

Exercise 1.1

Problem

Give a factor $\frac{1}{2}$ algorithm for the acyclic subgraph problem.

Acyclic Subgraph Problem. Given a directed graph $G = (V, E)$, pick a maximum cardinality set of edges from E so that the resulting subgraph is acyclic.

Solution

Arbitrarily label the vertices v_1, v_2, \dots, v_n . Define

$$F = \{(v_i, v_j) : i < j \text{ and } (v_i, v_j) \in E\}$$

and

$$B = \{(v_i, v_j) : i > j \text{ and } (v_i, v_j) \in E\}$$

so that F is the set of forwards edges and B is the set of backwards edges. Output the larger of the two sets, that is, output F if $|F| \geq |B|$ otherwise output B .

Lemma. *Let $G[F]$ be the subgraph induced by F and $G[B]$ be the subgraph induced by B . Then $G[F]$ and $G[B]$ are acyclic subgraphs of G .*

Proof. For the sake of contradiction, suppose $G[F]$ is cyclic. Since $G[F]$ is cyclic, there exists a sequence of vertices

$$v_{i_1}, v_{i_2}, v_{i_3}, \dots, v_{i_m}$$

such that the edges

$$(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots, (v_{i_{m-1}}, v_{i_m}), (v_{i_m}, v_{i_1})$$

are in F . Therefore, by definition of F ,

$$i_1 < i_2 < i_3 < \dots < i_m < i_1$$

which is a contradiction. Hence $G[F]$ is an acyclic subgraph of G . The proof for $G[B]$ is similar. \square

Theorem. *The largest set between F and B is a $\frac{1}{2}$ factor approximation.*

Proof. Without loss of generality, assume F is larger than B . Since F and B are a partition of E , $|F| \geq \frac{|E|}{2}$. Also $|E| \geq |\text{OPT}|$ so $\frac{|E|}{2} \geq \frac{|\text{OPT}|}{2}$. Hence

$$|F| \geq \frac{|E|}{2} \geq \frac{|\text{OPT}|}{2}$$

as desired. \square

Insights

To design a factor $\frac{1}{2}$ algorithm, partition the problem into 2 viable solutions and take the better of the two. One could also try finding any small set of viable solutions such that there must be at least one solution of the desired weight. For example, if the average of the viable solutions is at least $\frac{OPT}{3}$, then taking the best viable solution gives a factor $\frac{1}{3}$ algorithm.