Probability and Statistics, Spring 2018

Homework 6

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6.1.2

$$
\begin{aligned}
P_W(-4) &= P_{X,Y}(-2,-1) = 3/14, \\
P_W(-2) &= P_{X,Y}(-2,0) + P_{X,Y}(0,-1) = 3/14, \\
P_W(0) &= P_{X,Y}(-2,1) + P_{X,Y}(2,-1) = 2/14, \\
P_W(2) &= P_{X,Y}(0,1) + P_{X,Y}(2,0) = 3/14, \\
P_W(4) &= P_{X,Y}(2,1) = 3/14.
\end{aligned}
$$

6.3.6 Since $\mathrm{E}[T] = \frac{1}{\lambda} = 200 \rightarrow \lambda = \frac{1}{200}$.

We have

$$
f_X(x) = \begin{cases} \frac{1}{200} e^{-x/200} & x \geq 0, \\ 0 & x < 0. \end{cases}
$$

(a)

$$
\begin{aligned}
\mathrm{P}[C = 30] &= \int_0^{300} \frac{1}{200} e^{-x/200} dx \\
&= -(e^{-1.5} - 1) = 1 - e^{-1.5} \\
&= 1 - 0.223130 \\
&= 0.776869.
\end{aligned}
$$

(b)

$$
\begin{aligned}
C &= 30 + 0.5(T - 300) \\
T &= 2(C - 30) + 300 \\
&= 2C + 240.
\end{aligned}
$$

We have

$$
f_C(c) = \frac{1}{200} e^{-\frac{1}{200}(2c+240)}.
$$

Therefore,

$$
f_C(c) = \begin{cases} 0.776869 & c = 30, \\ \frac{1}{200} e^{-\frac{1}{200}(2c+240)} & c > 30. \\ 0 & \text{otherwise.} \end{cases}
$$

(c)

$$\begin{aligned}
\mathrm{E}[C] &= \int_{-\infty}^{\infty} c f_C(c) dc \\
&= \int_{30}^{\infty} c \frac{1}{200} e^{-\frac{1}{200}(2c+240)} dc \\
&= \left[100 \cdot 30 e^{-(60+240)/200} + 10000 e^{-(60+240)/200}\right] \\
&= 13000 e^{-1.5} \\
&= 13000(0.223130) \\
&\approx 2900.69.
\end{aligned}$$

6.3.8 (a)

$$\begin{aligned}
\mathrm{P}[Y = 0.5] &= \mathrm{P}[0 \le X \le 1] \\
&= \int_0^1 f_X(x) dx \\
&= \int_0^1 x/2 dx \\
&= \left.\frac{x^2}{4}\right|_0^1 \\
&= \frac{1}{4}.
\end{aligned}$$

(b) Split limit of x in the form of y in two parts as, $\frac{1}{2} < y \le 1$ and $1 < y \le 2$. Since $Y \ge \frac{1}{2}$, there fore it can conclude that for $y < \frac{1}{2}$,

$$F_Y(y) = 0.$$

And for $y \ge 2$,

$$F_Y(y) = 1.$$

CDF $F_Y(y)$ for limits $1 < y \le 2$ is

$$\begin{aligned}
F_Y(y) &= P[X \le y] \\
&= \int_0^y f_X(x) dx \\
&= \int_0^y \frac{x}{2} dx \\
& \frac{y^2}{4}.
\end{aligned}$$

Hence the CDF of Y can be written as,

$$F_Y(y) = \begin{cases} 0 & y < \frac{1}{2}, \\ \frac{1}{4} & \frac{1}{2} \le y \le 1, \\ \frac{y^2}{4} & 1 < y < 2, \\ 1 & y \ge 2. \end{cases}$$

6.3.10 (a) For $y < 0$, $F_Y(y) = 0$.
For $y > 36$, $F_Y(y) = 1$.
For $0 \leq y \leq 36$,

$$\begin{aligned}
F_Y(y) &= \mathrm{P}[Y \leq y] = \mathrm{P}[9X^2 \leq y] \\
&= \mathrm{P}[X^2 \leq \frac{y}{9}] \\
&= \mathrm{P}[-\frac{\sqrt{y}}{3} \leq X \leq \frac{\sqrt{y}}{3}] \\
&= \int_{-\frac{\sqrt{y}}{3}}^{\frac{\sqrt{y}}{3}} f_X(x)dx \\
&= \int_{-\frac{\sqrt{y}}{3}}^{\frac{\sqrt{y}}{3}} \frac{1}{4}dx \\
&= \frac{\sqrt{y}}{6}.
\end{aligned}$$

Thus, the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ \frac{\sqrt{y}}{6} & 0 \leq y \leq 36, \\ 1 & y > 36. \end{cases}$$

The PDF of Y is

$$f_Y(y) = \begin{cases} \frac{1}{12\sqrt{y}} & 0 \leq y \leq 36 \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$F_W(w) = \begin{cases} \frac{1}{12\sqrt{w}} & 0 \leq y \leq 16, \\ 16 & \text{otherwise.} \end{cases}$$

6.4.1

$$\begin{aligned}
F_V(v) &= \mathrm{P}[V \leq v] \\
&= \mathrm{P}[\max(X, Y) \leq v] \\
&= \mathrm{P}[X \leq v, Y \leq v] \\
&= \int_0^v \int_0^v 6xy^2 dxdy \\
&= v^5.
\end{aligned}$$

$$F_V(v) = \begin{cases} 0 & v < 0, \\ v^5 & 0 \leq v \leq 1, \\ 1 & v > 1. \end{cases}$$

PDF $f_V(v)$ is

$$f_V(v) = \begin{cases} 5v^4 & 0 \leq v \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$