CS345 Assignment 7

Chinmay Goyal(180206) and Supreeth Baliga(180801)

1 Atmospheric Science Experiment:

Solution:
We tried to approach this problem by constructing a network consisting of a bipartite graph based on the problem and finding the max-flow in this network. We state our algorithm as follows.

ALGORITHM:

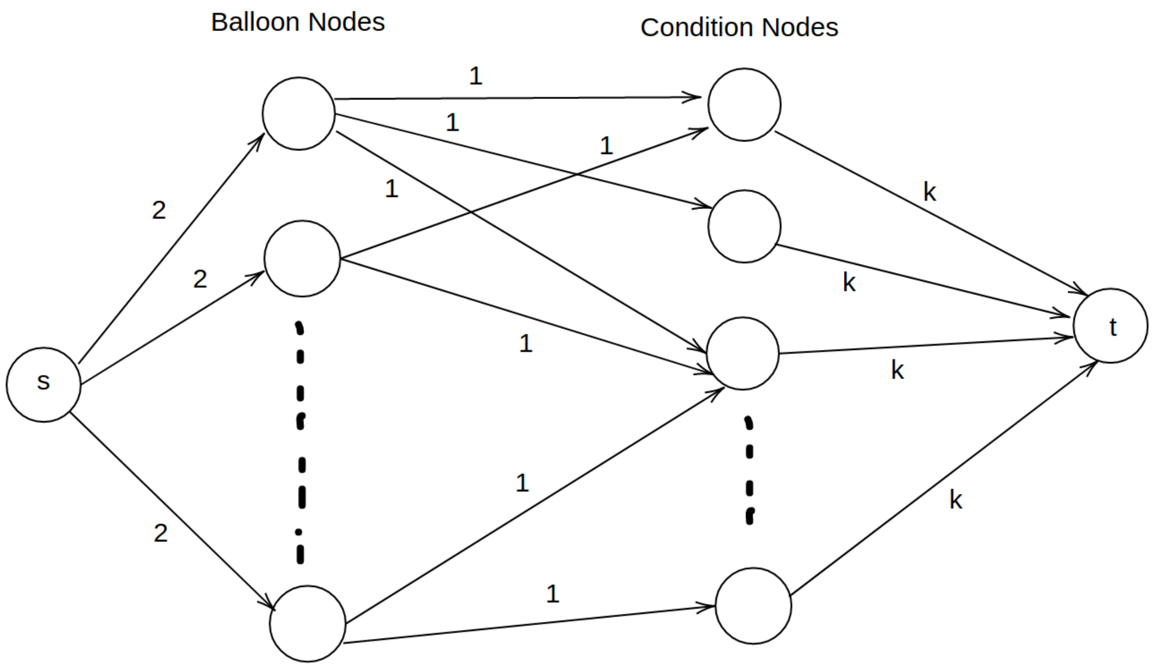
We will first explain how we are going to model this problem as a max-flow in a network. We will construct our network as follows. The set of n conditions is given as S. Let us say the set of m balloons is M. Like we saw in class, we construct a bipartite graph with the set of balloons M on one side and the set of conditions S on the other side. For each balloon node in M say M_i, we draw directed edges to the corresponding condition nodes belonging in the set S_i. Each of these edges will have capacity 1. This is because each balloon can measure a corresponding condition atmost once. This completes the bipartite part of the network.

Now, to model this as a network, we add a source vertex s and a sink vertex t. We draw directed edges from the source s to each of the balloon nodes in M. Each of these edges will have capacity 2 since each balloon can make at most 2 measurements. Lastly, we add directed edges from each of the condition nodes in set S to the sink vertex t. Each of these edges will have capacity k since they have planned to measure each measurement from at least k balloons. Here we are finding the flow which ensures that there are k balloons which can be used for each condition, in the given conditions more than k balloons for 1 condition might also be possible but our max flow just ensures that all the conditions are fulfilled. This finishes our job of constructing the network.

Now, we find the max-flow in this graph. From the algorithms discussed in class(Edmund-Karp algorithm), this can be done in polynomial time. Once we find the value of the max-flow, we check if it is equal to kn. If yes, then we say that there is a way to measure each condition by at least k different balloons such that each balloon measures at most 2 conditions. If not, we say it is not possible.

This completes our algorithm. We prove all the above arguments in the proof of correctness and then we discuss the time complexity of the given algorithm.

We present the diagrammatic representation of the construction of the network we have proposed for the algorithm in the following figure.



PROPOSED NETWORK CONSTRUCTION

PROOF OF CORRECTNESS:

Firstly, we see that the capacity constraint and the conservation constraint are trivially satisfied since we are always increasing flow along a valid path while finding max flow. Now, we need to prove that there is a way to measure each of the conditions under the given requirements and constraints if and only if the value of the max-flow is kn. We prove this by the following:

Theorem 1: Given that the flow in the network is kn, it is possible to measure each of the conditions under the given requirements and constraints.

Proof: If the max-flow in the network is kn, then we know that as there are only n incoming edges to the sink vertex t, and each edge has a capacity k, the flow in all these edges will be k. (We use integrality theorem to ensure that the max flow in a network with integer edge weights is achieved by integer flow in the edges). This in turn implies that the outgoing flow from each of the n condition vertices is k, which implies that the incoming flow in each vertex of set S is k(conservation of flow in an internal node). As each incoming vertex into set S is from a different node of set M and each incoming edge has a capacity 1(this is because we are constructing the network this way), we know that for each vertex of set S there are k incoming edges from k different vertices of set M. This says that each of the conditions is checked by k different balloons. Now we only have to prove that, each balloon in this situation would only be used in 2 conditions. For each vertex of set M, the capacity of incoming flow (here the only incoming edge for any vertex of set M is from the source vertex s), is 2, hence the outgoing flow can at max be 2 for any vertex. Each outgoing vertex has capacity 1 and from integrality theorem we know that the flow in any outgoing vertex can only be 1 or 0. This ensures that no balloon can be used in more than 2 conditions. For each edge from a node in M to a node in S, say M_i to S(j), if the flow along this edge is 1, then it means that the i^{th} balloon measures the j^{th} condition, else it does not. (Note S_j is different from S(j). S(j) denotes the j^{th} condition whereas S_j denotes the set of conditions the j^{th} balloon can measure). This gives us a valid way to measure the conditions. Hence, the above theorem holds.

Theorem 2: If the max flow in the network is not equal to kn, then it is not possible to measure each of the conditions under the given requirements and constraints.

Proof: Note that the flow in the graph cannot be more than kn. This is because we can construct a cut with only the sink vertex in the right set and all other vertices in the left set. The capacity of this cut is kn and we know that the flow in the network cannot exceed the capacity of any cut. If the max flow of the network is less than kn, it implies that there is an edge from a vertex of set S say p to the sink vertex t which has flow less than k, which implies that the incoming flow into vertex p is less than k. This means that the condition corresponding to the condition node p will be measured by less than k balloons which is not valid way to measure the conditions. Hence in case of max flow less than kn, the given instance of the problem can not have a valid way to measure each of the conditions under the given requirements and constraints.

The above two theorems complete our proof of correctness of our algorithm.

TIME COMPLEXITY:

We will give a rough argument about the time complexity. Firstly, let us see the time required for constructing the network. This will take time of the order of number of edges. The number of edges say $|E|$ is given by:

$|E| =$ no. of edges from s to M + no. of edges from M to S + no. of edges from S to t

Now, number of edges from s to M is basically the number of balloons. So, it is m. Number of edges from M to S is given by $\sum_{i=1}^{m} |S_i|$. And the number of edges from S to t is basically the number of conditions, so, it is n. Note that the above arguments are based on the method we applied for constructing the graph. So, we get the total number of edges is

$$|E| = m + n + \sum_{i=1}^{m} |S_i|$$

So, constructing the network takes time $O(m + n + \sum_{i=1}^{m} |S_i|)$.

Now, once the graph is constructed, we find the max-flow by using Edmond and Karp algorithm discussed in class which takes time $O(|E|^2|V|)$. Note that $|V|$ = m+n+2 since the network nodes are basically the balloon nodes, the condition nodes, the source vertex and the sink vertex.
Thus, the total time complexity is given by

$$T(\text{Atmospheric Science Experiment}) = O(|E|) + O(|E|^2|V|) = O(|E|^2|V|)$$

And we saw that both $|E|$ and $|V|$ are polynomial in the input size. Thus, the proposed algorithm takes polynomial-time.

Note: The input size here basically corresponds to the sum of the number of conditions (n here), the number of elements in each S_i ($\sum_{i=1}^{m} S_i$) and the number of balloons (m here).