Exercise 1.2

Problem

Design a factor 2 approximation for the problem of finding a minimum cardinality maximal matching in an undirected graph $G = (V, E)$.

Matching. A matching $M \subseteq E$ in a graph $G = (V, E)$ is a set of edges such that each $v \in V$ is incident to at most one edge $e \in M$.

Maximal Matching. A maximal matching is a set of edges $M \subseteq E$ such that

1. M is a matching.
2. Every $M' \supset M$ is not a matching.

Maximum Matching. A maximum matching $M^* \subseteq E$ is the largest possible maximal matching.

Solution

Start with an empty matching M. While M is not maximal, add some edge to M such that M is still a matching. Output the maximal matching M.

Lemma. *Every maximal matching has at least* $\frac{|M^*|}{2}$ *edges.*

Proof. We claim that one vertex from each edge in M^* must be matched in any maximal matching M. For sake of contradiction, suppose not. Then there exists some edge $(u, v) \in M^*$ such that no edges in M are incident to u or v. Then $M \cup \{(u, v)\}$ is a matching so M is not maximal. However, M is maximal so this is a contradiction. Thus one vertex from each edge in M^* must be in M so there must be at least $\frac{|M*|}{2}$ edges in M. □

Theorem. *Any maximal matching is a factor 2 approximation.*

Proof. By the lemma, $\frac{|M^*|}{2} \leq |\text{OPT}|$ so $|M^*| \leq 2|\text{OPT}|$. Since M^* is the largest possible maximal matching, $|M| \leq |M^*|$. Hence

$$|M| \leq |M^*| \leq 2|\text{OPT}|$$

as desired. □

Insights

Sometimes, all of the reasonable solutions are good approximations.