1 Вариант №1

№1

$$\int_0^{+\infty} \frac{\arcsin\left(\frac{1}{x+1}\right)}{1+x\sqrt{x}}\,dx$$

Пусть $F(x) = \frac{1}{1+x\sqrt{x}}$, $g(x) = \arcsin\left(\frac{1}{x+1}\right)$

По признаку Дирихле если $F(x)$ непрерывна на $[0;+\infty)$ и имеет ограниченную первообразную, а $g(x)$ непрерывно дифференцируемая на $[0;+\infty)$, монотонна и $\lim\limits_{x\to\infty} g(x) = 0$, то значит $\int_0^{+\infty} F(x)g(x)\,dx$ сходится

1) $F(x)$ непрерывна на $[0;+\infty)$ и имеет ограниченную первообразную

а) на $[0;+\infty)$ $\frac{1}{1+x\sqrt{x}}$ непрерывна

б) $\int_0^{+\infty} \frac{dx}{1+x\sqrt{x}} = \int_0^1 \frac{dx}{1+x\sqrt{x}} + \int_1^{+\infty} \frac{dx}{1+x\sqrt{x}}$

1) $\int_0^1 \frac{dx}{1+x\sqrt{x}}$; $\forall x \in (0;1]$ $\frac{1}{1+x\sqrt{x}} < \frac{1}{\sqrt{x}}$,

а $\int_0^1 \frac{dx}{\sqrt{x}}$ сходится $\Rightarrow$ из сходимости большего интеграла следует сходимость меньшего $\Rightarrow \int_0^1 \frac{dx}{1+x\sqrt{x}}$ сходится

2) $\int_1^{+\infty} \frac{dx}{1+x\sqrt{x}}$, $\forall x \in (1;+\infty)$ $\frac{1}{1+x\sqrt{x}} < \frac{1}{x\sqrt{x}}$

а $\int_1^{+\infty} \frac{dx}{x\sqrt{x}}$ сходится $\Rightarrow$ из сходимости большего интеграла следует сходимость меньшего $\Rightarrow \int_1^{+\infty} \frac{dx}{1+x\sqrt{x}}$ сходится

$\Rightarrow \int_0^{+\infty} \frac{dx}{1+x\sqrt{x}}$ сходится $\Rightarrow$ имеет ограниченную первообразиро.

II) $g(x)$ непрерывно дифференцируема на $[0;+\infty)$, монотонна и $\lim\limits_{x\to\infty} g(x)=0$

а) $\arcsin\left(\frac{1}{x+1}\right)$ непрерывно дифференцируема на $[0;+\infty)$

б) $\arcsin\left(\frac{1}{x+1}\right)$ монотонна на $[0;+\infty)$

в) $\lim\limits_{x\to\infty} g(x)=\lim\limits_{x\to\infty}\arcsin\frac{1}{x+1}=\cancel{\lim\limits_{x\to\infty}\arcsin\frac{1}{x+1}}=$

$=\lim\limits_{x\to\infty}\frac{1}{x+1}=0$

$$\Longrightarrow \int_0^{+\infty} F(x)g(x)\,dx=\int_0^{+\infty}\frac{\arcsin\left(\frac{1}{x+1}\right)}{1+x\sqrt{x}}\,dx \text{ сходится}$$

2 Вариант №1

Вариант №2

№1

$$\int_0^1 \frac{\sqrt[3]{x^4}}{e^{\frac{x^2}{2}} - \cos x}dx = \int_0^1 \frac{\sqrt[3]{x^4}}{e^{\frac{x^2}{2}} - 1 + \frac{x^2}{2} + o(x^2)} = \int_0^1 \frac{\sqrt[3]{x^4}}{x^2 + o(x^2)} = \int_0^1 x^{-\frac{2}{3}} = 3x^{\frac{1}{3}}\Big|_0^1 =$$

При $x \to 0$ можно сказать, что

$$\begin{cases} e^{\frac{x^2}{2}} - 1 \sim \frac{x^2}{2} \\ \cos x = 1 - \frac{x^2}{2} + o(x^2) \end{cases}$$

$= 3 \Rightarrow$ сходится

3 Вариант №1

№1

$$\int_1^{+\infty} \sqrt[3]{x^5}\left(\sin\frac{1}{x}-\operatorname{arctg}\frac{1}{x}\right)^2 dx$$

Пусть $a=1; b=+\infty$. Если на полуинтервале $[a;b)$ функции $f(x)$ и $g(x)$ неотрицательны и непрерывны, то если $\lim\limits_{x\to b-0}\frac{f(x)}{g(x)}=K<\infty$ и $K\neq 0$, то оба интеграла $\int_a^b f(x)dx$ и $\int_a^b g(x)dx$ одновременно сходятся, или одновременно расходятся

Пусть $f(x)=\sqrt[3]{x^5}\left(\sin\frac{1}{x}-\operatorname{arctg}\frac{1}{x}\right)^2$, а $g(x)=\frac{1}{\sqrt[3]{x}}$

Преобразуем $f(x)$: $\sqrt[3]{x^5}\left(\sin\frac{1}{x}-\operatorname{arctg}\frac{1}{x}\right)^2=$

$$=\frac{\sin^2\frac{1}{x}}{\frac{1}{\sqrt[3]{x^5}}}+\frac{2\sin\frac{1}{x}\cdot\operatorname{arctg}\frac{1}{x}}{\frac{1}{\sqrt[3]{x^5}}}+\frac{\operatorname{arctg}^2\frac{1}{x}}{\frac{1}{\sqrt[3]{x^5}}}.$$

Рассмотрим предел $\lim\limits_{x\to+\infty}\frac{\sin^2\frac{1}{x}}{\frac{1}{\sqrt[3]{x^5}}\cdot\frac{1}{\sqrt[3]{x}}}+\frac{2\sin\frac{1}{x}\operatorname{arctg}\frac{1}{x}}{\frac{1}{\sqrt[3]{x^5}}\cdot\frac{1}{\sqrt[3]{x}}}+$

$+\frac{\operatorname{arctg}^2\frac{1}{x}}{\frac{1}{\sqrt[3]{x^5}}\cdot\frac{1}{\sqrt[3]{x}}}$

При $x\to\infty$ $\sin\frac{1}{x}\sim\frac{1}{x}$, $\operatorname{arctg}\frac{1}{x}\sim\frac{1}{x}$

$\Rightarrow\lim\limits_{x\to+\infty}\frac{\sin^2\frac{1}{x}}{\frac{1}{\sqrt[3]{x^5}}\cdot\frac{1}{\sqrt[3]{x}}}+\frac{2\sin\frac{1}{x}\cdot\operatorname{arctg}\frac{1}{x}}{\frac{1}{\sqrt[3]{x^5}}\cdot\frac{1}{\sqrt[3]{x}}}+\frac{\operatorname{arctg}^2\frac{1}{x}}{\frac{1}{\sqrt[3]{x^5}}\cdot\frac{1}{\sqrt[3]{x}}}=\frac{\frac{1}{x^2}}{\frac{1}{x^2}}+\frac{2\frac{1}{x^2}}{\frac{1}{x^2}}+$

$+\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=1+2+1=4\neq 0$, но так как интеграл $\int_1^{+\infty}\frac{dx}{\sqrt[3]{x}}$ расходится, то расходится и интеграл $\int_1^{+\infty}\sqrt[3]{x^5}\left(\sin\frac{1}{x}-\operatorname{arctg}\frac{1}{x}\right)^2dx$

4 Вариант №1

№1

$$\int_0^1 \frac{1}{\sin x+\sqrt[3]{x}}\,dx = -\int_1^0 \frac{1}{\sin x+\sqrt[3]{x}}\,dx$$

Если $F(x)$ и $g(x)$ неотрицательны на $[a; b)$ и непрерывны на нём, то если $\lim\limits_{x\to b}\frac{f(x)}{g(x)} = K<\infty$, то $f(x)$ и $g(x)$ расходятся одновременно

Пусть $g(x)=\frac{1}{\sin x+\sqrt[3]{x}}$, а $f(x)=\frac{1}{\sqrt[3]{x}}$

$$\Rightarrow \lim_{x\to0+}\frac{\sin x+\sqrt[3]{x}}{\sqrt[3]{x}} = \lim_{x\to0+}\frac{\sin x}{x}\cdot\sqrt[3]{x^2}+\lim_{x\to0+}\frac{\sqrt[3]{x}}{\sqrt[3]{x}} =$$

$= 1\cdot0+1=1 \Rightarrow \frac{1}{\sqrt[3]{x}}$ и $\frac{1}{\sin x+\sqrt[3]{x}}$ одновременно или сходятся, или расходятся на $(0; 1]$, но $-\int_1^0 \frac{1}{\sqrt[3]{x}}dx$ сходится

$\Rightarrow$ сходится и $-\int_1^0\frac{dx}{\sin x+\sqrt[3]{x}} \Rightarrow$ сходится $\int_0^1 \frac{dx}{\sin x+\sqrt[3]{x}}$

Но этот способ, как по мне, не совсем легкий.

5 Вариант №1

№1

$$\int_1^{+\infty}(\sqrt{x^3+1}-\sqrt{x^3-1})\,dx$$

Пусть $a=1$, $b=+\infty$. Если на полуинтервале $[a;b)$ $f(x)$ и $g(x)$ неотрицательны и непрерывны, то если $\lim\limits_{x\to b-0}\frac{f(x)}{g(x)}=K<\infty$ и $K\neq 0$, то оба интеграла $\int_a^b f(x)dx$ и $\int_a^b g(x)dx$ одновременно сходятся или расходятся

Преобразуем $f(x)=\sqrt{x^3+1}-\sqrt{x^3-1}=$

$$=\frac{(\sqrt{x^3+1}-\sqrt{x^3-1})(\sqrt{x^3+1}+\sqrt{x^3-1})}{\sqrt{x^3+1}+\sqrt{x^3-1}}=\frac{2}{\sqrt{x^2+1}+\sqrt{x^3-1}}$$

Возьмём $g(x)=\frac{1}{\sqrt{x^3}}$

$$\Rightarrow\lim_{x\to\infty}\frac{2\sqrt{x^3}}{\sqrt{x^3+1}+\sqrt{x^3-1}}=\frac{2\sqrt{x^3}}{\sqrt{x^3}\left(\sqrt{1+\frac{1}{x^3}}+\sqrt{1-\frac{1}{x^3}}\right)}=1$$

$\Rightarrow$ более того, при $x\to\infty$ $\sqrt{x^3+1}-\sqrt{x^3-1}$ и $\frac{1}{\sqrt{x^3}}$ эквивалентны

$\int_1^{+\infty}\frac{1}{\sqrt{x^3}}dx$ сходится $\Rightarrow$ сходится и $\int_1^{+\infty}(\sqrt{x^3+1}-\sqrt{x^3-1})dx$

6 Вариант №1

№1

$$\int\limits_1^{+\infty} x\ln\left(\cos\frac{1}{x^2}\right)dx$$

Пусть $a=1$, $b=+\infty$. Если на полуинтервале $[a,b)$ $f(x)$ и $g(x)$ неотрицательны и непрерывны, то если $\lim\limits_{x\to b-0}\frac{f(x)}{g(x)}=K<\infty$ и $K\neq 0$, то оба интеграла

$\int\limits_a^b f(x)dx$ и $\int\limits_a^b g(x)dx$ одновременно сходятся или расходятся

Но $x\ln(\cos\frac{1}{x^2})<0$ на $[1;+\infty)$ $\Rightarrow$ рассмотрим интеграл $\int\limits_1^{+\infty}|x\ln(\cos\frac{1}{x^2})|dx$. По теореме об абсолютной сходимости, если сходится интеграл $\int\limits_a^b|f(x)|dx$, то сходится и интеграл $\int\limits_a^b f(x)dx$, причем абсолютно

Итак, пусть $f(x)=|x\ln(\cos\frac{1}{x^2})|$, а $g(x)=\frac{2}{x^3}$

Рассмотрим предел

$$\lim_{x\to\infty}\frac{|x\ln(\cos\frac{1}{x^2})|}{\frac{2}{x^3}}=\lim_{x\to\infty}\frac{|x\ln(1+\cos\frac{1}{x^2}-1)|}{\frac{2}{x^3}}$$

При $x\to\infty$ $\cos\frac{1}{x^2}\to 1\Rightarrow\cos\frac{1}{x^2}-1\to 0\Rightarrow$ при $x\to\infty$

$\ln(1+\cos\frac{1}{x^2}-1)\sim\cos\frac{1}{x^2}-1$

Так как $x\in[1;+\infty)$, то $|x|=x$

$$\Rightarrow\lim_{x\to\infty}\frac{|x\ln(1+\cos\frac{1}{x^2}-1)|}{\frac{2}{x^3}}=\lim_{x\to\infty}\frac{x\,|\cos\frac{1}{x^2}-1|}{\frac{2}{x^3}}=$$

$$= \lim_{x\to\infty} \frac{x\left|-2\sin^2\frac{1}{2x^2}\right|}{\frac{2}{x^3}} = \lim_{x\to\infty} \frac{2\sin^2\frac{1}{2x^2}}{\frac{2}{x^4}}$$

При $x\to\infty$ $\frac{1}{2x^2}\to 0 \Rightarrow$ при $x\to\infty$ $\sin\frac{1}{2x^2} \sim \frac{1}{2x^2}$

$$\Rightarrow \lim_{x\to\infty} \frac{2\sin^2\frac{1}{2x^2}}{\frac{2}{x^4}} = \lim_{x\to\infty} \frac{2\cdot\frac{1}{2x^2}\cdot\frac{1}{2x^2}}{\frac{2}{x^4}} = \frac{1}{4}$$

Интеграл $\int_1^{+\infty} \frac{2}{x^3}\,dx$ сходится $\Rightarrow$

интеграл $\int_1^{+\infty} \left|x\ln\left(\cos\frac{1}{x^2}\right)\right| dx$ сходится $\Rightarrow$ сходится и интеграл $\int_1^{+\infty} x\ln\left(\cos\frac{1}{x^2}\right) dx$, причём абсолютно

1 Вариант №2

Вариант 1

Задание 2

Найти:

а) $\frac{\partial z}{\partial u}$ и $\frac{\partial^2 z}{\partial u^2}$, если $z = f(x,y)$, $x = \frac{u}{v}$, $y = uv$

$$\boxed{\frac{\partial z}{\partial u}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \boxed{\frac{\partial f}{\partial x}\cdot\frac{1}{v} + \frac{\partial f}{\partial y}\cdot v}$$

$$\boxed{\frac{\partial^2 z}{\partial u^2}} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\cdot\frac{1}{v} + \frac{\partial f}{\partial y}\cdot v\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\cdot\frac{1}{v}\right) +$$

$$+ \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\cdot v\right) = \frac{1}{v}\cdot\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + v\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) =$$

$$= \frac{1}{v}\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial x\cdot\partial y}\cdot\frac{\partial y}{\partial u}\right) + v\left(\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial y\partial x}\cdot\frac{\partial x}{\partial u}\right) =$$

$$= \frac{1}{v}\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{1}{v} + \frac{\partial^2 f}{\partial y\partial x}\cdot v\right) + v\left(\frac{\partial^2 f}{\partial y^2}\cdot v + \frac{\partial^2 f}{\partial y\partial x}\cdot\frac{1}{v}\right) =$$

$$= \boxed{\frac{1}{v^2}\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial y\partial x} + v^2\frac{\partial^2 f}{\partial y^2}}$$

Ответ: $\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{1}{v} + \frac{\partial f}{\partial y}\cdot v$; $\frac{\partial^2 z}{\partial u^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial y\partial x} + v^2\frac{\partial^2 f}{\partial y^2}$.

б) $2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$ Найти: $\frac{\partial^2 z}{\partial x\partial y}$ — ?

Пусть $f = 2^{\frac{x}{z}} + 2^{\frac{y}{z}} - 8$, тогда

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{\frac{1}{z}\cdot 2^{\frac{x}{z}}\cdot\ln 2}{-\frac{x}{z^2}\cdot 2^{\frac{x}{z}}\cdot\ln 2 + \left(-\frac{y}{z^2}\cdot 2^{\frac{y}{z}}\cdot\ln 2\right)} = \frac{2^{x/z}}{\frac{x}{z}\cdot 2^{\frac{x}{z}} + \frac{y}{z}\cdot 2^{\frac{y}{z}}} = \frac{z\cdot 2^{\frac{x}{z}}}{x\cdot 2^{\frac{x}{z}} + y\cdot 2^{\frac{y}{z}}}.$$

$$f = 2^{\frac{x}{z}} + 2^{\frac{y}{z}} - 8 = 0$$

$$\frac{\partial f}{\partial x} = 2^{\frac{x}{z}} \cdot \ln 2 \cdot \frac{1}{z}.$$

$$\frac{\partial f}{\partial y} = 2^{\frac{y}{z}} \cdot \ln 2 \cdot \frac{1}{y}$$

$$\frac{\partial f}{\partial z} = 2^{\frac{x}{z}} \cdot \ln 2 \cdot \cancel{x}\left(-\frac{x}{z^2}\right) + 2^{\frac{y}{z}} \cdot \ln 2 \cdot \left(-\frac{y}{z^2}\right) =$$

$$= -\frac{\ln 2}{z^2}\left(2^{\frac{x}{z}} \cdot x + 2^{\frac{y}{z}} \cdot y\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\ln 2 \cdot \frac{1}{z}\right)^2 2^{\frac{x}{z}}$$

$$\frac{\partial^2 f}{\partial y^2} = \left(\ln 2 \cdot \frac{1}{y}\right)^2 \cdot 2^{\frac{y}{z}}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{2\ln 2}{z^3}\left(2^{\frac{x}{z}} \cdot x + 2^{\frac{y}{z}} \cdot y\right) - \frac{\ln 2}{z^2}\left(\left(-\frac{x}{z^2}\right) \cdot x \cdot 2^{\frac{x}{z}} \cdot \ln 2 + \right.$$

$$\left. + y \cdot \left(-\frac{y}{z^2}\right) \cdot \ln 2 \cdot 2^{\frac{y}{z}}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y \partial z} = \ln 2 \cdot \frac{1}{y} \cdot 2^{\frac{y}{z}} \cdot \ln 2 \cdot \left(-\frac{y}{z^2}\right)$$

$$\frac{\partial^2 f}{\partial x \partial z} = \ln 2\left(2^{\frac{x}{z}} \cdot \ln 2 \cdot \left(-\frac{x}{z^2}\right) \cdot \ln 2 + \left(-\frac{1}{z^2}\right) \cdot 2^{\frac{x}{z}} \cdot \ln 2\right)$$

2 Вариант №2

№2

а) $\frac{d^2y}{dx^2}$ для ф-ции, заданной неявно ур-нием $xe^y+e^x=0$

$$\frac{d^2y}{dx^2}=-\frac{\frac{\partial^2F}{\partial x^2}\cdot\left(\frac{\partial F}{\partial y}\right)^2-2\frac{\partial^2F}{\partial x\partial y}\cdot\frac{\partial F}{\partial x}\cdot\frac{\partial F}{\partial y}+\frac{\partial^2F}{\partial y^2}\cdot\left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}.\ =$$

1) $\frac{\partial F}{\partial x}=e^y+e^x$ 2) $\frac{\partial F}{\partial y}=xe^y$ 3) $\frac{\partial^2F}{\partial x^2}=e^x$ 4) $\frac{\partial^2F}{\partial y^2}=xe^y$

5) $\frac{\partial^2F}{\partial x\partial y}=e^y$

$$=-\frac{e^xx^2e^{2y}-2xe^{2y}\cdot(e^y+e^x)+xe^y(e^y+e^x)^2}{(xe^y)^3}$$

б) z''_{xx}, если $z=f(u,v)$, $u=x/y$, $v=x^2+y^2$

$$\frac{\partial^2z}{\partial x^2}=\frac{\partial^2z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+2\frac{\partial^2z}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial^2z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial z}{\partial u}\frac{\partial^2u}{\partial x^2}+\frac{\partial z}{\partial v}\frac{\partial^2v}{\partial x^2}$$

1) $\frac{\partial u}{\partial x}=\frac{1}{y}$ 2) $\frac{\partial^2u}{\partial^2x}=0$ 3) $\frac{\partial v}{\partial x}=2x$ 4) $\frac{\partial^2v}{\partial x^2}=2$

$$\frac{\partial^2z}{\partial x^2}=\frac{\partial^2z}{\partial u^2}\frac{1}{y^2}+2\frac{\partial^2z}{\partial u\partial v}\cdot\frac{2x}{y}+\frac{\partial^2z}{\partial v^2}\cdot4x^2+2\frac{\partial z}{\partial v}$$

+

3 Вариант №2

Вариант 3

№2

а) $\frac{\partial z}{\partial x}, \frac{dz}{dx}$, $z = \ln(3x - y^2)$, $y = e^{-x^2}$ $\quad z = \varphi(x, y)$, $y = y(x)$

$$\frac{\partial z}{\partial x} = \frac{3}{3x - y^2} \qquad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{3x - y^2} \qquad \frac{dy}{dx} = -2xe^{-x^2}$$

$$\frac{dz}{dx} = \frac{3}{3x - y^2} - \frac{2y}{3x - y^2} \cdot 2xe^{-x^2}$$

б) y', y'', если $y^3x + y^2 + 5x + 7 = 0$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \qquad \frac{\partial F}{\partial x} = y^3 + 5 \qquad \frac{\partial F}{\partial y} = 3xy^2 + 2y$$

$$y' = \frac{dy}{dx} = -\frac{y^3 + 5}{3xy^2 + 2y}$$

$$\frac{d^2y}{dx^2} = -\frac{\frac{\partial^2 F}{\partial x^2}\cdot\left(\frac{\partial F}{\partial y}\right)^2 - 2\frac{\partial^2 F}{\partial x \partial y}\cdot\frac{\partial F}{\partial x}\cdot\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2}\cdot\left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}.$$

$$\frac{\partial^2 F}{\partial x^2} = 0; \quad \frac{\partial^2 F}{\partial x \partial y} = 3y^2; \quad \frac{\partial^2 F}{\partial y^2} = 6xy + 2$$

$$y'' = \frac{d^2y}{dx^2} = -\frac{-2\cdot 3y^2\cdot(y^3 + 5)(3xy^2 + 2y) + (6xy + 2)(y^3 + 5)^2}{(3xy^2 + 2y)^3}$$

4 Вариант №2

В4

№2

а) $z = \varphi\left(\overset{u}{\frac{y}{x}};\ \overset{v}{\frac{x}{y}}\right)$ $\qquad z''_{yy} = ?$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 f}{\partial u \partial v}\cdot\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 +$$

$$+ \frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial f}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}$$

- $\frac{\partial u}{\partial y} = \frac{1}{x}$ $\qquad$ • $\frac{\partial v}{\partial y} = -\frac{x}{y^2}$

- $\frac{\partial^2 u}{\partial y} = 0$ $\qquad$ • $\frac{\partial^2 v}{\partial y^2} = \frac{2x}{y^3}$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 \varphi}{\partial u^2}\cdot\left(\frac{1}{x}\right)^2 + 2\frac{\partial^2 \varphi}{\partial u \partial v}\cdot\frac{1}{x}\left(-\frac{x}{y^2}\right) + \frac{\partial^2 \varphi}{\partial v^2}\cdot\left(-\frac{x}{y^2}\right)^2 +$$

$$+ \frac{\partial \varphi}{\partial u}\cdot 0 + \frac{\partial \varphi}{\partial v}\cdot\frac{2x}{y^3}$$

+

δ) $x + y + z = \sin(xyz) \quad z''_{xy}(0,0) = ?$

$$\frac{\partial^2 z}{\partial x \partial y} = \pm \frac{\frac{\partial^2 F}{\partial y \partial x} \cdot \left(\frac{\partial F}{\partial z}\right)^2 - \frac{\partial^2 F}{\partial z \partial x} \cdot \frac{\partial F}{\partial y} \cdot \frac{\partial F}{\partial z} - \frac{\partial^2 F}{\partial y \partial z} \cdot \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial z} + \frac{\partial^2 F}{\partial z^2} \cdot \frac{\partial F}{\partial y} \cdot \frac{\partial F}{\partial x}}{\left(\frac{\partial F}{\partial z}\right)^3}$$

$$\frac{\partial F}{\partial x} = 1 - \cos(xyz) \cdot yz$$

$$\left.\frac{\partial F}{\partial x}\right|_M = 1$$

$$\frac{\partial F}{\partial y} = 1 - \cos(xyz)xz \qquad \left.\frac{\partial F}{\partial y}\right|_M = 1$$

$$\frac{\partial F}{\partial z} = 1 - \cos(xyz)xy \qquad \left.\frac{\partial F}{\partial z}\right|_M = 1$$

$$\frac{\partial^2 F}{\partial z \partial x} = \sin(xyz) \cdot xy^2z - \cos(xyz)y$$

$$\left.\frac{\partial^2 F}{\partial z \partial x}\right|_M = 0$$

$$\frac{\partial^2 F}{\partial y \partial z} = \sin(xyz)x^2yz - \cos(xyz)x$$

$$\left.\frac{\partial^2 F}{\partial y \partial z}\right|_M = 0$$

$$\frac{\partial^2 F}{\partial z^2} = \sin(xyz)x^2y^2 \qquad \left.\frac{\partial^2 F}{\partial z^2}\right|_M = 0$$

$$\frac{\partial^2 F}{\partial x \partial y} = \sin(xyz) \cdot xyz^2 - \cos(xyz) \cdot z$$

$$\left.\frac{\partial^2 F}{\partial x \partial y}\right|_M = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = - \frac{0 - 0 - 0 - 0}{1}$$

$$= \underline{0}$$

5 Вариант №2

u) $z=\varphi(xy,\frac{x}{y})$ $\quad u=xy \quad v=\frac{x}{y} \qquad dz=? \quad z''_{xy}=?$

$$dz=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}dx+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}dy$$

$$\frac{\partial u}{\partial x}=y \quad \frac{\partial v}{\partial y}=-\frac{x}{y^2} \qquad dz=y\frac{\partial z}{\partial u}dx-\frac{x}{y^2}\cdot\frac{\partial z}{\partial v}dy$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x\partial y}$$

$$\frac{\partial u}{\partial y}=x \quad \cancel{\frac{\partial u}{\partial x}} \quad \cancel{\frac{\partial^2 z}{\partial u}} \quad \frac{\partial v}{\partial y}=-\frac{x}{y^2} \quad \frac{\partial v}{\partial x}=\frac{1}{y} \quad \frac{\partial^2 u}{\partial x\partial y}=1 \quad \frac{\partial^2 v}{\partial x\partial y}=-\frac{1}{y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial u^2}\cdot xy+\frac{\partial^2 z}{\partial u\partial v}\left(\cancel{x}-\frac{x}{y}+\frac{x}{y}\right)+\frac{\partial z}{\partial u}\quad+\frac{\partial z}{\partial v}\cdot\left(-\frac{1}{y^2}\right)$$

Вариант 5

№2

б) $z''_{xy}(1,-1)$

$x^2z^3+y^3z^2+z^2x^3=8$ $\qquad F=x^2z^3+y^3z^2+z^2x^3-8$

$z^3(1,-1)+z^2(1,-1)+z^2(1,-1)=8$

$z^3(1,-1)+\cancel{2z^2(1,-1)}=8$ $\qquad z(1,-1)=2$

$$z''_{xy}=-\frac{\frac{\partial^2F}{\partial y\partial x}\cdot\left(\frac{\partial F}{\partial z}\right)^2-\frac{\partial^2F}{\partial z\partial x}\cdot\frac{\partial F}{\partial y}\cdot\frac{\partial F}{\partial z}-\frac{\partial^2F}{\partial y\partial z}\cdot\frac{\partial F}{\partial x}\cdot\frac{\partial F}{\partial z}+\frac{\partial^2F}{\partial z^2}\cdot\frac{\partial F}{\partial y}\cdot\frac{\partial F}{\partial x}}{\left(\frac{\partial F}{\partial z}\right)^3}$$

$\frac{\partial F}{\partial x}\cancel{=2xz^3+3x^2z^2}$ $\qquad \frac{\partial F}{\partial x}=2xz^3+3x^2z^2$ $\qquad \frac{\partial F}{\partial y}=3y^2z^2$

$\frac{\partial F}{\partial z}=3z^2x^2+2y^3z+2zx^3$ $\qquad \frac{\partial F^2}{\partial y\partial x}=0$ $\qquad \frac{\partial F^2}{\partial y\partial z}=6y^2z$

$\frac{\partial^2F}{\partial z^2}=6zx^2+2y^3+2x^3$ $\qquad \frac{\partial^2F}{\partial x\partial z}=6xz^2+6x^2z$

$$z''_{xy}=-\frac{0-(6xz^2+6x^2z)\cdot3y^2z^2\cdot(3z^2x^2+2y^3z+2zx^3)-6y^2z(2x\cdot z^3+3x^2z^2)\cdot}{(3z^2x^2+2y^3z+2zx^3)^3}$$

$$\frac{\cdot(3z^2x^2+2y^3z+2zx^3)+(6zx^2+2y^3+2x^3)\cdot3y^2z^2\cdot(2xz^3+3x^2z^2)}{(3z^2x^2+2y^3z+2zx^3)^3}=$$

$$=\frac{(24+12)\cdot12\cdot(12-4+4)+12(16+12)\cdot(12-4+4)-(12-2+2)\cdot12(16+12)}{(12-4+4)^3}=3$$

6 Вариант №2

№ 2

а) Найти дифференциал и u''_{xz} функции

$u = \varphi(x+y+z, x^2+y^2+z^2)$

Пусть $\alpha = x+y+z$, $\beta = x^2+y^2+z^2$

1) I) $\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial x} + \frac{\partial\varphi}{\partial\beta}\cdot\frac{\partial\beta}{\partial x} = \frac{\partial\varphi}{\partial\alpha} + \frac{\partial\varphi}{\partial\beta}\cdot 2x$

II) $\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial y} + \frac{\partial\varphi}{\partial\beta}\cdot\frac{\partial\beta}{\partial y} = \frac{\partial\varphi}{\partial\alpha} + \frac{\partial\varphi}{\partial\beta}\cdot 2y$

III) $\frac{\partial\varphi}{\partial z} = \frac{\partial\varphi}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial z} + \frac{\partial\varphi}{\partial\beta}\cdot\frac{\partial\beta}{\partial z} = \frac{\partial\varphi}{\partial\alpha} + \frac{\partial\varphi}{\partial\beta}\cdot 2z$

$\Rightarrow du = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = \left(\frac{\partial\varphi}{\partial\alpha} + \frac{\partial\varphi}{\partial\beta}\cdot 2x\right)dx +$

$+ \left(\frac{\partial\varphi}{\partial\alpha} + \frac{\partial\varphi}{\partial\beta}\cdot 2y\right)dy + \left(\frac{\partial\varphi}{\partial\alpha} + \frac{\partial\varphi}{\partial\beta}\cdot 2z\right)dz$

2) $u''_{xz} = \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial\alpha} + \frac{\partial\varphi}{\partial\beta}\cdot 2x\right) = \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial\alpha}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial\beta}\cdot 2x\right) =$

$= \left(\frac{\partial^2\varphi}{\partial\alpha^2}\cdot\frac{\partial\alpha}{\partial z} + \frac{\partial^2\varphi}{\partial\alpha\partial\beta}\cdot\frac{\partial\beta}{\partial z}\right) + 2x\cdot\left(\frac{\partial^2\varphi}{\partial\beta\cdot\partial\alpha}\cdot\frac{\partial\alpha}{\partial z} + \frac{\partial^2\varphi}{\partial\beta^2}\cdot\frac{\partial\beta}{\partial z}\right) =$

$= \left(\frac{\partial^2\varphi}{\partial\alpha^2} + \frac{\partial^2\varphi}{\partial\alpha\partial\beta}\cdot 2z\right) + 2x\cdot\left(\frac{\partial^2\varphi}{\partial\beta\cdot\partial\alpha} + \frac{\partial^2\varphi}{\partial\beta^2}\cdot 2z\right) =$

$= \frac{\partial^2\varphi}{\partial\alpha^2} + 2(z+x)\frac{\partial^2\varphi}{\partial\alpha\partial\beta} + 4xz\frac{\partial^2\varphi}{\partial\beta^2}$

б) y'; y'', $yx^2+\sqrt{y}+3x^3+9=0$

$$y'x^2+2yx+\frac{y'}{2\sqrt{y}}+9x^2=0$$

$$y'\left(x^2+\frac{1}{2\sqrt{y}}\right)=-9x^2-2yx$$

$$\Rightarrow y'=\frac{-9x^2-2yx}{x^2+\frac{1}{2\sqrt{y}}}$$

$$y''=\left(\frac{-9x^2-2yx}{x^2+\frac{1}{2\sqrt{y}}}\right)'_y=\frac{(-9x^2-2yx)'(x^2+\frac{1}{2\sqrt{y}})+(9x^2+2yx)(x^2+\frac{1}{2\sqrt{y}})'}{(x^2+\frac{1}{2\sqrt{y}})^2}=$$

$$=\frac{(-18x-2y-2y'x)(x^2+\frac{1}{2\sqrt{y}})+(9x^2+2yx)(2x-\frac{y'}{4y\sqrt{y}})}{(x^2+\frac{1}{2\sqrt{y}})^2}=$$

$$=\frac{\left(-18x-2y+\frac{18x^3+4yx^2}{x^2+\frac{1}{2\sqrt{y}}}\right)(x^2+\frac{1}{2\sqrt{y}})+(9x^2+2yx)\left(2x+\frac{9x^2+2yx}{4y\sqrt{y}(x^2+\frac{1}{2\sqrt{y}})}\right)}{(x^2+\frac{1}{2\sqrt{y}})^2}=$$

$$=\frac{-18x^3-2yx^2-\frac{9x}{\sqrt{y}}-\sqrt{y}+18x^3+4yx^2+18x^3+4yx^2+\frac{(9x^2+2yx)^2}{4y\sqrt{y}(x^2+\frac{1}{2\sqrt{y}})}}{(x^2+\frac{1}{2\sqrt{y}})^2}=$$

$$=\frac{18x^3+6yx^2-\frac{9x}{\sqrt{y}}-\sqrt{y}+\frac{(9x^2+2yx)^2}{4y\sqrt{y}(x^2+\frac{1}{2\sqrt{y}})}}{(x^2+\frac{1}{2\sqrt{y}})^2}$$

1 Вариант №3

Задание 3 Найти производную скалярного поля $u = \cos^2 x - 2\sin y + z^2 x y^3$ в точке $M(\frac{\pi}{4}, \frac{\pi}{2}, 0)$ по направлению $\ell = 4\bar{i} + 3\bar{j}$, $\cos\alpha = \frac{4}{|\ell|} = \frac{4}{5}$ $\cos\gamma = \frac{0}{|\ell|} = 0$;

$\cos\beta = \frac{3}{|\ell|} = \frac{3}{5}$; $\vec{\ell_0} = (\frac{4}{5}, \frac{3}{5}, 0)$

Частные производные:

$$\frac{\partial u}{\partial x} = (2\cdot\cos x(-\sin x) + z^2 y^3)\Big|_M = -1$$

$$\frac{\partial u}{\partial y} = (-2\cos y + 3z^2 x y^2)\Big|_M = 0$$

$$\frac{\partial u}{\partial z} = (2zxy^3)\Big|_M = 0$$

$$\vec{\nabla} u = -\bar{i}$$

$$\frac{\partial u}{\partial \ell} = \vec{\nabla} u \cdot \vec{\ell_0} = \boxed{-\frac{4}{5}}$$

Ответ: $-\frac{4}{5}$

2 Вариант №3

№3

$u = 4\ln(3+x^2) - 8xyz \qquad M(1,1,1) \qquad \vec{e} = 2\vec{i} - 3\vec{j} + \vec{k}$

$\frac{\partial u}{\partial x} = \frac{8x}{3+x^2} - 8yz \qquad \left.\frac{\partial u}{\partial x}\right|_M = \frac{8}{3+1} - 8 = \underline{-6}$

$\frac{\partial u}{\partial y} = -8xz \qquad \left.\frac{\partial u}{\partial y}\right|_M = \underline{-8} \qquad \frac{\partial u}{\partial z} = -8xy \qquad \left.\frac{\partial u}{\partial z}\right|_M = \underline{-8}$

$|\vec{e}| = \sqrt{4+9+1} = \sqrt{14} \qquad \cos\alpha = \frac{2}{\sqrt{14}} \qquad \cos\beta = \frac{-3}{\sqrt{14}} \qquad \cos\gamma = \frac{1}{\sqrt{14}}$

$\frac{\partial u}{\partial e} = \frac{-12}{\sqrt{14}} + \frac{24}{\sqrt{14}} - \frac{8}{\sqrt{14}} = \frac{4}{\sqrt{14}}$

3 Вариант №3

В-3 (№3)

$\cancel{u(x,y,z) = (x^2+y^2+z^2)^{3/2}}$

$M\left(\sqrt{2}, \frac{1}{\sqrt{2}}; \frac{1}{\sqrt{3}}\right)$

$u = \frac{x^3}{3} + 6y^3 + 3\sqrt{6}\, z^6, \quad v = \frac{yz^2}{x^2}$

$\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}; \frac{\partial u}{\partial z}\right) = \left(x^2; 18y^2, 18\sqrt{6}\, z^5\right)$

$\operatorname{grad} v = \left(\frac{\partial v}{\partial x}; \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\right) = \left(\frac{-2yz^2}{x^3}; \frac{z^2}{x^2}; \frac{2yz}{x^2}\right)$

$\operatorname{grad} u(M) = \left(2, 9, \frac{18\sqrt{6}}{(\sqrt{3})^5}\right) \to 2\sqrt{2}$

$\operatorname{grad} v(M) = \left(\frac{-2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{3}}{(\sqrt{2})^3}; \frac{\frac{1}{3}}{2}; \frac{2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{3}}}{2}\right) =$

$= \left(-\frac{1}{6}; \frac{1}{6}; \frac{1}{\sqrt{6}}\right)$

$\cos\varphi = \frac{-\frac{1}{3} + \frac{3}{2} + \frac{2}{\sqrt{3}}}{\sqrt{\underbrace{(4+81+8)}_{93}\underbrace{\left(\frac{1}{36}+\frac{1}{36}+\frac{1}{6}\right)}_{\frac{8}{36}}}} = \frac{7+4\sqrt{3}}{6\cdot\sqrt{\frac{62}{3}}}$

$\cancel{\varphi = \arccos\left(\frac{7+4\sqrt{\ }}{4\cdot 31}\right)}$

$\varphi = \arccos\left(\frac{7+4\sqrt{3}}{2\sqrt{186}}\right)$

4 Вариант № 3

№3

$u = x + \ln(z^2 + y^2) \quad M(2, 1, 1) \quad \vec{l} = -2\vec{i} + \vec{j} - \vec{k}$

$\frac{\partial u}{\partial x} = 1 \quad \left.\frac{\partial u}{\partial x}\right|_M = 1 \quad \frac{\partial u}{\partial y} = \frac{2y}{z^2+y^2} \quad \left.\frac{\partial u}{\partial y}\right|_M = 1$

$\frac{\partial u}{\partial z} = \frac{2z}{z^2+y^2} \quad \left.\frac{\partial u}{\partial z}\right|_M = 1$

$|\vec{l}| = \sqrt{4+1+1} = \sqrt{6}$

$\cos\alpha = -\frac{2}{\sqrt{6}} \quad \cos\beta = \frac{1}{\sqrt{6}} \quad \cos\gamma = -\frac{1}{\sqrt{6}}$

$\frac{\partial u}{\partial l} = -\frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} = -\frac{2}{\sqrt{6}}$

5 Вариант №3

№3

$$u(x,y,z) = (x^2 + y^2 + z^2)^{3/2}$$

$$\vec{l} = i - j + k \quad ; \quad M(1,1,1)$$

$$|\vec{l}| = \sqrt{3}$$

$$\cos\alpha = \frac{1}{\sqrt{3}} \; ; \quad \cos\beta = -\frac{1}{\sqrt{3}} \; ; \quad \cos\gamma = \frac{1}{\sqrt{3}}$$

$$\frac{\partial u}{\partial x} = \frac{3}{2}\sqrt{x^2 + y^2 + z^2} \cdot 2x$$

$$\frac{\partial u}{\partial x}\Big|_M = \frac{\partial u}{\partial y}\Big|_M = \frac{\partial u}{\partial z}\Big|_M = 3 \cdot \sqrt{3}$$

$$\frac{\partial u}{\partial l}\Big|_M = 3 - 3 + 3 = 3$$

6 Вариант №3

№3

$u = \cos^2 x - 2\sin y + z^2 x y^3 \quad M\left(\frac{\pi}{4}, \frac{\pi}{2}, 0\right)$

$\vec{l} = 4\bar{i} + 3\bar{j} \Rightarrow \vec{l} = (4; 3; 0)$

$\frac{\vec{l}}{|\vec{l}|} = \left(\frac{4}{5}; \frac{3}{5}; 0\right)$

$\mathrm{grad}\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = \left(-\sin 2x + z^2 y^3; -2\cos y + 3y^2 z^2 x,\right.$

$\left. 2zxy^3\right)$

$\mathrm{grad}\,u(M) = (-1, 0, 0)$

$\Rightarrow$ Ответ: $\left((-1, 0, 0)\left(\frac{4}{5}, \frac{3}{5}, 0\right)\right) = -\frac{4}{5}$

1 Вариант №4

Задание 4

$$z = e^{3x}\cdot x + y^2\cdot e^{3x} + 2ye^{3x}$$

$$\frac{\partial z}{\partial x} = 3x\cdot e^{3x} + e^{3x} + 3e^{3x}\cdot y^2 + 6e^{3x}y$$

$$\frac{\partial z}{\partial y} = 2ye^{3x} + 2e^{3x}$$

$$\begin{cases} \frac{\partial z}{\partial x} = 0 \\ \frac{\partial z}{\partial y} = 0 \end{cases} \quad \begin{cases} e^{3x}(3x+1+3y^2+6y) = 0 \\ e^{3x}(2y+2) = 0 \end{cases} \quad \begin{cases} x = \frac{2}{3} \\ y = -1 \end{cases}$$

$$\frac{\partial^2 z}{\partial x^2} = 9x\cdot e^{3x} + 3e^{3x} + 3e^{3x} + 9e^{3x}\cdot y^2 + 18e^{3x}\cdot y$$

$$\frac{\partial^2 z}{\partial y^2} = 2e^{3x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6ye^{3x} + 6e^{3x}$$

Составим второй дифференциал ф-ии z:

$$d^2z = (9x\cdot e^{3x} + 6e^{3x} + 9e^{3x}\cdot y^2 + 18e^{3x}y)dx^2 + (2e^{3x})dy^2 + 2(6ye^{3x} + 6e^{3x})dxdy$$

$$A = \begin{pmatrix} e^{3x}(9x+6+9y^2+18y) & 6ye^{3x}+6e^{3x} \\ 6ye^{3x}+6e^{3x} & 2e^{3x} \end{pmatrix}$$

в точке, подозреваемой на экстремум, матрица имеет вид:

$$A = \begin{pmatrix} e^{2}\cdot 3 & 0 \\ 0 & 2e^{2} \end{pmatrix}$$

Критерий Сильвестра:

$\Delta_1 = 3e^2 > 0$

$\Delta_2 = 6e^4 > 0$, сл-но квадр форма положительно определена,

откуда следует, что точка $(\frac{2}{3}, -1)$ - точка минимума

Ответ: $(\frac{2}{3}, -1)$ - точка минимума.

2 Вариант №4

$$z = x^3 - 3xy + 3y^2 - 5 \qquad (x>0,\ y>0)$$

$$\begin{cases} \dfrac{\partial z}{\partial x} = 3x^2 - 3y = 0 \\ \dfrac{\partial z}{\partial y} = -3x + 6y = 0 \end{cases} \Leftrightarrow \begin{cases} y(4y-1) = 0 \\ x = 2y \end{cases} \Leftrightarrow \begin{cases} y = \dfrac{1}{4} \\ x = \dfrac{1}{2} \end{cases}$$

$$d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2 \cdot$$

$$M_1\left(\frac{1}{2};\ \frac{1}{4}\right)$$

$$d^2z = 6x\,dx^2 - 6\,dxdy + 6\,dy^2$$

$$d^2z = 3dx^2 - 6dxdy + 6dy^2$$

$$A = \begin{pmatrix} 3 & -3 \\ -3 & 6 \end{pmatrix}, \quad \Delta_1 = 3 > 0 \quad \Delta_2 = 9 \quad > 0$$

Ответ: $M_1\left(\frac{1}{2}; \frac{1}{4}\right)$ яв. точкой локального экстремума (мин)

3 Вариант №4

Вар 3

$\boxed{№ 4}$ $z = y^3 + 3x^2y - 12x - 15y$

1) $\frac{\partial z}{\partial x} = 6xy - 12$

$\frac{\partial z}{\partial y} = 3y^2 + 3x^2 - 15$

$\begin{cases} xy = 2 \\ y^2 + x^2 = 5 \end{cases} \Rightarrow \begin{cases} \left[\begin{array}{l} x = 1 \\ y = 2 \end{array}\right. & M_1 \\ \left[\begin{array}{l} x = 2 \\ y = 1 \end{array}\right. & M_2 \end{cases}$

2) $\frac{\partial^2 z}{\partial x^2} = 6y$; $\frac{\partial^2 z}{\partial y^2} = 6y$; $\frac{\partial^2 z}{\partial x \partial y} = 6x$

$A = \begin{pmatrix} 6y & 6x \\ 6x & 6y \end{pmatrix}$; для $M_1(2;1)$ $A_1 = \begin{pmatrix} 6 & 12 \\ 12 & 6 \end{pmatrix}$ знакопер. $\Delta_1 = 6 > 0$ $\Delta_2 = -108 < 0$

$\Rightarrow M_1$ – не т. экстр.

для $M_2(1;2)$ $A_2 = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}$ полож. опред. $\Delta_1 = 6 > 0$ $\Delta_2 = 108 > 0$

$\Rightarrow M_2$ – т. экстр.

Ответ: $M(1;2)$.

4 Вариант №4

N 4

$$z = y^2 \ln x - x^2 \quad (x>0;\ y>0).$$

1) $$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{y^2}{x} - 2x = 0 \\ \dfrac{\partial z}{\partial y} = 2y\ln x = 0 \end{cases}$$

$$y^2 - 2x^2 = 0 \Rightarrow y = \sqrt{2}\,x$$

$$2\sqrt{2}\,x \cdot \ln x = 0$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \text{ — не уд } x>0,\ y>0 \qquad \begin{cases} x = 1 \\ y = \sqrt{2} \end{cases} \Rightarrow M_1(1;\ \sqrt{2}).$$

2) $$\frac{\partial^2 z}{\partial x^2} = -\frac{y^2}{x^2} - 2$$

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_M = -\frac{2}{1} - 2 = -4$$

$$\frac{\partial^2 z}{\partial y^2} = 2\ln x;\quad \left.\frac{\partial^2 z}{\partial y^2}\right|_M = 0$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{2y}{x};\quad \left.\frac{\partial^2 z}{\partial x\partial y}\right|_M = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

$$\begin{pmatrix} -4 & 2\sqrt{2} \\ 2\sqrt{2} & 0 \end{pmatrix} \qquad \Delta_1 = -4 < 0, \quad \Delta_2 = -8 < 0$$

Значит M — не точка экстремума

$\rightarrow$ ф-ия не имеет экстремумов.

5 Вариант №4

№4

$$z = e^{x}(x + y^2 + 2y)$$

$2y e^{x} + 2e^{x} = 2e^{x}(y+1)$

1) $$\begin{cases} \frac{\partial z}{\partial x} = e^{x}(x + y^2 + 2y) + e^{x} = e^{x}(x + y^2 + 2y + 1) = 0 \\ \frac{\partial z}{\partial y} = e^{x} \cdot 2y + 2e^{x} = 2e^{x}(y + 1) = 0 \quad \Rightarrow y = -1 \end{cases}$$

$$e^{x}(x + 1 - 2 + 1) = 0$$

$$\begin{cases} x = 0 \\ y = -1 \end{cases} \Rightarrow M_2(0; -1)$$

2) $$\frac{\partial^2 z}{\partial x^2} = e^{x}(x + y^2 + 2y + 1) + e^{x} = e^{x}(x + y^2 + 2y + 2)$$

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_M = 0 + 1 - 2 + 2 = 1$$

$$\frac{\partial^2 z}{\partial y^2} = 2e^{x}; \quad \left.\frac{\partial^2 z}{\partial y^2}\right|_M = 2$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2e^{x}(y + 1)$$

$$\left.\frac{\partial^2 z}{\partial y \partial x}\right|_M = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \Delta_1 > 0; \quad \Delta_2 > 0$$

$\Rightarrow M$ – усл. минимум.

6 Вариант №4

№4

$z = -\frac{1}{2}y^2 \ln x + 3x^2$

$z'_x = -\frac{y^2}{2x} + 6x$

$z'_y = -\ln x \cdot y$

$\Rightarrow \begin{cases} -\frac{y^2}{2x} + 6x = 0 \\ -\ln x \cdot y = 0 \end{cases} \Rightarrow \begin{cases} y^2 = 12x^2 \\ y \ln x = 0 \end{cases}$

I) $y = 0 \Rightarrow x = 0$, но $\ln$ не определен в 0.

II) $x = 1 \Rightarrow y = \pm 2\sqrt{3}$

$z''_{xx} = \frac{y^2}{2x^2} + 6$

$z''_{xy} = -\frac{y}{x}$

$z''_{yy} = -\ln x$

а) $y = 2\sqrt{3}$

$A = z''_{xx}|_{(1, 2\sqrt{3})} = 12$

$B = z''_{xy}|_{(1, 2\sqrt{3})} = -2\sqrt{3}$

$C = z''_{yy}|_{(1, 2\sqrt{3})} = 0$

$\Rightarrow \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} 12 & -2\sqrt{3} \\ -2\sqrt{3} & 0 \end{vmatrix} = -12 < 0 \Rightarrow$ при $x = 1, y = 2\sqrt{3}$ у функции $z = -\frac{y^2 \ln x}{2} + 3x^2$ нет экстремума

б) $y = -2\sqrt{3}$

Здесь поменяется только $B = 2\sqrt{3}$, но $\begin{vmatrix} A & B \\ B & C \end{vmatrix} = -12 < 0$ и при $y = -2\sqrt{3} \Rightarrow$ нет экстремума и в точке $(1, -2\sqrt{3})$

$\Rightarrow$ у функции $-\frac{y^2 \ln x}{2} + 3x^2$ нет экстремумов

1 Вариант №5

Задание 5

Исследовать на экстремум ф-цию

$z=\frac{1}{x}+\frac{1}{y}$ при условии $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{2}$ $(x<0, y<0)$

Ф-ия Лагранжа:

$$L=\frac{1}{x}+\frac{1}{y}+\lambda\left(\frac{1}{x^2}+\frac{1}{y^2}-\frac{1}{2}\right)$$

Частные производные:

$$\begin{cases} \frac{\partial L}{\partial x}=-\frac{1}{x^2}-\frac{2\lambda}{x^3}=0 \\ \frac{\partial L}{\partial y}=-\frac{1}{y^2}-\frac{2\lambda}{y^3}=0 \\ \frac{\partial L}{\partial \lambda}=\frac{1}{x^2}+\frac{1}{y^2}-\frac{1}{2}=0 \end{cases} \qquad \begin{cases} -\frac{1}{x^2}-\frac{2\lambda}{x^3}=0 \;\Big|\cdot x^3 (x\neq 0) \\ -\frac{1}{y^2}-\frac{2\lambda}{y^3}=0 \;\Big|\cdot y^3 (y\neq 0) \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{2} \;\Big| \end{cases}$$

$$\begin{cases} x=-2\lambda \\ y=-2\lambda \\ \frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=\frac{1}{2} \;\Big|\cdot 4\lambda^2 \end{cases} \qquad \begin{cases} x=-2\lambda \\ y=-2\lambda \\ 2=2\lambda^2 \end{cases}$$

$$\left[\begin{array}{l} \begin{cases} \lambda=1 \\ x=-2 \\ y=-2 \end{cases} \\ \begin{cases} \lambda=-1 \\ x=2 \\ y=2 \end{cases} \end{array}\right.$$

Пусть имеет координаты $M(-2,-2)$

$\lambda=-1$ – не удовл. условию $x<0$ и $y<0$

Частные производные второго порядка

$$\frac{\partial^2 L}{\partial x^2}=\frac{2}{x^3}+\frac{6\lambda}{x^4}, \quad \frac{\partial^2 L}{\partial y^2}=\frac{2}{y^3}+\frac{6\lambda}{y^4}, \quad \frac{\partial^2 L}{\partial x\partial y}=0.$$

Составим второй дифференциал ф-ии Лагранжа

$$d^2L(M)=\left(\frac{2}{x^3}+\frac{6\lambda}{x^4}\right)\Big|_M dx^2+\left(\frac{2}{y^3}+\frac{6\lambda}{y^4}\right)\Big|_M dy^2=\left(-\frac{2}{8}+\frac{6}{16}\right)dx^2+\left(-\frac{2}{8}+\frac{6}{16}\right)dy^2=$$

$=\frac{1}{8}dx^2+\frac{1}{8}dy^2$, сл-но при dx^2 и $dy^2\neq 0 \Rightarrow$

$M(-2,2)$ – точка минимума

точка минимума ф-ии z.

Ответ: $(-2,-2)$ – точка минимума

2 Вариант №5

№5

$z = x + y \qquad \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{2} \qquad (x \geq 0,\ y > 0)$

$$L = x + y + \lambda\left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{2}\right)$$

$$\begin{cases} \frac{\partial L}{\partial x} = 1 - \frac{2\lambda}{x^3} \\ \frac{\partial L}{\partial y} = 1 - \frac{2\lambda}{y^3} \\ \frac{\partial L}{\partial \lambda} = \frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{2} \end{cases} \Longleftrightarrow \begin{cases} x^3 = 2\lambda \\ y^3 = 2\lambda \\ \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{2} \end{cases} \Rightarrow x = y \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda = \frac{x^3}{2} \\ x = y \\ x^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda = 4 \\ y = 2 \\ x = 2 \end{cases} M_1(2\ 2) \quad \begin{cases} \lambda = -4 \\ y = -2 \\ x = -2 \end{cases}$$

не подходит т.к. $(x \geq 0,\ y > 0)$

$$\frac{\partial^2 L}{\partial x^2} = \frac{6\lambda}{x^4};\ \frac{\partial^2 L}{\partial y^2} = \frac{6\lambda}{y^4};\ \frac{\partial^2 L}{\partial x \partial y} = 0 \qquad \left.\frac{\partial^2 L}{\partial x^2}\right|_M = \frac{3}{2} \quad \frac{\partial^2 L}{\partial y^2} = \frac{3}{2}$$

$$d^2L = \frac{3}{2}(dx^2 + dy^2) > 0$$

$M_1(2,2)$ – т. условного минимума

3 Вариант №5

№5

$z = x^2 + y^2 \qquad 3x + 4y = 12$

$L = x^2 + y^2 + \lambda(3x + 4y - 12)$

$$\begin{cases} \frac{\partial L}{\partial x} = 2x + 3\lambda = 0 \\ \frac{\partial L}{\partial y} = 2y + 4\lambda = 0 \\ \frac{\partial L}{\partial \lambda} = 3x + 4y - 12 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{3}{2}\lambda \\ y = -2\lambda \\ -\frac{9}{2}\lambda - 8\lambda - 12 = 0 \end{cases}$$

$-\frac{25}{2}\lambda = 12$

$\lambda = -\frac{24}{25}$

$x = \frac{36}{25} \quad y = \frac{48}{25}$

$M\left(\frac{36}{25}; \frac{48}{25}\right)$

$\frac{\partial^2 L}{\partial x^2} = 2 \qquad \frac{\partial^2 L}{\partial y^2} = 2 \qquad \frac{\partial^2 L}{\partial x \partial y} = 0$

$d^2L = 2(dx^2 + dy^2) > 0 \qquad M\left(\frac{36}{25}; \frac{48}{25}\right)$ – точка условного минимума

4 Вариант №5

№5 Найти точку экстремума ф-ии $u = 2x + z - 3y$ при условии $x^2 + y^2 + z^2 = 4$

$$L = 2x + z - 3y + \lambda(x^2 + y^2 + z^2 - 4)$$

$$\begin{cases} \frac{\partial L}{\partial x} = 2 + 2\lambda x = 0 \\ \frac{\partial L}{\partial y} = -3 + 2\lambda y = 0 \\ \frac{\partial L}{\partial z} = 1 + 2\lambda z = 0 \\ \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 4 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{1}{\lambda} \\ y = \frac{3}{2\lambda} \\ z = -\frac{1}{2\lambda} \\ \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 4 \end{cases}$$

$$\frac{14}{4\lambda^2} = 4 \qquad \lambda^2 = \frac{14}{16} \qquad \lambda = \pm\frac{\sqrt{14}}{4}$$

$$\begin{cases} x = -\frac{4}{\sqrt{14}} \\ y = \frac{\cancel{12}\ 6}{\cancel{2}\sqrt{14}} \\ z = -\frac{\cancel{4}\ 2}{\cancel{8}\sqrt{14}} \\ \lambda = \frac{\sqrt{14}}{4} \end{cases} \quad \text{и} \quad \begin{cases} x = \frac{4}{\sqrt{14}} \\ y = -\frac{6}{\sqrt{14}} \\ z = \frac{2}{\sqrt{14}} \\ \lambda = -\frac{\sqrt{14}}{4} \end{cases}$$

$$\frac{\partial^2 L}{\partial x^2} = 2\lambda \quad \frac{\partial^2 L}{\partial y^2} = 2\lambda \quad \frac{\partial^2 L}{\partial z^2} = 2\lambda$$

$$\frac{\partial^2 L}{\partial x \partial y} = 0 \quad \frac{\partial^2 L}{\partial y \partial z} = 0 \quad \frac{\partial^2 L}{\partial x \partial z} = 0$$

$$d^2 L = 2\lambda dx^2 + 2\lambda dy^2 + 2\lambda dz^2$$

1) $M_1\left(-\frac{4}{\sqrt{14}}; \frac{6}{\sqrt{14}}; -\frac{2}{\sqrt{14}}\right)$

$$d^2 L = \frac{\sqrt{14}}{2}dx^2 + \frac{\sqrt{14}}{2}dy^2 + \frac{\sqrt{14}}{2}dz^2 > 0$$

M_1 – условный минимум

2) $M_2\left(\frac{4}{\sqrt{14}}; -\frac{6}{\sqrt{14}}; \frac{2}{\sqrt{14}}\right)$

$$d^2 L = -\frac{\sqrt{14}}{2}dx^2 - \frac{\sqrt{14}}{2}dy^2 - \frac{\sqrt{14}}{2}dz^2 < 0$$

M_2 – условный максимум.

5 Вариант №5

$\boxed{№5}$ $z = x^2 - 2y^2$ $\quad x + 3y + 1 = 0$

$L = x^2 - 2y^2 + \lambda(x + 3y + 1)$

1) $\begin{cases} \frac{\partial z}{\partial x} = 2x + \lambda = 0 \\ \frac{\partial z}{\partial y} = -4y + 3\lambda = 0 \\ \frac{\partial z}{\partial \lambda} = x + 3y + 1 = 0 \end{cases}$ $\quad \begin{cases} x = -\frac{\lambda}{2} \\ y = \frac{3\lambda}{4} \\ -\frac{\lambda}{2} + \frac{9\lambda}{4} + 1 = 0 \\ \lambda = -\frac{4}{7} \end{cases}$

$\lambda = -\frac{4}{7}$; $x = \frac{2}{7}$; $y = -\frac{3}{7}$

$M\left(\frac{2}{7}; -\frac{3}{7}\right)$

• $x + 3y + 1 = 0 \Rightarrow dx = -3dy$

2) $\frac{\partial^2 z}{\partial x^2} = 2$; $\frac{\partial^2 z}{\partial y^2} = -4$; $\frac{\partial^2 z}{\partial x \partial y} = 0$

$d^2L = 2dx^2 - 4dy^2 = 18dy^2 - 4dy^2 = 14dy^2 > 0$

$\Rightarrow$ т. усл. минимума

$M\left(\frac{2}{7}; -\frac{3}{7}\right)$

6 Вариант №5

$\boxed{№5}$ $u = x + 2z + 2y$ $\quad x^2+y^2+z^2=1$ $\quad x>0$, $y>0$

1) $L = x + 2z + 2y + \lambda(x^2+y^2+z^2-1)$

$$\begin{cases} \frac{\partial L}{\partial x} = 1+2\lambda x = 0 \\ \frac{\partial L}{\partial y} = 2+2\lambda y = 0 \\ \frac{\partial L}{\partial z} = 2+2\lambda z = 0 \\ \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1=0 \end{cases} \qquad \begin{cases} x = -\frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \\ z = -\frac{1}{\lambda} \\ \frac{1}{4\lambda^2} + \frac{2}{\lambda^2} = 1 \Rightarrow \lambda = \pm\frac{3}{2} \end{cases}$$

$x = \frac{1}{3}$; $y = \frac{2}{3}$; $z = \frac{2}{3}$; $\lambda = -\frac{3}{2}$

$M\left(\frac{1}{3}; \frac{2}{3}; \frac{2}{3}\right)$

2) $\frac{\partial^2 L}{\partial x^2} = 2\lambda$; $\frac{\partial^2 L}{\partial y^2} = 2\lambda$; $\frac{\partial^2 L}{\partial z^2} = 2\lambda$

$\frac{\partial^2 L}{\partial x \partial y} = 0$; $\frac{\partial^2 L}{\partial x \partial z} = 0$; $\frac{\partial^2 L}{\partial y \partial z} = 0$

$d^2L = 2\lambda(dx^2+dy^2+dz^2) > 0$

$\Rightarrow M$ – т. условного миним.

$M\left(\frac{1}{3}; \frac{2}{3}; \frac{2}{3}\right)$