Probability and Statistics, Spring 2018

Homework 6

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7.1.2 The PMF of X is

$$\mathrm{P}_X(x) = \mathrm{P}[X = x] = \begin{cases} \frac{1}{6} & x = 0, 1, \ldots, 5 \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{E}[X] = \frac{0+5}{2} = 2.5.$

$$\begin{aligned} \mathrm{E}[X \mid X \geq \mathrm{E}[X]] &= \mathrm{E}[X \mid X \geq \frac{5}{2}] \\ &= \frac{3+4+5}{3} = 4. \end{aligned}$$

7.1.7 (a) Given that a person is healthy, the conditional PDF of Gaussian($\mu = 90$, $\sigma = 20$) random variable is

$$\begin{aligned} f_{X|H(x)} &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \\ &= \frac{1}{\sqrt{2\pi 400}} e^{-(x-90)^2/800} \\ &= \frac{1}{20\sqrt{2\pi}} e^{-(x-90)^2/800}. \end{aligned}$$

(b)

$$\begin{aligned} \mathrm{P}[T^+ \mid H] &= \mathrm{P}[x \geq 140 \mid H] \\ &= \mathrm{P}\Big[\frac{X-\mu}{\sigma} \geq \frac{140-\mu}{\sigma} \mid H\Big] \\ &= \mathrm{P}\Big[\frac{X-90}{20} \geq \frac{140-90}{20}\Big] \\ &= \mathrm{P}\Big[\frac{X-90}{20} \geq 2.5\Big] \\ &= \mathrm{P}[Z \geq 2.5] \\ &= 1 - \mathrm{P}[Z \leq 2.6] \\ &= 1 - \Phi(2.5) \\ &\approx 1 - 0.99379 \\ &= 0.00621. \end{aligned}$$

$$\begin{aligned} \mathrm{P}[T^- \mid H] &= \mathrm{P}[x \leq 110 \mid H] \\ &= \mathrm{P}\Big[\frac{X-\mu}{\sigma} \leq \frac{110-\mu}{\sigma} \mid H\Big] \\ &= \mathrm{P}\Big[\frac{X-90}{20} \leq \frac{110-90}{20}\Big] \\ &= \mathrm{P}[Z \leq 1] \\ &= \Phi(1) \\ &= 0.8413. \end{aligned}$$

(c) $\mathrm{P}[H \mid T^-] = \frac{\mathrm{P}[H]\cdot \mathrm{P}[T^- \mid H]}{\mathrm{P}[T^-]}$
where $\mathrm{P}[T^-] = \mathrm{P}[D] \cdot \mathrm{P}[T^- \mid D] + \mathrm{P}[H] \cdot \mathrm{P}[T^- \mid H]$.

$$\begin{aligned} P[T^- \mid D] &= \mathrm{P}[x \leq 110 \mid D] \\ &= \mathrm{P}\Big[\frac{X-\mu}{\sigma} \leq \frac{110-\mu}{\sigma} \mid D\Big] \\ &= \mathrm{P}\Big[\frac{X-60}{40} \leq \frac{110-60}{40}\Big] \\ &= \mathrm{P}[Z \leq 1.25] \\ &= \Phi(1.25) \\ &= 0.8944. \end{aligned}$$

$$\mathrm{P}[H \mid T^-] = \frac{0.9 \cdot 0.8413}{(0.1 \cdot 0.8944) + (0.9 \cdot 0.8413)} \approx 0.8943.$$

(d) Let

$$\begin{aligned} q &= \mathrm{P}[T^0 \mid H] \\ &= 1 - \mathrm{P}[T^- \mid H] - \mathrm{P}[T^+ \mid H] \\ &= 1 - 0.8413 - 0.0062 = 0.1525. \end{aligned}$$

$p = 1 - q = 1 - 0.1525 = 0.8475.$
We have
$P_{N|H}(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$

7.2.6 (a)

$$\begin{aligned} \mathrm{P}[\text{be sent to } A] &= \mathrm{P}[X=2] + \mathrm{P}[X=4] + \mathrm{P}[X=6] + \mathrm{P}[X=8] \\ &= 0.15 + 0.15 + 0.1 + 0.1 = 0.5. \end{aligned}$$

$$P_{X|A}(x) = \begin{cases} \frac{0.15}{0.5} = 0.3 & x = 2, 4, \\ \frac{0.1}{0.5} = 0.2 & x = 6, 8, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \mathrm{E}[X \mid A] &= \sum_{x=2,4,6,8} x \cdot P_{X|A}(x) \\ &= (2 \cdot 0.3) + (4 \cdot 0.3) + (6 \cdot 0.2) + (8 \cdot 0.2) \\ &= 0.6 + 1.2 + 1.2 + 1.6 = 4.6. \end{aligned}$$

$$\begin{aligned} \mathrm{E}[X^2 \mid A] &= \sum_{x=2,4,6,8} x^2 \cdot P_{X|A}(x) \\ &= (2^2 \cdot 0.3) + (4^2 \cdot 0.3) + (6^2 \cdot 0.2) + (8^2 \cdot 0.2) \\ &= 1.2 + 4.8 + 7.2 + 12.8 = 26. \end{aligned}$$

$$\begin{aligned}\mathrm{Var}[X \mid A] &= \mathrm{E}[X^2 \mid A] - (\mathrm{E}[X \mid A])^2\\ &= 26 - 4.6^2\\ &= 26 - 21.16 = 4.84.\end{aligned}$$

$\sigma_{X|A} = \sqrt{4.84} = 2.2.$

(b)

$$\begin{aligned}\mathrm{P}[\text{be sent to } B \text{ and } \leq 6] &= \mathrm{P}[X=1] + \mathrm{P}[X=3] + \mathrm{P}[X=5]\\ &= 0.15 + 0.15 + 0.1 = 0.4.\end{aligned}$$

$$P_{X|B}(x) = \begin{cases} \frac{0.15}{0.4} = 0.375 & x = 1, 3,\\ \frac{0.1}{0.4} = 0.25 & x = 5,\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{aligned}\mathrm{E}[X \mid B] &= \sum_{x=1,3,5} x \cdot P_{X|A}(x)\\ &= (1 \cdot 0.375) + (3 \cdot 0.375) + (5 \cdot 0.25)\\ &= 0.375 + 1.125 + 1.25 = 2.75.\end{aligned}$$

$$\begin{aligned}\mathrm{E}[X^2 \mid B] &= \sum_{x=1,3,5} x^2 \cdot P_{X|A}(x)\\ &= (1^2 \cdot 0.375) + (3^2 \cdot 0.375) + (5^2 \cdot 0.25)\\ &= 0.375 + 3.375 + 6.25 = 10.\end{aligned}$$

$$\begin{aligned}\mathrm{Var}[X \mid B] &= \mathrm{E}[X^2 \mid B] - (\mathrm{E}[X \mid B])^2\\ &= 10 - 2.75^2\\ &= 10 - 7.5625 = 2.4375.\end{aligned}$$

$\sigma_{X|B} = \sqrt{2.4375} \approx 1.56.$

7.3.5 (a)

$$\begin{aligned}\mathrm{P}[A] &= \mathrm{P}[Y \leq 1]\\ &= \int_0^1 \int_0^1 f_{X,Y}(x,y)dydx\\ &= \int_0^1 \int_0^1 \frac{x+y}{3}dydx\\ &= \frac{1}{3}.\end{aligned}$$

(b)

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{\mathrm{P}[A]} = \frac{(x+y)/3}{1/3} = (x+y) & 0 \leq x \leq 1; 0 \leq y \leq 1,\\ 0 & \text{otherwise.}\end{cases}$$

(c)

$$
\begin{aligned}
f_{X|A}(x) &= \int_0^1 f_{X,Y|A}(x,y)dy \\
&= \int_0^1 (x+y)dy \\
&= x + \frac{1}{2}.
\end{aligned}
$$

Thus,

$$
f_{X|A}(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
f_{Y|A}(y) &= \int_0^1 f_{X,Y|A}(x,y)dx \\
&= \int_0^1 (x+y)dx \\
&= y + \frac{1}{2}.
\end{aligned}
$$

Thus,

$$
f_{X|A}(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}
$$

7.4.10 (a) X and Y are independent if

$$
P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y).
$$

$\mathrm{P}_{X,Y}(x,y)$	$y=-1$	$y=0$	$y=1$	$\mathrm{P}_X(x)$
$x=-1$	3/16	1/16	0	$\mathrm{P}_X(-1) = 1/4$
$x=0$	1/6	1/6	1/6	$\mathrm{P}_X(0) = 1/2$
$x=1$	0	1/8	1/8	$\mathrm{P}_X(1) = 1/4$
$\mathrm{P}_Y(y)$	$\mathrm{P}_Y(-1) = 17/48$	$\mathrm{P}_Y(0) = 17/48$	$\mathrm{P}_Y(1) = 14/48$	

$P_{X,Y}(-1,1) = 0 \neq P_X(-1) \cdot P_Y(1) = \frac{1}{4} \cdot \frac{14}{48}$.
Hence, X and Y are not independent.

(b)