Probability and Statistics, Spring 2018

Homework 8

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8.2.1

$$\begin{aligned}
\mathrm{E}[X_1 X_2 \cdots X_i \ldots X_n] &= \mathrm{E}[X_1]\mathrm{E}[X_2] \cdots \mathrm{E}[X_i] \cdot \mathrm{E}[X_n] \\
&= 0
\end{aligned}$$

Thus, the i, j^{th} entry in the convariance matric C_X is defined as,

$$C_X(i,j) = \begin{cases} \sigma_i^2 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$C_X = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

8.5.1 (a)

$$\begin{aligned}
R_X &= C_X + \mu_X \mu_X' \\
&= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} \begin{bmatrix} 4 & 8 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 16 & 32 & 24 \\ 32 & 64 & 48 \\ 24 & 48 & 36 \end{bmatrix} \\
&= \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix}.
\end{aligned}$$

(b) Let $Y = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$. Since Y is a subset of the components of X, it is a Gaussian random vector. The expected value vector is,

$$\mu_Y = \begin{bmatrix} \mathrm{E}[X_1] \\ \mathrm{E}[X_2] \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}.$$

The convariance matrix is,

$$C_Y = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] \\ C_{X_1} X_2 & \mathrm{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}.$$

$$\det(C_Y) = 16 - 4 = 12.$$

The inverse of C_Y is,

$$C_Y^{-1} = \frac{1}{12}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{3}\end{bmatrix}.$$

Thus,

$$\begin{aligned}(y-\mu_Y)'C_Y^{-1}(y-\mu_Y) &= \begin{bmatrix}y_1-4 & y_2-8\end{bmatrix}\begin{bmatrix}1/3 & 1/6\\ 1/6 & 1/3\end{bmatrix}\begin{bmatrix}y_1-4\\ y_2-8\end{bmatrix}\\ &= \frac{y_1^2}{3}+\frac{y_1y_2}{3}-\frac{16y_1}{3}-\frac{20y_2}{3}+\frac{y_2^2}{3}+\frac{112}{3}.\end{aligned}$$

The PDF of Y is,

$$\begin{aligned}f_Y(y) &= f_{X_1,X_2}(x_1,x_2)\\ &= \frac{1}{(2\pi)^{n/2}(\det(C_X))^{1/2}}e^{-\frac{(y-\mu_Y)C_Y^{-1}(y-\mu_Y)}{2}}\\ &= \frac{1}{4\sqrt{3}\pi}e^{-\frac{y_1^2+y_1y_2-16y_1-20y_2+y_2^2+112}{6}}.\end{aligned}$$

(c) From the result of (b), the random variable X_1 is a Gaussian random variable with mean 4 and standard deviation 2.

Thus,

$$\begin{aligned}\mathrm{P}[X_1>8] &= 1-\mathrm{P}\left[\frac{X_1-4}{2}<\frac{8-4}{2}\right]\\ &= 1-\Phi(2)\\ &= 1-0.97725\\ &= 0.0228.\end{aligned}$$

9.1.3 (a) The PMF of N_1, the number of phone calls needed to obtain the correct answer, can be determined by observing that if the correct answer is given on the nth call, then the previous $n-1$ calls must have given wrong answers so that

$$P_{N_1}(n) = \begin{cases}(3/4)^{n-1}(1/4) & n=1,2,\ldots,\\ 0 & \text{otherwise.}\end{cases}$$

(b) $E[N_1] = \frac{1}{p} = 4$.

(c) Like (a),

$$P_{N_4}(n_4) = \begin{cases}\binom{n-1}{3}(3/4)^{n-4}(1/4)^4 & n=4,5,\ldots,\\ 0 & \text{otherwise.}\end{cases}$$

(d) By hint, $\mathrm{E}[N_4] = 4\mathrm{E}[N_1] = 16$.

9.2.2 (a)

$$P_J(j) = \begin{cases} 0.6 & j = -2 \\ 0.4 & j = -1. \end{cases}$$

The MGF of J is

$$\begin{aligned} \phi_J(s) &= \mathrm{E}[e^{Js}] \\ &= \sum_j e^{js} P_J(j) \\ &= 0.6e^{-2s} + 0.4e^{-s}. \end{aligned}$$

(b)

$$P_K(k) = \begin{cases} 0.7 & k = -1 \\ 0.2 & k = 0 \\ 0.1 & k = 1. \end{cases}$$

The MGF of K is

$$\begin{aligned} \phi_K(s) &= \mathrm{E}[e^{Ks}] \\ &= \sum_j e^{js} P_K(k) \\ &= 0.7e^{-s} + 0.2 + 0.1e^{s}. \end{aligned}$$

(c)

$$P_M(m) = \begin{cases} 0.42 & m = -3, \\ 0.40 & m = -2, \\ 0.14 & m = -1, \\ 0.04 & m = 0. \end{cases}$$

(d)

$$\begin{aligned} \mathrm{E}[M^4] &= \sum_m m^4 P_M(m) \\ &= (-3)^4 \cdot 0.42 + (-2)^4 \cdot 0.40 + (-1)^4 \cdot 0.14 + 0 \\ &= 40.56. \end{aligned}$$

10.1.1 (a) By $\sigma^2_{Mn(x)} = \sigma^2_X/n$. Realizing that $\sigma^2_X = 25$, we obtain

$$\mathrm{Var}[M_9(X)] = \frac{\sigma_X^2}{9} = \frac{25}{9}.$$

(b)

$$\begin{aligned} \mathrm{P}[X_1 \geq 7] &= 1 - \mathrm{P}[X_1 \leq 7] \\ &= 1 - F_X(7) \\ &= 1 - (1 - e^{-7/5}) \\ &= e^{-7/5} \approx 0.247. \end{aligned}$$

(c) First we express $\mathrm{P}[M_9(X) > 7]$ in terms of $X_1, \ldots, X_9$.

$$\begin{aligned}\mathrm{P}[M_9(X) > 7] &= 1 - \mathrm{P}[M_9(X) \leq 7] \\ &= 1 - \mathrm{P}[(X_1 + \cdots + X_9) \leq 63].\end{aligned}$$

Now the probability that $M_9(X) > 7$ can be approximated using the Central Limit Theorem (CLT).

$$\begin{aligned}\mathrm{P}[M_9(X) > 7] &= 1 - \mathrm{P}[(X_1 + \cdots + X_9) \leq 63] \\ &\approx 1 - \Phi\left(\frac{63 - 9\mu_X}{\sqrt{9}\sigma_X}\right) \\ &= 1 - \Phi(6/5) \approx 0.1151.\end{aligned}$$

10.2.1

$$\mathrm{P}[|W - \mathrm{E}[W]| \geq 200] \leq \frac{\mathrm{Var}[W]}{200^2} \leq \frac{100^2}{200^2} = 0.25.$$