Exercise 1.3

Problem

Consider the following factor 2 approximation algorithm for the cardinality vertex cover problem. Find a depth first search tree in the given graph, $G = (V, E)$, and output the set, say S, of all the non-leaf vertices of this tree. Show that S is indeed a vertex cover for G and $|S| \leq 2|\text{OPT}|$.

Vertex Cover. A vertex cover of a graph $G = (V, E)$ is a set of vertices S such that each edge $e \in E$ is incident to at least one vertex $v \in S$.

Cardinality Vertex Cover Problem. Given a graph $G = (V, E)$, compute the minimum cardinality vertex cover.

Solution

Define $v_1, v_2, ..., v_n \in V$ to be the depth first search order of the vertices. That is, if $i < j$, then v_i occurs before v_j in the depth first search tree.

Set $M \leftarrow \emptyset$ and $i \leftarrow 1$. While $i < n$

1. If v_i is not a leaf, set $M \leftarrow M \cup \{(v_i, v_{i+1})\}$ and $i \leftarrow i + 2$.
2. Otherwise, set $i \leftarrow i + 1$.

repeat the above.

Lemma. *In the depth first search tree of $G = (V, E)$, there are no edges between leaf nodes.*

Proof. Let L be the set of leaf vertices in the depth first search tree. For the sake of contradiction, suppose not, suppose there exist some edge $\{u, v\} \in E$ such that $u, v \in L$. Either u or v was explored first in the depth first search tree. Without loss of generality, assume u was explored first. Then v must be a child of u in the depth first search tree since, when u was encounted, v was unexplored and reachable from u. Thus u is not a leaf, which is a contradiction. There must be no edges between leaf nodes in the depth first search tree. □

Lemma. *In the depth first search tree of $G = (V, E)$, the set of non-leaf vertices S is a vertex cover of G.*

Proof. Fix some edge $(u, v) \in E$. By the lemma, u or v is a non-leaf vertex so $u \in S$ or $v \in S$. Hence (u, v) is covered so S is a vertex cover. □

Theorem. *The set of edges M is a maximal matching.*

Proof. Since each vertex is added at most once and the edge $\{v_i, v_{i+1}\}$ exists in the depth-first tree for non-leaf v_i, M is a matching. The only unmatched vertices are leaves, and by the lemma, there are no edges between them. Hence M is a maximal matching. □

Theorem. *The set of non-leaf vertices S in the depth first search tree of $G = (V, E)$ is a factor 2 approximation for the minimum cardinality vertex cover of G.*

Proof. By the theorem, there is a maximal matching of size at least $\frac{|S|}{2}$ edges. Since each vertex is incident to exactly one edge in M, taking one vertex from each edge in M yields a vertex cover of size at least $\frac{|S|}{2}$. Hence $\frac{|S|}{2} \leq \mathrm{OPT}$ so $|S| \leq 2\mathrm{OPT}$ as desired. □

Insights

There are no edges between the leaves in a depth-first search tree.

Once a lower bounding scheme is found, it can be used to prove bounds for new algorithms. In this example, maximal matching provided a known bound on cardinality vertex cover, allowing the approximation factor to be proven by constructing a maximal matching from the algorithm's output.