CSC 212: Data Structures and Abstractions

Balanced trees (part 1)

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THINK BIG WE DO℠

Practice

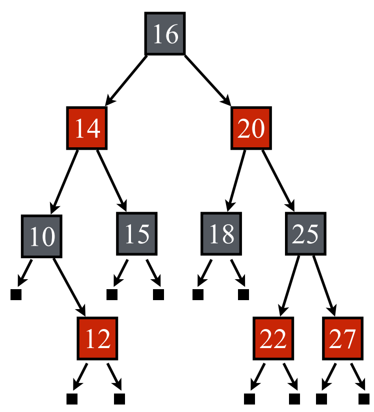
- Assume a dictionary has n keys, and a book has m words
  - ✓ What is the time complexity of identifying which words from the book do NOT appear in the dictionary?
    - dictionary is represented as a BST and assume that $h = O(\log n)$
    - book is represented as an array (vector) of strings, where each string is a word

Balanced search trees

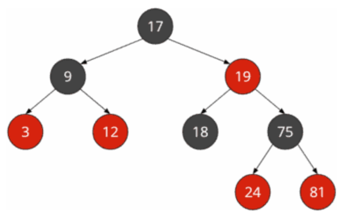
- **Balanced search trees** are a type of search trees that maintain specific structural invariants to ensure their height h remains $O(\log n)$ for n nodes
  - ✓ among the most important data structures in computer science
  - ✓ widely implemented in standard libraries:
    - Java: `TreeSet` and `TreeMap`,
    - C++: `std::set` and `std::map`
    - Python: no built-in implementation, but available through third-party libraries
- Examples of balanced trees:
  - ✓ AVL trees, **Red-Black trees**, B-trees, Treaps, etc.

Red-black trees

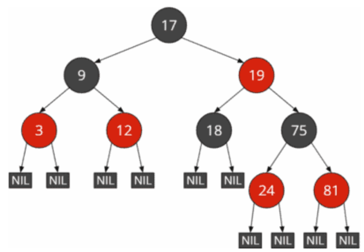
- Red-black trees are BSTs that maintain near-perfect balance by enforcing the following properties on the nodes:
  - ✓ every node is colored **red** or **black**
  - ✓ the root is always **black**
  - ✓ **red** nodes cannot have **red** children (no two consecutive red nodes)
  - ✓ *null* nodes are considered **black**
  - ✓ every *root-to-null* path must have the same number of **black** nodes



Examples



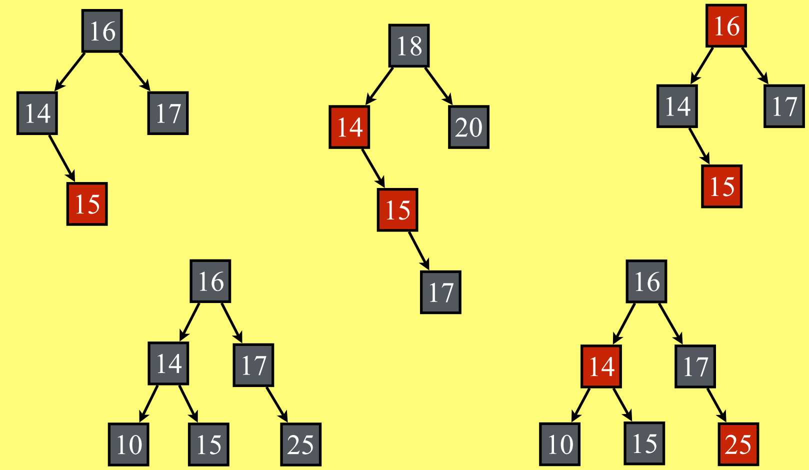
Red-black tree with implicit NIL leaves



Red-black tree with explicit NIL leaves

Practice

- Are these valid red-black trees? — (null nodes not shown)



Height of red-black trees

- A red-black tree with n nodes has height $h = O(\log n)$
  - ✓ after an insertion or deletion, the tree may temporarily violate one or more red-black properties
  - ✓ these violations are efficiently corrected through:
    - rotations (left or right) and recoloring of nodes
- Equivalence to **2-3-4 Trees**
  - ✓ red-black trees are conceptually equivalent to 2-3-4 trees (B-trees of order 4)
  - ✓ this correspondence provides intuition for:
    - how rebalancing maintains logarithmic height
    - why red-black tree operations run in $O(\log n)$ time

2-3-4 Trees (interlude)

Multi-way search trees

- A multi-way search tree is a generalization of a BST in which:
  - ✓ each node can store multiple keys (instead of just one)
  - ✓ each node can have more than two children
- Properties
  - ✓ the keys within each node are stored in **sorted** (increasing) order
  - ✓ for a node containing keys $[k_1, k_2, \ldots, k_m]$ and child subtrees $[T_0, T_1, \ldots, T_m]$:
    - all keys in T_0 are less than k_1
    - all keys in T_i (for $0 < i < m$) are between k_i and k_{i+1}
    - all keys in T_m are greater than k_m

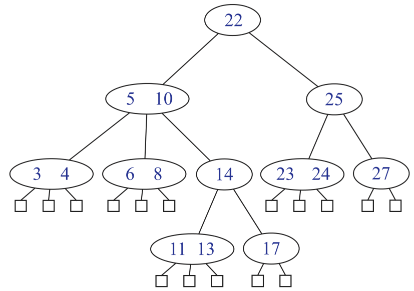


Image credit: Data Structures and Algorithms in C++ 2e

Example of a multi-way search tree

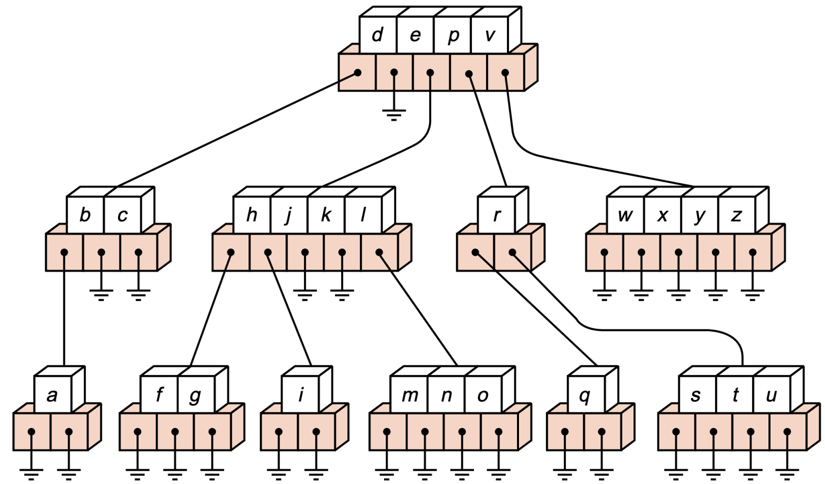
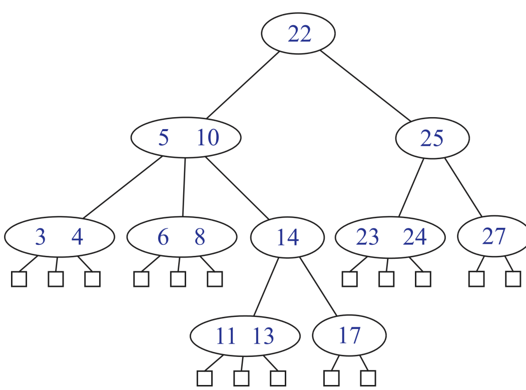


Image credit: Data Structures and Program Design In C++, Kruse and Ryba

Search on a multi-way search tree

- Perform **search** for 12, 17, 24, and 50 on the following tree



Assuming d denotes the maximum number of keys of any node of T, and h denotes the height of T. What is the cost of search?

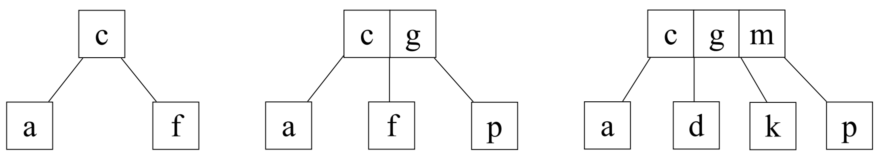
Image credit: Data Structures and Algorithms in C++ 2e

Balanced multi-way search trees

- A **balanced multi-way search tree** is a multi-way search tree that:
  - ✓ limits each node to a fixed maximum number of children
  - ✓ keeps all leaf nodes at the same depth, ensuring the tree remains perfectly balanced
- A **B-tree** is a specific type of balanced multi-way search tree
  - ✓ in a B-tree of order m (maximum number of children), every node, except the root, must have between $\lceil m/2 \rceil$ and m children
    - the term "order" can vary slightly between sources — some define it as the maximum number of keys, others as the maximum number of children
  - ✓ B-trees are heavily used in databases, filesystems, and storage systems because they minimize disk I/O by storing many keys per node (typical orders: 1024, 2048, 4096, …)

2-3-4 tree

- A 2–3–4 tree (also called a 2–4 tree) is a B-tree of order 4
  - each node can have 2, 3, or 4 children
  - all nodes (except the root, which can be empty) must have at least 1 key and at most 3 keys



2 node 3 node 4 node

Insertion (2-3-4 tree)

- Steps
  - **start at the root** and traverse downward to find the correct **<u>leaf</u>** for insertion
  - if the leaf has fewer than 3 keys
    - insert the new key into the node in sorted order
  - if the leaf already has 3 keys
    - temporarily insert the new key (so it contains 4 keys)
    - split the node into two nodes by promoting the middle key to the parent node and forming two new child nodes with the remaining keys
    - if the parent now has more than 3 keys, repeat this splitting process upward until the root if necessary
- Tree remains balanced after each insertion
  - all leaf nodes are at the same level

Practice

- Insert the following sequence into a 2-3-4 tree
  - 15, 10, 25, 5, 1, 30, 45, 60, 100, 70, 80, 40, 35, 90

https://www.cs.usfca.edu/~galles/visualization/BTree.html

Practice

- What is the max h of a 2-3-4 tree with n nodes?
  - to maximize the height, we want to minimize the number of keys per node (**instance of a worst-case**)
  - draw an example tree and express h in terms of n
- What is the cost of search and insert on a 2-3-4 tree?
  - worst-case scenario

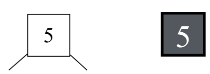
Analysis

- The cost of operations in a B-tree of order b is $O(b \log_b n)$
  - ✓ insert, search, remove
  - ✓ small values of b make this cost optimal
- In practice …
  - ✓ B-trees are <u>widely used in databases and file systems</u> to manage large amounts of data efficiently
  - ✓ useful for systems that read and write large blocks of data
    - B-trees can minimize the number of disk accesses required (much larger order values)

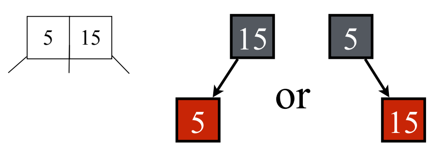
2-3-4 trees and red-black trees

Red-black trees <=> 2-3-4 trees

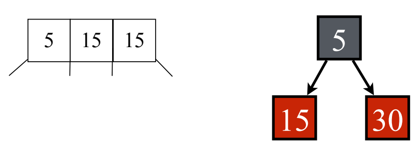
- A 2-node in a 2-3-4 tree corresponds to a **black** node in a red-black tree



- A 3-node corresponds to a **black** node with one **red** child

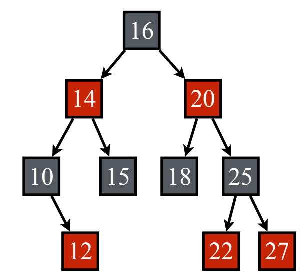


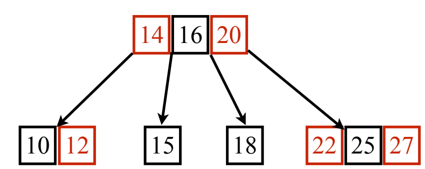
- A 4-node corresponds to a **black** node with two **red** children



Red-black trees <=> 2-3-4 trees

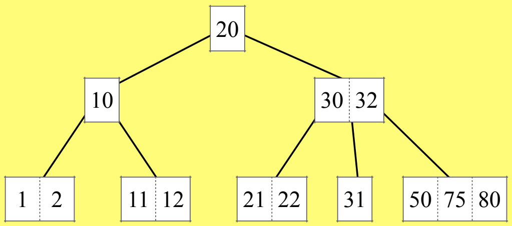
- Red-black trees are <u>**isomorphic**</u> to 2-3-4 trees
  - ✓ the <u>number of **black** nodes</u> on any *root-to-null* path corresponds to the <u>number of levels</u> of the 2-3-4 tree
  - ✓ every red-black tree can be transformed into an equivalent 2-3-4 tree and vice versa
  - ✓ the relationship is not bijective
    - a 3-node in a 2-3-4 tree can be represented in two ways in a red-black tree (leaning left or right)
    - each red-black tree corresponds to exactly one 2-3-4 tree (but not vice versa)





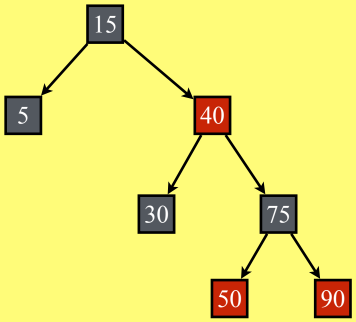
Practice

- Draw the red-black tree that corresponds to the following 2-3-4 tree



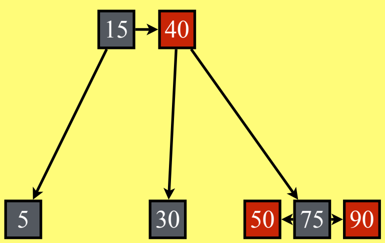
Practice

- Draw the 2-3-4 tree that corresponds to the following red-black tree



Practice

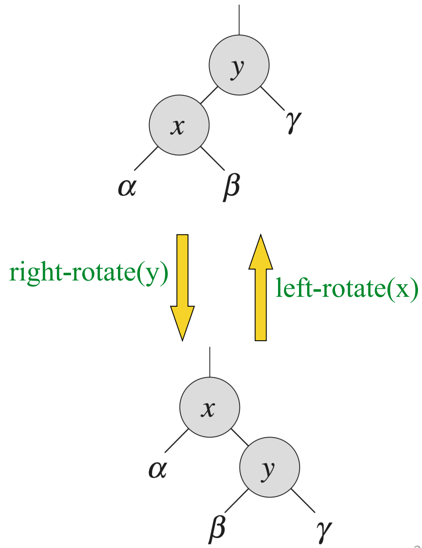
- Draw the 2-3-4 tree that corresponds to the following red-black tree



Rotations

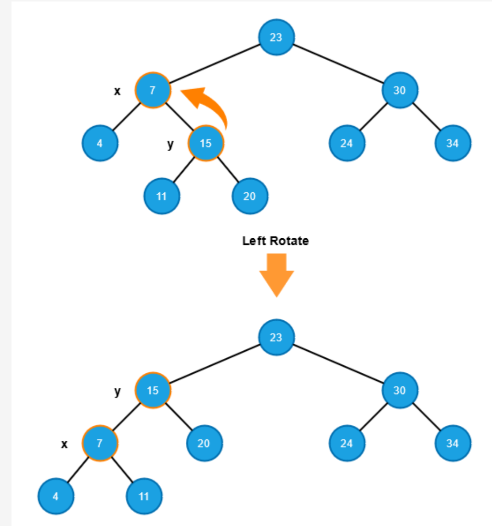
BST Rotations

- A **rotation** is a O(1)-time local operation that preserves the BST order property while changing the tree's structure
- **Right rotation** at node y
  - requires y's left child x to be *non-null*
  - elevates x to become the subtree root
  - y becomes x's right child
  - x's original right child becomes y's left child
- **Left rotation** at node x
  - requires x's right child y to be *non-null*
  - elevates y to become the subtree root
  - x becomes y's left child
  - y's original left child becomes x's right child



Example: left rotation

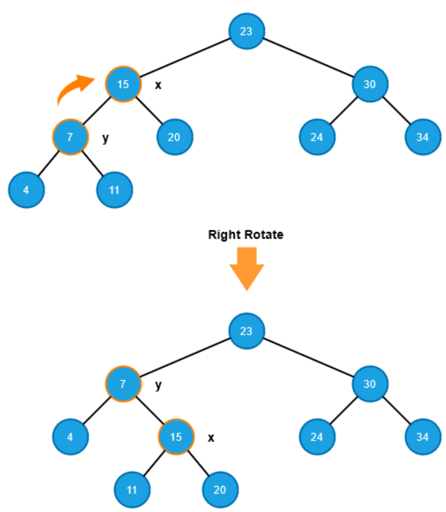
left-rotate(x)



https://www.formosa1544.com/2021/04/30/build-the-forest-in-python-series-red-black-tree/

Example: right rotation

right-rotate(x)



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Practice

- Perform the following operations in sequence
  - rotate-left(70)
  - rotate-left(50)
  - rotate-left(30)
  - rotate-right(50)