CSC 544 Homework 2

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As discussed in class, I assume that function juxtaposition is right-associative. I referred to `https://j-hui.com/pages/normal-forms/` for the definitions of beta normal form and weak normal form.

Problem 1

Find a beta normal form for $(\lambda(x, y).(x, y, z))(4, 3)$.

Solution

We have that

$$\begin{aligned}(\lambda(x, y).(x, y, z))(4, 3) &= ((\lambda x.\lambda y.(x, y, z))4)3 \\ &= (\lambda y.(4, y, z))3 \\ &= (4, 3, z)\end{aligned}$$

Problem 2

Find a beta normal form for $(\lambda(x, y).y)(5, 3)$.

Solution

We have that

$$\begin{aligned}(\lambda(x, y).y)(5, 3) &= ((\lambda x.\lambda y.y)5)3 \\ &= (\lambda y.y)3 \\ &= 3\end{aligned}$$

Problem 3

Find a beta normal form for $(\lambda x.x)(\lambda x.x)$

Solution

We have that

$$(\lambda x.x)(\lambda x.x) = \lambda x.x$$

Problem 4

Find a beta normal form for $(\lambda x.(xx))(\lambda x.(xx))$.

Solution

There is no beta normal form. After the k-th β-reduction, it is $(\lambda x.(xx))(\lambda x.(xx))$. Likewise, there is no weak normal form.

Problem 5

Let $::$ be the concatenation operator. Find a beta normal form for $(\lambda(x :: q).q)(f :: u :: n :: [])$.

Solution

We have that

$$(\lambda(x :: q).q)(f :: u :: n :: []) = u :: n :: []$$

Problem 6

Let $c\ ?\ t : f$ be the ternary operator. Find a beta normal form for $(\lambda xyz.x > 0\ ?\ y : z)3(\lambda q.q - 1)(\lambda p.p + 1)1$.

Solution

We have that

$$\begin{aligned}
(\lambda xyz.x > 0\ ?\ y : z)3(\lambda q.q - 1)(\lambda p.p + 1)1 &= ((\lambda x.\lambda y.\lambda z.x > 0\ ?\ y : z)3)((\lambda q.q - 1)((\lambda p.p + 1)1)) \\
&= ((\lambda x.\lambda y.\lambda z.x > 0\ ?\ y : z)3)((\lambda q.q - 1)2) \\
&= ((\lambda x.\lambda y.\lambda z.x > 0\ ?\ y : z)3)1 \\
&= (\lambda y.\lambda z.3 > 0\ ?\ y : z)1 \\
&= (\lambda y.\lambda z.y)1 \\
&= (\lambda y.y)1 \\
&= 1
\end{aligned}$$

Problem 7

Find a beta normal form for Ya where $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.

Solution

We have that

$$\begin{aligned} Ya &= (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))a \\ &= (\lambda x.a(xx))(\lambda x.a(xx)) \\ &= a((\lambda x.a(xx))(\lambda x.a(xx))) \\ &= a((\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))a) \\ &= aYa \end{aligned}$$

There is no beta normal form. After the k-th β-reduction, it is a^kYa. However, aYa is a weak normal form.

Problem 8

Find a beta normal form for $Y(\lambda f.(\lambda x.(x = 1)\ ?\ 1 : \text{mult}(x, f(x-1))))$ where $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.

Solution

Let $F = \lambda f.(\lambda x.(x = 1)\ ?\ 1 : \text{mult}(x, f(x-1)))$. By Problem 7, we have that

$$\begin{aligned} YF &= FYF \\ &= (\lambda f.(\lambda x.(x = 1)\ ?\ 1 : \text{mult}(x, f(x-1))))(YF) \\ &= \lambda x.(x = 1)\ ?\ 1 : \text{mult}(x, YF(x-1)) \end{aligned}$$

Like Problem 7, there is no beta normal form. However, $\lambda x.(x = 1)\ ?\ 1 : \text{mult}(x, YF(x-1))$ is a weak normal form.