AFFINE ABBILDUNGEN

① $A = \{3, 6, 9\}$, $B = \{1, 2\}$

② $y = 2x + 6$
$y - 6 = 2x$
$\frac{1}{2}y - 3 = x \quad \longrightarrow \quad f^{-1}(x) = \frac{1}{2}x - 3$

③ α^{-1} Drehung um $-30° = 330°$ um Z.

④ $x_s = \frac{4}{1} = 4 \quad \longrightarrow \quad D = [4, \infty)$ oder $(-\infty, 4]$

⑤ a) ist injektiv, keine Inverse
b) injektiv, surjektiv, bijektiv
Inverse: $-\vec{z}$ verknüpft mit zentrischer Streckung an Z mit $k = \frac{1}{2}$.

⑥ c) bijektiv, $g^{-1}(x) = x - 3$
d) surjektiv, keine Inverse
e) injektiv, keine Inverse
f) bijektiv, $j^{-1}(x) = \frac{1}{x}$
g) surjektiv, keine Inverse
h) injektiv für $A = \mathbb{R}^2$

⑦ a) $B = \{2x \mid x \in \mathbb{Z}\}$
d) $A = \mathbb{R}_0^+$
e) $B = \mathbb{R}_0^+$
g) $A = \{(x|y) \mid y = mx + b, m \neq 0\}$
h) $A = B = \mathbb{R}^2 \setminus \{(0|0)\}$

$r^{-1}: \vec{a} \longmapsto \frac{1}{\sqrt{|\vec{a}|}} \cdot \vec{a}$

⑧ a) $\alpha: \begin{cases} x' = \frac{3}{2}x \\ y' = \frac{3}{2}y \end{cases}$ b) $\alpha(2|6) = \underline{\underline{(3|9)}}$

c) $\alpha^{-1}: \begin{cases} x' = \frac{2}{3}x \\ y' = \frac{2}{3}y \end{cases}$ $\alpha^{-1}(4.5|-2.1) = \underline{\underline{(3|-1.4)}}$

⑨ a) ja, beide Komponenten sind affine Funktionen

b) $\tau^{-1}: \begin{cases} x' = x - 1 \\ y' = y + 2 \end{cases}$

c) $\alpha \circ \tau: \begin{cases} x' = (x+1) \cdot 3 = 3x + 3 \\ y' = (y-2) \cdot 3 = 3y - 6 \end{cases}$

$\tau \circ \alpha: \begin{cases} x' = 3x + 1 \\ y' = 3y - 2 \end{cases}$

d) $\alpha: \begin{cases} x' = (x-1) \cdot 3 + 1 = 3x - 2 \\ y' = (y-5) \cdot 3 + 5 = 3y - 10 \end{cases}$

$\alpha(0|0) = \underline{\underline{(-2|-10)}}$, $\alpha(2|-3) = \underline{\underline{(4|-19)}}$, $\alpha^{-1}(-5|8) = \underline{\underline{(-1|6)}}$

e) $\alpha: \begin{cases} x' = (x - s_x) \cdot k + s_x = kx + (1-k) s_x \\ y' = (y - s_y) \cdot k + s_y = ky + (1-k) s_y \end{cases}$

⑩ a) $\alpha^{-1} = \alpha$, $\beta^{-1} = \beta$

b) $\alpha \circ \beta = \beta \circ \alpha = \begin{cases} x' = -x \\ y' = -y \end{cases}$ (Punktspiegelung)

c) $\alpha: \begin{cases} x' = -(x-2) + 2 = -x + 4 \\ y' = y \end{cases}$ $\alpha: \begin{cases} x' = -(x-a) + a = -x + 2a \\ y' = y \end{cases}$

d) $\alpha: \begin{cases} x' = -x + 4 \\ y' = -y - 2 \end{cases}$ Punktspiegelung mit Translation

$\alpha(0|0) = (4|-2)$, $\alpha(1|1) = (3|-3)$, $\alpha(-1|-1) = (5|-1)$, $\alpha(0|2) = (4|-4)$
$\alpha(2|0) = (2|-2)$

e) $\alpha: \begin{cases} x' = -(x - s_x) + s_x = -x + 2s_x \\ y' = -(y - s_y) + s_y = -y + 2s_y \end{cases}$

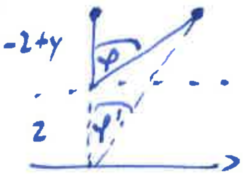
f) $\rho\circ\alpha: \begin{cases} x' = x \\ y' = -y \end{cases}$ Achsenspiegelung

g) $\beta\circ\alpha: \begin{cases} x' = -2x \\ y' = 2y \end{cases}$

11 a) $\alpha: \begin{cases} x' = x + y\tan(-60°) \\ y' = y \end{cases}$

b) $\alpha^{-1}: \begin{cases} x' = x - y\tan(-60°) = x + y\tan(60) \\ y' = y \end{cases}$

c)



$\alpha: \begin{cases} x' = x + (y-2)\tan\varphi \\ y' = y \end{cases}$

$\tan\varphi' = \frac{x}{y}$, $\tan\varphi = \frac{x}{y-2}$

12 a) $A'(3|5)$, $B'(-2|11)$ b) $A'(3|-5)$, $B'(-2|-11)$

c) $\alpha: \begin{cases} x' = -y \\ y' = x \end{cases}$ $A'(-5|-3)$, $B'(-11|2)$

d) $A'(5|-3)$, $B'(11|2)$

e) $P_a(5|15)$, $P_b(5|-15)$, $P_c(15|5)$, $P_d(15|-5)$

13 a) zentrische Streckung an O mit $k = 3$

Spiegelung an der Geraden $x = -2$

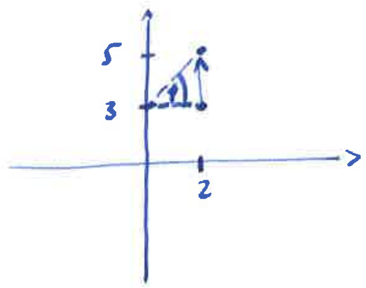
Punktspiegelung an $Z(3|1)$ (b))

c) $\alpha\circ\beta: \begin{cases} x' = 3(-x-4) = -3x - 12 \\ y' = 3y \end{cases}$

$\beta\circ\alpha: \begin{cases} x' = -3x - 4 \\ y' = 3y \end{cases}$ $\gamma\circ\beta: \begin{cases} x' = x + 10 \\ y' = -y + 2 \end{cases}$

$\gamma^{-1}: \begin{cases} x' = -x + 6 \\ y' = -y + 2 \end{cases}$ $\gamma\circ\beta\circ\gamma^{-1}: \begin{cases} x' = -x + 16 \\ y' = y \end{cases}$

(14)



$\alpha: \begin{cases} x' = x \\ y' = x \cdot \tan\varphi + y \end{cases}$

mit $\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{2}\right) = 45°$

$\Rightarrow k = 1$

also $\alpha: \begin{cases} x' = x \\ y' = x + y \end{cases}$

a) $Q'(2|1)$, $R'(-4|2)$, $S'(1|4)$

b) siehe oben

c) $\alpha^{-1}: \begin{cases} x' = x \\ y' = -x + y \end{cases}$

d) $\alpha \circ \gamma: \begin{cases} x' = -x + 6 \\ y' = -x - y + 8 \end{cases}$

e) $\alpha_x: \begin{cases} x' = x + ky \\ y' = y \end{cases}$ $\alpha_y: \begin{cases} x' = x \\ y' = \ell x + y \end{cases}$

$\alpha_y \circ \alpha_x: \begin{cases} x' = x + ky \\ y' = \ell x + k\ell y + y \end{cases}$ nein für $k, \ell \neq 0$

(15) a) $E_1'(1|0)$, $A'(a|0)$, $B'(0|b)$, $C_1'(2|2)$, $C_2'(0|0)$, $D_1'(6|6)$

$D_2'(2|2)$, $E_2'(0|1)$

b) $\alpha(a|0) = (a|0)$ und $\alpha(0|b) = (0|b)$, d.h. Achsen sind Fixpunktgeraden

c) $\alpha(a|1) = (2a|1+a) \sim (a'|1+\frac{1}{2}a')$

$\alpha(1|b) = (1+b|2b) \sim (b'|2b'-2)$

d) $\alpha(a|a) = (a+a^2|a+a^2) \sim (a'|a')$ $a' \geq -\frac{1}{4}$ (Scheitel von $a+a^2$)

e) $\alpha(a|2a) = (a+2a^2|2a+2a^2) \sim \left(x' \middle| x' + \frac{-1 \pm \sqrt{1+8x'}}{4}\right)$ → nein

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a) α : $E_1'(3|1)$, $E_2'(-1|2)$, $A'(3a|a)$, $B'(-b|2b)$, $C'(2|3)$, $D'(9|-4)$

β : $E_1'(0|1)$, $E_2'(1|1)$, $A'(-a+1|a)$, $B'(1|b)$, $C'(0|2)$, $D'(-1|-1)$

γ : $E_1'(0|1)$, $E_2'(-1|0)$, $A'(a-1|a^2)$, $B'(-1|0)$, $C'(\emptyset|1)$, $D'(1|4)$

c) $\alpha^{-1}: \begin{cases} x' = \frac{1}{7}(2x+y) \\ y' = \frac{1}{7}(-x+3y) \end{cases}$ $\qquad \gamma: \}$ n.d.

$\beta^{-1}: \begin{cases} x' = -x+1 \\ y' = x+y-1 \end{cases}$

b) α : $F(0|2)$, $G(1|-2)$, $H(\frac{13}{7}|\frac{25}{7})$

β : $F(3|2)$, $G(4|1)$, $H(-1|10)$

γ : –

d) $\alpha : (a|\frac{1}{3}a)$, $(b|-2b)$

$\beta : (a|-a+1)$, parallele zur y-Achse durch $x=1$

$\gamma : (a|(a+1)^2)$, $(-1|0)$

e) $\alpha(a|1) = (3a-1|a+2) \sim (a'|\frac{1}{3}a'+\frac{7}{3})$

$\alpha(1|b) = (3-b|1+2b) \sim (b'|-2b'+7)$

$\beta(a|1) = (-a+1|a+1) \sim (a'|-a'+2)$

$\beta(1|b) = (0|1+b)$ y-Achse

$\gamma(a|1) = (a-1|a^2) \sim (a'|(a'+1)^2)$

$\gamma(1|b) = (0|1)$ Punkt

f) $\alpha(x|x) = (2x|3x) \rightsquigarrow (x|\frac{3}{2}x)$

$\alpha(x, 2x-1) = (x+1 | 5x-2) \rightsquigarrow (x | 5x-7)$

$\beta(x|x) = (-x+1 | 2x) \rightsquigarrow (x | -2x+2)$

$\beta(x|2x-1) = (-x+1 | 3x-1) \rightsquigarrow (x | -3x+2)$

$\gamma(x|x) = (x-1 | x^2) \rightsquigarrow (x | (x+1)^2)$

$\gamma(x|2x-1) = (x-1 | (2x-1)^2) \rightsquigarrow (x | (2x+1)^2)$

g) $\alpha(x|mx+b) = ((3-m)x - b | (2m+1)x + 2b)$

$\beta(x|mx+b) = (-x+1 | (m+1)x + b)$

$\gamma(x|mx+b) = (x-1 | x^2)$ $\rightarrow$ nein

h) $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \xrightarrow{\text{det}} 6+1 = 7 \neq 0$

$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{det}} -1-0 = -1 \neq 0$

$\begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix} \xrightarrow{\text{det}} 0-0 = 0$ $\rightarrow$ nein

17 a) Streckung : Zentrum oder alle für $k=1$; alles Fixgeraden
Translation : keine ausser triviale ; alle Parallelen zur Verschiebung
Scherung : keine ausser triviale ; Parallelen zur Scherung

b) $\alpha(P) = P$: $\quad x = x + 4y \quad y = 2x - y + 8 \qquad y = 0 \quad x = -4 \qquad \underline{P(-4|0)}$

$$\alpha(x|mx+b) = (x + 4mx + 4b | 2x - mx - b + 8)$$
$$= ((4m+1)x + 4b | (2-m)x + 8 - b)$$
$$\rightsquigarrow \underline{\left(x' \,\middle|\, \frac{2-m}{4m+1}x' + \frac{4mb + 28m + 9b + 8}{4m+1}\right)}$$

Fixgeraden

$$m = \frac{2-m}{4m+1} \quad \Rightarrow \quad 4m^2+m = 2-m$$
$$4m^2+2m-2=0$$
$$2m^2+m-1=0$$
$$m_{1/2} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1\pm 3}{4} = \begin{cases} \frac{1}{2} \\ -1 \end{cases}$$

$m = \frac{1}{2}$: $3\cdot(8-b) = 2b+14+9b+8$
$$2 = 14b$$
$$\frac{1}{7} = b$$

$m=-1$: $-3(8-b) = -4b-28+9b+8$
$$-24+3b = 5b-20$$
$$-2 = b$$

Fixpunktgeraden

c) $\alpha(P)=P$: $x = x$, $y = x^2+y \Rightarrow x = 0 \wedge y=y \rightarrow y$-Achse

ii) $\alpha(a|y) = (a|a^2+y)$ mit a fix

iii) $\alpha(x|b) = (x|x^2+b) \rightarrow$ (Halbgerade) Parabel für freie x.

iv) $\alpha(x|mx+b) = (x|x^2+mx+b) \rightarrow$ Gerade für fixe x.

(18) $\alpha(P)=P$: $x = y$, $y = x+y \rightarrow x=0 \rightarrow y=0$ $\quad F(0|0)$

$\alpha(g)=g'$: $\alpha(x|mx+b) = (mx+b|x+mx+b) = (mx+b|(m+1)x+b)$
$$= (x'|x+x') = (x'|\frac{x'-b}{m}+x')$$
$$= (x'|(1+\frac{1}{m})x' - \frac{b}{m})$$

$$\Rightarrow m = 1+\frac{1}{m}$$
$$m^2 = m+1$$
$$0 = m^2-m-1 \qquad m_{1,2} = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$$

$b = -\frac{b}{m} \Rightarrow b = 0$, also $y = \frac{1\pm\sqrt{5}}{2}x$

$\beta(P) = P$: $x = 2x - y$ $\rightarrow x = x$ $\underline{\underline{F(x|x)}}$ (Fixpunktgerade $y=x$)

$y = x$

$\beta(g) = g'$: $\beta(x|mx+b) = (2x-(mx+b)|x) = ((2-m)x-b|x)$

$x' = (2-m)x - b$ $\sim (x' | \frac{1}{2-m}x' + \frac{b}{2-m})$

$x = \frac{x'+b}{2-m}$

$\Rightarrow m = \frac{1}{2-m}$

$2m - m^2 = 1$

$-m^2 + 2m - 1 = 0$ $m_{1/2} = 1$

$b = \frac{b}{2-1}$ $\Rightarrow b = b$ $y = \underline{\underline{x+b}}$

$\gamma(P) = P$: $x = 4y$ $\rightarrow x = 36x$ $\Rightarrow x = 0, y = 0$

$y = 9x$ $\underline{\underline{F(0|0)}}$

$\gamma(g) = g'$: $\gamma(x|mx+b) = (4mx+4b|9x) = (x' | \frac{9}{4m}x' - \frac{9b}{m})$

$x' = 4mx + b$

$x = \frac{x'-4b}{4m}$

$\Rightarrow m = \frac{9}{4m}$

$4m^2 = 9$

$m_{1/2} = \pm\frac{3}{2}$

$b = -\frac{9b}{m}$ $\Rightarrow b = 0$

$y = \underline{\underline{\pm\frac{3}{2}x}}$

$\delta(P) = P$: $x = x+1$ $\rightarrow$ kein Fixpunkt

$y = 2x - y$

$\delta(g)=g'$: $\delta(x|mx+b) = (x+1\,|\,2x-(mx+b)) = (x+1\,|\,(2-m)x-b)$

$= (x'\,|\,(2-m)(x'-1)-b)$

$= (x'\,|\,(2-m)x' + m-2-b)$

$\Rightarrow 2-m = m \;\rightarrow\; m=1$

$b = m-2-b$

$2b = -1 \;\Rightarrow\; b = -\frac{1}{2} \qquad y = \underline{\underline{x-\frac{1}{2}}}$

$\varepsilon(P)=P$: $x = x+1 \;\rightarrow\;$ <u>kein</u> Fixpunkt

$\varepsilon(g)=g'$: $\varepsilon(x|mx+b) = (x+1\,|\,2x+mx+b) = (x+1\,|\,(2+m)x+b)$

$= (x'\,|\,(2+m)(x'-1)+b) = (x'\,|\,(2+m)x'+b-m-2)$

$\Rightarrow m = m+2 \;\rightarrow\;$ <u>keine</u> Fixgerade.

$\zeta(P)=P$: $x = 5x-y$, $y=\frac{1}{x}$ $\;\rightarrow\; x = 5x-\frac{1}{x} \;\rightarrow\; x^2 = 5x^2-1$

$\frac{1}{4} = x^2 \qquad x_{1,2} = \pm\frac{1}{2}$

$y_{1,2} = \pm 2$

$\underline{\underline{F_1(\frac{1}{2}|2)}}$, $\underline{\underline{F_2(-\frac{1}{2}|-2)}}$

$\zeta(g)=g'$: $\zeta(x|mx+b) = (5x-mx+b\,|\,\frac{1}{x}) = ((5-m)x+b\,|\,\frac{1}{x})$

$= \left(x'\,\middle|\,\frac{5-m}{x'-b}\right)$

$x' = (5-m)x+b$

$x = \frac{x'-b}{5-m}$

$\rightarrow$ <u>keine</u> Fixgerade

(19) a) $\mathbb{D}=\mathbb{R}^2$, $\mathbb{W} = (0,1] \times \mathbb{R})$

b) $\alpha(a|y) = \left(\frac{1}{1+a^2}\,\middle|\,y\right)$

c) Wegen $\alpha(a|y) = \left(\frac{1}{1+a^2}\,\middle|\,y\right)$ muss $a = \frac{1}{1+a^2}$.

Die Graphen von $f(x)=x$ und $g(x) = \frac{1}{1+x^2}$ haben wegen der Stetigkeit auf

$(0|0) \in G_f$ sowie monoton wachsend, bzw.

$(0|1) \in G_g$ sowie monoton fallend

sicher einen Schnittpunkt, welcher die Lösung repräsentiert.

d) $\alpha(x|b) = \left(\frac{1}{1+x^2} \middle| b\right)$

Das Bild ist eine zur x-Achse parallele Strecke der Länge 1.

$$\mathbb{L} = \{(x|b) \mid x \in [0,1]\}$$

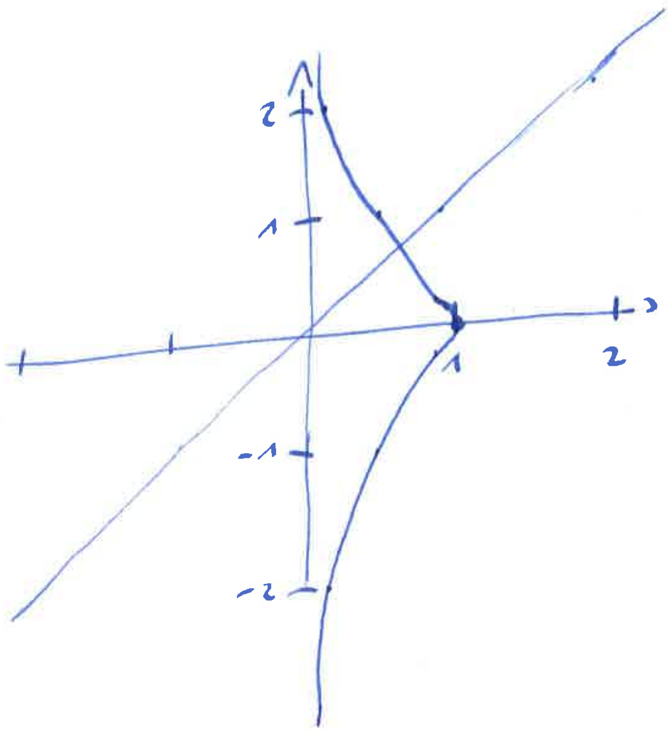
e) $\alpha(-3|-3) = (\frac{1}{10}|-3)$ $\quad\alpha(\frac{1}{2}|\frac{1}{2}) = (\frac{4}{5}|\frac{1}{2})$

$\alpha(-2|-2) = (\frac{1}{5}|-2)$ $\quad\alpha(\frac{1}{4}|\frac{1}{4}) = (\frac{16}{17}|\frac{1}{4})$

$\alpha(-1|-1) = (\frac{1}{2}|-1)$ $\quad\alpha(1|1) = (\frac{1}{2}|1)$

$\alpha(-\frac{1}{2}|-\frac{1}{2}) = (\frac{4}{5}|-\frac{1}{2})$ $\quad\alpha(2|2) = (\frac{1}{5}|2)$

$\alpha(0|0| = (1|0)$ $\quad\alpha(3|3) = (\frac{1}{10}|3)$



(26) a) $\mathbb{D} = \mathbb{R}^2 \setminus \{(0|0)\}$

b) $\alpha(P) = P$: $\quad x = \frac{x}{\sqrt{x^2+y^2}}$, $\quad y = \frac{y}{\sqrt{x^2+y^2}}$ $\rightarrow 1 = \sqrt{x^2+y^2} \Rightarrow 1 = x^2+y^2$

$y = \pm\sqrt{1-x^2}$ □

c) $\alpha(g) = g'$: $\alpha(x|mx+b) = \left(\frac{x}{x^2+m^2x^2} \middle| \frac{mx}{x^2+m^2x^2}\right)$

$= \left(\frac{1}{x(1+m^2)} \middle| \frac{m}{x(1+m^2)}\right)$

$= (x' | m x')$ □

d) $\frac{1}{d(O|P')} = \frac{1}{\sqrt{\frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}}} = \frac{1}{\sqrt{\frac{x^2+y^2}{(x^2+y^2)^2}}}$

$= \frac{1}{\sqrt{\frac{1}{x^2+y^2}}} = \sqrt{x^2+y^2} = d(O|P) \quad \square$

e) $\alpha(x|1) = \left(\frac{x}{x^2+1} \middle| \frac{1}{x^2+1}\right)$ $\quad x$ variabel

$d(O|P') = \frac{1}{d(O|P)} = \frac{1}{\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2+1}$

(21)

a) Drehung um α um Z

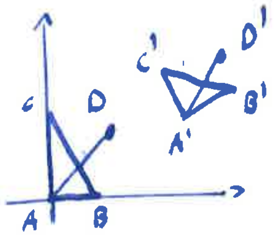
b) " $-\alpha$ "

c) $g_{A'B'} = \alpha(g_{AB})$

d) Auf den Mittelpunkt $\overline{A'B'}$.

e) nein, denn die Drehung ist eine Kongruenzabbildung und invertierbar. Hätten die Bilder zweier Parallelen $g \neq h$ einen Schnittpunkt, dann müsste dieser Schnittpunkt zwei Urbilder haben, eines auf g und eines auf h. Dann wäre die Abbildung aber nicht bijektiv, also nicht invertierbar.

(22)



$D' = (6|4.5)$

(23)

$A'(5|0)$, $B'(7|2)$, $C'(7|5)$

(24)

a) $\alpha: \begin{cases} x' = 2x + 4y + 1 \\ y' = -x + y + 2 \end{cases}$

b) $4 = c_1$

$6 = 1.5a_1 + 4 \qquad a_1 = \frac{4}{3}$

$3 = 3b_1 + 4 \qquad b_1 = -\frac{1}{3}$

$2 = c_2$

$3 = 1.5a_2 + 2 \qquad a_2 = \frac{2}{3}$

$4 = 3b_2 + 2 \qquad b_2 = \frac{2}{3}$

$$\alpha: \begin{cases} x' = \frac{4}{3}x - \frac{1}{3}y + 4 \\ y' = \frac{2}{3}x + \frac{2}{3}y + 2 \end{cases}$$

c) $6 = 2a_1 + b_1 + c_1 \quad (1) \qquad -2 = -a_1 + 3b_1 + c_1 \quad (2) \qquad 6 = a_1 - b_1 + c_1 \quad (3)$

$3 = 2a_2 + b_2 + c_2 \quad (4) \qquad 6 = -a_2 + 3b_2 + c_2 \quad (5) \qquad -4 = a_2 - b_2 + c_2 \quad (6)$

$(1)-(2): \; 8 = 3a_1 - 2b_1$

$(1)-(3): \; 0 = a_1 + 2b_1$

$8 = 4a_1$

$2 = a_1 \quad \Rightarrow b_1 = -1 \quad \Rightarrow c_1 = 3$

$(4)-(5): \; -3 = 3a_2 - 2b_2$

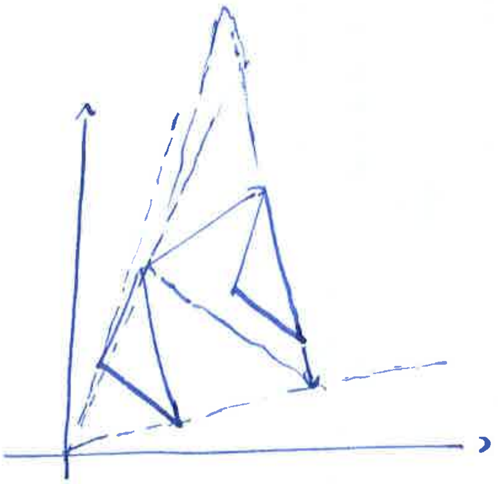
$(4)-(6): \; 7 = a_2 + 2b_2$

$4 = 4a_2$

$1 = a_2 \quad \Rightarrow b_2 = 3 \quad \Rightarrow c_2 = -2$

$$\alpha: \begin{cases} x' = 2x - y + 3 \\ y' = x + 3y - 2 \end{cases}$$

d)



(25) a) - d) Parallelogramm, e) Trapez, f) Viereck, g) Dreieck

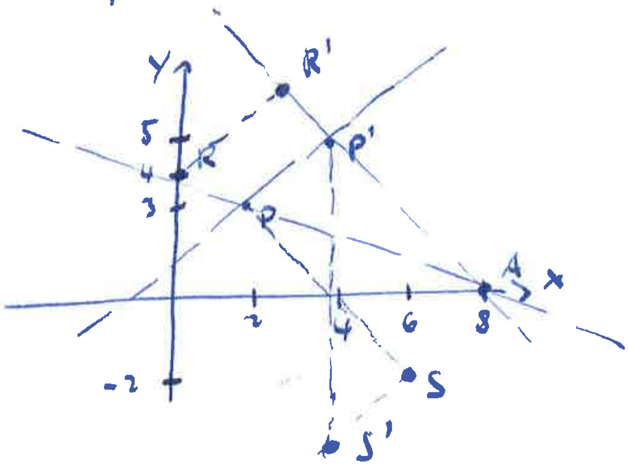
(26) Nein, betrachte den Schnittpunkt der Diagonalen.

(27) $g'_{P'R'} \quad h' = h \; ; \; R' = g' \cap h_{P'Q'} \; ; \; Z = g_{PP'} \cap h_{QQ'}$

$R' = i_{ZR} \cap g'$

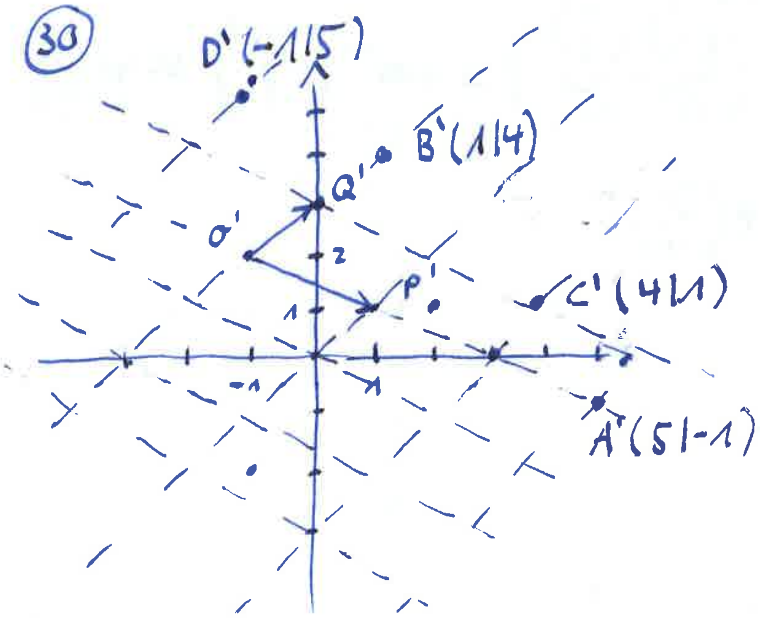
(28) a) ja, teilverhältnistreu b) nein, z.B.

(29)



$R'\left(\frac{8}{3} \middle| \frac{20}{3}\right)$ $S'\left(\frac{14}{3} \middle| -\frac{10}{3}\right)$

(30)



b) $\overrightarrow{OO'} = (-1|2)$

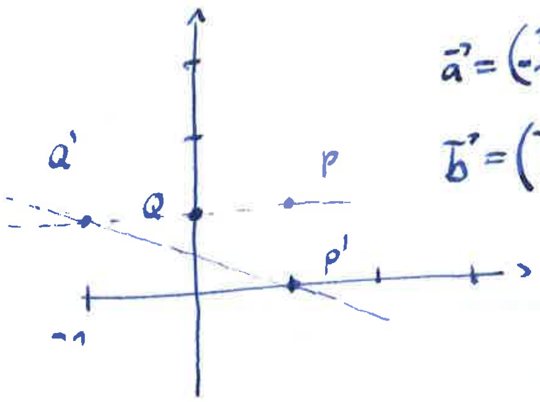
$\overrightarrow{O'E_x'} = (2|-1)$

$\overrightarrow{O'E_y'} = (1|1)$

$$\alpha: \begin{cases} x' = 2x + y - 1 \\ y' = -x + y + 2 \end{cases}$$

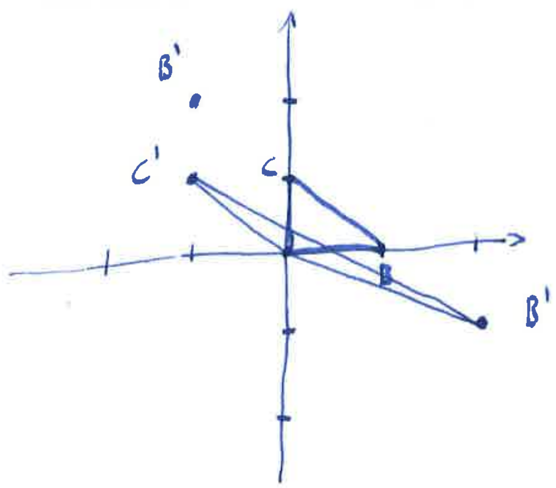
(31) a) $\alpha: \begin{cases} x' = 2x - y \\ y' = -x + y \end{cases}$

$B' = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$ $C' = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$

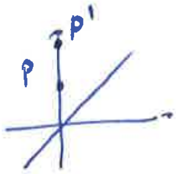


$\vec{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



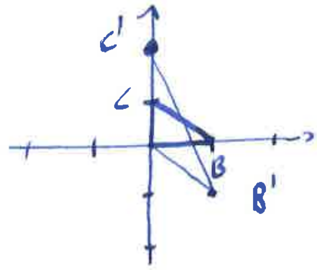
b)



$\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

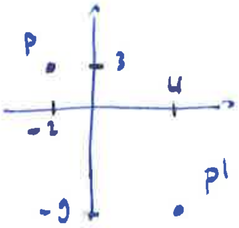
$\vec{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$$\alpha : \begin{cases} x' = x \\ y' = -x + 2y \end{cases}$$



$B' = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$ $\quad C' = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

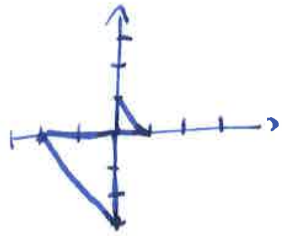
c)



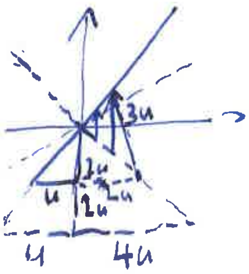
$$\alpha : \begin{cases} x' = -2x \\ y' = -3y \end{cases}$$

$B' = \begin{pmatrix} -10 \\ 0 \end{pmatrix}$

$C' = \begin{pmatrix} 0 \\ -15 \end{pmatrix}$



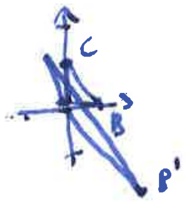
d)



$\vec{a} = \begin{pmatrix} \frac{5}{3} \\ -2 \end{pmatrix}$

$\vec{b} = \begin{pmatrix} -\frac{1}{3} \\ \frac{4}{3} \end{pmatrix}$

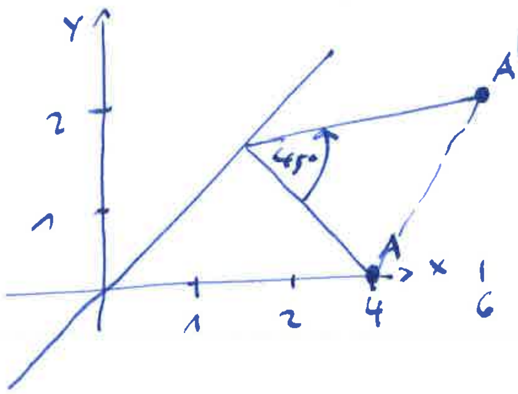
$$\alpha' : \begin{cases} x' = \frac{5}{3}x - \frac{1}{3}y \\ y' = -2x + \frac{4}{3}y \end{cases}$$



$B' = \begin{pmatrix} \frac{25}{3} \\ -10 \end{pmatrix}$ $\quad C' = \begin{pmatrix} -\frac{5}{3} \\ \frac{20}{3} \end{pmatrix}$

32

a)

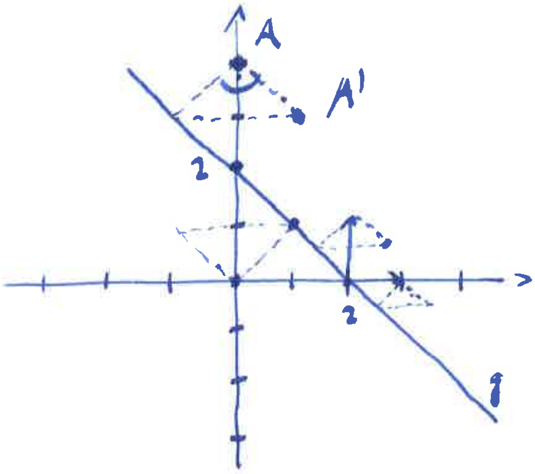


$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$$\alpha : \begin{cases} x' = 1\frac{1}{2}x - \frac{1}{2}y \\ y' = \frac{1}{2}x + \frac{1}{2}y \end{cases}$$

b)



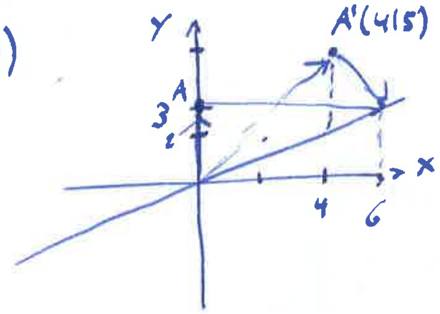
$$\alpha\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{3}{2}\\-\frac{1}{2}\end{pmatrix}$$

$$\alpha\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$$

$$\alpha\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

$$\alpha : \begin{cases} x' = \frac{3}{2}x + \frac{1}{2}y - 1 \\ y' = -\frac{1}{2}x + \frac{1}{2}y + 1 \end{cases}$$

c)



$$\tan\varphi = \frac{2}{1}$$

$$\alpha\begin{pmatrix}0\\3\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix} \quad \text{, also } \begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}4/3\\5/3\end{pmatrix}$$

$$\alpha\begin{pmatrix}6\\0\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix} \quad , \begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1/3\\-1/3\end{pmatrix}$$

$$\alpha : \begin{cases} x' = \frac{1}{3}x + \frac{4}{3}y \\ y' = -\frac{1}{2}x + \frac{5}{3}y \end{cases}$$

33)

$$-1 = 2a + b + c \quad (1) \qquad 6 = 2d + e + f \quad (4)$$
$$4 = a + 3b + c \quad (2) \qquad -2 = d + 3e + f \quad (5)$$
$$-3 = -2a - 2b + c \quad (3) \qquad 7 = -2d - 2e + f \quad (6)$$

$(1)-(2):\ -5 = a - 2b$

$(1)-(3):\ 2 = 4a + 3b$

$\Rightarrow\ 2 = 4(-5+2b) + 3b$

$22 = 11b$

$2 = b \qquad \Rightarrow a = -1, c = -1$

$(4)-(5):\ 8 = d - 2e$

$(4)-(6):\ -1 = 4d + 3e$

$\Rightarrow\ -1 = 4(8+2e) + 3e$

$-33 = 11e$

$-3 = e$

$\Rightarrow d = 2,\ f = 5$

$$\alpha : \begin{cases} x' = -x + 2y - 1 \\ y' = 2x - 3y + 5 \end{cases}$$

34 $\rightarrow \begin{matrix} x\cos(-\varphi) - y\sin(-\varphi) \\ x\sin(-\varphi) + y\cos(-\varphi) \end{matrix} \sim \begin{matrix} x\cos(\varphi) + y\sin(\varphi) \\ -x\sin(\varphi) + y\cos(\varphi) \end{matrix}$

$$\rightarrow \begin{matrix} x\cos(\varphi) + y\sin(\varphi) \\ x\sin(\varphi) - y\cos(\varphi) \end{matrix}$$

$$\rightarrow \begin{matrix} (x\cos(\varphi) + y\sin(\varphi))\cdot\cos\varphi - (x\sin(\varphi) - y\cos(\varphi))\cdot\sin\varphi \\ (x\cos(\varphi) + y\sin(\varphi))\cdot\sin\varphi + (x\sin(\varphi) - y\cos(\varphi))\cdot\cos\varphi \end{matrix}$$

$$x\cos^2(\varphi) + y\sin\varphi\cos\varphi + y\sin\varphi\cos\varphi - x\sin^2\varphi$$

$$x\sin(\varphi)\cos(\varphi) + y\sin^2(\varphi) + x\sin\varphi\cos\varphi - y\cos^2(\varphi)$$

$$x(\cos^2\varphi - \sin^2\varphi) + y\cdot 2\sin\varphi\cos\varphi$$

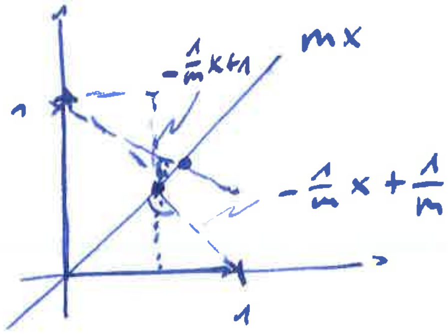
$$x(2\sin\varphi\cos\varphi) + y(\sin^2\varphi - \cos^2\varphi)$$

$$\sim \begin{matrix} x\cdot k + y\cdot l \\ x\cdot l - y\cdot k \end{matrix} \rightsquigarrow \begin{matrix} x\cdot\cos(2\varphi) + y\cdot\sin(2\varphi) \\ x\cdot\sin(2\varphi) + y\cos(2\varphi) \end{matrix}$$

vgl. mit Gerade mx

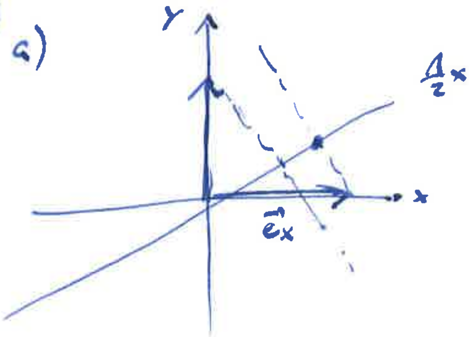
$$\vec{e_x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \vec{e'_x} \begin{pmatrix} \frac{1-m^2}{1+m^2} \\ \frac{2m}{1+m^2} \end{pmatrix} \sim \begin{pmatrix} k \\ l \end{pmatrix}$$

$$\vec{e_y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \vec{e'_y} \begin{pmatrix} \frac{2m}{1+m^2} \\ \frac{m^2-1}{1+m^2} \end{pmatrix} \sim \begin{pmatrix} l \\ -k \end{pmatrix}$$



$$\Rightarrow \gamma: \begin{cases} kx + ly \\ lx - ky \end{cases}$$

35 a)



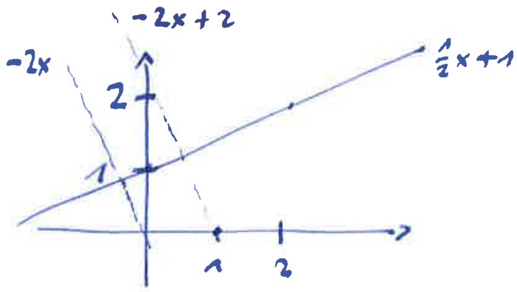
$$-2x + 2 = \frac{1}{2}x$$
$$2 = \frac{5}{2}x$$
$$\frac{4}{5} = x \quad , \quad y = \frac{2}{5}$$
$$\alpha(\vec{e_x}) = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$-2x + 1 = \frac{1}{2}x$$
$$1 = \frac{5}{2}x$$
$$\frac{2}{5} = x \quad , \quad y = \frac{1}{5}$$

$$\alpha(\vec{e_y}) = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$$

$$\alpha : \begin{cases} x' = \frac{3}{5}x + \frac{4}{5}y \\ y' = \frac{4}{5}x - \frac{3}{5}y \end{cases}$$

b)



$$-2x = \frac{1}{2}x + 1$$
$$-\frac{5}{2}x = 1$$
$$x = -\frac{2}{5} \Rightarrow y = \frac{4}{5}$$
$$\sigma = \begin{pmatrix} -\frac{4}{5} \\ \frac{8}{5} \end{pmatrix} = \vec{OO'}$$

$$-2x + 2 = \frac{1}{2}x + 1$$
$$-\frac{5}{2}x = -1$$
$$x = \frac{2}{5} \Rightarrow y = \frac{6}{5}$$
$$E_1' = \left(-\frac{1}{5} \middle| \frac{12}{5}\right)$$
$$\vec{O'E_1'} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$E_2 = E_2'$$
$$\vec{O'E_2'} = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$$

$$\Rightarrow \alpha : \begin{cases} x' = \frac{3}{5}x + \frac{4}{5}y - \frac{4}{5} \\ y' = \frac{4}{5}x - \frac{3}{5}y + \frac{8}{5} \end{cases}$$

36 $\cos\varphi = \frac{12}{13}$

$$1-\cos^2\varphi = \frac{169-144}{169} = \frac{25}{169} = \sin^2\varphi$$

Also $\sin\varphi = \frac{5}{13}$ und $\varphi = \sin^{-1}\left(\frac{5}{13}\right) \approx \underline{\underline{22.62^\circ}}$

37 $\cos^{-1}\left(-\frac{12}{13}\right) = 157.38^\circ$ $\sin^{-1}\left(-\frac{5}{13}\right) = -22.62$

$\to 2\varphi = 360 - 157.38^\circ = 202.62^\circ$

$\to \varphi = 101.31$ $m = \tan\varphi = -5$ $\to y = -5x$

39 Neutrales $E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Inverse zu $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ ist $-A = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$

Assoziativität vererbt von $(\mathbb{R},+)$ weil komponentenweise.

40

$A\cdot\vec{u} = A\cdot\vec{v} \iff A\vec{u} - A\vec{v} = 0 \iff A(\vec{u}-\vec{v}) = 0$

$\iff A = 0$ A muss die Nullmatrix sein.

41 $\det(A) = 1-0 = 1$, $\det(B) = -4-3 = -7$, $\det(C) = 12-(-12)$

42 $A\cdot B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\cdot\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$

$$\det(A\cdot B) = (a_{11}b_{11}+a_{12}b_{21})(a_{21}b_{12}+a_{22}b_{22}) - (a_{21}b_{11}+a_{22}b_{21})\cdot(a_{11}b_{12}+a_{12}b_{22})$$

$$= \cancel{a_{11}a_{21}b_{11}b_{12}} + a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{21}b_{12} + \cancel{a_{12}a_{22}b_{21}b_{22}}$$
$$- (\cancel{a_{11}a_{21}b_{11}b_{12}} + a_{12}a_{21}b_{11}b_{22} + a_{11}a_{22}b_{12}b_{21} + \cancel{a_{12}a_{22}b_{21}b_{22}})$$

$$= a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22}$$
$$= (a_{11}a_{22}-a_{12}a_{21})(b_{11}b_{22}-b_{12}b_{21}) = \det(A)\cdot\det(B) \square$$

(43) $id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \det(id) = \underline{\underline{1}}$

$S_k = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ $\quad \det(S_k) = \underline{\underline{k^2}}$

$S_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \det(S_x) = \underline{\underline{-1}}$

$S_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \det(S_y) = \underline{\underline{-1}}$

$S_\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \det(S_\sigma) = \underline{\underline{1}}$

$V_x = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ $\quad \det(V_x) = \underline{\underline{1}}$

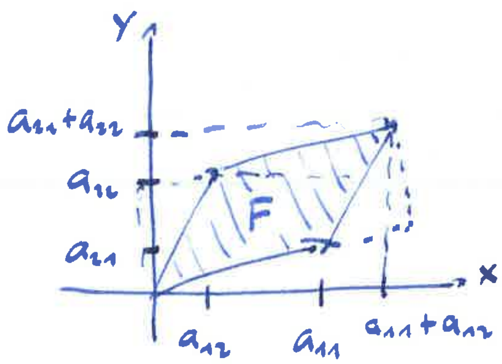
$V_y = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ $\quad \det(V_x) = \underline{\underline{1}}$

$d_\sigma = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ $\quad \det(d_\sigma) = \cos^2\varphi + \sin^2\varphi = \underline{\underline{1}}$

$S_{mx} = \begin{pmatrix} k & \ell \\ \ell & -k \end{pmatrix}$ $\quad \det(S_{mx}) = -k^2 - \ell^2 = -(k^2+\ell^2)$

$= \underline{\underline{-1}}$

(44) $A \cdot \vec{e_1} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ $\quad A \cdot \vec{e_2} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$



$|\vec{e_1'} \times \vec{e_2'}| = \det(A)$

oder

$$F = (a_{11}+a_{12})(a_{21}+a_{22}) - a_{12}a_{21} - a_{12}\cdot a_{21} - 2\cdot\tfrac{1}{2}a_{12}a_{22} - 2\cdot\tfrac{1}{2}a_{11}a_{21}$$

$$= \cancel{a_{11}a_{21}} + a_{11}a_{22} + a_{12}a_{21} + \cancel{a_{12}a_{22}} - 2a_{12}a_{21} - \cancel{a_{12}a_{22}} - \cancel{a_{11}a_{21}}$$

$$= a_{11}a_{22} - a_{12}a_{21} = \det(A)$$

(45) $A = \begin{pmatrix} 3 & 4 \\ 2 & \lambda \end{pmatrix}$ $\quad \det(A) = 3\lambda - 8$

a) $3\lambda - 8 = 0$
$\lambda = \frac{8}{3}$ $\quad$ Für $\lambda \in \mathbb{R} \setminus \{\frac{8}{3}\}$

b) $A_{\frac{8}{3}} = \begin{pmatrix} 3 & 4 \\ 2 & \frac{8}{3} \end{pmatrix}$

c) $A\vec{u} = A\vec{v}$ für $\vec{u} \neq \vec{v}$

$A(\vec{u} - \vec{v}) = \vec{0}$

(46) $\alpha \cdot \beta = A \cdot B$: $\det(AB) = \det(A) \cdot \det(B) = 0$
$\rightarrow$ singulär.

(47) $A \cdot A^{-1} = E$

a) $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} \frac{a_{22}}{\det A} & -\frac{a_{12}}{\det A} \\ -\frac{a_{21}}{\det A} & \frac{a_{11}}{\det A} \end{pmatrix} = \begin{pmatrix} \frac{+a_{11}a_{22} - a_{12}a_{21}}{\det A} & \frac{-a_{11}a_{12} + a_{11}a_{12}}{\det A} \\ \frac{a_{21}a_{22} - a_{22}a_{21}}{\det A} & \frac{-a_{21}a_{12} + a_{22}a_{11}}{\det A} \end{pmatrix}$

$= E \quad \square$

b) $\det(A) = 6 + 6 = 12$
$A^{-1} = \frac{1}{12}\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{12} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix}$

c) $1 = \det(E) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}) \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} \quad \square$

(48) $\vec{PS} \times \vec{PQ} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6+1 \end{pmatrix} \Rightarrow F = 7 \quad R = (2|4)$

$A = \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \quad \det(A) = -2 - 1 = -3 \rightarrow$ gegensinnig , $F' = 3 \cdot 7 = 21$

$P' = A \cdot P = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $Q' = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$, $R' = \begin{pmatrix} -8 \\ 2 \end{pmatrix}$, $S' = \begin{pmatrix} -7 \\ -2 \end{pmatrix}$

(49) $A = \begin{pmatrix} \cos(60) & \sin(60) \\ \sin(60) & -\cos(60) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

$\det(A-E) = (-\frac{1}{2})\cdot(-\frac{3}{2}) - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3}{4} - \frac{3}{4} = 0$

$\Rightarrow \exists$ eine Fixpunktgerade:

$-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 0$

$y = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}x = \frac{\sin(30)}{\cos(30)}x = \tan(30)\,x$ ✓

(50) $\det(A-E) = 2\cdot(-2) - 1 = -5$ $\Rightarrow$ Ursprung ist einziger Fixpunkt.

(51) $\det(A-E) = 1\cdot(-6) - 3\cdot(-2) = 0$ $\Rightarrow \exists$ eine Fixpunktgerade

$x = 1x - 2y$

$-x = -2y$ $\qquad y = \frac{1}{2}x$ $\qquad F_1(0|0),\ F_2(2|1),\ F_3(-2|-1)$

(52) $A = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ $\qquad A-E = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ $\qquad \det(A-E) = -4-1 = -5$

$2x + y - 1 = 0$

$x - 2y + 2 = 0$ $\quad \Rightarrow 4y - 4 + y - 1 = 0 \Leftrightarrow y = 1 \Rightarrow x = 0$

$F(0|1)$ ist einziger Fixpunkt.

(53) Ein lineares Glsys mit 2 Gleichungen und 2 Unbekannten hat keine, eine oder unendlich viele Lösungen, hier werden Fixpunkte repräsentiert. Kennt man also zwei Fixpunkte, so muss es wegen der Linearität unendlich viele geben, die auf einer Geraden liegen.

(54) Gäbe es mehrere Fixpunktgeraden, dann sicher 3 Fixpunkte, die nicht auf ein und derselben Geraden liegen. Dies ist aber nur für die Identität möglich.
Im R^3 kann es unendlich viele Fixpunktgeraden geben, die eine Fixebene bilden.

(55) siehe (54)

(56) Eine $n \times n$-Matrix hat höchstens n Eigenwerte, weil n der Grad des charakteristischen Polynoms ist.

Kein Eigenwert → keine Fixgeraden

Bei einer Streckung hat man $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

$$\det(A - \lambda E) = \det\begin{pmatrix} k-\lambda & 0 \\ 0 & k-\lambda \end{pmatrix} = (k-\lambda)^2 = 0$$

⇒ $k = \lambda$ → es gibt einen Eigenwert, aber unendlich viele Fixgeraden.

(57) $A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$ $\qquad \det(A - \lambda E) = \lambda^2 - (0-1)\lambda + 0\cdot(-1) - 2\cdot 1$

$$= \lambda^2 + \lambda - 2 = 0$$

$$(\lambda - 1)(\lambda + 2) = 0$$

$\lambda_1 = 1$, $\lambda_2 = -2$

$$\Rightarrow \begin{aligned} -x + y &= 0 \\ 2x - 2y &= 0 \end{aligned} \qquad \Rightarrow y = x \quad \text{(Affinitätsachse)}$$

$$\Rightarrow \begin{aligned} 2x + y &= 0 \\ 2x + y &= 0 \end{aligned} \qquad \Rightarrow y = -2x + b$$

(58) a) $S = \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}$

$$\lambda^2 - (\cos 2\varphi - \cos 2\varphi)\lambda + (-\cos^2 2\varphi - \sin^2 2\varphi) = 0$$

$$\lambda^2 = 1 \iff \lambda = \pm 1$$

$$(\cos 2\varphi - 1)x + \sin 2\varphi \cdot y = 0$$

$$y = \frac{1 - \cos 2\varphi}{\sin 2\varphi} x = \frac{1 - 1 + 2\sin^2\varphi}{2\sin\varphi\cos\varphi} x = \frac{\sin\varphi}{\cos\varphi} x = \tan\varphi \, x$$

→ Spiegelachse selber.

$$(\cos 2\varphi + 1)x + \sin 2\varphi = 0$$

$$y = \frac{-1 - \cos 2\varphi}{\sin 2\varphi} x = \frac{-1 - 1 + 2\sin^2\varphi}{2\sin\varphi\cos\varphi} x = \frac{-\cos^2\varphi}{\sin\varphi\cos\varphi} x = -\frac{\cos\varphi}{\sin\varphi} x = -\frac{1}{\tan\varphi} x$$

→ Senkrechte auf der Spiegelachse.

b) $\lambda^2 - (-1+10)\lambda + (-10+24) = 0$

$\lambda^2 - 9\lambda + 14 = 0$

$(\lambda - 7)(\lambda - 2) = 0 \qquad \lambda_1 = 7, \quad \lambda_2 = 2$

$(-1-7)x - 4y = 0$

$6x + (10-7)y = 0$

$-8x - 4y = 0$

$6x + 3y = 0$

$y = -2x$

$y = -2x$

$(-1-2)x - 4y = 0$

$6x + (10-2)y = 0$

$-3x - 4y = 0$

$6x + 8y = 0$

$y = -\frac{3}{4}x$

$y = -\frac{3}{4}x$

c) $\lambda^2 - (5+1)\lambda + (5-0) = 0$

$\lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 5)(\lambda - 1) = 0 \qquad \lambda_1 = 5, \quad \lambda_2 = 1$

$(5-5)x + 0y = 0$

$-3x + (1-5)y = 0$

$0 = 0$

$y = -\frac{3}{4}x$

$(5-1)x + 0y = 0$

$-3x + (1-1)y = 0$

$4x = 0$

$-3x = 0$

d) $\lambda^2 - (5+3)\lambda + (15+1) = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0 \qquad \lambda = 4$

$(5-4)x + y = 0$

$-x + (3-4)y = 0$

$y = -x$

$y = -x$

(59) $\lambda^2-(2+3)\lambda+(6+p)=0$

$\lambda^2-5\lambda+6+p=0$

$D=b^2-4ac=25-24-p=1-p$

$\Rightarrow$ keine : $p>1$

eine : $p=1$

zwei : $p<1$ $\quad (p=-2 \to$ viele$)$

$(2-\lambda)x-y=0$ $\qquad (2-\lambda)x=\frac{2}{5}$ $\qquad x=\frac{2}{5(2-\lambda)}$

$px+(3-\lambda)y=0$ $\qquad \frac{px}{\lambda-3}=\frac{2}{5}$

$\frac{2p}{5(2-\lambda)(\lambda-3)}=\frac{2}{5}$ $\qquad p=(2-\lambda)(\lambda-3)$

$\det A=6+p$ $\quad \to p>-6,\ p<-6$

(60) $\lambda^2-(1+2)\lambda+(2-6)=0$

$\lambda^2-3\lambda-4=0$

$(\lambda-4)(\lambda+1)=0$ $\qquad \lambda_1=4,\ \lambda_2=-1$

$(1-4)x+2y=0$ $\qquad (1-(-1))x+2y=0$

$3x+(2-4)y=0$ $\qquad (3x)+(2-(-1))y=0$

$-3x+2y=0$ $\qquad 2x+2y=0$

$3x-2y=0$ $\qquad 3x+3y=0$

$\vec{v}_1=\begin{pmatrix}2\\3\end{pmatrix}$ $\qquad \vec{v}_2=\begin{pmatrix}1\\-1\end{pmatrix}$

$\vec{X}=\begin{pmatrix}2&1\\3&-1\end{pmatrix} \to \vec{X}^{-1}=\begin{pmatrix}-1&-1\\-3&2\end{pmatrix}\cdot\frac{1}{-2-3}=\frac{1}{5}\begin{pmatrix}-1&-1\\-3&2\end{pmatrix}$

$D_A=\frac{1}{5}\begin{pmatrix}-1&-1\\-3&2\end{pmatrix}\cdot\begin{pmatrix}1&2\\3&2\end{pmatrix}\cdot\begin{pmatrix}2&1\\3&-1\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-1&-1\\-3&2\end{pmatrix}\cdot\begin{pmatrix}8&-1\\12&1\end{pmatrix}$

$=\frac{1}{5}\begin{pmatrix}-20&0\\0&5\end{pmatrix}=\underline{\underline{\begin{pmatrix}-4&0\\0&1\end{pmatrix}}}$

(61) google

(62) $\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$ $\rightarrow$ A nicht diagonalisierbar

$\lambda^2 - \lambda + 1 = 0$

$\lambda_{1,2} = +1 \pm \sqrt{1-4}$ $\rightarrow$ B nicht diagonalisierbar

$\lambda^2 - 1 = 0$ $\lambda_{1,2} = \pm 1$ $\rightarrow$ C diagonalisierbar

$(1+1)x = 0$

$6x + (-1+1)y = 0$

$0 = 0$

$6x - 2y = 0$

$3x = y$

$\vec{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$(1+1)x = 0$

$6x + (-1+1)y = 0$

$2x = 0$

$6x = 0$

$\vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow X = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ $X^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \cdot \frac{1}{1}$

$$D_C = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}}$$

(63) $A^2 = \begin{pmatrix} 2 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$$A^{20} = (A^2)^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}}}$$

(64) a) Es ist $(AB)(AB)^{-1} = E = B \cdot B^{-1} = A \cdot (B \cdot B^{-1}) \cdot A^{-1} = (AB)(B^{-1}A^{-1})$
und $(AB)^{-1}(AB) = E$

b) $A = X \cdot D \cdot X^{-1}$

$A^{-1} = (X \cdot D \cdot X^{-1})^{-1} = X \cdot D^{-1} \cdot X^{-1}$

wegen $D = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$ ist $D^{-1} = \begin{pmatrix} d_{22} & 0 \\ 0 & d_{11} \end{pmatrix}$ auch wieder eine Diagonalmatrix

65 a) $\lambda^2 - (1-2)\lambda - 2 - 4 = 0$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda+3)(\lambda-2) = 0$ $\qquad \lambda_1 = -3, \ \lambda_2 = 2$

$(1+3)x + 2y = 0$ $\qquad (1-2)x + 2y = 0$

$2x + (-2+3)y = 0$ $\qquad 2x + (-2-2)y = 0$

$4x + 2y = 0$ $\qquad -x + 2y = 0$

$2x + y = 0$ $\qquad 2x - 4y = 0$

$\vec{v_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\qquad \vec{v_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\vec{v_1} \cdot \vec{v_2} = 0$ ✓

c) Seien λ, μ zwei verschiedene Eigenwerte zu den Eigenvektoren $\vec{v}, \vec{w}$ und A ihre symmetrische Matrix.

$\lambda v \cdot w = (\lambda v) \cdot w = (Av) \cdot w = (v \cdot A^t) \cdot w \underset{A \text{ symm}}{=} v \cdot (Aw)$

$= v(\mu w) = \mu v w$

$\Rightarrow \lambda v w - \mu v w = 0$

$(\lambda - \mu) v w = 0$, d.h. $v \cdot w = 0$

d) Sei $Av = \lambda v$ und $Aw = \mu w$ und $v \cdot w = 0$

$\Rightarrow (\lambda - \mu) v w = 0$

$\lambda v w = \mu v w = v \mu w$

$A v w = v A w$

$A v w = A^t v w$

$A(vw) = A^t(vw)$

(61)

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

diagonalisieren:

$$\lambda^2 - (1+0)\lambda + 0 - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0 \qquad \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\left(1 - \frac{1+\sqrt{5}}{2}\right)x + y = 0 \qquad \frac{1-\sqrt{5}}{2}x + y = 0 \qquad y = \frac{-1+\sqrt{5}}{2}x$$

$$x + \left(0 - \frac{1+\sqrt{5}}{2}\right)y = 0 \qquad x - \frac{1+\sqrt{5}}{2}y = 0 \qquad y = \frac{2}{1+\sqrt{5}}x = \frac{-1+\sqrt{5}}{2}x$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix} \qquad \frac{2}{-1+\sqrt{5}} = \frac{2(\sqrt{5}+1)}{4} = \frac{1+\sqrt{5}}{2}$$

$$\left(1 - \frac{1-\sqrt{5}}{2}\right)x + y = 0 \qquad \frac{1+\sqrt{5}}{2}x + y = 0 \qquad y = \frac{-1-\sqrt{5}}{2}x$$

$$x + \left(0 - \frac{1-\sqrt{5}}{2}\right)y = 0 \qquad x - \frac{1-\sqrt{5}}{2}y = 0 \qquad y = \frac{2}{1-\sqrt{5}}x = \frac{-1-\sqrt{5}}{2}x$$

$$-\frac{2(1-\sqrt{5})}{1-5} = \frac{1-\sqrt{5}}{2}$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \end{pmatrix} \qquad -\frac{2}{1+\sqrt{5}}$$

$$\Rightarrow X = \begin{pmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \end{pmatrix} \quad \rightarrow X^{-1} = \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix} \cdot \frac{1}{-\sqrt{5}}$$

$$X^{-1}A = \frac{1}{-\sqrt{5}} \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{-\sqrt{5}} \begin{pmatrix} \frac{-3+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

$$\rightarrow \cdot X = \frac{1}{-\sqrt{5}} \begin{pmatrix} \frac{-3+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \end{pmatrix} = \frac{1}{-\sqrt{5}} \Big($$

$$= \frac{1}{-\sqrt{5}} \begin{pmatrix} \frac{-3+\sqrt{5}}{2} + \frac{+1-5}{4} & \frac{-3+\sqrt{5}}{2} + \frac{+1+2\sqrt{5}+5}{4} \\ \frac{3-\sqrt{5}}{2} + \frac{-1+2\sqrt{5}-5}{4} & \frac{3-\sqrt{5}}{2} + \frac{-1+5}{4} \end{pmatrix} = \frac{1}{-\sqrt{5}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \end{pmatrix} \longrightarrow X^{-1} = \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix} \cdot \frac{1}{-\sqrt{5}}$$

$$\det X = -\frac{1+\sqrt{5}}{2} - \frac{-1+\sqrt{5}}{2} = \frac{-2\sqrt{5}}{2} = -\sqrt{5}$$

$$X^{-1} \cdot A = \frac{1}{-\sqrt{5}} \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{-\sqrt{5}} \begin{pmatrix} \frac{-3+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

$$\xrightarrow{\cdot X} \frac{1}{-\sqrt{5}} \begin{pmatrix} \frac{-3+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \end{pmatrix} =$$

$$= \frac{1}{-\sqrt{5}} \begin{pmatrix} \frac{-3+\sqrt{5}}{2} + \frac{1-5}{4} & \frac{-3+\sqrt{5}}{2} + \frac{1+2\sqrt{5}+5}{4} \\ \frac{3-\sqrt{5}}{2} + \frac{-1+2\sqrt{5}-5}{4} & \frac{3-\sqrt{5}}{2} + \frac{-1+5}{4} \end{pmatrix}$$

$$= \frac{1}{-\sqrt{5}} \begin{pmatrix} \frac{-5+\sqrt{5}}{2} & \sqrt{5} \\ 0 & \frac{5-\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}+1}{2} & 0 \\ 0 & \frac{-\sqrt{5}+1}{2} \end{pmatrix} = D$$

Wegen $A^n = (X \cdot D \cdot X^{-1})^n = X \cdot D^n \cdot X^{-1}$ gilt:

$$A^n = \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{\sqrt{5}+1}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \frac{1}{-\sqrt{5}} \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix}$$

$$= \frac{1}{-\sqrt{5}} \begin{pmatrix} \left(\frac{\sqrt{5}+1}{2}\right)^n & \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \frac{\sqrt{5}-1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^n & -\frac{1+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix}$$

$$= \frac{1}{-\sqrt{5}} \begin{pmatrix} -\frac{1+\sqrt{5}}{2}\left(\frac{\sqrt{5}+1}{2}\right)^n + \frac{1-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n & -\left(\frac{\sqrt{5}+1}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \\ -2\left(\frac{\sqrt{5}+1}{2}\right)^n + 2\left(\frac{1-\sqrt{5}}{2}\right)^n & -\frac{\sqrt{5}-1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^n - \frac{1+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix}$$

$$\Rightarrow F_n = \frac{1}{-\sqrt{5}}\left(-\frac{1+\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \frac{1-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right)$$

(66)

$$\begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{k-1} \\ a_{k-2} \end{pmatrix} \qquad 0 \quad 1 \quad 3 \quad 7 \quad 15 \ldots$$

$$= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^{k-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda^2 - (3-0)\lambda + (0+2) = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda-1)(\lambda-2) = 0 \qquad \lambda_1 = 1, \; \lambda_2 = 2$$

$$(3-1)x - 2y = 0 \qquad y = x \qquad \vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$1x + (0-1)y = 0 \qquad y = x$$

$$(3-2)x - 2y = 0 \qquad y = \tfrac{1}{2}x \qquad \vec{v_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$1x + (0-2)y = 0 \qquad y = \tfrac{1}{2}x$$

$$X = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \qquad X^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$X^{-1} \cdot A = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\cdot X = \begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = D$$

$$A^n = X \cdot D^n \cdot X^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2^{n+1} \\ 1 & 2^n \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+2^{n+1} & 2-2^{n+1} \\ -1+2^n & 2-2^n \end{pmatrix}$$

$$A^{n-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a_k = -1 + 2^n$$

(67) $\begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k-1} \\ a_{k-2} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}^{k-1} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\lambda^2 - (3+0)\lambda + (0-4) = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0 \qquad \lambda_1 = 4, \lambda_2 = -1$

$(3-4)x + 4y = 0 \qquad -x + 4y = 0 \qquad \vec{v_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$1x + (0-4)y = 0 \qquad x - 4y = 0$

$(3+1)x + 4y = 0 \qquad 4x + 4y = 0 \qquad \vec{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$1x + (0+1)y = 0 \qquad x + y = 0$

$X = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \qquad X^{-1} = \frac{1}{-5} \begin{pmatrix} -1 & -1 \\ -1 & 4 \end{pmatrix}$

$X^{-1} \cdot A = \frac{1}{-5} \begin{pmatrix} -1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} -4 & -4 \\ 1 & -4 \end{pmatrix}$

$\cdot X = +\frac{1}{5} \begin{pmatrix} +4 & +4 \\ -1 & +4 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 8 & 0 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} \frac{8}{5} & 0 \\ 0 & -1 \end{pmatrix}$

$A^n = X \cdot D^n \cdot X^{-1} = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} (\frac{8}{5})^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{4}{5} \end{pmatrix}$

$= \begin{pmatrix} 4 \cdot (\frac{8}{5})^n & (-1)^n \\ (\frac{8}{5})^{n} & (-1)^{n+1} \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5}(\frac{8}{5})^n + (-1)^n \cdot \frac{1}{5} & \frac{4}{5}(\frac{8}{5})^n - \frac{4}{5}(-1)^n \\ \frac{1}{5}(\frac{8}{5})^n + \frac{1}{5}(-1)^{n+1} & \frac{1}{5}(\frac{8}{5})^n - \frac{4}{5}(-1)^{n+1} \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow a_k = \frac{4}{5}\left((\frac{8}{5})^{k-1} - (-1)^{k-1} \right)$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \longrightarrow a_k = -\frac{4}{5}(\frac{8}{5})^{n-1} - (-1)^{n-1} \cdot \frac{1}{5} + \frac{4}{5}(\frac{8}{5})^{n-1} - \frac{4}{5}(-1)^{n-1}$

$= -\frac{1}{5}(-1)^{n-1} - \frac{4}{5}(-1)^{n-1} = (-1)^k$

(68) $$\begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{k-1} \begin{pmatrix} a_{k-1} \\ a_{k-2} \end{pmatrix}$$

$$\lambda^2 - (p+0)\lambda + (0+q) = 0$$
$$\lambda^2 - p\lambda - q = 0 \qquad \lambda_{1,2} = \frac{p \pm \sqrt{p^2+4q}}{2}$$

(69) $$\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}^5 = \begin{pmatrix} -1 & 11 \\ 0 & 32 \end{pmatrix}$$

$$b = 1 \cdot \left((-1)^4 + (-1)^3 \cdot 2 + (-1)^2 \cdot 2^2 + (-1) \cdot 2^3 + 2^4 \right)$$
$$= 1 - 2 + 4 - 8 + 16 = 11$$

(70) $$\lambda^2 - (1+3)\lambda + (3+1) = 0$$
$$\lambda^2 - 4\lambda + 4 = 0$$
$$(\lambda - 2)^2 = 0 \qquad \lambda_{1,2} = 2$$

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$ (Jordan decomposition)

(71) a) $$\begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k-1} \\ a_{k-2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^{k-1} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\lambda^2 - (2+0)\lambda + (0+1) = 0$$
$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0 \qquad \lambda = 1$$

$$(2-1)x - y = 0 \qquad x - y = 0 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$1x + (0-1)y = 0 \qquad x - y = 0$$

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad X^{-1} = \text{n.d.}$$

$$D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad X^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Bem: Um X zu basteln nimmt man den Eigenvektor und irgend einen andern Vektor, der mit dem Eigenvektor eine Basis des $\mathbb{R}^2$ bildet.

(68) $A = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$

$$\lambda^2 - p\lambda - q = 0 \qquad \lambda_{1,2} = \frac{p \pm \sqrt{p^2+4q}}{2} \qquad \begin{array}{l} +p = \lambda_1 + \lambda_2 \\ -q = \lambda_1 \lambda_2 \end{array}$$

$$x - \lambda_1 y = 0 \qquad \vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

$$x = \lambda_1 y$$

$$X = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \qquad \det X = \lambda_1 - \lambda_2$$

$$X^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \qquad \lambda_1 - \lambda_2 = \sqrt{p^2+4q}$$

$$AX = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 p + q & \lambda_2 p + q \\ \lambda_1 & \lambda_2 \end{pmatrix}$$

$$X^{-1}AX = \frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1 p + q & \lambda_2 p + q \\ \lambda_1 & \lambda_2 \end{pmatrix} = \frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} \lambda_1 p + q - \lambda_1\lambda_2 & \lambda_2 p + q - \lambda_2^2 \\ -\lambda_1 p - q + \lambda_1^2 & -\lambda_2 p - q + \lambda_1 \lambda_2 \end{pmatrix}$$

$$= \cancel{\frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} \lambda_1(-\lambda_1-\lambda_2) + \lambda_1\lambda_2 + \lambda_1\lambda_2 & 0 \\ 0 & +\lambda_2(\lambda_1+\lambda_2) - \lambda_1\lambda_2 + \lambda_1\lambda_2 \end{pmatrix}}$$

$$\cancel{= \frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} -\lambda_1^2 - \lambda_1\lambda_2 & 0 \\ 0 & \lambda_2^2 + \lambda_1\lambda_2 \end{pmatrix}}$$

$$\cancel{= \frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} -\lambda_1(\lambda_1+\lambda_2) & 0 \\ 0 & \lambda_2(\lambda_2+\lambda_1) \end{pmatrix}} \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\uparrow$$

$$= \frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} \lambda_1(\lambda_1+\lambda_2) - \lambda_1\lambda_2 - \lambda_1\lambda_2 & 0 \\ 0 & -\lambda_2(\lambda_1+\lambda_2) + \lambda_1\lambda_2 + \lambda_1\lambda_2 \end{pmatrix}$$

$$= \frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} \lambda_1^2 - \lambda_1\lambda_2 & 0 \\ 0 & -\lambda_2^2 + \lambda_1\lambda_2 \end{pmatrix} = \frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} \lambda_1(\lambda_1-\lambda_2) & 0 \\ 0 & +\lambda_2(\lambda_1-\lambda_2) \end{pmatrix}$$

$$A^n = X D^n X^{-1}$$

$$D^n X^{-1} = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \cdot \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^n & -\lambda_1^n \lambda_2 \\ -\lambda_2^n & \lambda_2^n \lambda_1 \end{pmatrix}$$

$$X D^n X^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & -\lambda_1^n \lambda_2 \\ -\lambda_2^n & \lambda_2^n \lambda_1 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & -\lambda_1^{n+1}\lambda_2 + \lambda_2^{n+1}\lambda_1 \\ \lambda_1^n - \lambda_2^n & -\lambda_1^n \lambda_2 + \lambda_2^n \lambda_1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} & + \cancel{\lambda_1^n \lambda_2} \\ \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} & \end{pmatrix}$$

b) $A = \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}$ $\qquad \lambda^2 - 4\lambda + 1 = 0 \qquad \lambda_{1,2} = 2 \pm \sqrt{3}$

$D = \begin{pmatrix} 2-\sqrt{3} & 0 \\ 0 & 2+\sqrt{3} \end{pmatrix} \qquad X = \begin{pmatrix} 2-\sqrt{3} & 2+\sqrt{3} \\ 1 & 1 \end{pmatrix} \qquad X^{-1} = \begin{pmatrix} -\frac{1}{2\sqrt{3}} & -\frac{-2-\sqrt{3}}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & -\frac{2-\sqrt{3}}{2\sqrt{3}} \end{pmatrix}$

72

$\det(A - \lambda E) = (-\lambda)(2-\lambda)(3-\lambda) + 0 + 0 \cancel{+4} \overset{-(-2)(2-\lambda)\cdot 1}{} - 0 - 0$

$\cancel{= (-\lambda)(6 - 5\lambda + \lambda^2) + 4 = -\lambda^3 + 5\lambda^2 - 6\lambda + 4 \overset{!}{=} 0}$

$= -\lambda^3 + 5\lambda^2 - 6\lambda + 2\lambda + 4 = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$

$-\lambda^3 + \lambda^2$

$4\lambda^2$

$-(4\lambda^2 - 4\lambda)$

-4λ

$-4\lambda + 4$

0

$\lambda = 1: \; -1^3 + 5 - 8 + 4 = 0$ ✓

$:(\lambda - 1) = -\lambda^2 + 4\lambda - 4$

$\rightarrow \lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0 \quad \Rightarrow \quad \lambda_{2,3} = 2$

$\lambda = 1: \quad -1x + 0y - 2z = 0$

$1x + (2-1)y + z = 0$

$1x + 0y + (3-1)z = 0$

$-x - 2z = 0$

$x + y + z = 0$

$x + 2z = 0$

$y - z = 0$

$x = -2z$

$\vec{v_1} = \begin{pmatrix} -2 \\ +1 \\ 1 \end{pmatrix}$

$\lambda = 2: \quad -2x - 2z = 0$

$x + z = 0$

$x + z = 0$

$z = -x$

y beliebig

$\vec{v_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \vec{v_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$X = \begin{pmatrix} -2 & 1 & 0 \\ +1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \rightarrow \quad X^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ +1 & 0 & +2 \\ +1 & 1 & +1 \end{pmatrix}$

$AX = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ +1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}$

$X^{-1} \cdot AX = \begin{pmatrix} -1 & 0 & -1 \\ +1 & 0 & +2 \\ +1 & 1 & +1 \end{pmatrix} \begin{pmatrix} -2 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$$

$$\det(B - \lambda E) = (1-\lambda)(2-\lambda)^2 + 0 + 0 - 0 - 0 - 0 = 0$$

$\Rightarrow \lambda_1 = 1, \ \lambda_{2,3} = 2$

$\lambda = 1$: $0 = 0$ $\qquad -x = y$ $\qquad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 8 \end{pmatrix}$

$x + y = 0$

$-3x + 5y + z = 0 \qquad -8x + z = 0 \qquad 8x = z$

$\lambda = 2$: $-x = 0$ $\qquad \vec{v}_{2,3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.2 \\ 0 \end{pmatrix}$

$x = 0$

$-3x + 5y = 0$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & \frac{1}{5} \\ 8 & 1 & 0 \end{pmatrix} \underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}} \begin{pmatrix} 1 & 0 & 0 \\ -8 & 0 & 1 \\ 5 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$$

73 a) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ c) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} \cos(90°) & 0 & +\sin(90°) \\ 0 & 1 & 0 \\ -\sin(90°) & 0 & \cos(90°) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

74 $\det(A) = -1, \ \det(B) = -1, \ \det(C) = 1, \ \det(D) = 1, \ \det(E) = 0$

75 $\det(A - \lambda E) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix} =$

$$= \underbrace{(a_{11}-\lambda)(a_{22}-\lambda)}_{(a_{11}a_{22} - (a_{11}+a_{22})\lambda + \lambda^2)}(a_{33}-\lambda) + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}(a_{22}-\lambda)a_{31}$$

$$- (a_{11}-\lambda)a_{23}a_{32} - a_{12}a_{21}(a_{33}-\lambda)$$

$$= a_{11}a_{22}a_{33} - \lambda a_{11}a_{22} - (a_{11}+a_{22})a_{33}\lambda + (a_{11}+a_{22})\lambda^2 + a_{33}\lambda^2 - \lambda^3 + \ldots$$

$$= -\lambda^3 + (a_{11}+a_{22}+a_{33})\lambda^2 - (a_{22}a_{33} - a_{32}a_{23} + a_{11}a_{33} - a_{31}a_{13} + a_{11}a_{22} - a_{21}a_{12}) \cdot \lambda$$

$$+ \det(A) = 0 \Rightarrow \chi_A(\lambda) = \lambda^3 - \mathrm{Spur}(A)\lambda^2 + (\det(A_1) + \det(A_2) + \det(A_3)) \cdot \lambda - \det(A)$$

76 a) $\lambda^3+\lambda^2+\lambda+1=0$

$\to \underline{\underline{\lambda_1=-1}} \quad \to \ :(\lambda+1) = \lambda^2+1$

$\to \lambda^2+1=0 \quad \to \underline{\underline{\lambda_{2,3}=\pm i}}$

$$\lambda_1=-1: \quad \left.\begin{aligned} x+y&=0\\ -x+y&=0 \end{aligned}\right\} x=0=y \qquad \underline{\underline{\vec{v}_1=\begin{pmatrix}0\\0\\1\end{pmatrix}}}$$

$$0\cdot z=0 \qquad z \text{ beliebig}$$

$$\lambda_2=i: \quad \begin{aligned} -ix+y&=0\\ -x-iy&=0\\ (-1-i)z&=0 \end{aligned} \qquad \begin{aligned} y&=ix\\ y&=ix\\ z&=0 \end{aligned} \qquad \underline{\underline{\vec{v}_2=\begin{pmatrix}1\\i\\0\end{pmatrix}}}$$

$$\lambda_3=-i: \quad \begin{aligned} ix+y&=0\\ -x+iy&=0\\ (-1+i)z&=0 \end{aligned} \qquad \begin{aligned} y&=-ix\\ y&=-ix\\ z&=0 \end{aligned} \qquad \underline{\underline{\vec{v}_3=\begin{pmatrix}1\\-i\\0\end{pmatrix}}}$$

b) $A\cdot A^T=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&-1\end{pmatrix}\cdot\begin{pmatrix}0&-1&0\\1&0&0\\0&0&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \to$ orthogonal

$$\begin{pmatrix}x'\\y'\\z'\end{pmatrix}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}y\\-x\\-z\end{pmatrix}$$

c) Drehung $90°$ um z-Achse verkettet mit Spiegelung an xy-Ebene.

B: a) $1, \frac{1+i}{\sqrt2}, \frac{1-i}{\sqrt2}$ $\qquad \vec{v}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{v}_2=\begin{pmatrix}0\\-i\\1\end{pmatrix}, \vec{v}_3=\begin{pmatrix}0\\i\\1\end{pmatrix}$

$$B\cdot B^T=\begin{pmatrix}1&0&0\\0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\0&-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \to \text{orthogonal}$$

Drehung um x-Achse um $45°$ verkettet mit Spiegelung an xy-Ebene

C: a) $1, -0{,}2\pm0.97\ldots i$, $\begin{pmatrix}2\\1\\1\end{pmatrix}$, $\big(-2+i$

$$C\cdot C^T=\begin{pmatrix}0.6&0.8&0\\0&0&1\\0.8&-0.6&0\end{pmatrix}\begin{pmatrix}0.6&0&0.8\\0.8&0&-0.6\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \to \text{orthogonal}$$

Drehung um z-Achse um $23.6°$, Spiegelung an xz-Ebene, Spiegelung an der Winkelhalbierenden Ebene durch x-Achse

(77)

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(78) $y = \frac{1}{2}x$

im $\mathbb{R}^2$: $A = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$ aus Üb 35

z bleibt unverändert

$$A = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & -\frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(79) $\lambda^3 - \lambda^2 - \lambda + 1 = 0$

$\rightarrow \underline{\underline{\lambda_1 = 1}}$ $\rightarrow$ $:(\lambda - 1) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$

$\Rightarrow \lambda_{2,3} = \underline{\underline{\pm 1}}$

λ_3: $\frac{18}{17}x - \frac{12}{17}y - \frac{12}{17}z = 0$

$-\frac{12}{17}x + \frac{25}{17}y - \frac{9}{17}z = 0$ $\rightarrow$ $2y - 2z = 0$

$-\frac{12}{17}x - \frac{9}{17}y + \frac{25}{17}z = 0$ $\quad y = z$

$\rightarrow \frac{18}{17}x - \frac{24}{17}y = 0$

$\frac{18}{17}x = \frac{24}{17}y$

$x = \frac{4}{3}y$

$$\underline{\underline{\vec{v}_3 = \begin{pmatrix} \frac{4}{3} \\ 1 \\ 1 \end{pmatrix}}}$$

$\lambda_{1,2}$: $\left(\frac{1}{17}-1\right)x - \frac{12}{17}y - \frac{12}{17}z = 0$

$-\frac{12}{17}x + \left(\frac{8}{17}-1\right)y - \frac{9}{17}z = 0$

$-\frac{12}{17}x - \frac{9}{17}y + \left(\frac{8}{17}-1\right)z = 0$

$-\frac{16}{17}x - \frac{12}{17}y - \frac{12}{17}z = 0$

$-\frac{12}{17}x - \frac{9}{17}y - \frac{9}{17}z = 0$

$-\frac{12}{17}x - \frac{9}{17}y - \frac{9}{17}z = 0$

$$-\frac{16}{17}x = \frac{12}{17}(y+z)$$

$$x = -\frac{3}{4}(y+z)$$

$$\vec{v}_1 = \begin{pmatrix} -\frac{3}{4} \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -\frac{3}{4} \\ 0 \\ 1 \end{pmatrix}$$

~~det (A) = -1~~

$$A = \underbrace{\begin{pmatrix} \frac{4}{3} & -\frac{3}{4} & -\frac{3}{4} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\uparrow} \underbrace{\begin{pmatrix} \frac{6}{17} & \frac{9}{34} & \frac{9}{34} \\ -\frac{6}{17} & -\frac{9}{34} & \frac{25}{34} \\ -\frac{6}{17} & \frac{25}{34} & -\frac{9}{34} \end{pmatrix}}_{X^{-1}}$$

Spiegelung an yz-Ebene

$x = 0$

$$\det(A) = \det(X) \cdot \det(D) \cdot \det(X^{-1})$$

$$= \det(X) \cdot (-1) \cdot \frac{1}{\det(X)}$$

$$\underline{\underline{= -1}}$$