12.8.1.2

EE24BTECH11033 - KOLLURU SURAJ

Question: Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

Solution:

Variable	Description	values
V	Quadratic form of the matrix	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
u	Linear coefficient vector	$\begin{pmatrix} -\frac{9}{2} \\ 0 \end{pmatrix}$
m	The direction vector of line	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
h	Point on line	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

TABLE 0: Variables used

Theoretical Solution: Calculating point of intersection The point of intersection of the line with the circle is $x_i = h + k_i m$,
where, k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top Vm}\left(-m^\top (Vh + u) \pm \sqrt{\left[m^\top (Vh + u)\right]^2 - g(h)(m^\top Vm)}\right)$$

Substituting the input parameters into k_i,

$$k_i = \frac{1}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}\left(-\begin{pmatrix} 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{-9}{2} \\ 0 \end{pmatrix}\right)\right) \pm$$

$$\sqrt{\begin{pmatrix} 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{-9}{2} \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 0 \end{pmatrix}^2 - g\begin{pmatrix} 2 \\ 0 \end{pmatrix}\left(\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)} \quad (0.1)$$

We get,
$k_i = \pm\sqrt{18}$
Substituting k_i into $x_i = h + k_i m$ we get

$$x_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \left(\sqrt{18}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.2}$$

$$\implies x_1 = \begin{pmatrix} 2 \\ \sqrt{18} \end{pmatrix} \tag{0.3}$$

$$x_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \left(-\sqrt{18}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.4}$$

$$\implies x_2 = \begin{pmatrix} 2 \\ -\sqrt{18} \end{pmatrix} \tag{0.5}$$

Here we need only x_1 because the point above x-axis is only we want.
Similarly for line x=4 if we calculate we get $\mathbf{x} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$
If the area is calulated directly using integration it would be

$$A = \int_2^4 3\sqrt{x}dx \tag{0.6}$$

$$A = \left[3 \cdot \frac{2}{3} x^{3/2}\right]_{x=2}^{x=4} \tag{0.7}$$

$$A = 10.343146 \tag{0.8}$$

Let's try to verify this computationally using Trapezoidal rule
Some theory about Trapezoidal rule:
The idea behind the trapezoidal rule is to approximate the region under the curve of the function as a series of trapezoids, rather than using rectangles. The area of each trapezoid is then computed and summed to estimate the total area under the curve (the integral).
First we need to set up the integral
Area under curve is given by

$$A = \int_2^4 3\sqrt{x}dx \tag{0.9}$$

We will approximate this using trapezoidal rule. Divide the interval [2,4] into 100 subintervals with each of width $h = \frac{4-2}{100} = \frac{1}{50}$, Let:

$$x_0 = 2, x_1 = x_0 + h, x_2 = x_1 + h, ...x_n = 4 \tag{0.10}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$ be equidistant points with step-size h. Then,

$$A(x_n + h) = A(x_n) + \frac{1}{2}h\left(y(x_n + h) + y(x_n)\right) \tag{0.11}$$

where $\frac{1}{2}h\left(y(x_n + h) + y(x_n)\right)$ is area of difference trapezium We can repeat this till we get the required area.

Let $A(x_n) = A_n$ and $y(x_n) = y_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1} + y_n\right) \tag{0.12}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left((y_n + hy'_n) + y_n\right) \tag{0.13}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{0.14}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{0.15}$$

$$\tag{0.16}$$

from 0.5

$$x_{n+1} = x_n + h \tag{0.17}$$

In the given question, $y_n^2 = 9x_n$ and $y'_n = \frac{3}{2\sqrt{x_n}}$
General Difference Equation is given by, from 0.10

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{0.18}$$

$$= A_n + h\left(3\sqrt{x_n}\right) + \frac{1}{2}h^2\left(\frac{3}{2\sqrt{x_n}}\right) \tag{0.19}$$

$$x_{n+1} = x_n + h \tag{0.20}$$

Upon iterating the equations from $x_0 = 2$ to x_n=4 using a C code we get the final area as

$$A \approx 10.34317 \tag{0.21}$$

which is closer to the theoritical solution hence this is correct

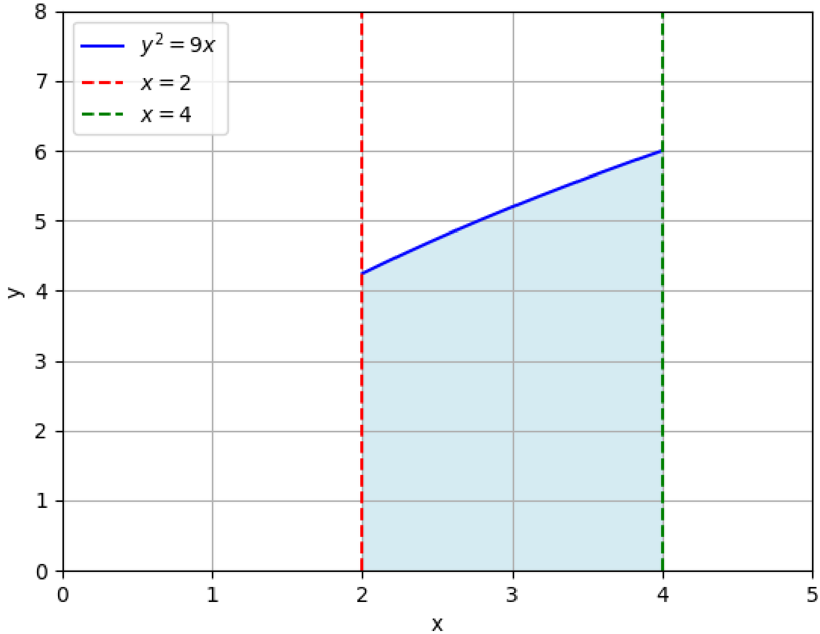

Fig. 0.1