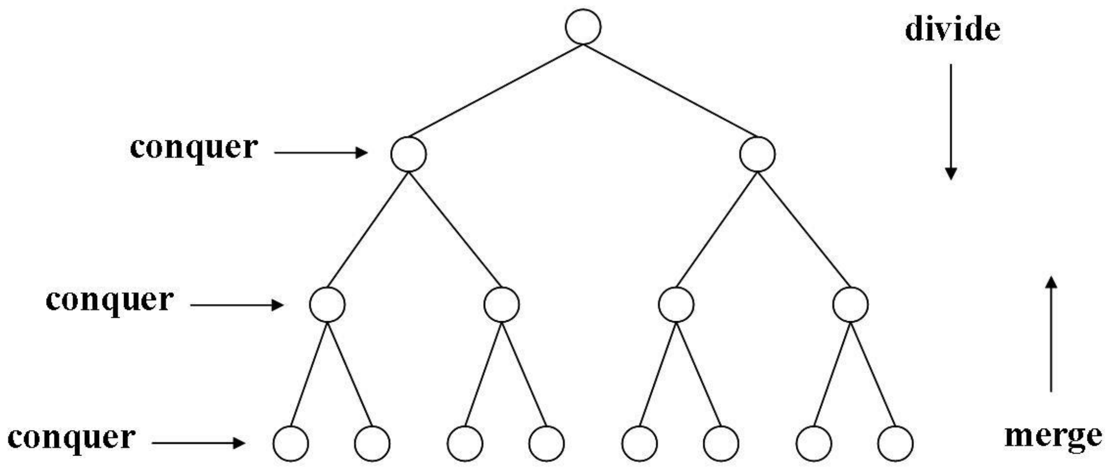
Divide-and-Conquer

Three main steps：

1. **divide;**
2. **conquer;**
3. **merge.**



Let I denote the (sub)problem instance and S be its solution. The divide-and-conquer strategy can be described as follows.

Procedure divide-and-conquer (I, S);

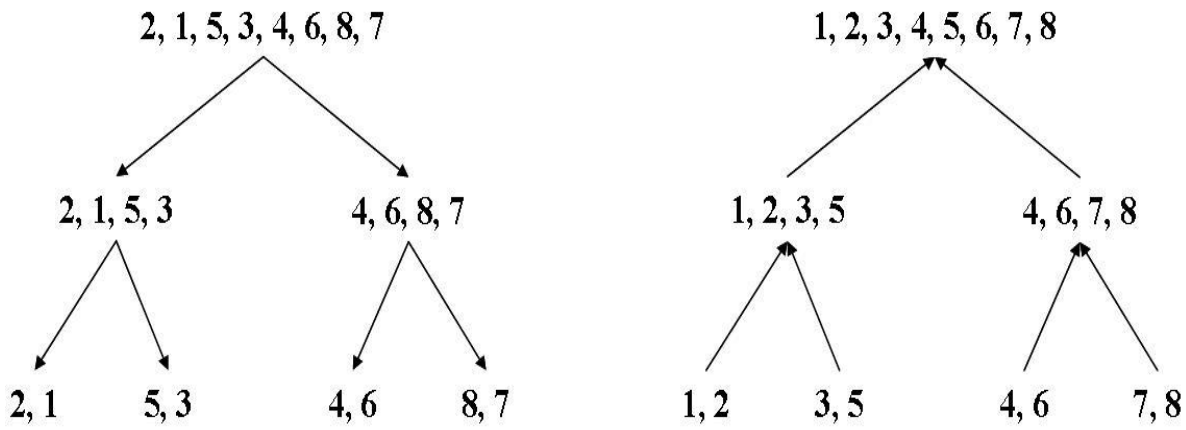
$O(1)$ **if size(I) is small enough, then $S \leftarrow$ solution(I)**
else begin
T_D **split I into I_1, I_2 , …, I_k;**
for $i \leftarrow 1$ to k
T_i **divide-and-conquer(I_i, S_i);**
T_M **$S \leftarrow$ merge(S_1, S_2 , …, S_k)**
end.

$$T = T_D + \sum_{i=1}^{k} T_i + T_M$$

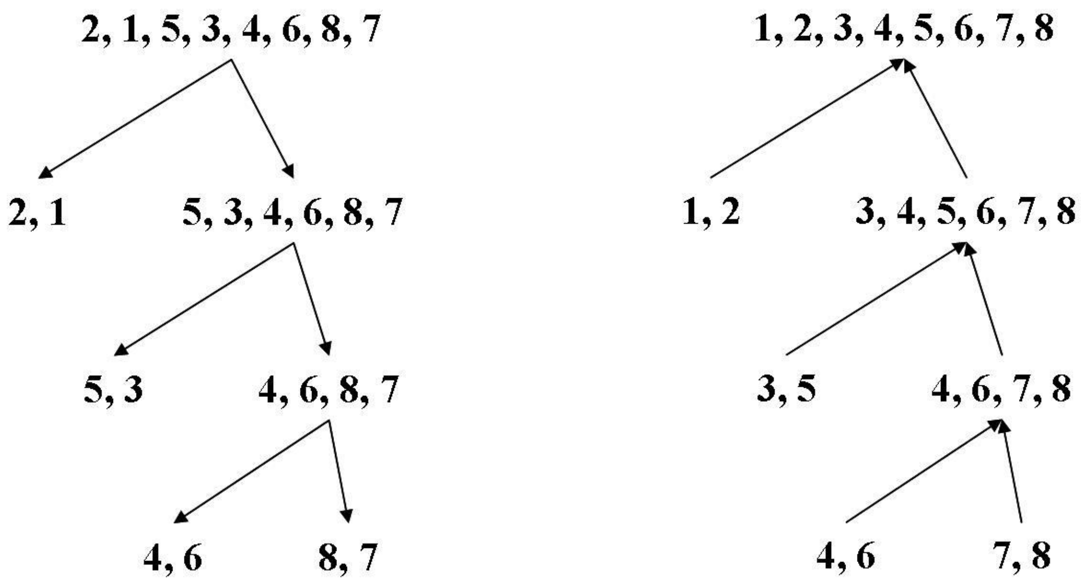
If size(I) = n, k = 2 and size(I_1) = size(I_2), then $T(n) = 2T(n/2) + T_D + T_M$.

Ex.   Merge sort.

(Assume that $m + n$ time steps are required to merge two sorted lists of lengths m and n, respectively.)



16 time steps are required for merging.



18 time steps are required for merging.

$\Rightarrow$ **a problem instance is always split into equal-size subproblem instances.**

Ex. Compute $a_1 + a_2 + \ldots + a_n$.

$$\begin{aligned} T(n) &= 2T(n/2) + O(1) \\ &= O(n) \end{aligned}$$

Ex. Merge sort of $a_1, a_2, \ldots, a_n$.

$$\begin{aligned} T(n) &= 2T(n/2) + O(n) \\ &= O(n\log n) \end{aligned}$$

Ex. Matrix multiplication

$$C_{n\times n} = A_{n\times n} \times B_{n\times n}$$

- **divide**

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$A_{ij}, B_{ij}, C_{ij}, : \quad n/2 \times n/2$$

- **conquer**

$$T_1 = A_{11}B_{11}, \quad T_2 = A_{12}B_{21}, \quad T_3 = A_{11}B_{12}, \quad T_4 = A_{12}B_{22}$$
$$T_5 = A_{21}B_{11}, \quad T_6 = A_{22}B_{21}, \quad T_7 = A_{21}B_{12}, \quad T_8 = A_{22}B_{22}$$

- **merge**

$$C_{11} = T_1 + T_2, \quad C_{12} = T_3 + T_4, \quad C_{21} = T_5 + T_6,$$
$$C_{22} = T_7 + T_8$$

$$\begin{aligned} T(n) &= 8T(n/2) + O(n^2) \\ &= O(n^3) \end{aligned}$$

Strassen's Method

- **conquer**

$$P = (A_{11} + A_{22}) \times (B_{11} + B_{22}) \qquad Q = (A_{21} + A_{22}) \times B_{11}$$

$$R = A_{11} \times (B_{12} - B_{22}) \qquad S = A_{22} \times (B_{21} - B_{11})$$

$$T = (A_{11} + A_{12}) \times B_{22} \qquad U = (A_{21} - A_{11}) \times (B_{11} + B_{12})$$

$$V = (A_{12} - A_{22}) \times (B_{21} + B_{22})$$

- **merge**

$$C_{11} = P + S - T + V \qquad C_{12} = R + T$$

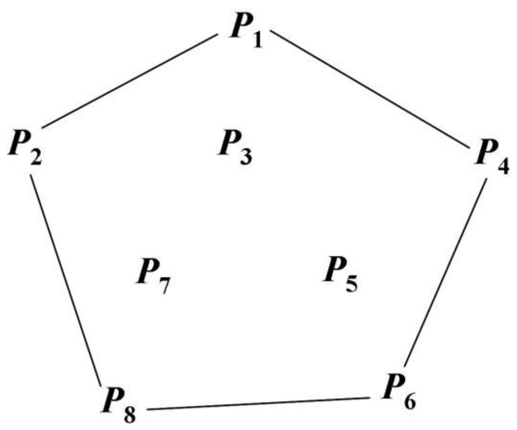
$$C_{21} = Q + S \qquad C_{22} = P + R - Q + U$$

$$\begin{aligned} T(n) &= 7T(n/2) + O(n^2) \\ &= O(n^{\log_2 7}) \\ &\approx O(n^{2.81}) \end{aligned}$$

Ex. The Convex Hull Problem.

Given a set *S* of *n* points in the plane, find its convex hull.

The convex hull of *S* is the smallest convex set that contains all the points of *S*.



The problem is required to report the vertices of the convex hull in either a clockwise sequence or a counterclockwise sequence, e.g., p_2, p_1, p_4, p_6, p_8 or p_6, p_4, p_1, p_2, p_8.

Graham's Scan

Step 1. **Find an interior point O arbitrarily.** $O(1)$

Step 2. **Sort all the points in angular order about O.** $O(n\log n)$

Step 3. **Select a point (assume p_1) that is on the convex hull.** $O(n)$

Step 4. **Start from p_1 and scan the sorted list of points. Three contiguous points are examined at a time.** $O(n)$

Case 1.

P_i

P_{i+1} P_{i-1}

0

Do nothing, and examine (p_{i+2}, p_{i+1}, p_i) next.

Case 2.

P_{i+1} P_{i-1}

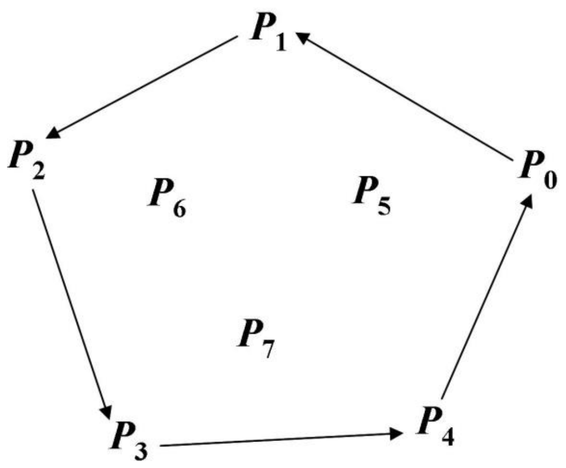
P_i

0

p_i is deleted, and examine (p_{i+1}, p_{i-1}, p_{i-2}) next.

$T(n) = O(n\log n)$

Jarvis' March



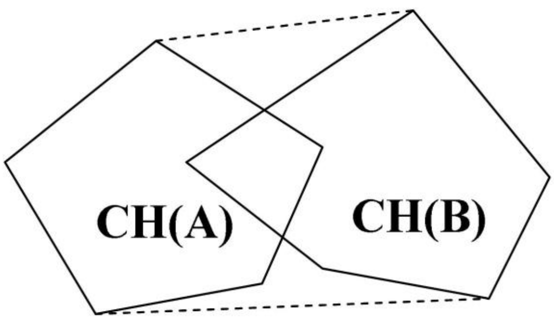
Step 1. Select a point (assume p_0) that is on the convex hull. $O(n)$

Step 2. Start from p_0 and find all the other convex hull vertices sequentially in a clockwise or counterclockwise order. $O(hn)$

$T(n) = O(hn)$, where h is the number of convex hull vertices.

Divide-and-Conquer Approach

- **divide :** **Divide the point set into two subsets *A*, *B* of approximately equal size.** $O(1)$
- **conquer :** **Find the convex hulls CH(*A*), CH(*B*) of *A* and *B*, recursively.** $2T(n/2)$
- **merge :** **Find the convex hull of the original point set from CH(*A*) and CH(*B*).** $O(n)$



$$T(n) = 2T(n/2) + O(n) = O(n\log n)$$

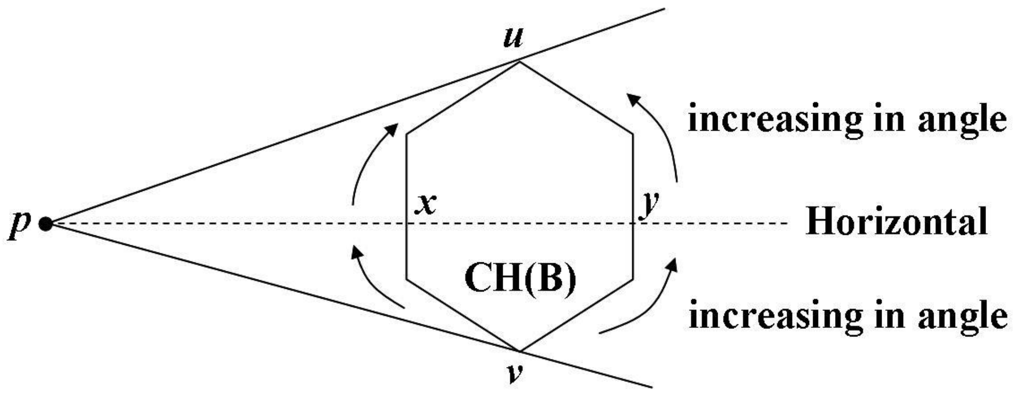
(1) Find an interior point (assume p) of CH(A) arbitrarily. $O(1)$

(2) Determine whether p is interior to CH(B) or not. If not, go to (4). $O(n)$

(3) (p is interior to CH(B)) Since the vertices of CH(A) and the vertices of CH(B) are arranged in a sorted angular order about p, we may obtain a sorted list of their union by merging their respective sorted lists. Go to (5). $O(n)$

(4) (p is not interior to CH(B)) Find two vertices u, v of CH(B) so that the convex wedge defined by u, p and v contains CH(B). $O(n)$

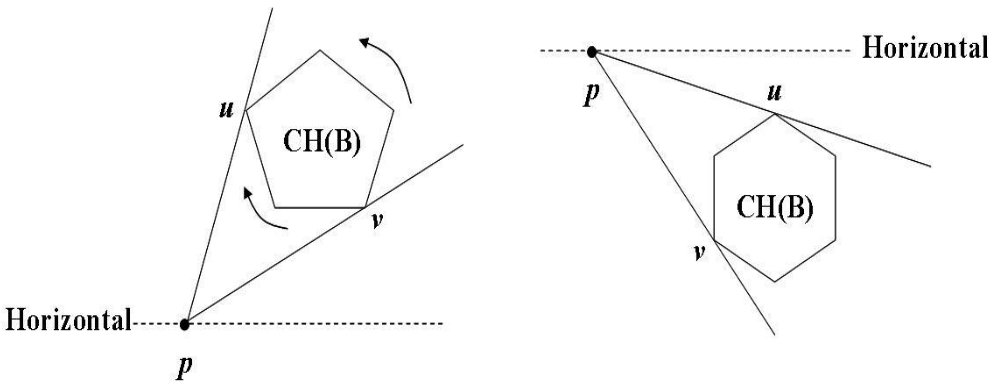
Case 1.



List 1 : list of vertices arranged in their occurring sequence from *x* to *u* and then from *v* to *x*. (Assume that the degrees are measured between 0 and 2π.)

List 2 : list of vertices arranged in their occurring sequence from *y* to *u* and then from *v* to *y*.

Case 2.



List 1 : list of vertices arranged in their clockwise occurring sequence from *v* to *u*.

List 2 : list of vertices arranged in their counterclockwise occurring sequence from *v* to *u*.

Since List 1 and List 2 are increasing in polar angle about *p*, we may merge them and vertices of CH(*A*) into a sorted list.

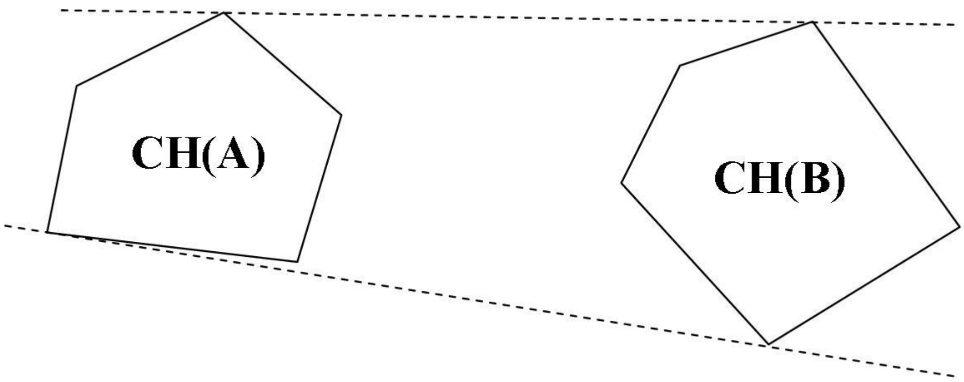
(5) Perform Step 3 and Step 4 of Graham's Scan to obtain CH($A \cup B$). $O(n)$

(Graham's Scan :

Step 3. Select a point (assume p_1) that is on the convex hull.

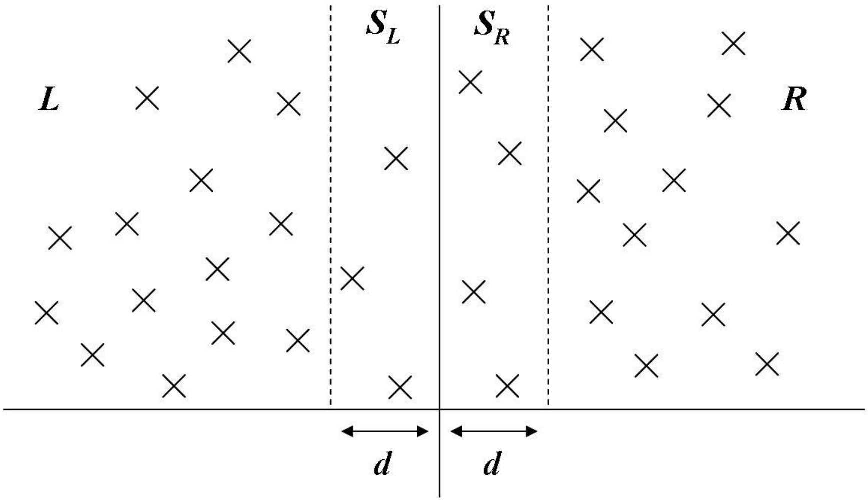
Step 4. Start from p_1 and scan the sorted list of points. Three contiguous points are examined at a time.)

N.B. **There is another divide-and-conquer approach to the convex hull problem, which divides the point set into two approximately equal-size subsets *A* and *B* by means of a vertical line. CH(*A*∪*B*) is obtained by determining two common supporting lines of CH(*A*) and CH(*B*). Refer to "Convex hulls of finite sets of points in two and three dimensions", *Comm. ACM*, vol. 20, no. 2, 1977, 87-93. (by Preparata and Hong)**

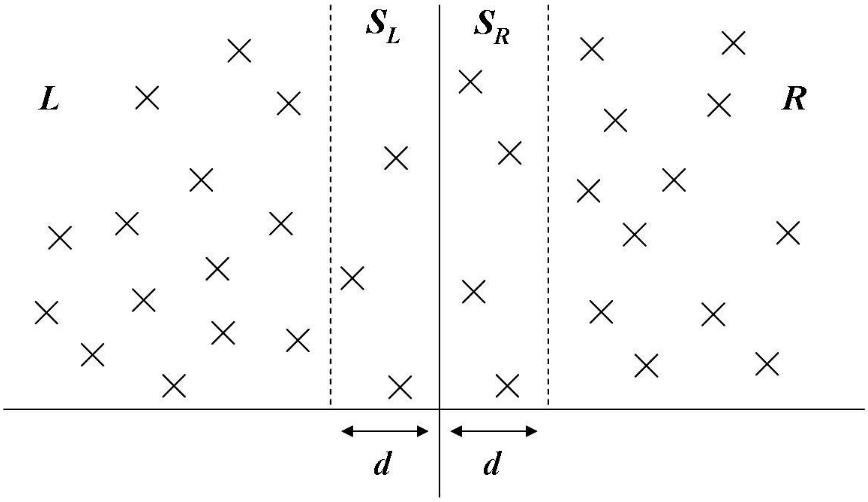


Ex. The 2-dimensional closest pair problem.

Given a set of *n* points in the plane, find the pair of points that are closest.

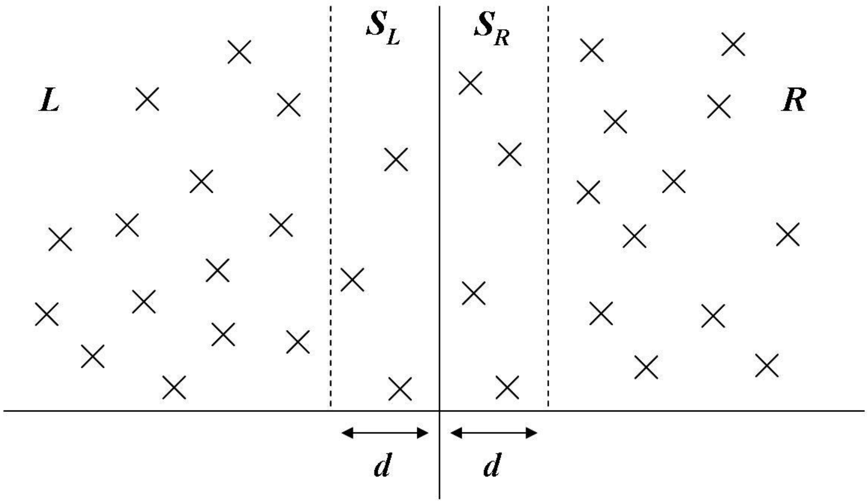


- **divide :** **Find a line perpendicular to the *x*-axis which divides the point set into two equal-size subsets *L* and *R* (i.e., find the median of all the *x*-coordinates).** $O(n)$



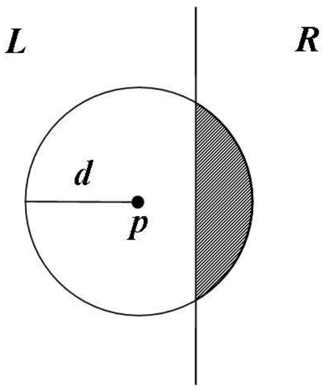
- **conquer :** **Find the closest pairs in *L* and *R*, respectively. Let d_L and d_R be their respective distances.** $2T(n/2)$

- **merge :** **Let $d = \min\{d_L, d_R\}$. If the closest pair of $L \cup R$ consists of a point in L and a point in R, then they must lie within a slab of width $2d$ centered at the dividing line. Thus, we only have to determine for each point p in L whether there exist points in R that are within d from p. $O(n)$**

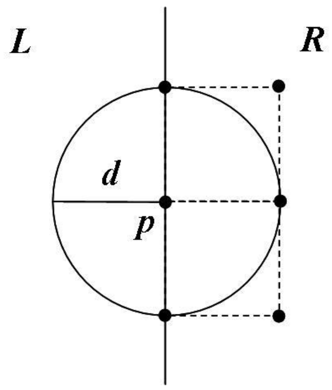


(1) Presort all the given points according to their y-coordinates (before the divide-and-conquer procedure is initiated). $O(n\log n)$

(2) Examine the two sorted lists of points that are located in S_L and S_R, respectively. For each point p in L, we only need to examine at most five points in R. $O(n)$



Only the shaded area has to be examined.



At most six points fit in a $d\times 2d$ rectangle.

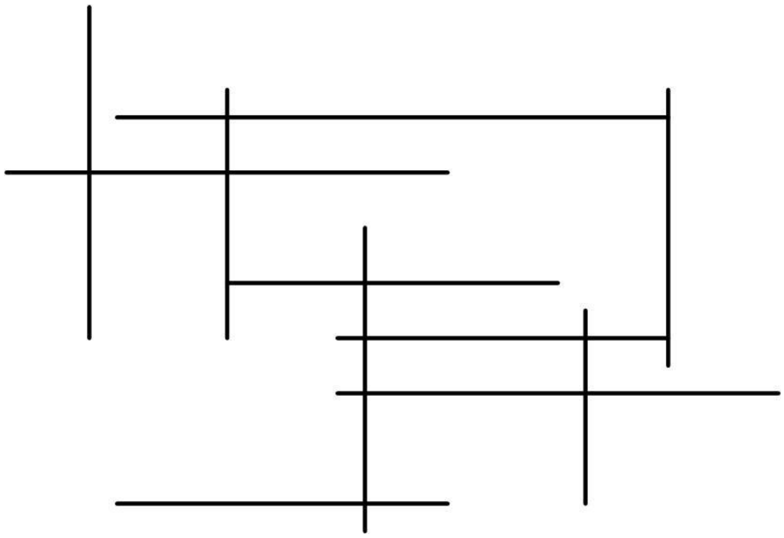
presorting time : $O(n\log n)$

$$
\begin{aligned}
T(n) &= 2T(n/2) + O(n) \\
&= O(n\log n)
\end{aligned}
$$

$$
\begin{aligned}
\textbf{total time complexity} &= \textbf{presorting time} + T(n) \\
&= O(n\log n)
\end{aligned}
$$

Ex. The Iso-Oriented Line Segment Intersection Problem.

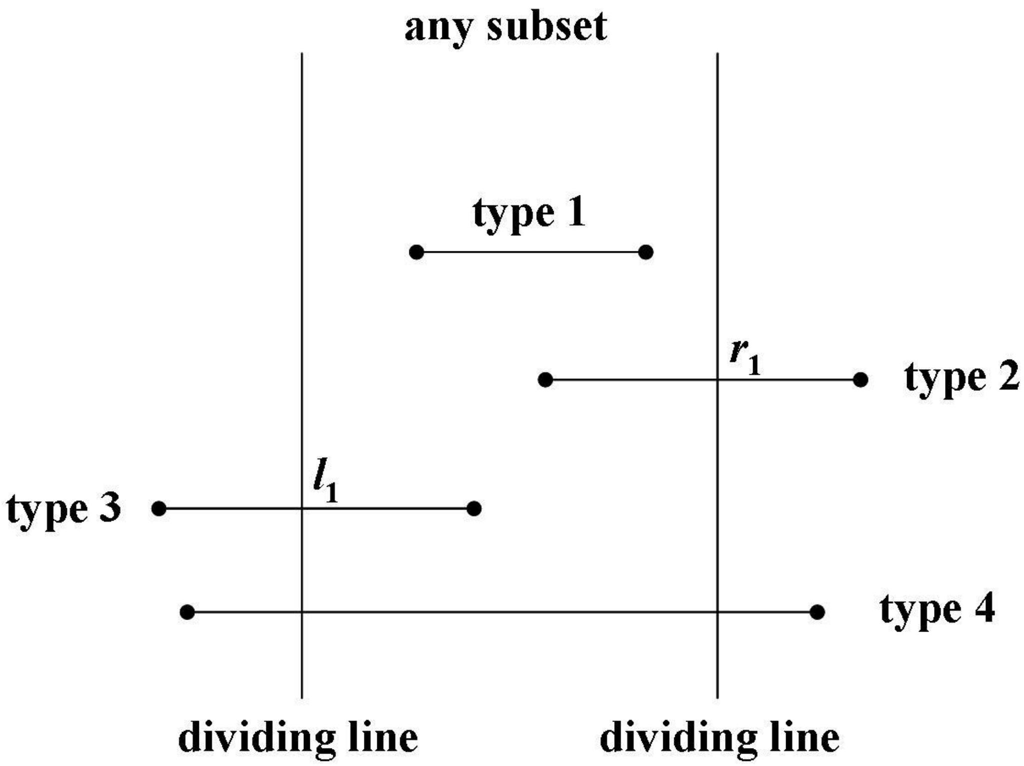
Given a set of m horizontal and vertical line segments $L_1, L_2, \ldots, L_m$, find all pairs of intersecting line segments.



Let objects refer to either vertical line segments or endpoints of horizontal line segments. Assume that there are *n* objects initially.

- **Presort all the endpoints of vertical line segments according to their *y*-coordinates.** $O(n\log n)$

- **divide : Divide the set of objects into two approximately equal-size subsets *L* and *R* by a vertical line. Also split the sorted list above in accordance with *L* and *R*.** $O(n)$


any subset
type 1
r_1
type 2
l_1
type 3
type 4
dividing line
dividing line

- **conquer : 2T(n/2)**

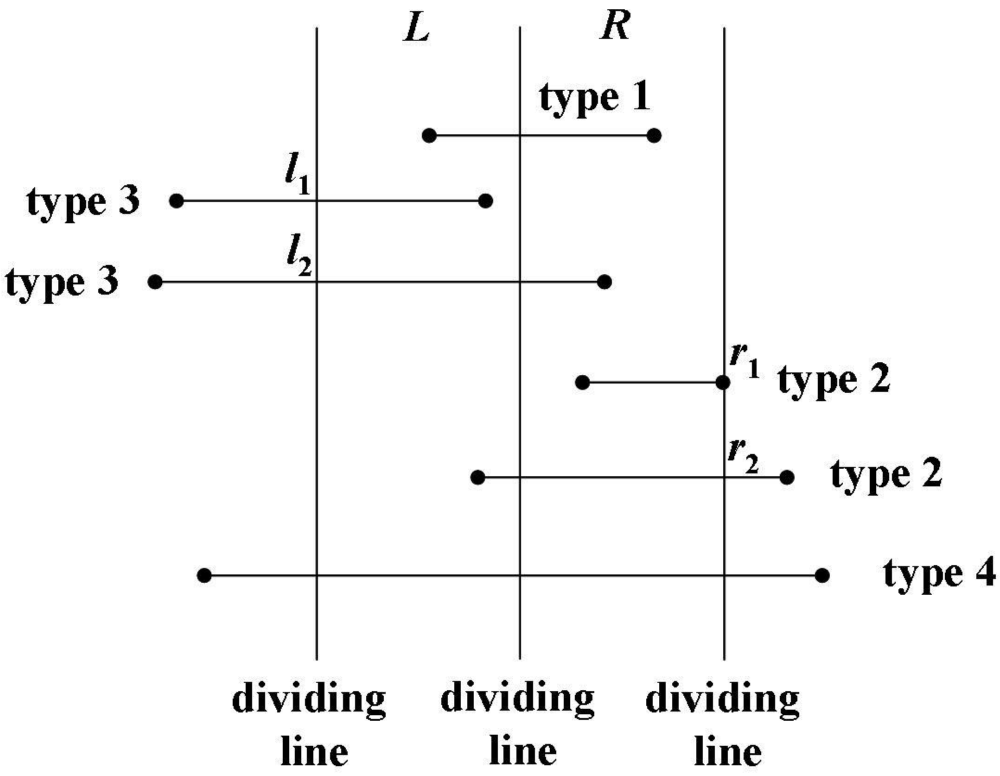
For each subset (the existence of line segments of type 4 is unknown),

(1) report all intersecting pairs in which only type 1, type 2 and type 3 horizontal line segments are included;

(2) record right boundary points r_1, r_2, … (denoted by list $\tilde{R}$) in sorted y-coordinate sequence;

(3) record left boundary points l_1, l_2, … (denoted by list $\tilde{L}$) in sorted y-coordinate sequence.

• **merge :** $O(n)+s$



$\tilde{R}_L$ ($\tilde{R}_R$) : **list $\tilde{R}$ of the left (right) subset**

$\tilde{L}_L$ ($\tilde{L}_R$) : **list $\tilde{L}$ of the left (right) subset**

$\tilde{L}\tilde{R}$: **list of y-coordinates of type 1 line segments (across the middle dividing line)**

V_L (V_R) : **set of vertical line segments in the left (right) subset**

$$\tilde{L}\tilde{R} \leftarrow \tilde{R}_L \cap \tilde{L}_R$$

$$\tilde{R} \leftarrow \tilde{R}_R + (\tilde{R}_L - \tilde{L}\tilde{R})$$

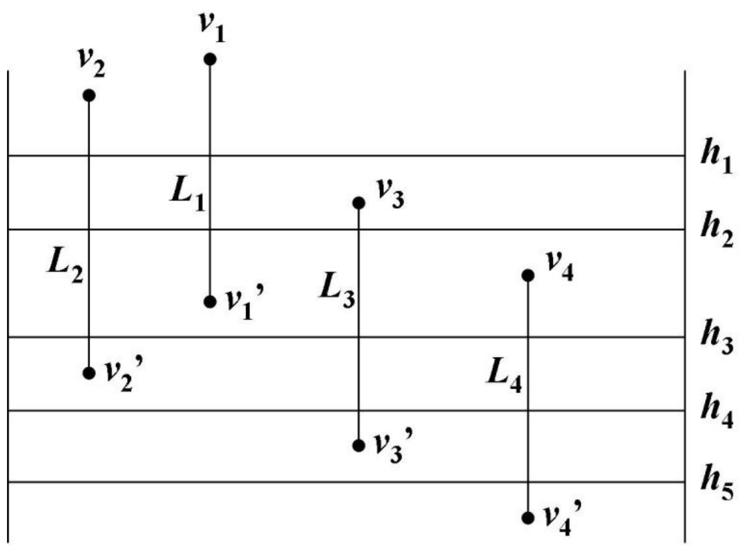
$$\tilde{L} \leftarrow \tilde{L}_L + (\tilde{L}_R - \tilde{L}\tilde{R})$$

Report all intersecting pairs induced by

- ♣ V_L **and the horizontal line segments passing** $\tilde{L}_R - \tilde{L}\tilde{R}$**;**
- ♣ V_R **and the horizontal line segments passing** $\tilde{R}_L - \tilde{L}\tilde{R}$

Since $\tilde{R}_L$, $\tilde{R}_R$, $\tilde{L}_L$ **and** $\tilde{L}_R$ **are sorted in** ***y*****-coordinate, we may obtain** $\tilde{L}\tilde{R}$, $\tilde{L}$ **and** $\tilde{R}$ **(sorted in** ***y*****-coordinate) in** $O(n)$ **time by performing merge-like operations.**

A method to report all intersecting pairs in $O(n+s)$ time, where s is the number of intersecting pairs reported :



Given two sorted lists :

$$H = (h_1, h_2, h_3, h_4, h_5)$$

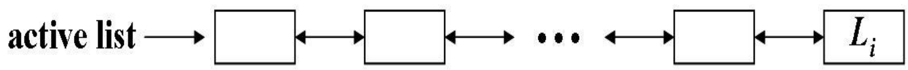
$$V = (v_1, v_2, v_3, v_4, v_1', v_2', v_3', v_4')$$

Step 1. **Merge *H* and *V* into *L*.**

$$L = (v_1, v_2, h_1, v_3, h_2, v_4, v_1', h_3, v_2', h_4, v_3', h_5, v_4')$$

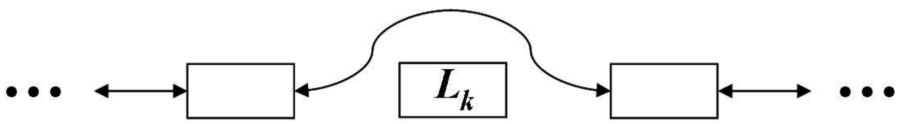
Step 2. **Scan *L* from the left to the right.**

Case 1. **Add L_i to the active list whenever a v_i is met.**



Case 2. **Report the active list whenever an h_j is met. Each vertical line segment in the active list intersects with h_j.**

Case 3. **Delete L_k from the active list whenever a v_k' is met.**



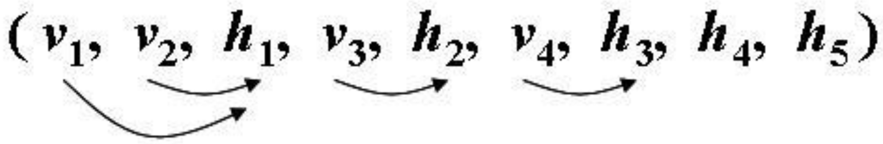
Another method to report all intersecting pairs in $O(n + s)$ time :

Given two sorted lists :

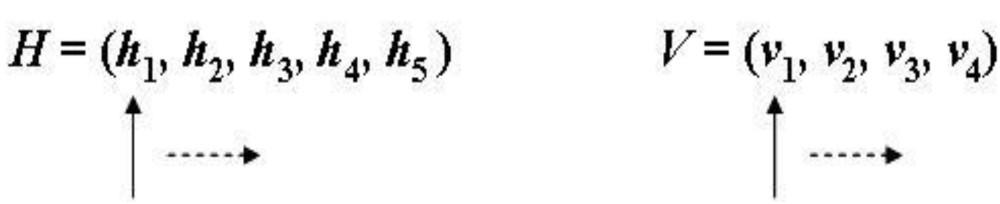
$$H = (h_1, h_2, h_3, h_4, h_5)$$
$$V = (v_1, v_2, v_3, v_4)$$

Step 1. **Merge H and V into $L = (v_1, v_2, h_1, v_3, h_2, v_4, h_3, h_4, h_5)$.**

Step 2. **Scan L from the right to the left.**
For each v_i, find the h_j that is succeeding and closest to it.



Step 3. **Report all intersecting pairs by scanning H and V simultaneously.**



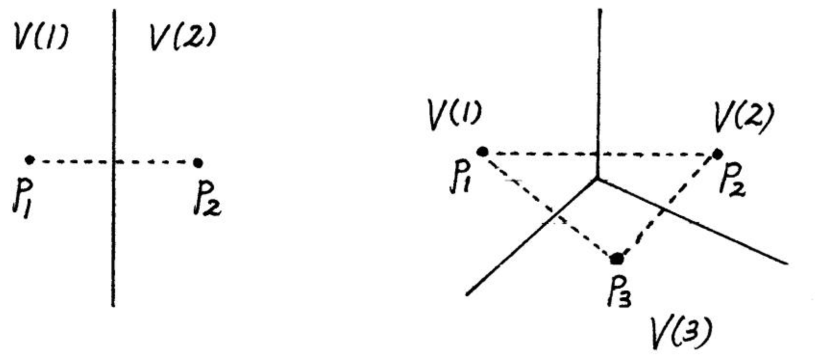
$$T(n) = 2T(n/2) + O(n) = O(n\log n)$$

$$\begin{aligned}\text{total time complexity} &= T(n) + O(n\log n) + O(N) \\ &= O(n\log n + N),\end{aligned}$$

where $T(n)$ is the time for D&C, $O(n\log n)$ is the time for presorting, and N is the total number of intersecting pairs.

Ex. Voronoi Diagram Construction.

Given n points $p_1, p_2, \ldots, p_n$ in the plane, we denote by $V(i)$ the locus of points that are closer to p_i than to any other point.

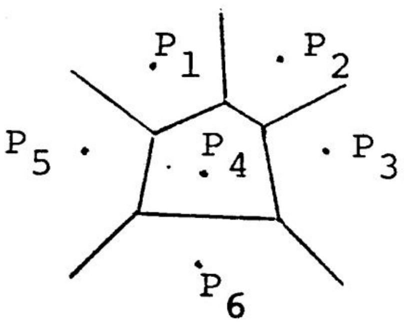


Let $H(p_i, p_j)$ denote the half-plane where all the points are closer to p_i than to p_j. Thus, $V(i)$ is a convex polygonal region having no more than $n - 1$ sides, defined by

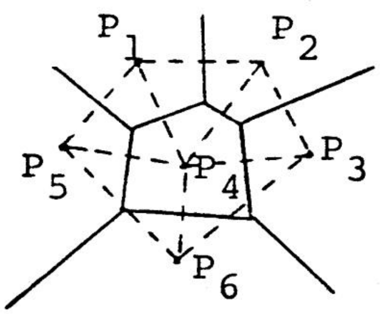
$$V(i) = \bigcap_{j \neq i} H(p_i, p_j).$$

$V(i)$ is called the *Voronoi polygon* associated with p_i. The partition of the plane by n Voronoi polygons is called the *Voronoi diagram*.

The straight-line dual of a Voronoi diagram on a set *S* of *n* points is a graph whose *n* vertices correspond to the points of *S*, in which there is an edge from p_i to p_j iff *V*(*i*) and *V*(*j*) share an edge.



A Voronoi diagram of six points



A Delaunay Triangulation

The straight-line dual of a Voronoi diagram is a planar graph, called the *Delaunay graph*. If no four points of the original set are cocircular, the straight-line dual is a triangulations, called the *Delaunay triangulation* (proved by Delaunay at 1934).

Theorem. **A Voronoi diagram on n points has at most $2n-4$ vertices and $3n-6$ edges.**

Proof.

(1) Each edge in the Delaunay graph corresponds to a unique Voronoi edge.

(2) A planar graph of n vertices has at most $3n-6$ edges.

(3) Each Voronoi vertex has degree at least 3.

(1), (2) $\Rightarrow$ there are at most $3n-6$ Voronoi edges

$\Rightarrow$ (with (3)) there are at most $2*(3n-6)/3 = 2n-4$ Voronoi vertices.

Theorem. **A Voronoi polygon $V(i)$ is unbounded iff p_i lies on the boundary of the convex hull (of given points).**

Proof. **Refer to "Computational Geometry" by Shamos, Ph.D. dissertation, Yale Univ., 1978.**

Constructing Voronoi diagram

By "constructing" Voronoi diagram of a set of points, we mean obtaining all of the following data :

(1) coordinates of Voronoi vertices;

(2) Voronoi edges (pairs of Voronoi vertices) incident with each Voronoi vertex;

(3) the two points that determine each Voronoi edge;

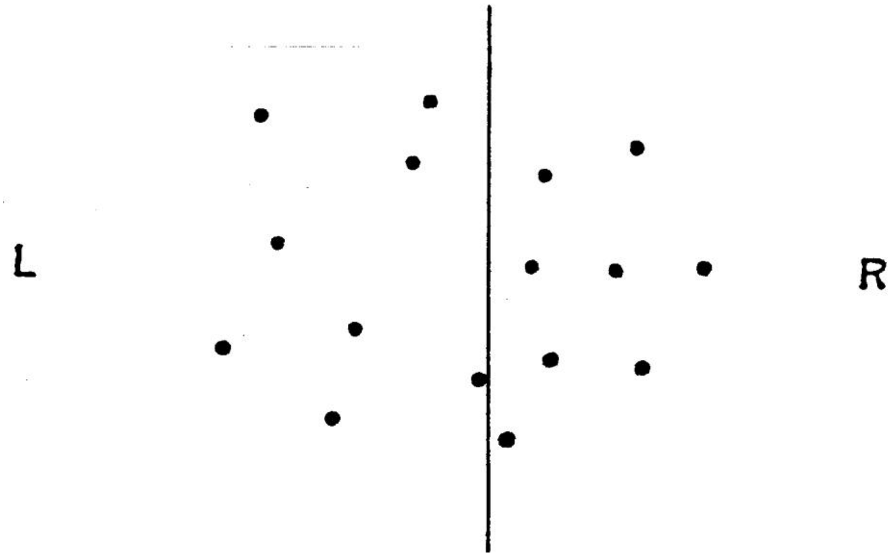
(4) the edges of each Voronoi polygon in a cyclic order.

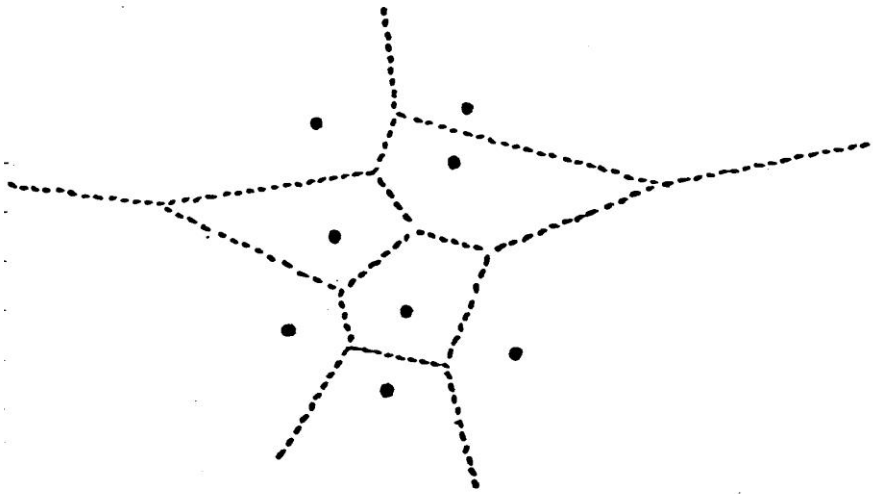
- **divide :** **Divide the given set of points into two subsets L and R of approximately equal size by a vertical line.** $O(n)$

- **conquer :** **Construct the Voronoi diagram, denoted by $VD(L)$, for L and the Voronoi diagram, denoted by $VD(R)$, for R.** $2T(n/2)$

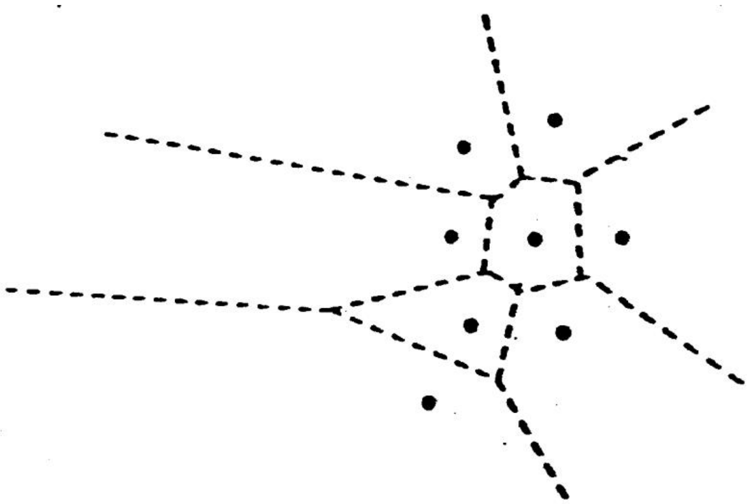
- **merge :** **Find P, which is the locus simultaneously closest to a point in L and a point in R, and discard all line segments of $VD(R)$ ($VD(L)$) that lie to the left (right) of P.** $O(n)$

dividing line

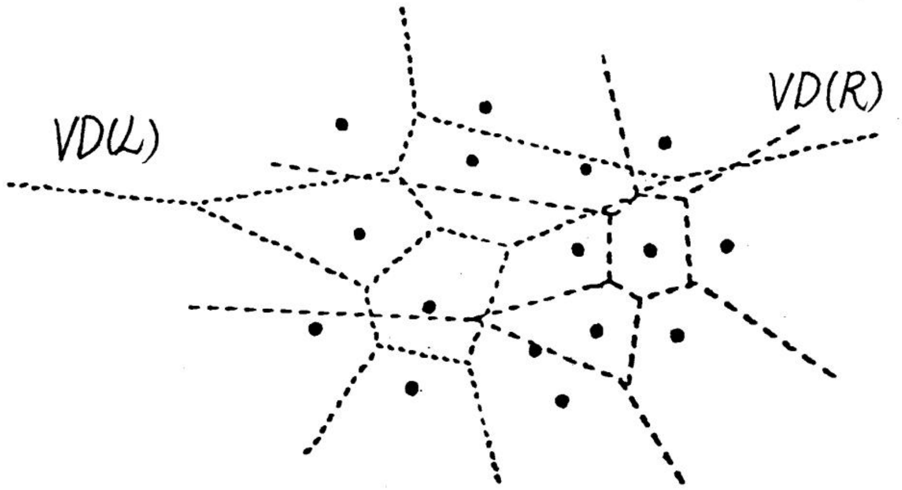


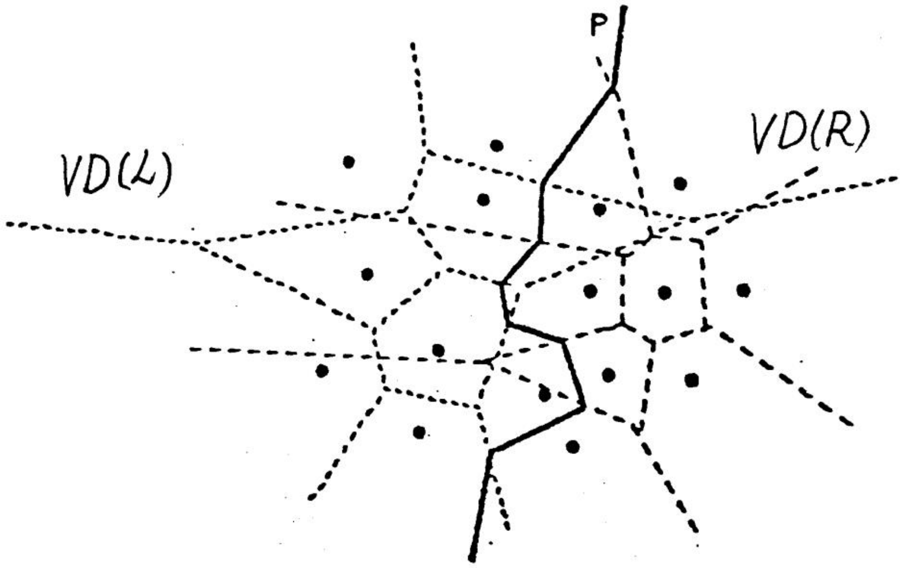


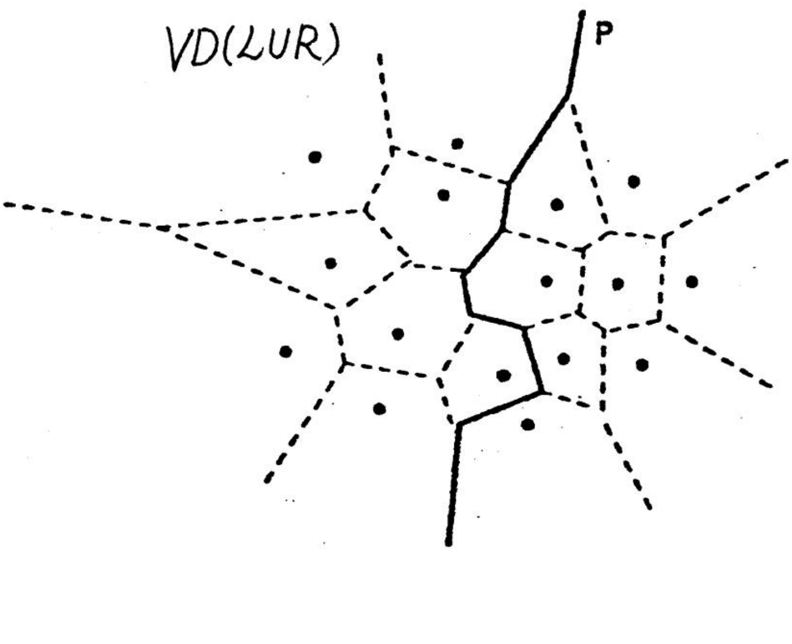
VD(L)



VD(R)


VD(L)
VD(R)


P
VD(L)
VD(R)


VD(L∪R)
P

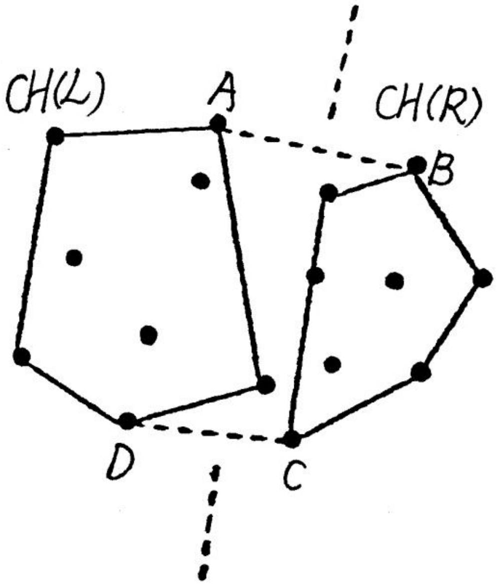
Theorem. ***P*** **is monotonic and continuous in *y*-coordinate and consists of two infinite rays and some piecewise line segments.**

Proof. **Refer to "Computational Geometry" by Shamos, Ph.D. dissertation, Yale Univ., 1978.**

Two steps to find *P* in *O*(*n*) time :

Step 1. Find the two infinite rays. *O*(*n*)

Since each ray is associated with two adjacent vertices in CH(*L*∪*R*), we find the two edges of CH(*L*∪*R*) that are not present in either CH(*L*) or CH(*R*).



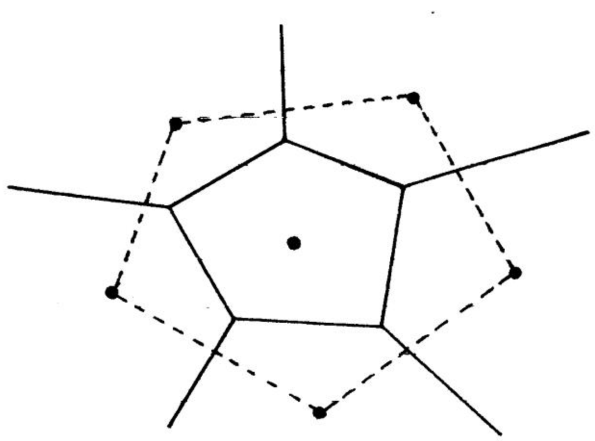
The infinite rays of *P* are the perpendicular bisectors of the two segments joining CH(*L*) and CH(*R*).

1-1. Find CH(*L*) and CH(*R*). $O(n)$

(a) Find a ray arbitrarily.

(b) Find another ray belonging to the same Voronoi polygon.

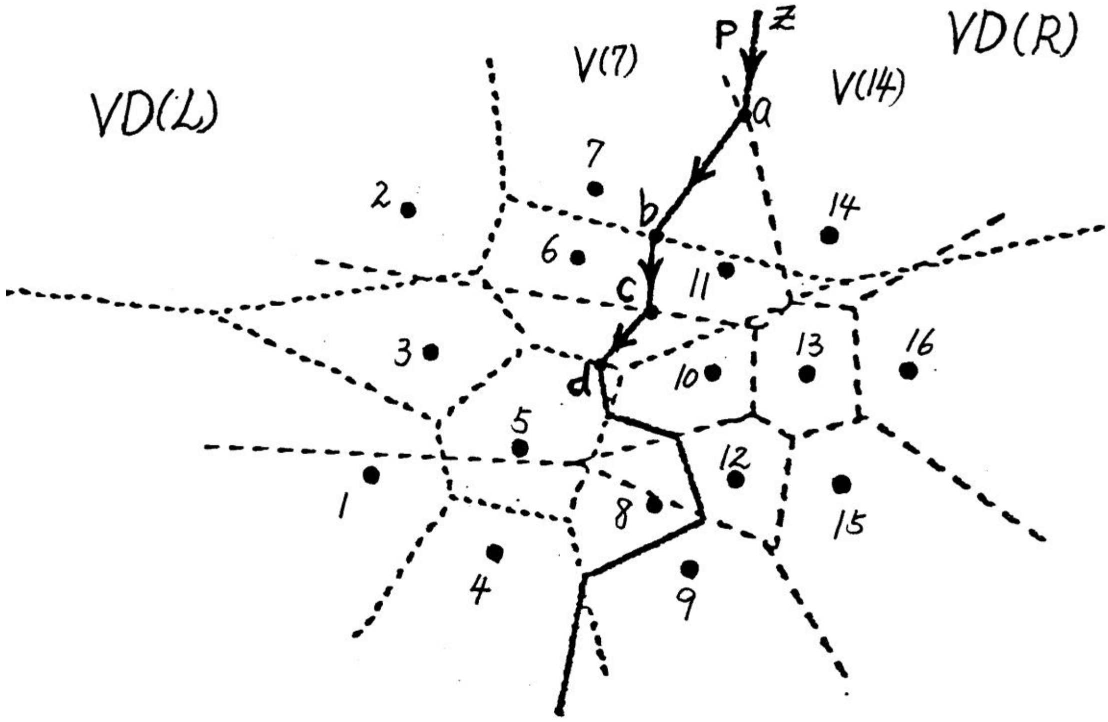
(c) Perform (b) in clockwise or counter-clockwise orientation until the initial ray is met.



1-2. Find *A, B, C, D*. $O(n)$.

The merge algorithm for the convex hull problem can be applied here.

Step 2. A zigzag walk to construct *P*. $O(n)$



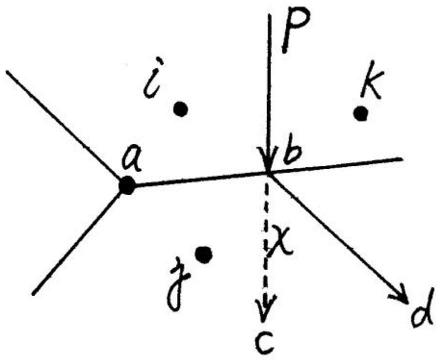
za **has equal distance to 7 and 14 (*za* in $V(7) \cap V(14)$).**

ab **has equal distance to 7 and 11 (*ab* in $V(7) \cap V(11)$).**

bc **has equal distance to 6 and 11 (*bc* in $V(6) \cap V(11)$).**

Two observations :

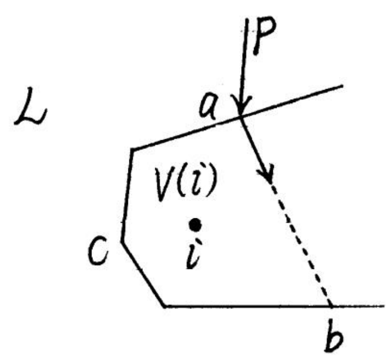
1. ***P*** **always moves downward (since *P* is monotonic).**
2. ***P*** **will bend toward the right (left) when it passes through an edge belonging to *VD*(*L*) (*VD*(*R*)).**

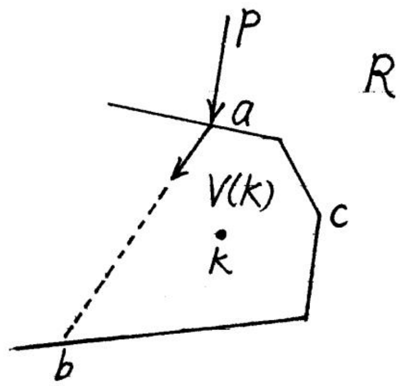


pb **has equal distance to *i* and *k*.**

bd **has equal distance to *j* and *k*.**

For each $x \in bc$, $jx < ix = xk$**.**





P **moves out of** ***V***(***i***) **at** ***b*** **(if no edge belonging to VD(*****R*****) is met before** ***b*****).**

P **moves out of** ***V***(***k***) **at** ***b*** **(if no edge belonging to VD(*****L*****) is met before** ***b*****).**

(1) When *P* enters *V*(*i*), we mark the edge containing *b*. If *P* meets edges belonging to VD(*R*) before leaving *V*(*i*), we mark the edge of *V*(*i*) where *P* will leave *V*(*i*) if *P* continues in its changed direction. The newly marked edge can be found as we traverse the edges of *V*(*i*) in clockwise orientation, starting from the last marked edge.

(2) When *P* enters *V*(*k*), we mark the edge containing *b*. If *P* meets edges belonging VD(*L*) before leaving *V*(*k*), we mark the edge of *V*(*k*) where *P* will leave *V*(*k*) if *P* continues in its changed direction. The newly marked edge can be found if we traverse the edges of *V*(*k*) in counterclockwise orientation, starting from the last marked edge.

Therefore, whenever *P* passes through an edge, we can find the edge that *P* will pass through next as follows :

> **determine the two currently marked edges that belong to *VD*(*L*) and *VD*(*R*), respectively, and choose the one that *P* will meet first.**

Thus, *O*(*n*) time is enough to construct *P*.

Applications with Voronoi diagram：

1. **All nearest neighbors problem.**

   Given *n* points in the plane, find the nearest neighbor of each.

Theorem. The nearest neighbor of each p_i defines an edge of $V(i)$.

Proof. Refer to *Computational Geometry*, Springer-Verlag, by Preparata & Shamos.

Thus, given the Voronoi diagram, this problem can be solved in $O(n)$ time.

2. Nearest neighbor search problem.

Given *n* point in the plane, with preprocessing allowed, find the nearest neighbor of a query point.

(1) Construct the Voronoi diagram of the given *n* points.

(2) Find the Voronoi polygon where the query point is located.

Theorem. Point location in an *n*-vertex planar subdivision can be effected in $O(\log n)$ time using $O(n)$ storage, given $O(n\log n)$ preprocessing time.

Proof. Refer to *Computational Geometry*, Springer-Verlag, by Preparata & Shamos.

Thus, nearest-neighbor search for a query point can be performed in $O(\log n)$ time, using $O(n)$ storage, with $O(n\log n)$ preprocessing time.

3. Euclidean minimum spanning tree problem.

Given n point in the plane, construct a tree of minimum total length whose vertices are the given points.

If we construct a complete graph of n vertices, each corresponding to a point, then $O(n^2)$ time (by Prim's method) or $O(n^2\log n)$ time (by Kruskal's method) is needed.

N.B. Prim's algorithm takes $O(|V|^2)$ time, and Kruskal's algorithm takes $O(|E|\cdot\log|E|)$ time.

Theorem. **Every Euclidean minimum spanning tree is a subgraph of the Delaunay graph.**

Proof. **Refer to "Computational Geometry" by Shamos, Ph.D. Dissertation, Yale Univ., 1978.**

Theorem. **A minimum spanning tree of a planar graph can be found in $O(n)$ time.**

Proof. **Refer to "Finding minimum spanning trees" by Cheriton & Tarjan, *SIAM J. Computing*, vol. 5, no. 4, 1976, 724-742.**

1. **Construct the Voronoi diagram.** $O(n\log n)$

2. **Construct the Delaunay graph.** $O(n)$

3. **Find a minimum spanning tree on the Delaunay graph.** $O(n)$

A related work :

H. Imai, M. Iri, and K. Murota, "Voronoi diagram in the Laguerre geometry and its applications," ***SIAM J. Computing*****, vol. 14, no. 1, 1985, 93-105.**

Program Assignment 4：

Write an executable program to solve the 2-dimensional closest pair problem.

Exercise 4：

Imai, Iri, and Murota, "Voronoi diagram in the Laguerre geometry and its applications," *SIAM Journal on Computing*, vol. 14, no. 1, 1985, pp. 93-105.