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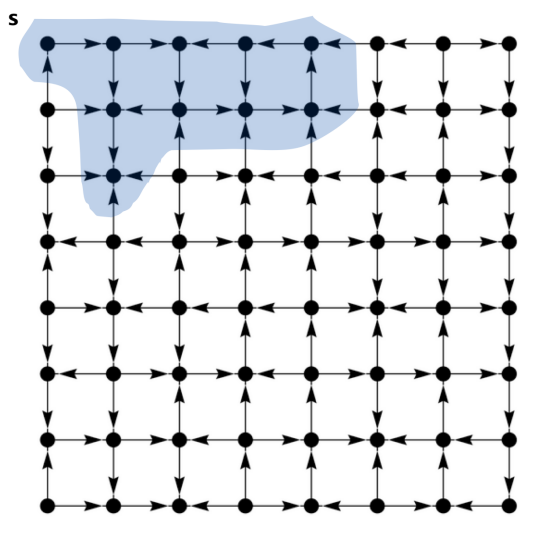
4. Graphs and Digraphs I

- introduction
- graph representation
- depth-first search
- path finding
- undirected graphs

Reachability problem in a digraph

Reachability problem. Given a digraph G and vertex s, find all vertices reachable from s.



Reachability problem in a digraph

Reachability problem. Given a digraph G and vertex s, find all vertices reachable from s.

Depth-first search. A systematic method to explore all vertices reachable from s.

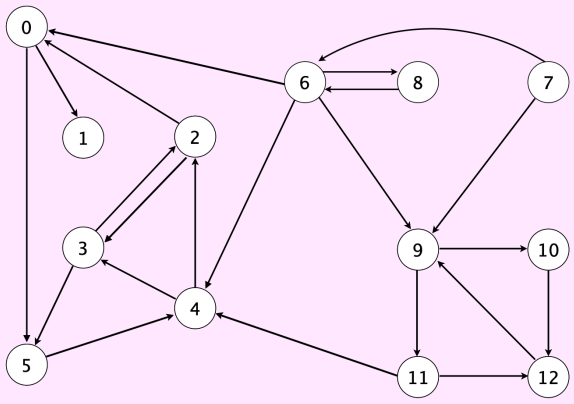
DFS (to visit a vertex v)

Mark vertex v.
Recursively visit all unmarked vertices w adjacent from v.

Depth-first search (in a digraph) demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent from v.



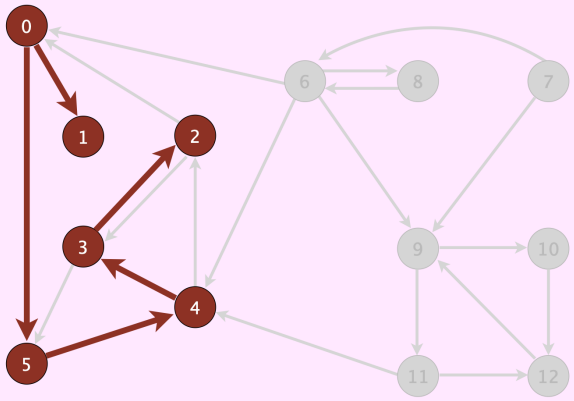
a directed graph

4→2
2→3
3→2
6→0
0→1
2→0
11→12
12→9
9→10
9→11
8→9
10→12
11→4
4→3
3→5
6→8
8→6
5→4
0→5
6→4
6→9
7→6

Depth-first search (in a digraph) demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent from v.



reachable from 0

v	marked[]
0	T
1	T
2	T
3	T
4	T
5	T
6	F
7	F
8	F
9	F
10	F
11	F
12	F

reachable from vertex 0

Graphs and digraphs I: poll 3

Run DFS using the given adjacency-lists representation of digraph G, starting at vertex 0. In which order is `dfs(digraph, v)` called?

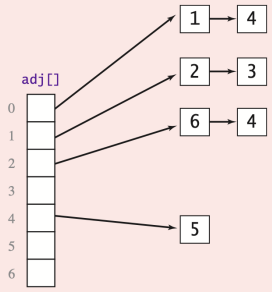
DFS preorder

A. 0 1 2 4 5 3 6

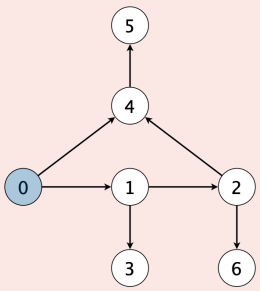
B. 0 1 2 4 5 6 3

C. 0 1 3 2 6 4 5

D. 0 1 2 6 4 5 3



adjacency-lists representation



digraph G

Depth-first search: analysis

Proposition. DFS uses $\Theta(V)$ extra space (not including the digraph itself).

Pf.

- The `marked[]` array uses $\Theta(V)$ space.
- The function-call stack uses $O(V)$ space. ← *`dfs()` called at most once per vertex*

Proposition. DFS marks all vertices reachable from s in $\Theta(E + V)$ time in the worst case.

Pf.

- Initializing the `marked[]` array takes $\Theta(V)$ time.
- Each vertex is visited at most once.
- Visiting a vertex takes time proportional to its outdegree:

$$outdegree(v_0) + outdegree(v_1) + outdegree(v_2) + \ldots = E$$

↑ *in the worst case, all vertices are reachable from s*

Note. If all vertices are reachable from s, then $E \geq V - 1$ and running time simplifies to $\Theta(E)$.

Graphs and digraphs I: poll 4

What could happen if we marked a vertex at the end of the DFS call (instead of beginning)?

A. Marks a vertex not reachable from s.

B. Compile-time error.

C. Infinite loop / stack overflow.

D. None of the above.

```
private void dfs(Digraph digraph, int v) {
 marked[v] = true;
  for (int w : digraph.adj(v))
     if (!marked[w])
        dfs(digraph, w);
 marked[v] = true;
}
```

Reachability application: program control-flow analysis

Every program is a digraph.

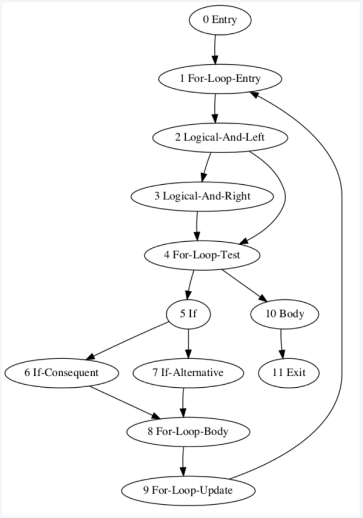
- Vertex = basic block of instructions (straight-line program).
- Edge = jump.

Dead-code elimination.

Find (and remove) unreachable code.

Infinite-loop detection.

Determine whether exit is unreachable.



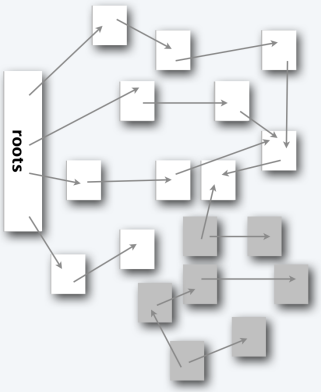
Reachability application: mark-sweep garbage collector

Every data structure is a digraph.

- Vertex = object.
- Edge = reference/pointer.

Roots. Objects known to be directly accessible by program (e.g., stack frame).

Reachable objects. Objects indirectly accessible by program (starting at a root and following a chain of pointers).

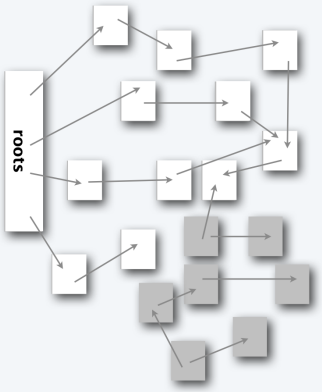


Reachability application: mark-sweep garbage collector

Mark-sweep algorithm. [McCarthy, 1960]

- Mark: mark all reachable objects.
- Sweep: if object is unmarked, it is garbage (so add to free list).

Memory cost. Uses one extra mark bit per object (plus DFS function-call stack).



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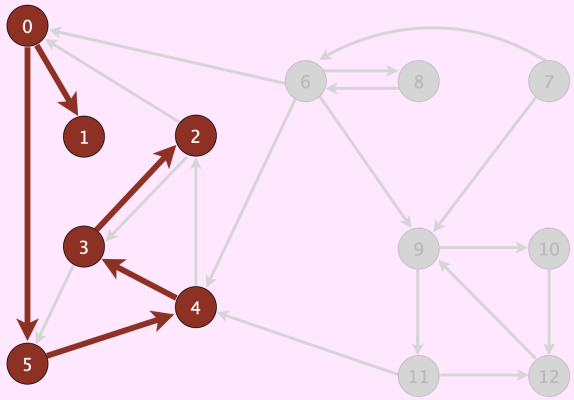
4. Graphs and Digraphs I

- *introduction*
- *graph representation*
- *depth-first search*
- *path finding*
- *undirected graphs*

Directed paths DFS demo

Goal. DFS determines which vertices are reachable from s. How to reconstruct paths?

Solution. Use parent-link representation.



vertices reachable from 0 (and directed paths)

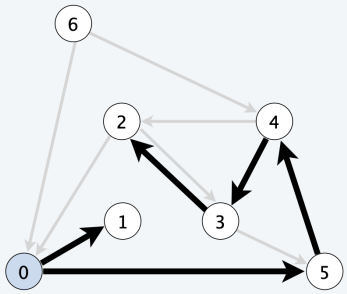
v	marked[]	edgeTo[]
0	T	-
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	-
7	F	-
8	F	-
9	F	-
10	F	-
11	F	-
12	F	-

parent-link representation of paths from vertex 0

Depth-first search: path finding

Parent-link representation of paths from vertex s.

- Maintain an integer array `edgeTo[]`.
- Interpretation: `edgeTo[v]` is the next-to-last vertex on a directed path from s to v.
- To reconstruct path from s to v, trace `edgeTo[]` backward from v to s (and reverse).



v	marked[]	edgeTo[]
0	T	-
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	-

```
public Iterable<Integer> pathTo(int v) {
   if (!marked[v]) return null;
   Stack<Integer> path = new Stack<>();
   for (int x = v; x != s; x = edgeTo[x])
      path.push(x);
   path.push(s);
   return path;
}
```

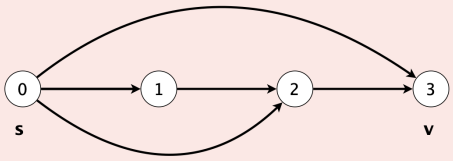
Graphs and digraphs I: poll 5

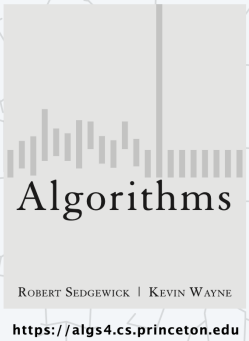
Suppose there are many paths from s to v. Which one does `DepthFirstDirectedPaths` find?

A. A shortest path (fewest edges).

B. A longest path (most edges).

C. Depends on digraph representation.





4. Graphs and Digraphs I

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Flood fill

Problem. Implement flood fill (Photoshop magic wand).

Depth-first search in undirected graphs

Connectivity problem. Given an undirected graph G and vertex s, find all vertices connected to s.

Solution. Use DFS. ← *but now, for each undirected edge v–w: v is adjacent with w (and w is adjacent with v)*

DFS (to visit a vertex v)

Mark vertex v.

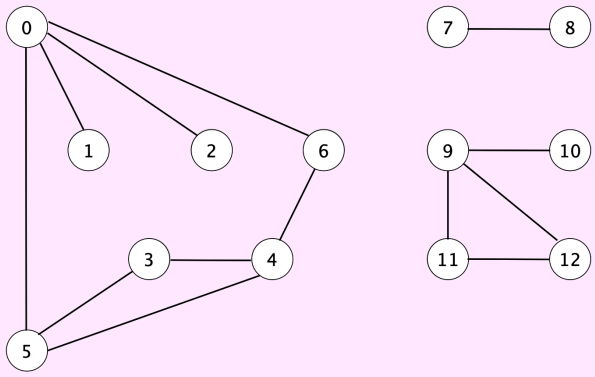
Recursively visit all unmarked vertices w adjacent with v.

Proposition. DFS marks all vertices connected to s in $\Theta(E + V)$ time in the worst case.

Depth-first search (in an undirected graph) demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent with v.



graph G

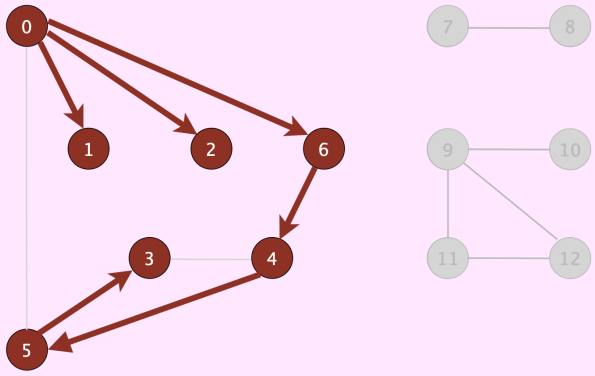
tinyG.txt

V → 13
E ← 13
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3

Depth-first search (in an undirected graph) demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent with v.



vertices connected to 0
(and associated paths)

v	marked[]	edgeTo[]
0	T	-
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	-
8	F	-
9	F	-
10	F	-
11	F	-
12	F	-

Graphs and digraphs I: poll 6

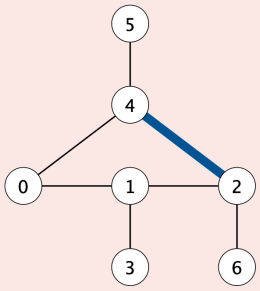
How to represent an undirected edge v–w using adjacency lists?

A. Add w to adjacency list for v.

B. Add v to adjacency list for w.

C. Both A and B.

D. None of the above.



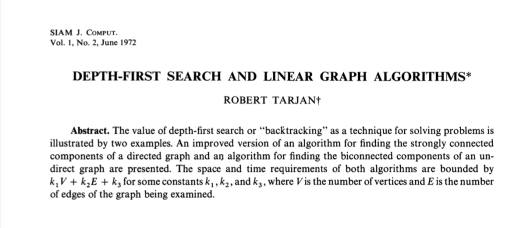
Depth-first search summary

DFS enables direct solution of several elementary graph and digraph problems.

- Reachability (in a digraph). ✓
- Connectivity (in a graph). ✓
- Path finding (in a graph or digraph). ✓
- Topological sort. ← *next lecture*
- Directed cycle detection. ← *precept*

DFS is also core of solution to more advanced problems.

- Euler cycle.
- Biconnectivity.
- 2-satisfiability.
- Planarity testing.
- Strong components.
- Nonbipartite matching.
- ...



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DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

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