Exercise 1.3

Problem

Consider the following factor 2 approximation algorithm for the cardinality vertex cover problem. Find a depth first search tree in the given graph, $G = (V, E)$, and output the set, say S, of all the nonleaf vertices of this tree. Show that S is indeed a vertex cover for G and $|S| \leq 2|\text{OPT}|$.

Vertex Cover. A vertex cover of a graph $G = (V, E)$ is a set of vertices S such that each edge $e \in E$ is incident to at least one vertex $v \in S$.

Cardinality Vertex Cover Problem. Given a graph $G = (V, E)$, compute the minimum cardinality vertex cover.

Solution

Lemma. *In the depth first search tree of $G = (V, E)$, there are no edges between leaf nodes.*

Proof. Let L be the set of leaf vertices in the depth first search tree. For the sake of contradiction, suppose not, suppose there exist some edge $\{u, v\} \in E$ such that $u, v \in L$. Either u or v was explored first in the depth first search tree. Without loss of generality, assume u was explored first. Then v must be a child of u in the depth first search tree since it is unexplored and reachable from u. Thus u is not a leaf, which is a contradiction. There must be no edges between leaf nodes in the depth first search tree. □

Lemma. *The set of nonleaf vertices S in the depth first search tree of $G = (V, E)$ is a vertex cover of G.*

Proof. Any edge $e \in E$ with a nonleaf endpoint is covered by S. Then, the only possible uncovered edges have two leaf endpoints. However, by the lemma, there are no such edges. Hence S is a vertex cover. □

Theorem. *The set of nonleaf vertices S in the depth first search tree of $G = (V, E)$ is a factor 2 approximation for the minimum cardinality vertex cover of G.*

Proof. We will show that there is maximal matching of size $|S|$. Construct a matching M as follows. Starting with the root of the tree r, add (r, u) to M where u is the first child of r. Then recurse on the children of u and the children of r except u. Only recurse on nodes in S. This matching is incident to every vertex in S. Moreover, by the lemma, this matching is maximal, since there are no edges between vertices outside of S and these are the only unmatched vertices.

By the approximate min-max relation between minimum maximal matching and cardinality vertex cover, $|S|$ is a 2 approximation. □

Insights