Preparation for Vega-Quantathon: set 1

Contents

While implementing the functions below, you need to account for the singularities of the type 0/0.

Yield from maturity and discount factor

Input:

t_0 : the initial time given as year fraction.

Output: continuously compounded yield surface $\gamma = (\gamma(t, D(t)))_{t>t_0}$, where t is the maturity and $D(t)$ is the discount factor.

We recall that

$$D(t) = e^{-\gamma(t)(t-t_0)}, \quad t > t_0.$$

Yield curve computed from discount curve

Input:

$D = (D(t))_{t \geq t_0}$: the discount curve.

t_0 : the initial time given as year fraction.

Output: continuously compounded yield curve $\gamma = (\gamma(t))_{t \geq t_0}$.

We recall that

$$D(t) = e^{-\gamma(t)(t-t_0)}, \quad t \geq t_0.$$

Algorithm. To handle the singularity at $t \approx t_0$, $t \geq t_0$, we take sufficiently small ϵ (say, $\epsilon = 10^{-10}$) and consider two cases:

1. If $t_0 < t < t_0 + \epsilon$, then

$$\gamma(t) = -\frac{1}{t-t_0} \log D(t) \approx -\frac{1}{\epsilon} \log D(t_0 + \epsilon) \approx \frac{1 - D(t_0 + \epsilon)}{\epsilon}.$$

   Effectively, we return the interest rate for period ϵ.

2. If $t \geq t_0 + \epsilon$, then

$$\gamma(t) = -\frac{1}{t-t_0} \log D(t).$$

Yield shape curve 1

Input:

$\lambda \geq 0$: the mean-reversion rate.

t_0 : the initial time given as year fraction.

Output: function

$$\Gamma(t) = \frac{1 - e^{-\lambda(t-t_0)}}{\lambda(t-t_0)}, \quad t \geq t_0,$$

appearing in the descriptions of yield curves for various financial models (Hull-White, Nelson-Siegel, Svensson, Vasicek).

Algorithm. To handle the singularity at $\lambda(t - t_0) \approx 0$, we introduce function

$$x(t) = \lambda(t - t_0), \quad t \geq t_0,$$

We have Taylor's expansion:

$$f(x) \triangleq \frac{1}{x}\left(1 - e^{-x}\right) = \frac{1}{x}\left(1 - \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{(-x)^k}{(k+1)!}.$$

For small values of x we get the quadratic approximation:

$$f(x) \approx g(x) \triangleq 1 - \frac{x}{2} + \frac{x^2}{6}, \quad x \to 0.$$

Thus, we can approximate $\Gamma = \Gamma(t)$ as

$$\Gamma(t) \approx f(x(t))1_{\{x(t)>\epsilon\}} + g(x(t))1_{\{x(t)\leq\epsilon\}}, \quad t \geq t_0.$$

where ϵ is a sufficiently small number (say, $\epsilon = 10^{-10}$).

Yield shape curve 2

Input:

$\lambda \geq 0:$ the mean-reversion rate.

$t_0:$ the initial time given as year fraction.

Output: function

$$\Gamma(t) = \frac{1 - \exp(-\lambda(t - t_0))}{\lambda(t - t_0)} - \exp(-\lambda(t - t_0)), \quad t \geq t_0,$$

appearing in the descriptions of yield curves for various financial models (Nelson-Siegel, Svensson).

Algorithm. To handle the singularity at $\lambda(t - t_0) \approx 0$, we introduce function

$$x(t) = \lambda(t - t_0), \quad t \geq t_0,$$

We have Taylor's expansion:

$$\begin{aligned} f(x) &\triangleq \frac{1}{x}\left(1-e^{-x}\right)-e^{-x} = \frac{1}{x}\left(1-\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\right)-\sum_{k=0}^{\infty}\frac{(-x)^k}{k!} \\ &= \sum_{k=0}^{\infty}(-x)^k\left(\frac{1}{(k+1)!}-\frac{1}{k!}\right) = \sum_{k=1}^{\infty}(-x)^k\left(\frac{1}{(k+1)!}-\frac{1}{k!}\right). \end{aligned}$$

For small values of x we get the quadratic approximation:

$$f(x) \approx g(x) \triangleq \frac{x}{2}-\frac{x^2}{3}, \quad x \to 0.$$

Thus, we can approximate $\Gamma = \Gamma(t)$ as

$$\Gamma(t) \approx f(x(t))1_{\{x(t)>\epsilon\}} + g(x(t))1_{\{x(t)\leq\epsilon\}}, \quad t \geq t_0.$$

where ϵ is a sufficiently small number (say, $\epsilon = 10^{-10}$).

The Nelson-Siegel yield curve

Input:

$(c_i)_{i=0,1,2}$: the constant coefficients of the Nelson-Siegel model of interest rates.

$\lambda > 0$: the mean-reversion rate.

t_0 : the initial time given as year fraction.

Output: the Nelson-Siegel yield curve $\gamma = (\gamma(t))_{t\geq t_0}$. It has the form:

$$\gamma(t) = c_0 + c_1\frac{1-e^{-\lambda(t-t_0)}}{\lambda(t-t_0)} + c_2\left(\frac{1-e^{-\lambda(t-t_0)}}{\lambda(t-t_0)} - e^{-\lambda(t-t_0)}\right), \quad t \geq t_0.$$

The Nelson-Siegel discount curve

Input:

$(c_i)_{i=0,1,2}$: the constant coefficients of the Nelson-Siegel model of interest rates.

$\lambda > 0$: the mean-reversion rate.

t_0 : the initial time given as year fraction.

Output: the discount curve

$$D(t) = e^{-\gamma(t)(t-t_0)}, \quad t \geq t_0,$$

where the yield curve has the Nelson-Siegel form:

$$\gamma(t) = c_0 + c_1 \frac{1 - e^{-\lambda(t-t_0)}}{\lambda(t-t_0)} + c_2 \left(\frac{1 - e^{-\lambda(t-t_0)}}{\lambda(t-t_0)} - e^{-\lambda(t-t_0)} \right), \quad t \geq t_0.$$

Forward exchange rate from domestic and foreign discount factors

Input:

$S(t_0)$: the spot exchange rate (the number of units of domestic currency per one unit of foreign currency).

Output:

$F = F(D^d, D^f)$: the forward FX rates. Here D^d and D^f are domestic and foreign discount factors.

Algorithm. The value of the long position at expiration date t is

$$V(t) = 1 \text{ unit of foreign currency} - F(t) \text{ units of domestic currency.}$$

By the financial meaning of forward contract, the value at initial time t_0 is zero: $V(t_0) = 0$. On the other hand, by replication,

$$V(t_0) = S(t_0)D^f(t) - D^d(t)F(t).$$

It follows that

$$F(t) = S(t_0)\frac{D^f(t)}{D^d(t)}, \quad t \geq t_0.$$

Forward price curve for a cash flow

Input:

$(P_i)_{i=1,\ldots,M}$: the cash payments.

$(t_i)_{i=1,\ldots,M}$: the payment times, $t_1 > t_0$.

$D = (D(t))_{t \geq t_0}$: the discount curve.

t_0 : the initial time.

Output:

$F = (F(t))_{t \in [t_0, t_M]}$: the forward prices for the cash flow.

The buyer pays forward price $F(t)$ at delivery time t and then receives payments P_i at payment times t_i, $i = 1, \ldots, M$, such that $t_i > t$. The forward price $F(t)$ is set at t_0 to make the value of the forward contract to be zero.

Forward price curve for a coupon bond

Input:

q : the coupon rate.

δt : the time interval between coupon payments.

T : the maturity.

$D = (D(t))_{t \geq t_0}$: the discount curve.

t_0 : the initial time = the issue time for the forward.

bClean : the boolean parameter specifying the type of the prices: "clean" or "dirty". The dirty price is the actual amount paid in a transaction. The clean price is the difference between the dirty price and the accrued interest. If t is the settlement time and t_i is the previous coupon time (or the issue time if no coupons were paid before t), then the accrued interest is given by

$$A(t) = q(t - t_i).$$

Output:

$F = (F(t))_{t \in [t_0, T]}$: the forward prices for the bond.

The bond pays coupons $q\delta t$ at times $(t_i)_{i=1,\dots,M}$ such that

$$t_0 < t_1 \leq t_0 + \delta t, \quad t_{i+1} - t_i = \delta t, \quad t_M = T.$$

The bond also pays notional $N = 1$ at maturity T. The buyer of the forward contract pays "dirty" forward price $F(t)$ (bClean = false) at delivery time t and receives the bond, that is, gets the notional $N = 1$ at maturity T and coupons $q\delta t$ at times t_i, $i = 1, \dots, M$, such that $t_i > t$.

Discount curve obtained by linear interpolation of yields

Input:

$(t_i)_{i=1,\dots,M}$: the strictly increasing vector of maturities, $t_1 > t_0$.

$(d_i)_{i=1,\dots,M}$: the market discount factors.

r_0 : the initial short-term interest rate.

t_0 : the initial time given as year fraction.

Output: the discount curve

$$d(t) = \exp(-\gamma(t)(t - t_0)), \quad t \in [t_0, t_M],$$

where yield curve $\gamma = \gamma(t)$ is obtained by the linear interpolation of the market yields to maturity:

$$\gamma(t) = LinInterp\left((t_i)_{i=0,1,\dots,M}, (\gamma_i)_{i=0,\dots,M}\right), \quad t \in [t_0, t_M],$$
$$\gamma_0 = r_0, \quad \gamma_i = -\frac{\log(d_i)}{t_i - t_0}, \quad i = 1, \dots, M,$$