different interpolation methods, estimating the yield curve by Nelson-Siegel-Svensson model and also interpolation method, dynamics of the Nelson-Siegel-Svensson model and pricing the interest rate derivatives are among the subjects of the articles that are used to cover all Chapters of this thesis. At the research level, studying the time series of the parameters of the Nelson-Siegel-Svensson model, as well as studying the comparative advantages between the classic model of Nelson-Siegel and its extensions, seems to be a good directions for further researches and analysis that is beyond the scope of this thesis. Although the classic model of Nelson-Siegel and its extended version by Svensson are fully covered in Chapter 3.

7.2 Objective 2 – Methodological knowledge

In Chapter 3, the method of Nelson-Siegel-Svensson, and in Chapter 4, interpolation and bootstrapping method for building the term structure of interest rate are given in details. Furthermore the author has used the graphical, numerical and analytical research in different parts of this thesis. The visual presentation of the spot curves are used, to modify and adjust the vector of parameters of the Nelson-Siegel-Svensson model. The optimal value of the vector of parameters of the Nelson-Siegel-Svensson model is obtained numerically. And finally the equation for forward rate in Section 3.2, as well as the proofs in all Chapters are given by analytical research.

7.3 Objective 3 – Critically and systematically integrate knowledge

The sources used for this thesis are primarily: the lecture notes in Analytical Finance II by Jan Röman, papers and articles found through the Google, and also the thesis done by students in Mälardalen Högskolan and other universities. Those resources that I have directly used in this thesis are mentioned in Reference Section, however to better understanding some concepts at the first step, the author has used some sources such as Wikipedia, Investopedia, etc. Naturally not all the sources that are used to acquire information and knowledge to accomplish this thesis made it through to the final text.

7.4 Objective 4 – Independently and creatively identify and carry out advanced tasks

In Section 3.2 the equation for the forward rate is driven by the author from Nelson-Siegel-Svensson model. This equation is used to find the forward rates for any future date, and helps to avoid using interpolation and bootstrapping method to derive the forward rate for underlying derivatives. Furthermore this work is a practical effort to use the theoretical knowledge to value the interest rate swaps, and using the Nelson-Siegel-Svensson model. The author believes that the result of this thesis can be used by the Banks in Sweden in order to revise their methods to build the term structure of the interest rate (which by the knowledge of the author has been mostly interpolation method).

7.5 Objective 5 – Present and discuss conclusions and knowledge

I believes that the text in this thesis is easily understandable by readers with moderate knowledge in mathematical finance and familiar with financial terms. The author has tried to explain all the terms used in this thesis in details and in simple English understandable by readers with modest knowledge in English. In the conclusion Section the author has summarized the advantages of using the Nelson-Siegel-Svensson model over the Interpolation and bootstrapping method to build the yield curve, furthermore some limitations in using the Nelson-Siegel-Svensson model are summarized in Chapter 6, which can be the direction for future studies.

7.6 Objective 7 – Scientific, social and ethical aspects

In Chapter 5, the author has used the Euro currency and Swedish calendar to do the calculations, although all the methods are applicable via small modifications to other calendars and currencies. All the resources that have been used in this thesis are cited in References. I believes that science owes its growth to the collective effort of all scientists in different areas and in different parts of the world. I have benefited the researches done by the other students and scientists to accomplish this thesis and I sure hope that others will benefit the result of my work.

References

Barrett, W. R., T. F. Jr. Gosnell, and A. J. Heuson. 1995. *Yield Curve Shifts and the Selection of Immunization Strategies*, Journal of Fixed Income **5**, 53-64.

Bianchetti, Marco. 2010. *Two Curves, One Price: Pricing & Hedging Interest Rate Derivatives Decoupling Forwarding and Discounting Yield Curves*, Risk Magazine.

Diebold Francis X. and Canlin Li. 2006. *Forecasting the Term Structure of Government Bond Yields*, Journal of Econometrics **130**, 337-364.

Donald, J.Smith. 2012. *Valuing Interest Rate Swaps Using OIS Discounting*, Boston U. School of Management Research Paper No. 2012-11.

Dullmann, K. and M. Uhrig-Homburg.2000. *Profiting Risk Structure of Interest Rates: An Empirical Analysisfor Deutschemark-Denominated Bonds*, European Financial Management **6**, 367-388.

Fabozzi, F. J., L. Martellini, and P. Priaulet. 2005. *Predictability in the Shape of the Term Structure of Interest Rates*, Journal of Fixed Income **15**, 40-53.

Hagan, Patrick S. and Graeme West. 2006. *Interpolation Methods for Curve Construction*, Applied Mathematical Finance **13**, 89-129.

Hodges, Stewart D. and Naru Parekh. 2006. *Term Structure Slope Risk Convexity Revisited*, The Journal of Fixed Income 16, 54-59.

Hull, John C. 2009. *Options, Futures, and other Derivatives*, 7th ed., Prentice Hall India.

Johannes, Michael and Suresh Sundaresan. 2007. *The Impact of Collateralization on Swap Rates*, Journal of Finance.

Martellini, L. and Meyfredi, J. C. 2007. *A Copula Approach to Value-at-Risk Estimation for Fixed-Income Portfolios*, Journal of Fixed Income **17**, 5-15.

Nelson, Charles R. and Andrew F.Siegel. 1987. *Parsimonious Modeling of Yield Curves*. The Journal of Business **60**, 473-489.

Tobler, J. 1999. *Ein zweistufiges Verfahren zur Schätzung der Zinskurve*, Schweizerische Zeitschrift fur Volkswirtschaft und Statistik **135**, 41-74.

Appendix 1

➢ Visual Basic code for Nelson-Siegel spot rate function:

```
' ---------------------------DISCRIPTION--------------------------------------------
' NS calculates the spot rates according to Nelson-Siegel model
' m is time to maturity
' tau1 is time and shows the location of the possible hump in the spot curve
' beta0, beta1 and beta2 are constants and parameters of the Nelson-Siegel model
' ---------------------------------------------------------------------------------

Function NS(m As Double, beta0 As Double, beta1 As Double, beta2 As Double, _
            tau1 As Double) As Double
  If m = 0 Then m = m1
  If tau1 = 0 Then tau1 = m1

  NS = _
    beta0 + _
    beta1 * ((1 - Exp(-m / tau1)) / (m / tau1)) + _
    beta2 * (((1 - Exp(-m / tau1)) / (m / tau1)) - Exp(-m / tau1))
End Function
```

➢ Visual Basic code for Nelson-Siegel instantaneous forward rate function:

```
' ---------------------------DISCRIPTION--------------------------------------------
' NS_f calculates the instantaneous forward rates according to Nelson-Siegel model
' m is time to maturity
' tau1 is time and shows the location of the possible hump in the forward curve
' beta0, beta1 and beta2 are constants and parameters of the Nelson-Siegel model
' ---------------------------------------------------------------------------------

Function NS_f(m As Double, beta0 As Double, beta1 As Double, beta2 As Double, _
               tau1 As Double) As Double
  If m = 0 Then m = m1
  If tau1 = 0 Then tau1 = m1

  NS_f = _
    beta0 + _
    beta1 * Exp(-m / tau1) + _
    beta2 * (m / tau1) * Exp(-m / tau1)
End Function
```

- Visual Basic code for Nelson-Siegel- Svensson spot rate function:

```
' ---------------------------DISCRIPTION-------------------------------------------
' NSS calculates the spot rates according to extended model of Nelson-Siegel
' m is time to maturity
' tau1 is time and shows the location of the possible hump in the forward curve
' tau2 is time and shows the location of the second possible hump in the forward
'curve
' beta0, beta1, beta2 and beta3 are constants and parameters of the Nelson-Siegel-
'Svensson model
' ---------------------------------------------------------------------------------

Function NSS(m As Double, beta0 As Double, beta1 As Double, beta2 As Double, _
             beta3 As Double, tau1 As Double, tau2 As Double) As Double
  Dim m1 As Double
  m1 = 0.00000001

  If m = 0 Then m = m1
  If tau1 = 0 Then tau1 = m1
  If tau2 = 0 Then tau2 = m1
  If tau2 < tau1 Then tau2 = tau1

  NSS = _
    beta0 + _
    beta1 * ((1 - Exp(-m / tau1)) / (m / tau1)) + _
    beta2 * (((1 - Exp(-m / tau1)) / (m / tau1)) - Exp(-m / tau1)) + _
    beta3 * (((1 - Exp(-m / tau2)) / (m / tau2)) - Exp(-m / tau2))
End Function
```

➢ Visual Basic code for Nelson-Siegel- Svensson instantaneous forward rate function:

```
' ---------------------------DISCRIPTION-------------------------------------------
' NSS_f calculates the instantaneous forward rates according to extended model of
' Nelson-Siegel
' m is time to maturity
' tau1 is time and shows the location of the possible hump in the forward curve
' tau2 is time and shows the location of the second possible hump in the forward
'curve
' beta0, beta1, beta2 and beta3 are constants and parameters of the Nelson-Siegel-
'Svensson model
' ---------------------------------------------------------------------------------

Function NSS_f(m As Double, beta0 As Double, beta1 As Double, beta2 As Double, _
               beta3 As Double, tau1 As Double, tau2 As Double) As Double

Dim m1 As Double
  m1 = 0.00000001
  If m = 0 Then m = m1
  If tau1 = 0 Then tau1 = m1
  If tau2 = 0 Then tau2 = m1
  If tau2 < tau1 Then tau2 = tau1

  NSS_f = _
    beta0 + _
    beta1 * Exp(-m / tau1) + _
    beta2 * (m / tau1) * Exp(-m / tau1) + _
    beta3 * (m / tau2) * Exp(-m / tau2)
End Function
```

- Visual Basic code for Nelson-Siegel- Svensson forward rate function:

```
' ---------------------------DISCRIPTION--------------------------------------------
' NSS_ft calculates the forward rates according to extended model of Nelson-Siegel
' m, m1 and m2 are time
' tau1 is time and shows the location of the possible hump in the forward curve
' tau2 is time and shows the location of the second possible hump in the forward
'curve
' beta0, beta1, beta2 and beta3 are constants and parameters of the Nelson-Siegel-
'Svensson model
' ---------------------------------------------------------------------------------

Function NSS_ft(m1 As Double, m2 As Double, beta0 As Double, beta1 As Double, beta2
As Double, beta3 As Double,tau1 As Double, tau2 As Double) As Double
      Dim m As Double
      m = 0.0000001

      If m1 = 0 Then m1 = m
      If m2 = 0 Then m2 = m
      If tau1 = 0 Then tau1 = m
      If tau2 = 0 Then tau2 = m
      If tau2 < tau1 Then tau2 = tau1

      NSS_ft = _
      beta0 + _
      beta1 * tau1 / (m2 - m1) * (Exp(-m1 / tau1) - Exp(-m2 / tau1)) + _
      beta2 * tau1 / (m2 - m1) * (Exp(-m1 / tau1) - Exp(-m2 / tau1) - (m2 / tau1) *
      Exp(-m2 / tau1) + (m1 / tau1) * Exp(-m1 / tau1)) + _
      beta3 * tau2 / (m2 - m1) * (Exp(-m1 / tau2) - Exp(-m2 / tau2) - (m2 / tau2) *
      Exp(-m2 / tau2) + (m1 / tau2) * Exp(-m1 / tau2))

End Function
```

- Visual Basic code for Linear Interpolation:

```
' ---------------------------DISCRIPTION-------------------------------------------
' IPOLY is the interpolation function
' n   number of interpolating points
' x   interpolating points
' f   function values to be interpolated
' ---------------------------------------------------------------------------------

Function IPOLy(X As Double, pX As Range, pY As Range) As Double
  Dim j, i, N As Integer
  Dim lX() As Date
  Dim lY() As Double

  N = Application.Count(pX)  ' Length of the ranges
  ReDim lX(N) As Date
  ReDim lY(N) As Double

  For i = 1 To N
    lX(i) = pX.Cells(i)
    lY(i) = pY.Cells(i)
  Next i

  For i = 2 To N
    If (X = lX(i - 1)) Then
      IPOLy = lY(i - 1)
      Exit Function
    ElseIf (X = lX(i)) Then
      IPOLy = lY(i)
      Exit Function
    End If

    If (X > lX(i - 1) And X < lX(i)) Then
      ' Här interpolerar man
      IPOLy = lY(i - 1) + (X - lX(i - 1)) * (lY(i) - lY(i - 1)) / _
              (lX(i) - lX(i - 1))
      Exit Function
    End If
  Next i

  If (X > lX(N - 1)) Then
    IPOLy = lY(N) + (X - lX(N)) * (lY(N) - lY(N - 1)) / (lX(N) - lX(N - 1))
  End If

End Function
```

Appendix 2

Method of Bootstrapping and Interpolation for Missing Discount Factors

Denote the reset days for any swap as: 0= $T_0, T_1, \ldots, T_n$ and define τ_i as the time interval $T_i - T_{i-1}$, $(i=1,\ldots,n)$. (For sake of simplicity in calculations, assume that the payment dates for both floating side and fixed side are the same). As is mentioned in Section 4.4 the discount function for a swap is given by the following formula

$$R \cdot \sum_{i=1}^{n} \tau_i \cdot D_i + D_n = 1$$

$$R \cdot \sum_{i=1}^{n-1} \tau_i \cdot D_i + R \cdot \tau_n \cdot D_n = 1 - D_n$$

$$D_n \left(1 + R \cdot \tau_n\right) = 1 - R \cdot \sum_{i=1}^{n-1} \tau_i \cdot D_i$$

$$D_n = \frac{1 - R \cdot \sum_{i=1}^{n-1} \tau_i \cdot D_i}{1 + R \cdot \tau_n}$$

Here R is the swap's par rate (fixed rate), D_i is the discount factor at T_i , and D_n is the discount factor at maturity of the swap.

Suppose that n = 12 and the discount factor for $n = 11$, (D_{11}), is missing[27]. We use interpolation to find the D_{11} by using the value of D_{12} which is still unknown.

$$D_{11} = D_{10} + \frac{D_{12} - D_{10}}{T_{12} - T_{10}}\left(T_{11} - T_{10}\right) = D_{10} + \frac{D_{12} - D_{10}}{\tau_{12} + \tau_{11}}\tau_{11} \approx D_{10} + \frac{D_{12} - D_{10}}{2}$$
$$= \frac{1}{2}\left(D_{12} + D_{10}\right)$$

[27] Usually in the swap market, rates for the first year up to the 10th year are given. For swaps with maturities more than 10 years, rates for the 12th, 15th, 20th, 25th, 30th years and so on, are given and other rates are missing.

Then

$$D_{12} = \frac{1 - R \cdot \sum_{i=1}^{11} \tau \cdot D_i}{1 + R \cdot \tau} = \frac{1 - R \cdot \tau \cdot \left(\sum_{i=1}^{10} D_i + \frac{1}{2}\left(D_{10} + D_{12}\right) \right)}{1 + R \cdot \tau}$$

$$= \frac{1 - R \cdot \tau \cdot \sum_{i=1}^{10} D_i - R \cdot \tau \cdot \left(\frac{D_{10}}{2} + \frac{D_{12}}{2} \right)}{1 + R \cdot \tau}$$

$$= \frac{1 - R \cdot \tau \cdot \sum_{i=1}^{10} D_i - R \cdot \tau \cdot \frac{D_{10}}{2}}{1 + R \cdot \tau} - \frac{R \cdot \tau \cdot \frac{D_{12}}{2}}{1 + R \cdot \tau}$$

We can write the above equation as

$$D_{12}\left(1 + R \cdot \tau\right) + R \cdot \tau \cdot \frac{D_{12}}{2} = 1 - R \cdot \tau \cdot \sum_{i=1}^{10} D_i - R \cdot \tau \cdot \frac{D_{10}}{2}$$

$$D_{12}\left(1 + 3\frac{R \cdot \tau}{2}\right) = 1 - R \cdot \tau \cdot \sum_{i=1}^{10} D_i - R \cdot \tau \cdot \frac{D_{10}}{2}$$

$$D_{12} = \frac{1 - R \cdot \tau \cdot \sum_{i=1}^{10} D_i - R \cdot \tau \cdot \frac{D_{10}}{2}}{1 + \frac{3 \cdot R \cdot \tau}{2}}$$

or

$$D_{12} = \frac{1 - R \cdot \sum_{i=1}^{11} \tau_i \cdot D_i}{1 + R \cdot \tau_{12}} = \frac{1 - R \cdot \left(\sum_{i=1}^{10} \tau_i \cdot D_i + D_{10} \cdot \tau_{11} + \frac{D_{12} - D_{10}}{\tau_{12} + \tau_{11}} \tau_{11}{}^2 \right)}{1 + R \cdot \tau_{12}}$$

$$= \frac{1 - R \cdot \sum_{i=1}^{10} \tau_i \cdot D_i - R \cdot \tau_{11} \cdot \left(D_{10} - \frac{D_{10} \cdot \tau_{11}}{\tau_{12} + \tau_{11}} + \frac{D_{12} \cdot \tau_{11}}{\tau_{12} + \tau_{11}} \right)}{1 + R \cdot \tau_{12}}$$

$$= \frac{1 - R \cdot \sum_{i=1}^{10} \tau_i \cdot D_i - R \cdot \tau_{11} \cdot D_{10} \left(1 - \frac{\tau_{11}}{\tau_{12} + \tau_{11}} \right)}{1 + R \cdot \tau_{12}} - \frac{R \cdot D_{12} \cdot \frac{\tau_{11}{}^2}{\tau_{12} + \tau_{11}}}{1 + R \cdot \tau_{12}}$$

then we get the following equation

$$D_{12}+\frac{R\cdot D_{12}\dfrac{{\tau_{11}}^2}{\tau_{12}+\tau_{11}}}{1+R\cdot\tau_{12}}=\frac{1-R\cdot\sum_{i=1}^{10}\tau_i\cdot D_i-R\cdot\tau_{11}\cdot D_{10}\left(1-\dfrac{\tau_{11}}{\tau_{12}+\tau_{11}}\right)}{1+R\cdot\tau_{12}}$$

$$D_{12}\left(1+\frac{R\cdot{\tau_{11}}^2/\left(\tau_{12}+\tau_{11}\right)}{1+R\cdot\tau_{12}}\right)\cdot\left(1+R\cdot\tau_{12}\right)=1-R\cdot\sum_{i=1}^{10}\tau_i\cdot D_i-R\cdot\tau_{11}\cdot D_{10}\left(1-\frac{\tau_{11}}{\tau_{12}+\tau_{11}}\right)$$

$$D_{12}\left(1+R\cdot\left(\tau_{12}+\frac{{\tau_{11}}^2}{\tau_{12}+\tau_{11}}\right)\right)=1-R\cdot\sum_{i=1}^{10}\tau_i\cdot D_i-R\cdot\tau_{11}\cdot D_{10}\left(1-\frac{\tau_{11}}{\tau_{12}+\tau_{11}}\right)$$

$$D_{12}=\frac{1-R\cdot\sum_{i=1}^{10}\tau_i\cdot D_i-R\cdot\tau_{11}\cdot D_{10}\left(1-\dfrac{\tau_{11}}{\tau_{12}+\tau_{11}}\right)}{1+R\cdot\left(\tau_{12}+\dfrac{{\tau_{11}}^2}{\tau_{12}+\tau_{11}}\right)}$$

Next, suppose n=15 and discount factors for n=13 and 14 are missing.

$$\begin{aligned}D_{13}&=D_{12}+\frac{D_{15}-D_{12}}{T_{15}-T_{12}}\left(T_{13}-T_{12}\right)=D_{12}+\frac{D_{15}-D_{12}}{\tau_{15}+\tau_{14}+\tau_{13}}\tau_{13}\approx D_{12}+\frac{D_{15}-D_{12}}{3}\\&=\frac{1}{3}\left(D_{15}+2D_{12}\right)\\D_{14}&=D_{12}+\frac{D_{15}-D_{12}}{T_{15}-T_{12}}\left(T_{14}-T_{12}\right)=D_{12}+\frac{D_{15}-D_{12}}{\tau_{15}+\tau_{14}+\tau_{13}}\left(\tau_{14}+\tau_{13}\right)\approx D_{12}+2\frac{D_{15}-D_{12}}{3}\\&=\frac{2D_{15}-2D_{12}+3D_{12}}{3}=\frac{1}{3}\left(2D_{15}+D_{12}\right)\end{aligned}$$

then we have

$$\begin{aligned}D_{15}&=\frac{1-R\cdot\sum_{i=1}^{14}\tau\cdot D_i}{1+R\cdot\tau}=\frac{1-R\cdot\tau\cdot\left(\sum_{i=1}^{12}D_i+\dfrac{1}{3}\left(D_{15}+2D_{12}+2D_{15}+D_{12}\right)\right)}{1+R\cdot\tau}\\&=\frac{1-R\cdot\tau\cdot\sum_{i=1}^{12}D_i-R\cdot\tau\left(D_{12}+D_{15}\right)}{1+R\cdot\tau}\\&=\frac{1-R\cdot\tau\cdot\sum_{i=1}^{12}D_i-R\cdot\tau\cdot D_{12}}{1+R\cdot\tau}-\frac{R\cdot\tau\cdot D_{15}}{1+R\cdot\tau}\end{aligned}$$

so

$$D_{15}+\frac{R\cdot\tau\cdot D_{15}}{1+R\cdot\tau}=\frac{1-R\cdot\tau\cdot\sum_{i=1}^{12}D_i-R\cdot\tau\cdot D_{12}}{1+R\cdot\tau}$$

$$D_{12}\left(1+\frac{R\cdot\tau}{1+R\cdot\tau}\right)\cdot\left(1+R\cdot\tau\right)=1-R\cdot\tau\cdot\sum_{i=1}^{12}D_i-R\cdot\tau\cdot D_{12}$$

$$D_{12}\left(1+2R\cdot\tau\right)=1-R\cdot\tau\cdot\sum_{i=1}^{12}D_i-R\cdot\tau\cdot D_{12}$$

$$D_{15}=\frac{1-R\cdot\tau\cdot\sum_{i=1}^{12}D_i-R\cdot\tau\cdot D_{12}}{1+2\cdot R\cdot\tau}$$

The above equation can be written as

$$\begin{aligned}
D_{15}&=\frac{1-R\cdot\sum_{i=1}^{14}\tau_i\cdot D_i}{1+R\cdot\tau_{15}}\\
&=\frac{1-R\cdot\left(\sum_{i=1}^{12}\tau_iD_i+\tau_{13}D_{12}+\frac{D_{15}-D_{12}}{\tau_{15}+\tau_{14}+\tau_{13}}\tau_{13}{}^2+\tau_{14}D_{12}+\frac{D_{15}-D_{12}}{\tau_{15}+\tau_{14}+\tau_{13}}\left(\tau_{14}+\tau_{13}\right)\tau_{14}\right)}{1+R\cdot\tau_{15}}\\
&=\frac{1-R\cdot\sum_{i=1}^{12}\tau_iD_i-R\cdot D_{12}\left(\tau_{13}+\tau_{14}-\frac{\tau_{13}{}^2+\left(\tau_{14}+\tau_{13}\right)\tau_{14}}{\tau_{15}+\tau_{14}+\tau_{13}}\right)}{1+R\cdot\tau_{15}}-R\cdot D_{15}\frac{\frac{\tau_{13}{}^2+\left(\tau_{14}+\tau_{13}\right)\tau_{14}}{\tau_{15}+\tau_{14}+\tau_{13}}}{1+R\cdot\tau_{15}}
\end{aligned}$$

$$D_{15}\left(1+R\cdot\frac{\frac{\tau_{13}{}^2+\left(\tau_{14}+\tau_{13}\right)\tau_{14}}{\tau_{15}+\tau_{14}+\tau_{13}}}{1+R\cdot\tau_{15}}\right)=\frac{1-R\cdot\sum_{i=1}^{12}\tau_iD_i-R\cdot D_{12}\left(\tau_{13}+\tau_{14}-\frac{\tau_{13}{}^2+\left(\tau_{14}+\tau_{13}\right)\tau_{14}}{\tau_{15}+\tau_{14}+\tau_{13}}\right)}{1+R\cdot\tau_{15}}$$

$$D_{12}\left(1+R\cdot\left\{\tau_{15}+\frac{\tau_{13}{}^2+\left(\tau_{14}+\tau_{13}\right)\tau_{14}}{\tau_{15}+\tau_{14}+\tau_{13}}\right\}\right)=1-R\cdot\sum_{i=1}^{12}\tau_iD_i-R\cdot D_{12}\left(\tau_{13}+\tau_{14}-\frac{\tau_{13}{}^2+\left(\tau_{14}+\tau_{13}\right)\tau_{14}}{\tau_{15}+\tau_{14}+\tau_{13}}\right)$$

$$D_{15}=\frac{1-R\cdot\sum_{i=1}^{12}\tau_iD_i-R\cdot D_{12}\left(\tau_{13}+\tau_{14}-\frac{\tau_{13}{}^2+\left(\tau_{14}+\tau_{13}\right)\tau_{14}}{\tau_{15}+\tau_{14}+\tau_{13}}\right)}{\left(1+R\cdot\left\{\tau_{15}+\frac{\tau_{13}{}^2+\left(\tau_{14}+\tau_{13}\right)\tau_{14}}{\tau_{15}+\tau_{14}+\tau_{13}}\right\}\right)}$$

Suppose now that n=20 and discount factors for n=16, 17, 18 and 19 are missing, following the same procedure we have

$$D_{16}=D_{15}+\frac{D_{20}-D_{15}}{T_{20}-T_{15}}\left(T_{16}-T_{15}\right)=D_{15}+\frac{D_{20}-D_{15}}{\tau_{20}+\tau_{19}+\tau_{18}+\tau_{17}+\tau_{16}}\tau_{16}\approx D_{15}+\frac{D_{20}-D_{15}}{5}$$
$$=\frac{1}{5}\left(D_{20}+4\cdot D_{15}\right)$$
$$D_{17}=D_{15}+\frac{D_{20}-D_{15}}{T_{20}-T_{15}}\left(T_{17}-T_{15}\right)=D_{15}+\frac{\left(D_{20}-D_{15}\right)\left(\tau_{17}+\tau_{16}\right)}{\tau_{20}+\tau_{19}+\tau_{18}+\tau_{17}+\tau_{16}}\approx D_{15}+2\frac{D_{20}-D_{15}}{5}$$
$$=\frac{1}{5}\left(2D_{20}+3\cdot D_{15}\right)$$
$$D_{18}=D_{15}+\frac{D_{20}-D_{15}}{T_{20}-T_{15}}\left(T_{18}-T_{15}\right)=D_{15}+\frac{\left(D_{20}-D_{15}\right)\left(\tau_{18}+\tau_{17}+\tau_{16}\right)}{\tau_{20}+\tau_{19}+\tau_{18}+\tau_{17}+\tau_{16}}\approx D_{15}+3\frac{D_{20}-D_{15}}{5}$$
$$=\frac{1}{5}\left(3D_{20}+2\cdot D_{15}\right)$$
$$D_{19}=D_{15}+\frac{D_{20}-D_{15}}{T_{20}-T_{15}}\left(T_{19}-T_{15}\right)=D_{15}+\frac{\left(D_{20}-D_{15}\right)\left(\tau_{19}+\tau_{18}+\tau_{17}+\tau_{16}\right)}{\tau_{20}+\tau_{19}+\tau_{18}+\tau_{17}+\tau_{16}}\approx D_{15}+4\frac{D_{20}-D_{15}}{5}$$
$$=\frac{1}{5}\left(4D_{20}+D_{15}\right)$$

then

$$D_{20}=\frac{1-R\cdot\sum_{i=1}^{19}\tau\cdot D_i}{1+R\cdot\tau}=\frac{1-R\cdot\tau\cdot\left(\sum_{i=1}^{15}D_i+\frac{1}{5}\left(D_{20}+4D_{15}+2D_{20}+3D_{15}+3D_{20}+2D_{15}+4D_{20}+D_{15}\right)\right)}{1+R\cdot\tau}$$
$$=\frac{1-R\cdot\tau\cdot\sum_{i=1}^{15}D_i-R\cdot\tau\cdot\frac{1}{5}\left(10D_{20}+10D_{15}\right)}{1+R\cdot\tau}$$
$$=\frac{1-R\cdot\tau\cdot\sum_{i=1}^{15}D_i-R\cdot\tau\cdot 2D_{15}}{1+R\cdot\tau}-\frac{R\cdot\tau\cdot 2D_{20}}{1+R\cdot\tau}$$
$$D_{20}+\frac{R\cdot\tau\cdot 2D_{20}}{1+R\cdot\tau}=\frac{1-R\cdot\tau\cdot\sum_{i=1}^{15}D_i-R\cdot\tau\cdot 2D_{15}}{1+R\cdot\tau}$$
$$D_{20}\left(1+2\cdot\frac{R\cdot\tau}{1+R\cdot\tau}\right)\cdot\left(1+R\cdot\tau\right)=1-R\cdot\tau\cdot\sum_{i=1}^{15}D_i-R\cdot\tau\cdot 2D_{15}$$
$$D_{20}\left(1+3\cdot R\cdot\tau\right)=1-R\cdot\tau\cdot\sum_{i=1}^{15}D_i-R\cdot\tau\cdot 2D_{15}$$

$$D_{20} = \frac{1 - R \cdot \tau \cdot \sum_{i=1}^{15} D_i - R \cdot \tau \cdot 2D_{15}}{1 + 3 \cdot R \cdot \tau}$$

or more exact

$$\begin{aligned}
D_{20}(1 + R.\tau_{20}) &= 1 - R.\sum_{i=1}^{19} \tau_i.D_i \\
&= 1 - R.\sum_{i=1}^{15} \tau_i.D_i \\
&-R.\tau_{16}D_{15} - R.\frac{D_{20} - D_{15}}{\tau_{20} + \tau_{19} + \tau_{18} + \tau_{17} + \tau_{16}}\tau_{16}^2 \\
&-R.\tau_{17}D_{15} - R.\frac{(D_{20} - D_{15})(\tau_{17} + \tau_{16})\tau_{17}}{\tau_{20} + \tau_{19} + \tau_{18} + \tau_{17} + \tau_{16}} \\
&-R.\tau_{18}D_{15} - R.\frac{(D_{20} - D_{15})(\tau_{18} + \tau_{17} + \tau_{16})\tau_{18}}{\tau_{20} + \tau_{19} + \tau_{18} + \tau_{17} + \tau_{16}} \\
&-R.\tau_{19}D_{15} - R.\frac{(D_{20} - D_{15})(\tau_{19} + \tau_{18} + \tau_{17} + \tau_{16})\tau_{19}}{\tau_{20} + \tau_{19} + \tau_{18} + \tau_{17} + \tau_{16}}
\end{aligned}$$

$$D_{20}(1 + R.\tau_{20}) = 1 - R.\sum_{i=1}^{15} \tau_i.D_i - R.D_{15}(\tau_{19} + \tau_{18} + \tau_{17} + \tau_{16})$$
$$-R.(D_{20} - D_{15})\frac{(\tau_{16}^2 + (\tau_{17} + \tau_{16})\tau_{17} + (\tau_{18} + \tau_{17} + \tau_{16})\tau_{18} + (\tau_{19} + \tau_{18} + \tau_{17} + \tau_{16})\tau_{19})}{\tau_{20} + \tau_{19} + \tau_{18} + \tau_{17} + \tau_{16}}$$

$$D_{20}\left(1 + R.\left\{\tau_{20} - \frac{(\tau_{16}^2 + (\tau_{17} + \tau_{16})\tau_{17} + (\tau_{18} + \tau_{17} + \tau_{16})\tau_{18} + (\tau_{19} + \tau_{18} + \tau_{17} + \tau_{16})\tau_{19})}{\tau_{20} + \tau_{19} + \tau_{18} + \tau_{17} + \tau_{16}}\right\}\right)$$
$$= 1 - R.\sum_{i=1}^{15} \tau_i.D_i - R.D_{15}(\tau_{19} + \tau_{18} + \tau_{17} + \tau_{16})$$
$$-R.D_{15}\frac{(\tau_{16}^2 + (\tau_{17} + \tau_{16})\tau_{17} + (\tau_{18} + \tau_{17} + \tau_{16})\tau_{18} + (\tau_{19} + \tau_{18} + \tau_{17} + \tau_{16})\tau_{19})}{\tau_{20} + \tau_{19} + \tau_{18} + \tau_{17} + \tau_{16}}$$

let

$$T = \frac{\left(\tau_{16}{}^2 + \left(\tau_{16} + \tau_{17}\right)\tau_{17} + \left(\tau_{16} + \tau_{17} + \tau_{18}\right)\tau_{18} + \left(\tau_{16} + \tau_{17} + \tau_{18} + \tau_{19}\right)\tau_{19}\right)}{\tau_{16} + \tau_{17} + \tau_{18} + \tau_{19} + \tau_{20}}$$

then

$$D_{20} = \frac{1 - R.\sum_{i=1}^{15} \tau_i.D_i - R.D_{15}(\tau_{16} + \tau_{17} + \tau_{18} + \tau_{19} - T)}{(1 + R.\{\tau_{20} - T\})}$$

Suppose now that n=25 and discount factors for n=21, 22, 23 and 24 are missing.

$$D_{21} = D_{20} + \frac{D_{25} - D_{20}}{T_{15} - T_{20}}\left(T_{21} - T_{20}\right) = D_{21} + \frac{D_{25} - D_{20}}{5 \cdot \tau}\tau = D_{20} + \frac{D_{25} - D_{20}}{5}$$
$$= \frac{1}{5}\left(D_{25} + 4 \cdot D_{20}\right)$$

then

$$D_{25} = \frac{1 - R \cdot \tau \cdot \sum_{i=1}^{20} D_i - R \cdot \tau \cdot 2D_{20}}{1 + R \cdot \tau} - \frac{R \cdot \tau \cdot 2D_{25}}{1 + R \cdot \tau}$$

or

$$D_{25} = \frac{1 - R \cdot \tau \cdot \sum_{i=1}^{20} D_i - R \cdot \tau \cdot 2D_{20}}{1 + 3 \cdot R \cdot \tau}$$

$$D_{25} = \frac{1 - R.\sum_{i=1}^{20} \tau_i . D_i - R.D_{20}(\tau_{21} + \tau_{22} + \tau_{23} + \tau_{24} - T)}{(1 + R.\{\tau_{25} - T\})}$$

Here T is equal to

$$T = \frac{\left(\tau_{21}{}^2 + \left(\tau_{21} + \tau_{22}\right)\tau_{22} + \left(\tau_{21} + \tau_{22} + \tau_{23}\right)\tau_{23} + \left(\tau_{21} + \tau_{22} + \tau_{23} + \tau_{24}\right)\tau_{24}\right)}{\tau_{21} + \tau_{22} + \tau_{23} + \tau_{24}}$$

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