ASSIGNMENT-I [ELECTROSTATICS]

① state and Explain Coulomb's Law.

Ans:-

Statement :-

Coulomb's law states that the force F between two point charges Q_1 and Q_2 is

i. Along the line joining them.

ii. Directly proportional to the product of $Q_1 Q_2$.

iii. Inversly proportional to the square of the distance (R) between them.

$$\boxed{F = \frac{K . Q_1 Q_2}{R^2}} \quad — ①$$

where, K is proportnality Constant.

Q_1 & Q_2 in coulombs (c)

R in meters

F in Newtons

$$\boxed{K = \frac{1}{4\pi\varepsilon_0}}$$

where, ε_0 - permittivity of free space (farads/meter) (F/m)

$$\varepsilon_0 = 8.854 \times 10^{-12} = \frac{10^{-9}}{36\pi} \text{ F/m}$$

(or)

$$K = \frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \text{ m/F}$$

eqn ① becomes

$$\boxed{F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}}$$

② State Gauss Law?

Ans:

Statement:- The total electric flux ψ through any closed surface is equal to the total change enclosed by that surface.

Thus, $\boxed{\psi = Q_{enc}}$

i.e $\psi = \oint d\psi = \oint_s D \cdot ds$

= Total change enclosed $Q = \int \rho_v dv$.

$$\boxed{Q = \oint_s D \cdot ds = \int_v \rho_v dv}$$

↳ Integral form of Gauss law.

③ Define Linear, Isotropic and Homogeneous dielectrics?

Ans:

Linear Dielectric:- A material is said to be linear if D varies linearly with E & non-linear otherwise.

Homogeneous Dielectrics:- materials for which E (or σ) does not vary in the region being consider and is therefore same at all points (ie independent of x, y, z) are said to be homogeneous.

Isotropic Dielectric :-

materials for which D & E are in the same direction are said to be isotrophic i.e Isotropic dielectrics are those which have the same properties in all directions.

4. Define poisson's and Laplace equations?

Ans: The poisson's and Laplace equations are derived easily from gauss law. (for a linear material medium).

$$\nabla \cdot D = \nabla \cdot \epsilon E = \rho_v \quad — ①$$

$$\& \quad E = -\nabla V \quad — ②$$

substituting ② in ①

$$\boxed{\nabla \cdot (-\epsilon \cdot \nabla V) = \rho_v}$$ for a Inhomogeneous medium

For homogeneous medium,

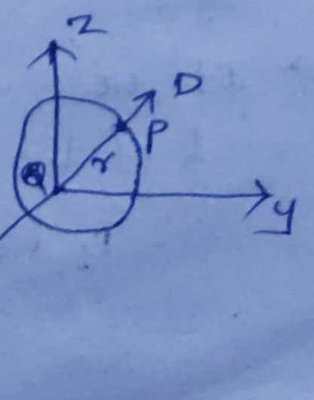
$$\boxed{\nabla^2 V = \frac{-\rho_v}{\epsilon}}$$ - poisson's Eqn

For a change free region, $\rho_v = 0$

$$\boxed{\nabla^2 V = 0}$$ - Laplace's equation.

7 Derive D at a point P, due to a point charge and an infinite Line charge?

Ans:- a) point charge :-



→ suppose a point charge Q is located at the origin. To determine D at a point P, it is easy if we choose a spherical surface containing P will satisfy symmetry conditions.

→ A spherical surface centered at the origin is the gaussian surface in this case as shown in the fig.

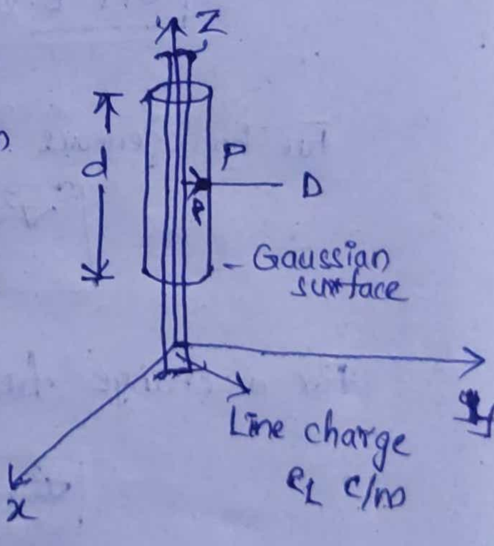
→ some D is normal every where normal to the gaussian surface i.e $D = D_r a_r$. then applying gauss law.

$$Q = \oint D \cdot ds = D_r \oint ds = D_r (4\pi r^2)$$

where $\left[\oint ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2\right]$

$$\boxed{D = \frac{Q}{4\pi r^2} = a_r}$$

b) Infinite Line charge :-



→ suppose infinite line of uniform change ρ_L c/m lies along the z-axis. To find D at a point P, we choose a cylindrical surface containing P to satisfy symmetry condition as shown in fig.

→ D is Constant on & normal to the cylindrical gaussian surface, i.e $D = D_\rho a_\rho$

→ If we apply Gauss law to an arbitrary length l of the line,

$$\rho_L l = Q = \oint D \cdot ds = D_\rho \oint ds$$

$\rho_L l = D_\rho (2\pi \rho L)$

$$\boxed{D_\rho = \frac{\rho_L}{2\pi\rho} a_\rho}$$

$[\because ds = \rho \, d\phi \, dz \, a_\rho$

$$\oint ds = \int_{z=0}^{l} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz$$

$$= (2\pi)(\rho L)$$

$$= 2\pi\rho L]$$

[Note : $\int D.ds$ evaluated on the top & bottom surfaces of the cylinder is zero. since D has no z-component i.e D is tangential to those surfaces.].

⑨ Derive the expression for equation of continuity and relaxation time?

Ans: Continuity Equation:-

→ Due to the principle of charge conservation the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume.

→ Thus current I_{out} coming out of the closed surface is

$$I_{out} = \oint J.ds = \frac{-dQ_{in}}{dt} \quad -①$$

→ where Q_{in} is the total charge enclosed by the closed surface.

→ Applying divergence theorem,

$$\oint J.ds = \int_v \nabla.J \, dv \quad -②$$

But

$$\frac{-dQ_{in}}{dt} = \frac{-d}{dt} \int \rho_v dv = -\int_v \frac{\partial \rho_v}{\partial t} dv \quad -③$$

substituting ② & ③ in ①

$$\int_V \nabla \cdot J \, dv = -\int_V \frac{\partial \rho_v}{\partial t} dv$$

$$\boxed{\nabla \cdot J = -\frac{\partial \rho_v}{dt}}$$ continuity of current equation.

Relaxation time :-

$J = \sigma E$

$D = \epsilon E$

$E = \frac{D}{\epsilon}$, $J = \sigma \frac{D}{\epsilon}$

The point form of the continuity equation states that

$$\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}$$

$$\nabla \cdot \left(\sigma \frac{D}{\epsilon}\right) = \frac{-\partial \rho_v}{\partial t}$$

$$\frac{\sigma}{\epsilon}(\nabla \cdot D) = \frac{-\partial \rho_v}{\partial t}$$ But $\nabla \cdot D = \rho_v$

$$\frac{\sigma}{\epsilon} \rho_v = -\frac{\partial \rho_v}{\partial t}$$

$$\therefore \frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon} \rho_v = 0$$

This is the 1st order differential equation in ρ_v whose solution is given by

$$\rho_v = A \cdot e^{-\sigma t/\epsilon}$$

where A is a constant

Let the initial value at $t=0$ be $\rho_v = \rho_{v0}$

so that $A = \rho_{v0}$

$$\therefore \rho_v = \rho_{v0}\, e^{-\sigma t/\epsilon}$$

The time constant $\epsilon/\sigma = T_r$ is the relaxation time in seconds.

$$\therefore \rho_v = \rho_{v0}\, e^{-t/T_r}$$

At $t = T_r$, $\rho_v = \frac{\rho_{v0}}{e} = 0.368\rho_{v0}$

$\therefore T_r$ (relation time $T_r = \epsilon/\sigma$ is the time taken for the charge density to decay to 36.8 % (or) (1/e) time its initial value

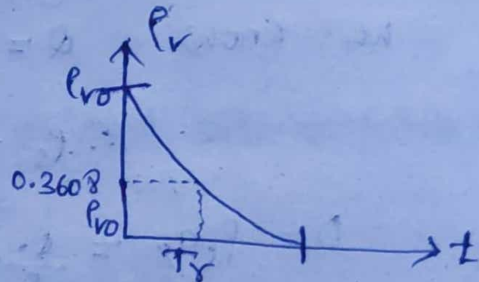


Fig. charge density & relaxation time

(10) Define Capacitance of a capacitor in electrostatic fields? And obtain the expression for the capacitance for coaxial and parallel plate capacitors?

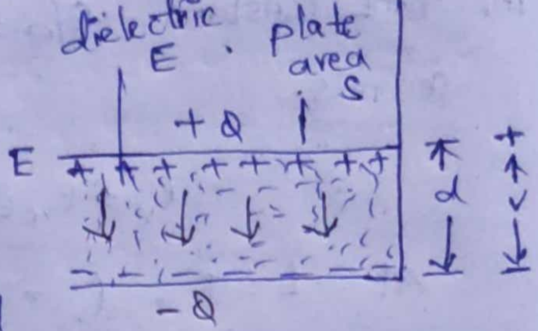
Ans:- <u>Capacitance of a capacitor</u>:-

capacitance C of a capacitor is the ratio of magnitude of the change one one of the plates to the potential difference b/w them

$$\boxed{C = \frac{Q}{V} = \frac{\epsilon \oint E\, ds}{\int E \cdot dl}}$$ unit - farads

→ The negative sign for $V = -\int E \cdot dl$ has been dropped because we consider only absolute value of V.

① ⇒ parallel - plate Capacitor :-



→ Consider the parallel plate capacitor shown.

→ Each of the plates has an area S & they are separated by a distance d. we assume that plates 1 & 2 respectively carry charges +Q & -Q uniformly distributed on them, so

we know, $Q = \rho_s S$

$$\therefore \rho_s = Q/S$$

$$D = \rho_s a_z = \frac{Q}{S} a_z$$

$$E = \frac{D}{\varepsilon} = \frac{Q}{\varepsilon s} a_z$$

$$V = \int_L E\, dl = \frac{Q}{\varepsilon s} a_z . d . a_z$$

$$= \frac{Q.d}{\varepsilon s} a_z . a_z = \frac{Qd}{\varepsilon s} \text{ volts}$$

$$C = \frac{Q}{V} = \frac{\cancel{Q}.\varepsilon s}{\cancel{Q}d} = \frac{\varepsilon s}{d} \text{ volts,}$$

$$\therefore \boxed{C = \frac{\varepsilon s}{d} \text{ volts}}$$

$$\boxed{\varepsilon_r = \frac{C}{C_0}}$$

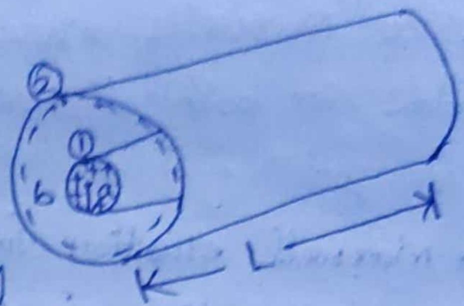
C - cap of ||el plates with dielectric field

C_0 - cap of ||el plates with air field

→ Energy stored in a capacitor

$$\boxed{W_E = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}}$$

Coaxial Capacitor:-



- This is a coaxial cable or coaxial cylindrical capacitor. Consider length L of two coaxial conductors of inner radius 'a' & outer radius b (b>a).

→ E is directed from inner conductor to the outer conductor. The potential difference is work done in moving unit charge against E i.e from r=b to r=a.

→ space b/t the conductors is filled with a homogeneous dielectric with permittivity ε.

→ we assume that conductors 1 & 2 respectively carry +Q & -Q.

→ Applying Gauss's Law

$$Q = \varepsilon \oint E \cdot ds$$

$$= \varepsilon E_\rho 2\pi \rho L$$

$$\therefore ds = \int_{\phi=0}^{2\pi} \int_{z=0}^{L} \rho \, d\phi \, dz$$

$$= \rho(2\pi)(L)$$

$$= \rho 2\pi L$$

$$E = \frac{Q}{2\pi \varepsilon \rho L} a_\rho$$

$$V = -\int_a^b E \, dl = -\int_a^b \left[\frac{Q}{2\pi \varepsilon \rho L} a_\rho\right] d\rho \, a_\rho$$

$$= \frac{-Q}{2\pi \varepsilon L} \left[\ln(\rho)\right]_a^b = \frac{Q}{2\pi \varepsilon L} \ln\left(\frac{b}{a}\right)$$

∴ capacitance of co-axial cylinder is,

$$\boxed{C = \frac{Q}{V} = \frac{2\pi \epsilon L}{\ln(b/a)}}$$

⑤ state Maxwells equations for static electrostatics fields in both differential and integral form?

Ans:

→ Maxwell equation for static electrostatics fields in differential form:

$$\rho_v = \nabla \cdot D$$

→ It is also called point first form maxwell equation.

→ maxwell equation for static electrostatics fields in integral form:

$$Q = \oint_s D \cdot ds = \int_v \rho_v \, dv.$$

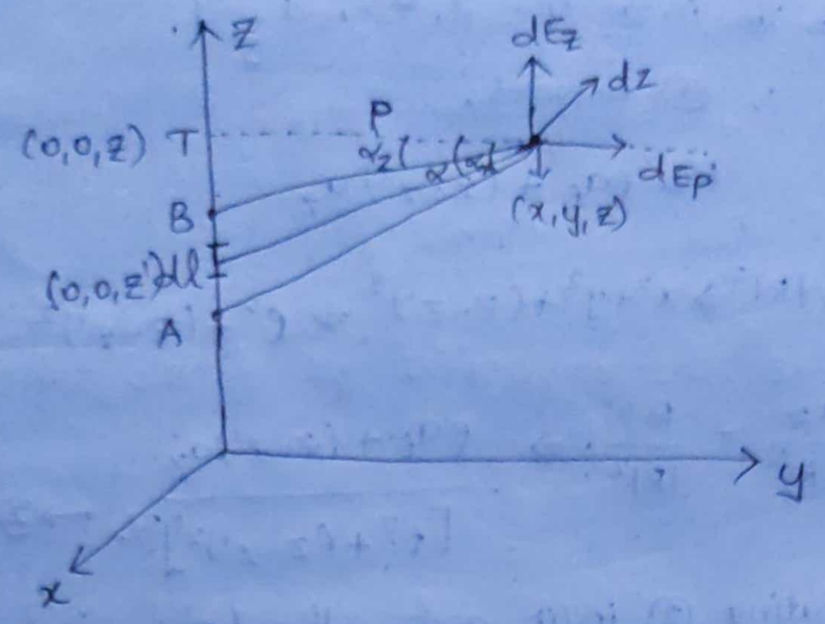
⑥ obtain an expression for electric field at a point due to an infinite Line charge?

Ans:

Consider a line charge with uniform charge density ρ_L extending from A to B along z direction as shown.

→ The charge element dQ associated with the element $dl = dz$ of the line is,

$$dQ = \rho_L \, dl = \rho_L \, dz$$



→ The total charge Q is, $Q = \int_{z_A}^{z_B} \rho_L \, dz$

→ The electric field intensity E at an arbitrary point p(x, y, z) can be found by eqn,

$$E = \int \frac{\rho_L \, dl}{4\pi \epsilon_0 R^2} a_R \quad \text{—— (1)}$$

→ we denote field point by (x, y, z) and the source point by (x', y', z').

From figure,

$$dl = dz'$$

$$R = (x, y, z) - (0, 0, z')$$

$$\Rightarrow x a_x + y a_y + (z - z') a_z \longrightarrow (a)$$

put
$$\left.\begin{aligned} a_x &= \cos\phi \, a_\rho - \sin\phi \, a_\phi \\ a_y &= \sin\phi \, a_\rho + \cos\phi \, a_\phi \\ x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{aligned}\right\} \rightarrow (b)$$

sub ⑥ in ⓐ for x, a_x, y, a_y

$\therefore$ ⓐ becomes,

$$R = \rho a_\rho + (z-z')a_z$$

$$R^2 = |R|^2 \Rightarrow x^2+y^2+(z-z')^2 \Rightarrow \rho^2+(z-z')^2$$

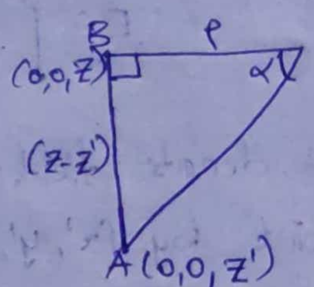
$$\frac{a_R}{R^2} = \frac{R}{|R|^2} \Rightarrow \frac{\rho a_\rho + (z-z')a_z}{[\rho^2+(z-z')^2]^{3/2}} \rightarrow ②$$

substituting ② in ① and $dl = dz'$

$$E = \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{\rho a_\rho + (z-z')a_z}{[\rho^2+(z-z')^2]^{3/2}}\, dz' \rightarrow ③$$

Let we define α, α_1 and α_2 in the figure,

$$\cos\alpha x = \frac{\rho}{R} \qquad \frac{1}{\cos\alpha} = \frac{R}{\rho}$$



$$\boxed{\rho = R\cos\alpha}, \quad \sec\alpha = \frac{R}{\rho}$$

$$\boxed{R = \rho\sec\alpha}$$

$$R^2 = \rho^2 + (z-z')^2$$

$$\boxed{R = \sqrt{\rho^2+(z-z')^2} = \rho\sec\alpha}$$

$$\sin\alpha = \frac{z-z'}{R}$$

$$\boxed{z-z' = R\sin\alpha}$$

$$\tan\alpha = \frac{z-z'}{\rho}$$

$$z - z' = \rho \tan\alpha$$

$$\frac{dz}{d\alpha} - \frac{dz'}{d\alpha} = \rho \sec^2\alpha$$

$$\boxed{dz' = -\rho \sec^2\alpha \, d\alpha}$$

$\therefore \frac{dz}{d\alpha}$ is a constant $= 0$

$\therefore$ eq^n (3) becomes,

$$E = \frac{-\rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{(R\cos\alpha \, a_\rho + R\sin\alpha \, a_z)}{\rho^3 \sec^3\alpha} \rho \sec^2\alpha \, d\alpha$$

$$= -\frac{\rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{(\cos\alpha \, a_\rho + \sin\alpha \, a_z)}{\rho^3 \sec^3\alpha} \rho^2 \sec^2\alpha \, d\alpha$$

$$= -\frac{\rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} (\cos\alpha \cdot a_\rho + \sin\alpha \, a_z) \, d\alpha$$

For a finite line charge,

$$E = \frac{\rho_L}{4\pi\epsilon_0\rho} \left[-\sin\alpha \, a_\rho + \cos\alpha \, a_z\right]_{\alpha_1}^{\alpha_2}$$

$$E = \frac{\rho_L}{4\pi\epsilon_0\rho} \left[-(\sin\alpha_2 - \sin\alpha_1) a_\rho + (\cos\alpha_2 - \cos\alpha_1) a_z\right]$$

For an infinite line charge,

point B is at $(0,0,\infty)$ and point A is at $(0,0,-\infty)$

so, that

$\alpha_1 = \frac{\pi}{2}$ and $\alpha_2 = -\frac{\pi}{2}$

$$\therefore \boxed{E = \frac{\rho_L}{2\pi\epsilon_0 \rho} a_\rho}$$

8. Derive the expression for electric potential in electrostatic fields ?

Ans:

→ To move a point charge Q from point A to point B in an electric field E, the work done in displacing the charge by dl is,

$$dW = -F\,dl \Rightarrow -QE\cdot dl$$

where 'F' is the force on Q.

→ The (-ve) sign indicates that the work is being done by an external agent.

→ The total work done or the potential energy required in moving Q from A to B is,

$$W = -Q\int_A^B E\cdot dl$$

→ The potential energy per unit charge or the potential difference b/w points A and B is

$$V_{AB} = \frac{W}{Q} = -\int_A^B E\cdot dl$$

→ Electric potential at any point is the negative gradient of line integral of electric field from ∞ to given point.

Note:-

i. In determining V_{AB}, 'A' is the initial point while 'B' is the final point.

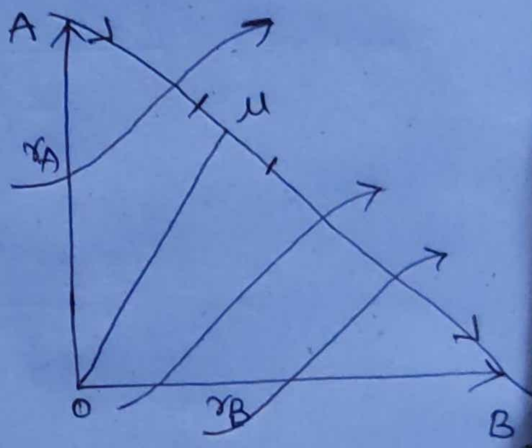
ii. V_{AB} is independent of the path taken.

iii. V_{AB} is measured in Joules/coloumb or volts.

→ If E is the field due to a point charge Q located at the origin then,

$$E = \frac{Q}{4\pi\epsilon_0 r^2} a_r$$

substituting E in V_{AB} eqn



$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} a_r dr\, a_r$$

$$= \frac{-Q}{4\pi\epsilon_0}\left[\frac{-1}{r}\right]_{r_A}^{r_B}$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$\boxed{V_{AB} = V_B - V_A}$ where V_A & V_B are potentials at A & B.

→ V_{AB} is potential at B w.r.t A.

→ In problems involving point charges, it is customary to choose infinity as reference i.e potential at infinity is zero.

→ In general, the potential at any point Q is defined as the potential difference b/w that point and a chosen point at which the potential is zero.

→ If the point charge Q is not located at the origin but at a point where position vector is r', the potential V(x,y,z) or V(r) is,

$$\boxed{V(r) = \frac{Q}{4\pi\epsilon_0 |r-r'|}}$$