Introduction to Computational Logic

Homework 3

DUE DATE: NOVEMBER 8, 2017 學號：**b03902129** 系級：資工四 姓名：陳鵬宇

Let S be a binary predicate symbol, P and Q unary predicate symbols.

(1) Find a natural deduction proof to show

$$\exists x \exists y (S(x,y) \lor S(y,x)) \vdash \exists x \exists y S(x,y).$$

1		$\exists x \exists y (S(x,y) \lor S(y,x))$	premise
2	x_0		
3		$\exists y (S(x_0,y) \lor S(y,x_0))$	assumption
4	y_0		
5		$S(x_0,y_0) \lor S(y_0,x_0)$	assumption
6		$S(x_0,y_0)$	assumption
7		$\exists y S(x_0,y)$	$\exists y$i 6
8		$\exists x \exists y S(x,y)$	$\exists x$i 7
9		$S(y_0,x_0)$	assumption
10		$\exists y S(y_0,y)$	$\exists y$i 9
11		$\exists x \exists y S(x,y)$	$\exists x$i 10
12		$\exists x \exists y S(x,y)$	$\lor$e $5, 6-8, 9-11$
13		$\exists x \exists y S(x,y)$	$\exists y$e $3, 4-12$
14		$\exists x \exists y S(x,y)$	$\exists x$e $1, 2-13$

(2) Find a natural deduction proof to show

$$\forall x \forall y \forall z (S(x,y) \land S(y,z) \Rightarrow S(x,z)), \forall x \neg S(x,x) \vdash \forall x \forall y (S(x,y) \Rightarrow \neg S(y,x)).$$

1		$\forall x \forall y \forall z (S(x,y) \land S(y,z) \Rightarrow S(x,z))$	premise
2		$\forall x \neg S(x,x)$	premise
3	a		
4		$\forall y \forall z (S(a,y) \land S(y,z) \Rightarrow S(a,z))$	$\forall x$e 1
5	b		
6		$\forall z (S(a,b) \land S(b,z) \Rightarrow S(a,z))$	$\forall y$e 4
7		$S(a,b) \land S(b,a) \Rightarrow S(a,a)$	$\forall z$e 6
8		$\neg S(a,a)$	$\forall x$e 2
9		$S(a,b)$	assumption
10		$S(b,a)$	assumption
11		$S(a,b) \land S(b,a)$	$\land$i $9, 10$
12		$S(a,a)$	$\Rightarrow$ e $11, 7$
13		$\perp$	$\neg$e $12, 8$
14		$\neg S(b,a)$	$\neg$i $10-13$
15		$S(a,b) \Rightarrow \neg S(b,a)$	$\Rightarrow$ i $9-14$
16		$\forall y S(a,y) \Rightarrow \neg S(y,a)$	$\forall y$i $5-15$
17		$\forall x \forall y S(x,y) \Rightarrow \neg S(y,x)$	$\forall x$i $3-16$

(3) Find a natural deduction proof to show

$$\exists x \exists y (S(x,y) \vee S(y,x)), \neg \exists x S(x,x) \vdash \exists x \exists y \neg (x = y).$$

1		$\exists x \exists y (S(x,y) \vee S(y,x))$	premise
2		$\neg \exists S(x,x)$	premise
3	x_0		
4		$\exists y (S(x_0,y) \vee S(y,x_0))$	assumption
5	y_0		
6		$S(x_0,y_0) \vee S(y_0,x_0)$	assumption
7		$x_0 = y_0$	assumption
8		$S(x_0,x_0)$	assumption
9		$S(y_0,y_0)$	$=$ e $7, 8$
10		$\exists x\ S(x,x)$	$\exists x$i 9
11		$\bot$	$\neg$e $10, 2$
12		$S(y_0,x_0)$	assumption
13		$S(y_0,y_0)$	$=$ e $7, 12$
14		$\exists x\ S(x,x)$	$\exists x$i 13
15		$\bot$	$\neg$e $14, 2$
16		$\bot$	$\vee$e $6, 8-11, 12-15$
17		$\neg(x_0 = y_0)$	$\neg$i $7-16$
18		$\exists y \neg (x_0 = y)$	$\exists y$i 17
19		$\exists x \exists y \neg (x = y)$	$\exists x$i 18
20		$\exists x \exists y \neg (x = y)$	$\exists y$e $4, 5-19$
21		$\exists x \exists y \neg (x = y)$	$\exists x$e $1, 3-20$

(4) Show that there is no natural deduction proof for

$$\forall x (P(x) \vee Q(x)) \vdash \forall x P(x) \vee \forall x Q(x).$$

Consider the outcomes from flipping a coin: $P(x)$: "x is heads," $Q(x)$: "x is tails", with x belonging to the domain of all coins.

Then we can say truthfully that:

$$\forall x (P(x) \vee Q(x)).$$

But there's a problem with the RHS:

$$\forall x P(x) \vee \forall x Q(x).$$

That is, the right hand side claims that every coin tossed turns of heads, or else, every coin tossed turns of tails.

(5) Semantically show

$$\forall x \neg \phi \vDash \neg \exists x \phi.$$

Let $\mathcal{M}$ be a model that $\mathcal{M} \vDash \forall x \neg \phi$ (in words, an interpretation satisfying the formula).

And assume that $\mathcal{M} \nvDash \neg \exists x \phi$, this means that $\mathcal{M} \vDash \exists x \phi$, i.e., these are some $t \in \mathcal{M}$, $\phi(t/x)$ holds.

But we have: $\mathcal{M} \vDash \forall x \neg \phi$, i.e., $\forall t \in \mathcal{M}$, $\neg \phi(t/x)$ holds.

There is a contradiction, thus $\mathcal{M} \vDash \neg \exists x \phi$.