Exercise 1.6

Problem

Give a lower bounding scheme for the arbitrary cost version of the vertex cover problem.

Vertex Cover Problem. Given an undirected graph $G = (V, E)$ and a cost function $c : V \to \mathbb{Q}^+$ find the minimum cost vertex cover, where the cost of a cover U is $c(U) = \sum_{u \in U} c(u)$.

Solution

The vertex cover problem is equivalent to the following

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} x_v c(v) & \\ \text{subject to} \quad & x_u + x_v \geq 1 & \forall (u, v) \in E \\ & x_v \in \{0, 1\} & \forall v \in V \end{aligned}$$

integer linear program. The variable x_v takes the value 1 if and only if v is in the computed vertex cover.

The ILP has the following

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} x_v c(v) & \\ \text{subject to} \quad & x_u + x_v \geq 1 & \forall (u, v) \in E \\ & x_v \geq 0 & \forall v \in V \end{aligned}$$

linear programming relaxation. Define

$$U = \{v : x_v \geq \frac{1}{2} \text{ and } v \in V\}$$

to be the solution to vertex cover based on the LP-relaxation.

Theorem. *U is a vertex cover.*

Proof. Fix some arbitrary edge $(u, v) \in E$. Since $x_u + x_v \geq 1$ and $x_u, x_v \geq 0$, $x_u \geq \frac{1}{2}$ or $x_v \geq \frac{1}{2}$. Then $u \in V$ or $v \in V$ so (u, v) is covered. Hence U is a vertex cover. □

Theorem. *U is a factor 2 approximation for the LP-relaxation.*

Proof. We have that

$$c(U) = \sum_{u \in U} c(u) \leq 2 \sum_{u \in U} c(u) x_u \leq 2 \sum_{v \in V} c(v) x_v = 2\mathrm{OPT}_{LP}$$

as desired. □

Theorem. *U is a factor 2 approximation for vertex cover.*

Proof. We have that

$$c(U) \leq 2\mathrm{OPT}_{LP} \leq 2\mathrm{OPT}$$

as desired. □

Insights

To obtain a lower bounding scheme for a problem, formulate the problem as an integer linear program and compute the linear programming relaxation. From the solution to the LP-relaxation, re-construct a solution to the original problem via rounding. Consider the approximation factor, say α, the rounded solution obtains with respect to the LP-relaxation. Since $\alpha\mathrm{OPT}_{LP} \leq \alpha\mathrm{OPT}$, analyzing an algorithm with respect to the LP-relaxation can prove approximation factor no better than α.

Taking the rounded LP-relaxation as the approximation algorithm yields a polynomial time approximation algorithm with factor α. However, non-LP algorithms might be faster in practice (e.g. lower, but still polynomial, time complexity).