Chapter 9

Inverse Trigonometric Functions

1. If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then **[JEE (Main)-2013]**

(1) $x = y = z$ (2) $2x = 3y = 6z$

(3) $6x = 3y = 2z$ (4) $6x = 4y = 3z$

2. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to **[JEE (Main)-2013]**

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$

(3) 1 (4) $\sqrt{2}$

3. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is **[JEE (Main)-2015]**

(1) $\frac{3x-x^3}{1-3x^2}$ (2) $\frac{3x+x^3}{1-3x^2}$

(3) $\frac{3x-x^3}{1+3x^2}$ (4) $\frac{3x+x^3}{1+3x^2}$

4. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right)$.

A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point **[JEE (Main)-2016]**

(1) $\left(0, \frac{2\pi}{3}\right)$ (2) $\left(\frac{\pi}{6}, 0\right)$

(3) $\left(\frac{\pi}{4}, 0\right)$ (4) $(0, 0)$

5. If $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}\left(x > \frac{3}{4}\right)$, then x is equal to **[JEE (Main)-2019]**

(1) $\frac{\sqrt{146}}{12}$ (2) $\frac{\sqrt{145}}{12}$

(3) $\frac{\sqrt{145}}{10}$ (4) $\frac{\sqrt{145}}{11}$

6. If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to **[JEE (Main)-2019]**

(1) 7π (2) 10

(3) 0 (4) π

7. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is **[JEE (Main)-2019]**

(1) $\frac{19}{21}$ (2) $\frac{23}{22}$

(3) $\frac{22}{23}$ (4) $\frac{21}{19}$

8. All x satisfying the inequality $(\cot^{-1}x)^2 - 7(\cot^{-1}x) + 10 > 0$, lie in the interval **[JEE (Main)-2019]**

(1) $(\cot 2, \infty)$

(2) $(\cot 5, \cot 4)$

(3) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$

(4) $(-\infty, \cot 5) \cup (\cot 2, \infty)$

9. If $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$, $\beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to **[JEE (Main)-2019]**

(1) $\tan^{-1}\left(\frac{9}{14}\right)$ (2) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

(3) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (4) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

10. If $\cos^{-1}x - \cos^{-1}\dfrac{y}{2} = \alpha$, where $-1 \le x \le 1$, $-2 \le y \le 2$, $x \le \dfrac{y}{2}$, then for all x, y, $4x^2 - 4xy\cos\alpha + y^2$ is equal to : **[JEE (Main)-2019]**

(1) $2\sin^2\alpha$ (2) $4\sin^2\alpha - 2x^2y^2$

(3) $4\cos^2\alpha + 2x^2y^2$ (4) $4\sin^2\alpha$

11. The value of $\sin^{-1}\left(\dfrac{12}{13}\right) - \sin^{-1}\left(\dfrac{3}{5}\right)$ is equal to

[JEE (Main)-2019]

(1) $\dfrac{\pi}{2} - \sin^{-1}\left(\dfrac{56}{65}\right)$ (2) $\pi - \sin^{-1}\left(\dfrac{63}{65}\right)$

(3) $\pi - \cos^{-1}\left(\dfrac{33}{65}\right)$ (4) $\dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{9}{65}\right)$

12. If $f'(x) = \tan^{-1}(\sec x + \tan x)$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, and $f(0) = 0$, then $f(1)$ is equal to **[JEE (Main)-2020]**

(1) $\dfrac{\pi+1}{4}$ (2) $\dfrac{1}{4}$

(3) $\dfrac{\pi+2}{4}$ (4) $\dfrac{\pi-1}{4}$

13. The domain of the function

$f(x) = \sin^{-1}\left(\dfrac{|x|+5}{x^2+1}\right)$ is

$(-\infty, -a] \cup [a, \infty)$. Then a is equal to

[JEE (Main)-2020]

(1) $\dfrac{1+\sqrt{17}}{2}$ (2) $\dfrac{\sqrt{17}}{2}+1$

(3) $\dfrac{\sqrt{17}-1}{2}$ (4) $\dfrac{\sqrt{17}}{2}$

14. $2\pi - \left(\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{16}{65}\right)$ is equal to

[JEE (Main)-2020]

(1) $\dfrac{\pi}{2}$ (2) $\dfrac{7\pi}{4}$

(3) $\dfrac{3\pi}{2}$ (4) $\dfrac{5\pi}{4}$

15. If S is the sum of the first 10 terms of the series

$\tan^{-1}\left(\dfrac{1}{3}\right) + \tan^{-1}\left(\dfrac{1}{7}\right) + \tan^{-1}\left(\dfrac{1}{13}\right) + \tan^{-1}\left(\dfrac{1}{21}\right) + \ldots$,

then $\tan(S)$ is equal to **[JEE (Main)-2020]**

(1) $-\dfrac{6}{5}$ (2) $\dfrac{5}{11}$

(3) $\dfrac{10}{11}$ (4) $\dfrac{5}{6}$

16. The derivative of $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to

$\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = \dfrac{1}{2}$ is

[JEE (Main)-2020]

(1) $\dfrac{2\sqrt{3}}{3}$ (2) $\dfrac{2\sqrt{3}}{5}$

(3) $\dfrac{\sqrt{3}}{10}$ (4) $\dfrac{\sqrt{3}}{12}$

17. If $y = \sum\limits_{k=1}^{6} k\cos^{-1}\left\{\dfrac{3}{5}\cos kx - \dfrac{4}{5}\sin kx\right\}$, then

$\dfrac{dy}{dx}$ at $x = 0$ is ________. **[JEE (Main)-2020]**

18. $\lim\limits_{n\to\infty} \tan\left\{\sum\limits_{r=1}^{n} \tan^{-1}\left(\dfrac{1}{1+r+r^2}\right)\right\}$ is equal to ______. **[JEE (Main)-2021]**

19. A possible value of $\tan\left(\dfrac{1}{4}\sin^{-1}\dfrac{\sqrt{63}}{8}\right)$ is :

[JEE (Main)-2021]

(1) $2\sqrt{2}-1$ (2) $\sqrt{7}-1$

(3) $\dfrac{1}{\sqrt{7}}$ (4) $\dfrac{1}{2\sqrt{2}}$

20. $\operatorname{cosec}\left[2\cot^{-1}(5) + \cos^{-1}\left(\dfrac{4}{5}\right)\right]$ is equal to :

[JEE (Main)-2021]

(1) $\dfrac{65}{56}$ (2) $\dfrac{65}{33}$

(3) $\dfrac{75}{56}$ (4) $\dfrac{56}{33}$

21. If $\frac{\sin^{-1}x}{a}=\frac{\cos^{-1}x}{b}=\frac{\tan^{-1}y}{c}$; $0 < x < 1$, then the value of $\cos\left(\frac{\pi c}{a+b}\right)$ is : **[JEE (Main)-2021]**

(1) $1-y^2$ (2) $\frac{1-y^2}{y\sqrt{y}}$

(3) $\frac{1-y^2}{1+y^2}$ (4) $\frac{1-y^2}{2y}$

22. If $0 < a, b < 1$, and $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$, then the value of

$(a + b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) +$ ... is : **[JEE (Main)-2021]**

(1) $e^2 - 1$ (2) $\log_e\left(\frac{e}{2}\right)$

(3) e (4) $\log_e 2$

23. Given that the inverse trigonometric function take principal values only. Then, the number of real values of x which satisfy $\sin^{-1}\left(\frac{3x}{5}\right)+\sin^{-1}\left(\frac{4x}{5}\right)=\sin^{-1}x$ is equal to :

[JEE (Main)-2021]

(1) 3 (2) 1

(3) 0 (4) 2

24. If $\cot^{-1}(\alpha) = \cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32$ +... upto 100 terms, then α is :

[JEE (Main)-2021]

(1) 1.01 (2) 1.02

(3) 1.03 (4) 1.00

25. The sum of possible values of x for $\tan^{-1}(x+1)+\cot^{-1}\left(\frac{1}{x-1}\right)=\tan^{-1}\left(\frac{8}{31}\right)$ is

[JEE (Main)-2021]

(1) $-\frac{30}{4}$ (2) $-\frac{31}{4}$

(3) $-\frac{32}{4}$ (4) $-\frac{33}{4}$

26. The number of solutions of the equation $\sin^{-1}\left[x^2+\frac{1}{3}\right]+\cos^{-1}\left[x^2-\frac{2}{3}\right]=x^2$, for $x\in[-1,1]$ and [x] denotes the greatest integer less than or equal to x, is : **[JEE (Main)-2021]**

(1) Infinite (2) 2

(3) 4 (4) 0

27. The real valued function $f(x)=\frac{\operatorname{cosec}^{-1}x}{\sqrt{x-[x]}}$, where [x] denotes the greatest integer less than or equal to x, is defined for all x belonging to :

[JEE (Main)-2021]

(1) all non-integers except the interval [–1, 1]

(2) all integers except 0, –1, 1

(3) all reals except integers

(4) all reals except the interval [–1, 1]

28. The number of real roots of the equation $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{4}$ is :

[JEE (Main)-2021]

(1) 2 (2) 1

(3) 4 (4) 0

29. The value of $\tan\left(2\tan^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)\right)$ is equal to

[JEE (Main)-2021]

(1) $\frac{220}{21}$ (2) $\frac{151}{63}$

(3) $\frac{-181}{69}$ (4) $\frac{-291}{76}$

30. The domain of the function $\operatorname{cosec}^{-1}\left(\frac{1+x}{x}\right)$ is :

[JEE (Main)-2021]

(1) $\left[-\frac{1}{2},0\right)\cup[1,\infty)$ (2) $\left[-\frac{1}{2},\infty\right)-\{0\}$

(3) $\left(-1,-\frac{1}{2}\right]\cup(0,\infty)$ (4) $\left(-\frac{1}{2},\infty\right)-\{0\}$

31. If $\left(\sin^{-1}x\right)^2-\left(\cos^{-1}x\right)^2=a$; $0 < x < 1$, $a \neq 0$, then the value of $2x^2 - 1$ is **[JEE (Main)-2021]**

(1) $\cos\left(\frac{2a}{\pi}\right)$ (2) $\sin\left(\frac{4a}{\pi}\right)$

(3) $\cos\left(\frac{4a}{\pi}\right)$ (4) $\sin\left(\frac{2a}{\pi}\right)$

32. If $y(x)=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), x\in\left(\frac{\pi}{2},\pi\right)$, then $\frac{dy}{dx}$ at $x=\frac{5\pi}{6}$ is **[JEE (Main)-2021]**

(1) $-\frac{1}{2}$ (2) 0

(3) –1 (4) $\frac{1}{2}$

33. Let M and m respectively be the maximum and minimum values of the function $f(x) = \tan^{-1}(\sin x + \cos x)$ in $\left[0,\frac{\pi}{2}\right]$. Then the value of $\tan(M-m)$ is equal to **[JEE (Main)-2021]**

(1) $2-\sqrt{3}$ (2) $2+\sqrt{3}$

(3) $3+2\sqrt{2}$ (4) $3-2\sqrt{2}$

34. The domain of the function

$$f(x)=\sin^{-1}\left(\frac{3x^2+x-1}{(x-1)^2}\right)+\cos^{-1}\left(\frac{x-1}{x+1}\right) \text{ is}$$

[JEE (Main)-2021]

(1) $\left[0,\frac{1}{4}\right]$ (2) $\left[0,\frac{1}{2}\right]$

(3) $\left[\frac{1}{4},\frac{1}{2}\right]\cup\{0\}$ (4) $[-2,0]\cup\left[\frac{1}{4},\frac{1}{2}\right]$

35. $\cos^{-1}(\cos(-5))+\sin^{-1}(\sin(6))-\tan^{-1}(\tan(12))$ is equal to (The inverse trigonometric functions take the principal values) **[JEE (Main)-2021]**

(1) $3\pi + 1$ (2) $3\pi - 11$

(3) $4\pi - 11$ (4) $4\pi - 9$

36. Let $S_k=\sum_{r=1}^{k}\tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$. Then $\lim_{k\to\infty} S_k$ is equal to : **[JEE (Main)-2021]**

(1) $\tan^{-1}(3)$ (2) $\tan^{-1}\left(\frac{3}{2}\right)$

(3) $\cot^{-1}\left(\frac{3}{2}\right)$ (4) $\frac{\pi}{2}$

37. If $\sum_{r=1}^{50}\tan^{-1}\frac{1}{2r^2}=p$, then the value of $\tan p$ is

[JEE (Main)-2021]

(1) 100 (2) $\frac{50}{51}$

(3) $\frac{101}{102}$ (4) $\frac{51}{50}$

38. Considering only the principal values of inverse functions, the set **[JEE (Main)-2021]**

$$A=\left\{x\ge 0: \tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}\right\}$$

(1) Is a singleton

(2) Contains two elements

(3) Contains more than two elements

(4) Is an empty set

39. The set of all values of k for which $\left(\tan^{-1}x\right)^3+\left(\cot^{-1}x\right)^3=k\pi^3$, $x\in\mathrm{R}$, is the interval

[JEE (Main)-2022]

(1) $\left[\frac{1}{32},\frac{7}{8}\right)$ (2) $\left(\frac{1}{24},\frac{13}{16}\right)$

(3) $\left[\frac{1}{48},\frac{13}{16}\right]$ (4) $\left[\frac{1}{32},\frac{9}{8}\right)$

40. The domain of the function

$$f(x)=\frac{\cos^{-1}\left(\frac{x^2-5x+6}{x^2-9}\right)}{\log_e(x^2-3x+2)} \text{ is}$$ **[JEE (Main)-2022]**

(1) $(-\infty,1)\cup(2,\infty)$

(2) $(2,\infty)$

(3) $\left[-\frac{1}{2},1\right)\cup(2,\infty)$

(4) $\left[-\frac{1}{2},1\right)\cup(2,\infty)-\left\{\frac{3+\sqrt{5}}{2},\frac{3-\sqrt{5}}{2}\right\}$

41. Let $x * y = x^2 + y^3$ and $(x * 1) * 1 = x * (1 * 1)$. Then a value of $2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$ is

[JEE (Main)-2022]

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{3}$

(3) $\frac{\pi}{2}$ (4) $\frac{\pi}{6}$

42. The value of $\tan^{-1}\left(\dfrac{\cos\left(\dfrac{15\pi}{4}\right)-1}{\sin\left(\dfrac{\pi}{4}\right)}\right)$ is equal to

[JEE (Main)-2022]

(1) $-\dfrac{\pi}{4}$ (2) $-\dfrac{\pi}{8}$

(3) $-\dfrac{5\pi}{12}$ (4) $-\dfrac{4\pi}{9}$

43. Let $f(x)=2\cos^{-1}x+4\cot^{-1}x-3x^2-2x+10,\ \chi\in[-1,1]$, If $[a, b]$ is the range of the function f, then $4a - b$ is equal to **[JEE (Main)-2022]**

(1) 11 (2) $11 - \pi$

(3) $11 + \pi$ (4) $15 - \pi$

44. If the inverse trigonometric functions take principal values, then

$$\cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)+\frac{2}{5}\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right)$$

is equal to **[JEE (Main)-2022]**

(1) 0 (2) $\dfrac{\pi}{4}$

(3) $\dfrac{\pi}{3}$ (4) $\dfrac{\pi}{6}$

45. $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)+\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)+\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$ is equal to **[JEE (Main)-2022]**

(1) $\dfrac{11\pi}{12}$ (2) $\dfrac{17\pi}{12}$

(3) $\dfrac{31\pi}{12}$ (4) $-\dfrac{3\pi}{4}$

46. The value of $\cot\left(\displaystyle\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right)$ is

[JEE (Main)-2022]

(1) $\dfrac{26}{25}$ (2) $\dfrac{25}{26}$

(3) $\dfrac{50}{51}$ (4) $\dfrac{52}{51}$

47. The value of $\displaystyle\lim_{n\to\infty} 6\tan\left\{\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{r^2+3r+3}\right)\right\}$ is equal to **[JEE (Main)-2022]**

(1) 1 (2) 2

(3) 3 (4) 6

48. The domain of the function

$\cos^{-1}\left(\dfrac{2\sin^{-1}\left(\dfrac{1}{4x^2-1}\right)}{\pi}\right)$ is **[JEE (Main)-2022]**

(1) $\mathbf{R}-\left\{-\dfrac{1}{2},\dfrac{1}{2}\right\}$

(2) $(-\infty,-1]\cup[1,\infty)\cup\{0\}$

(3) $\left(-\infty,\dfrac{-1}{2}\right)\cup\left(\dfrac{1}{2},\infty\right)\cup\{0\}$

(4) $\left(-\infty,\dfrac{-1}{\sqrt{2}}\right]\cup\left[\dfrac{1}{\sqrt{2}},\infty\right)\cup\{0\}$

49. $50\tan\left(3\tan^{-1}\left(\dfrac{1}{2}\right)+2\cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right)\right)$

$+4\sqrt{2}\tan\left(\dfrac{1}{2}\tan^{-1}\left(2\sqrt{2}\right)\right)$ is equal to ______.

[JEE (Main)-2022]

50. $\tan\left(2\tan^{-1}\dfrac{1}{5}+\sec^{-1}\dfrac{\sqrt{5}}{2}+2\tan^{-1}\dfrac{1}{8}\right)$ is equal to

[JEE (Main)-2022]

(1) 1 (2) 2

(3) $\dfrac{1}{4}$ (4) $\dfrac{5}{4}$

51. For $k \in R$, let the solution of the equation

$$\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}\right)\right)\right)\right)\right)$$

$$=k, 0<|x|<\frac{1}{\sqrt{2}}$$

Inverse trigonometric functions take only principal values. If the solutions of the equation $x^2 - bx - 5 = 0$ are $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ and $\dfrac{\alpha}{\beta}$, then $\dfrac{b}{k^2}$ is equal to ______.

[JEE (Main)-2022]

52. Considering only the principal values of the inverse trigonometric functions, the domain of the function

$f(x) = \cos^{-1}\left(\dfrac{x^2 - 4x + 2}{x^2 + 3}\right)$ is **[JEE (Main)-2022]**

(1) $\left(-\infty, \dfrac{1}{4}\right]$ (2) $\left[-\dfrac{1}{4}, \infty\right)$

(3) $\left(\dfrac{-1}{3}, \infty\right)$ (4) $\left(-\infty, \dfrac{1}{3}\right]$

53. Considering the principal values of the inverse trigonometric functions, the sum of all the solutions of the equation $\cos^{-1}(x) - 2\sin^{-1}(x) = \cos^{-1}(2x)$ is equal to **[JEE (Main)-2022]**

(1) 0 (2) 1

(3) $\dfrac{1}{2}$ (4) $-\dfrac{1}{2}$

54. The sum of the absolute maximum and absolute minimum values of the function

$f(x) = \tan^{-1}(\sin x - \cos x)$ in the interval $[0, \pi]$ is

(1) 0 (2) $\tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right) - \dfrac{\pi}{4}$

(3) $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right) - \dfrac{\pi}{4}$ (4) $\dfrac{-\pi}{12}$

[JEE (Main)-2022]

55. The domain of the function

$f(x) = \sin^{-1}\left(\dfrac{x^2 - 3x + 2}{x^2 + 2x + 7}\right)$ is

(1) $[1, \infty)$ (2) $[-1, 2]$

(3) $[-1, \infty)$ (4) $(-\infty, 2]$

[JEE (Main)-2022]

56. If $0 < x < \dfrac{1}{\sqrt{2}}$ and $\dfrac{\sin^{-1} x}{\alpha} = \dfrac{\cos^{-1} x}{\beta}$, then a value of $\sin\left(\dfrac{2\pi\alpha}{\alpha + \beta}\right)$ is

(1) $4\sqrt{(1-x^2)}(1-2x^2)$ (2) $4x\sqrt{(1-x^2)}(1-2x^2)$

(3) $2x\sqrt{(1-x^2)}(1-4x^2)$ (4) $4\sqrt{(1-x^2)}(1-4x^2)$

[JEE (Main)-2022]

57. The domain of the function

$f(x) = \sin^{-1}\left[2x^2 - 3\right] + \log_2\left(\log_{\frac{1}{2}}\left(x^2 - 5x + 5\right)\right)$,

where $[t]$ is the greatest integer function, is

[JEE (Main)-2022]

(1) $\left(-\sqrt{\dfrac{5}{2}}, \dfrac{5-\sqrt{5}}{2}\right)$ (2) $\left(\dfrac{5-\sqrt{5}}{2}, \dfrac{5+\sqrt{5}}{2}\right)$

(3) $\left(1, \dfrac{5-\sqrt{5}}{2}\right)$ (4) $\left(1, \dfrac{5+\sqrt{5}}{2}\right)$

58. Let $x = \sin(2\tan^{-1}\alpha)$ and $y = \sin\left(\dfrac{1}{2}\tan^{-1}\dfrac{4}{3}\right)$. If $S = \{\alpha \in \mathbb{R} : y^2 = 1 - x\}$, then $\sum_{\alpha \in S} 16\alpha^3$ is equal to _______.

[JEE (Main)-2022]

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Chapter 9

Inverse Trigonometric Functions

1. Answer (1)

$\because\ 2y = x + z$ and $2\tan^{-1}y = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$

$\Rightarrow \frac{2y}{1-y^2} = \frac{x+z}{1-xy}$

$\Rightarrow \boxed{y^2 = xz}$

$\Rightarrow$ x, y, z are in GP

$\therefore\ x = y = z$

2. Answer (1)

$y = \sec\sec^{-1}\left(\sqrt{1+x^2}\right) = \sqrt{1+x^2}$

$\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$

$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{\sqrt{2}}$

3. Answer (1)

$\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$y = \frac{3x-x^3}{1-3x^2}$

4. Answer (1)

$f(x) = \tan^{-1}\sqrt{\frac{1+\sin x}{1-\sin x}}$

$= \tan^{-1}\sqrt{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$

$= \frac{\pi}{4}+\frac{x}{2}$

$\Rightarrow f'(x) = \frac{1}{2}$ and at $x = \frac{\pi}{6}$, $f(x) = \frac{\pi}{3}$

So, equation of normal is

$y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right) \Rightarrow y + 2x = \frac{2\pi}{3}$

5. Answer (2)

$\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2} \qquad \left(x > \frac{3}{4}\right)$

$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{3}{4x}\right)$

$\because\ \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \sin^{-1}\left(\frac{3}{4x}\right)$

$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \cos^{-1}\left(\frac{\sqrt{16x^2-9}}{4x}\right)$

$\because\ \sin^{-1}\left(\frac{3}{4x}\right) = \cos^{-1}\left(\frac{\sqrt{16x^2-9}}{4x}\right)$

$\Rightarrow \frac{2}{3x} = \frac{\sqrt{16x^2-9}}{4x}$

$\Rightarrow x^2 = \frac{64+81}{9\times 16}$

$\boxed{x = \frac{\sqrt{145}}{12}} \qquad \left(\because\ x > \frac{3}{4}\right)$

6. Answer (4)

$x = \sin^{-1}(\sin 10)$

$x = 3\pi - 10$ $\quad \begin{cases} 3\pi - \dfrac{\pi}{2} < 10 < 3\pi + \dfrac{\pi}{2} \\ \Rightarrow 3\pi - x \end{cases}$

and $y = \cos^{-1}(\cos 10)$ $\quad \begin{cases} 3\pi < 10 < 4\pi \\ \Rightarrow 4\pi - x \end{cases}$

$y = 4\pi - 10$

$\therefore \quad y - x = (4\pi - 10) - (3\pi - 10) = \pi$

7. Answer (4)

$$\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$$

$$= \cot\left(\sum_{n=1}^{19}\cot^{-1}(1+n(n+1))\right)$$

$$= \cot\left(\sum_{n=1}^{19}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)\right)$$

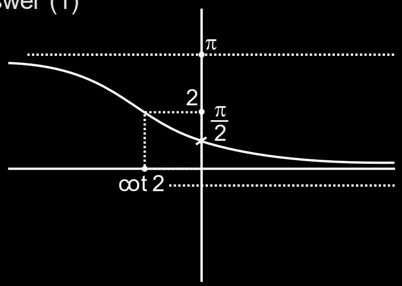
$$= \cot\left(\sum_{n=1}^{19}\left(\tan^{-1}(n+1)-\tan^{-1}n\right)\right)$$

$$= \cot\left(\tan^{-1}20-\tan^{-1}1\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{20-1}{1+20\times 1}\right)\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{19}{21}\right)\right) = \cot\cot^{-1}\left(\frac{21}{19}\right) = \frac{21}{19}$$

8. Answer (1)



$(\cot^{-1}x - 5)\ (\cot^{-1} - 2) > 0$

$\cot^{-1}x \in (-\infty, 2) \cup (5, \infty)$ ...(i)

But $\cot^{-1}x$ lies in $(0, \pi)$

From equation (i)

So, $\cot^{-1}x \in (0, 2)$

By graph,

$x \in (\cot 2, \infty)$

9. Answer (3)

$\because \quad \cos\alpha = \dfrac{3}{5}$

$\Rightarrow \quad \tan\alpha = \dfrac{4}{3}$

$\Rightarrow$ and $\tan\beta = \dfrac{1}{3}$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\cdot\tan\beta}$$

$$= \frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{9}} = \frac{1}{\frac{13}{9}}$$

$$= \frac{9}{13}$$

$$\alpha - \beta = \tan^{-1}\left(\frac{9}{13}\right) = \sin^{-1}\left(\frac{9}{5\sqrt{10}}\right) = \cos^{-1}\left(\frac{13}{5\sqrt{10}}\right)$$

10. Answer (4)

$$\cos^{-1}x - \cos^{-1}\ \frac{y}{2} = \alpha$$

$$\Rightarrow \cos^{-1}\left(\frac{xy}{2}+\sqrt{1-x^2}.\sqrt{1-\frac{y^2}{4}}\right) = \alpha$$

$$\Rightarrow \frac{xy}{2}+\frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2} = \cos\alpha$$

$$\Rightarrow xy + \sqrt{1-x^2}\sqrt{4-y^2} = 2\cos\alpha$$

$(xy - 2\cos\alpha)^2 = (1 - x^2)\ (4 - y^2)$

$x^2y^2 + 4\cos^2\alpha - 4xy\cos\alpha = 4 - y^2 - 4x^2 + x^2y^2$

$4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

11. Answer (1)

$$-\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{12}{13}\right)$$

$$= -\sin^{-1}\left(\frac{3}{5}\times\frac{5}{13}-\frac{12}{13}\times\frac{4}{5}\right)$$

$(\because xy \geq 0$ and $x^2 + y^2 \leq 1)$

$$= -\sin^{-1}\left(\frac{-33}{65}\right) \quad = \sin^{-1}\left(\frac{33}{65}\right)$$

$$= \cos^{-1}\left(\frac{56}{65}\right) \quad = \frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$$

12. Answer (1)

$$f'(x) = \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$$

$\because \ \frac{\pi}{4}+\frac{x}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ in the neighbourhood of $x = 1$

$$\Rightarrow \ f'(x) = \frac{\pi}{4}+\frac{x}{2}$$

$$\Rightarrow \ f(x) = \frac{\pi}{4}x + \frac{x^2}{4} + c$$

$\because \ f(0) = 0 \qquad \Rightarrow c = 0$

So $f(1) = \frac{\pi}{4}+\frac{1}{4} = \frac{\pi+1}{4}$

13. Answer (1)

$$\because \ f(x) = \sin^{-1}\left(\frac{|x|+5}{x^2+1}\right)$$

$$\therefore \ -1 \le \frac{|x|+5}{x^2+1} \le 1$$

$$\therefore \ x^2 - |x| - 4 \ge 0$$

$$\left(|x| - \frac{1-\sqrt{17}}{2}\right)\left(|x| - \frac{1+\sqrt{17}}{2}\right) \ge 0$$

$$\therefore \ |x| \ge \frac{1+\sqrt{17}}{2}$$

$$\therefore \ x \in \left(-\infty, -\frac{1+\sqrt{17}}{2}\right] \cup \left[\frac{1+\sqrt{17}}{2}, \infty\right)$$

14. Answer (3)

$$2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$$

$$= 2\pi - \left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{16}{63}\right)$$

$$= 2\pi - \left\{\tan^{-1}\left(\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}\cdot\frac{5}{12}}\right) + \tan^{-1}\frac{16}{63}\right\}$$

$$= 2\pi - \left(\tan^{-1}\frac{63}{16} + \tan^{-1}\frac{16}{63}\right)$$

$$= 2\pi - \left(\tan^{-1}\frac{63}{16} + \cot^{-1}\frac{63}{16}\right)$$

$$= 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$$

15. Answer (4)

$$S = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \ldots\ldots\ldots$$

$$= \tan^{-1}\left(\frac{1}{1+1\cdot 2}\right) + \tan^{-1}\left(\frac{1}{1+2\cdot 3}\right) + \tan^{-1}\left(\frac{1}{1+3\cdot 4}\right) + \ldots\ldots\ldots$$

$$= \tan^{-1}\left(\frac{2-1}{1+2\cdot 1}\right) + \tan^{-1}\left(\frac{3-2}{1+3\cdot 2}\right) + \tan^{-1}\left(\frac{4-3}{1+3\cdot 4}\right) + \ldots\ldots\ldots + \tan^{-1}\left(\frac{11-10}{1+11\cdot 10}\right)$$

$= (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + (\tan^{-1}4 - \tan^{-1}3) + \ldots\ldots + (\tan^{-1}11 - \tan^{-1}10)$

$= \tan^{-1}11 - \tan^{-1}1$

$$= \tan^{-1}\left(\frac{11-1}{1+11\cdot 1}\right)$$

$$= \tan^{-1}\left(\frac{5}{6}\right)$$

$$\tan(S) = \frac{5}{6}$$

16. Answer (3)

$$\frac{d\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)}{d\left(\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)\right)} = \frac{\dfrac{d\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)}{dx}}{\dfrac{d\left(\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)\right)}{dx}}$$

Simplifying $\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)$ Put $x = \tan\theta$

$$\Rightarrow \ \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\left(1-2\sin^2\theta/2\right)}{2\sin\theta/2\cos\theta/2}\right) = \frac{\theta}{2}$$

$$\therefore \ \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{\tan^{-1}x}{2}$$

& similarly $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ Put $x = \sin\theta$

$$\Rightarrow \ \tan^{-1}\left(\frac{\sin 2\theta}{\cos 2\theta}\right) = 2\theta = 2\sin^{-1}x$$

Hence required derivative

$$\frac{\frac{1}{2(1+x^2)}}{\frac{2}{\sqrt{1-x^2}}} = \left.\frac{\sqrt{1-x^2}}{4(1+x^2)}\right|_{x=\frac{1}{2}} = \frac{\sqrt{3}}{10}$$

17. Answer (91)

$$y = \sum_{k=1}^{6} k \cos^{-1}\left\{\frac{3}{5}\cos kx - \frac{4}{5}\sin kx\right\}$$

Let $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$

$$\therefore \quad y = \sum_{k=1}^{6} k \cos^{-1} \{\cos\alpha\cos kx - \sin\alpha\sin kx\}$$

$$= \sum_{k=1}^{6} k \cos^{-1}(\cos(kx+\alpha))$$

$$= \sum_{k=1}^{6} k\ (kx+\alpha) = \sum_{k=1}^{6} (k^2x+\alpha k)$$

$$\frac{dy}{dx} = \sum_{k=1}^{6} k^2 = \frac{6(7)(13)}{6} = 91$$

18. Answer (1)

$$\tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}\left(\frac{(r+1)-r}{1+(r+1)r}\right)$$

$$= \tan^{-1}(r+1) - \tan^{-1} r$$

So $\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r+r^2}\right) = (\tan^{-1}2 - \tan^{-1}1)$

$+ (\tan^{-1}3 - \tan^{-1}2) + \dots\dots\dots\dots + (\tan^{-1}(n+1) - \tan^{-1}n)$

$= \tan^{-1}(n+1) - \tan^{-1}1$

$$\Rightarrow \quad \lim_{n\to\infty} \tan\left\{\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)\right\}$$

$$\lim_{n\to\infty} \tan\left(\tan^{-1}(n+1) - \tan^{-1}1\right)$$

$$= \tan\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 1$$

19. Answer (3)

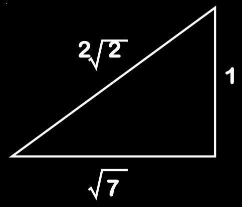
$$\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right) \Rightarrow \tan\left(\frac{\theta}{4}\right) = ?$$

Let

$$\sin^{-1}\frac{\sqrt{63}}{8} = \theta \text{ and } \cos\theta = \frac{1}{8} \Rightarrow \sin\theta = \frac{\sqrt{63}}{8} \Rightarrow \tan\theta = \sqrt{63} \text{ and } 6$$

$$\Rightarrow \quad 2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8} \Rightarrow 2\cos^2\frac{\theta}{2} = \frac{9}{8} \Rightarrow \cos^2\frac{\theta}{2} = \frac{9}{16}$$

$$\Rightarrow \quad \cos\frac{\theta}{2} = \frac{3}{4}$$



$$\Rightarrow \quad 2\cos^2\frac{\theta}{4} - 1 = \frac{3}{4} \qquad \Rightarrow \quad 2\cos^2\frac{\theta}{4} = \frac{7}{4}$$

$$\Rightarrow \quad \cos^2\frac{\theta}{4} = \frac{7}{8} \Rightarrow \cos\frac{\theta}{4} = \frac{\sqrt{7}}{2\sqrt{2}}$$

$$\Rightarrow \quad \tan\frac{\theta}{4} = \frac{1}{\sqrt{7}}$$

20. Answer (1)

$$\operatorname{cosec}\left(2\cot^{-1}5 + \cos^{-1}\left(\frac{4}{5}\right)\right)$$

$$\Rightarrow \quad \operatorname{cosec}\left(\tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{3}{4}\right)\right)$$

$$\Rightarrow \quad \operatorname{cosec}\left(\tan^{-1}\left(\frac{56}{33}\right)\right)$$

$$\Rightarrow \quad \operatorname{cosec}\left(\operatorname{cosec}^{-1}\left(\frac{65}{56}\right)\right) = \frac{65}{56}$$

21. Answer (3)

$$\because \quad \frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b} = \frac{\tan^{-1}y}{c} = k \ \text{(say)}$$

$\therefore \quad \sin^{-1} x = ak,\ \cos^{-1}x = bk$ and $\tan^{-1}y = ck$

Now,

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$(a+b)x = \frac{\pi}{2}$$

$$\therefore \quad k = \frac{\pi}{2(a+b)}$$

$$\text{Now } \tan^{-1}y = \frac{\pi c}{2(a+b)}$$

$$\therefore \quad \cos\left(\frac{\pi c}{ab}\right) = \cos(2\tan^{-1}y)$$

$$= \cos\left(\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right) \qquad [\text{if } y > 0]$$

$$= \frac{1-y^2}{1+y^2}$$

22. Answer (4)

$$\tan^{-1}a+\tan^{-1}b=\tan^{-1}\left(\frac{a+b}{a-ab}\right)=\frac{\pi}{4}$$

$\Rightarrow$ a + b + ab = 1

$\Rightarrow$ (1 + a)(1 + b) = 2

Given

$$\left(a-\frac{a^2}{2}+\frac{a^3}{3}+...\right)+\left(b-\frac{b^2}{2}+\frac{b^3}{3}+...\right)$$

ln(1 + a) + ln(1 + b)

$\Rightarrow$ ln(1 + a)(1 + b) = ln 2

23. Answer (1)

$$\sin^{-1}\left(\frac{3x}{5}\right)+\sin^{-1}\left(\frac{4x}{5}\right)=\sin^{-1}x$$

$$\sin^{-1}\left(\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right)=\sin^{-1}x$$

$$\sin^{-1}\left(\frac{3x\sqrt{25-16x^2}+4x\sqrt{25-9x^2}}{25}\right)=\sin^{-1}x$$

$$3x\sqrt{25-16x^2}+4x\sqrt{25-9x^2}=25x$$

x = 0 or $3\sqrt{25-16x^2}+4\sqrt{25-9x^2}=25$

Squaring both sides

$$x^2=\frac{1}{2}$$

$$x=\pm\frac{1}{\sqrt{2}}$$

$$\therefore\quad x=0,\ \pm\frac{1}{\sqrt{2}}$$

24. Answer (1)

$$\cot^{-1}(\alpha)=\cot^{-1}2+\cot^{-1}8+\cot^{-1}18+\cot^{-1}32+...100\text{ terms}$$

$$=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}+\tan^{-1}\frac{1}{32}+...100\text{ term}$$

$$=\sum_{k=1}^{100}\tan^{-1}\frac{1}{2k^2}$$

$$=\sum_{k=1}^{100}\tan^{-1}\frac{2}{4k^2}=\sum_{k=1}^{n}\tan^{-1}\frac{(2k+1)-(2k-1)}{1+(2k-1)(2k+1)}$$

$$=\sum_{k=1}^{100}\left(\tan^{-1}(2k+1)-\tan^{-1}(2k-1)\right)$$

$= \tan^{-1}201 - \tan^{-1}1$

$$=\tan^{-1}\frac{200}{202}$$

$= \cot^{-1}(1.01)$

Hence α = 1.01

25. Answer (3)

$$\tan^{-1}(x+1)+\cot^{-1}\left(\frac{1}{x-1}\right)=\tan^{-1}\left(\frac{8}{31}\right)$$

$$\Rightarrow\quad \tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow\quad \tan^{-1}\left(\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right)=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow\quad \frac{x}{2-x^2}=\frac{4}{31}$$

$\Rightarrow$ $4x^2 + 31x - 8 = 0 \Rightarrow x=\frac{1}{4}$ or $x = -8$

$x=\frac{1}{4}$ does not satisfy

Hence, sum of possible values of $x=-8=\frac{-32}{4}$

26. Answer (4)

$$\sin^{-1}\left(\left[x^2+\frac{1}{3}\right]\right)+\cos^{-1}\left(\left[x^2+\frac{1}{3}\right]-1\right)=x^2$$

$\because\quad x^2+\frac{1}{3}\in\left[\frac{1}{3},\frac{4}{3}\right]$; so $\left[x^2+\frac{1}{3}\right]=0$ or 1

Hence L.H.S. is always equal to π.

and $x^2 = \pi$ has no solution in [–1, 1].

27. Answer (1)

$\text{cosec}^{-1}x$ defined for $x\in(-\infty, -1] \cup [1, \infty)$

also $\sqrt{\{x\}}>0\Rightarrow x\neq Z$

$\therefore$ f(x) is defined for all non-integer except interval [–1, 1]

28. Answer (4)

$x(x + 1) \geq 0$ ...(i)

$x^2 + x + 1 \leq 1$

$\Rightarrow$ $x^2 + x \leq 0$ ...(ii)

$\Rightarrow$ $x^2 + x = 0 \Rightarrow x = 0, -1$

When x = 0 or –1

$\text{LHS}=\frac{\pi}{2}\Rightarrow$ No solution

29. Answer (1)

$$2\tan^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{6/5}{1-9/25}\right)=\tan^{-1}\left(\frac{6/5}{16/25}\right)=\tan^{-1}\frac{15}{8}$$

$\therefore$

$$2\tan^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)=\tan^{-1}\left(\frac{15}{8}\right)+\tan^{-1}\left(\frac{5}{12}\right)$$

$$=\tan^{-1}\left(\frac{\frac{15}{8}+\frac{5}{12}}{1-\frac{15}{8},\frac{5}{12}}\right)$$

$$=\tan^{-1}\left(\frac{180+40}{21}\right)=\tan^{-1}\left(\frac{220}{21}\right)$$

30. Answer (2)

For domain

$$\frac{1+x}{x}\le -1 \text{ or } \frac{1+x}{x}\ge 1$$

$$\Rightarrow \frac{1+2x}{x}\le 0 \text{ or } \frac{1}{x}\ge 0$$

$$\Rightarrow x\in\left[-\frac{1}{2},0\right) \text{ or } x\in(0,\infty)$$

$$\therefore \text{ domian } x\in\left[-\frac{1}{2},0\right)\cup(0,\infty)$$

$$\text{i.e } x\in\left[-\frac{1}{2},\infty\right)-\{0\}$$

31. Answer (4)

$(\sin^{-1} x)^2 - (\cos^{-1} x)^2 = (\sin^{-1} x + \cos^{-1} x)\ (\sin^{-1} x - \cos^{-1} x) = a$

$$=\frac{\pi}{2}\left(\frac{\pi}{2}-2\cos^{-1}x\right)=a$$

$$\Rightarrow =\frac{\pi}{2}-2\cos^{-1}x=\frac{2a}{\pi}$$

take sine both sides

$$\therefore \sin\left(\frac{\pi}{2}-2\cos^{-1}x\right)=\sin\left(\frac{2a}{\pi}\right)$$

$$\Rightarrow \cos\left(2\cos^{-1}x\right)=\sin\left(\frac{2a}{\pi}\right)$$

$$\Rightarrow 2\cos^2\left(\cos^{-1}x\right)-1=\sin\left(\frac{2a}{\pi}\right)$$

$$\Rightarrow 2x^2-1=\sin\left(\frac{2a}{\pi}\right)$$

32. Answer (1)

$$\because \sqrt{1+\sin x}=\sin\frac{x}{2}+\cos\frac{x}{2} \text{ and}$$

$$\sqrt{1-\sin x}=\sin\frac{x}{2}-\cos\frac{x}{2}$$

$$y(x)=\cot^{-1}\left(\frac{2\sin\frac{x}{2}}{2\cos\frac{x}{2}}\right)=\cot^{-1}\left(\tan\frac{x}{2}\right)=\frac{\pi}{2}-\frac{x}{2}$$

$$\frac{dy}{dx}=-\frac{1}{2}$$

33. Answer (4)

Range of $\sin x + \cos x$ for $x\in\left[0,\frac{\pi}{2}\right]$ is $[1,\sqrt{2}]$

So, $M=\tan^{-1}\sqrt{2}$ and $m=\tan^{-1}1$

$$\Rightarrow M-m=\tan^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$\Rightarrow \tan(M-m)=\frac{\sqrt{2}-1}{\sqrt{2}+1}=3-2\sqrt{2}$$

34. Answer (3)

$$\because \left(\frac{3x^2+x-1}{(x-1)^2}\right)\in[-1,1] \text{ and } \frac{x-1}{x+1}\in[-1,1]$$

$$\Rightarrow x\in\left[-2,\frac{1}{2}\right] \text{ and } x\in(-\infty,0]\cup\left[\frac{1}{4},\infty\right)-\{1\} \text{ and}$$
$$x\in[0,\infty)$$

finally $x\in\{0\}\cup\left[\frac{1}{4},\frac{1}{2}\right]$

35. Answer (3)

$$\cos^{-1}(\cos(-5))=-5+2\pi=a(\text{say})$$

$$\sin^{-1}(\sin 6)=6-2\pi=b(\text{say})$$

$$\tan^{-1}(\tan 12)=12-4\pi=c(\text{say})$$

$\therefore$ $a + b - c = 2\pi - 5 + 6 - 2\pi - 12 + 4\pi$
$= 4\pi - 11$

36. Answer (3)

$$\text{Let } T_k=\tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)=\tan^{-1}\left(\frac{\frac{2^r}{3^{r+1}}}{1+\left(\frac{2}{3}\right)^{2r+1}}\right)$$

$$=\tan^{-1}\left(\frac{\left(\frac{2}{3}\right)^r-\left(\frac{2}{3}\right)^{r+1}}{1+\left(\frac{2}{3}\right)^{2r+1}}\right)=\tan^{-1}\left(\frac{2}{3}\right)^r-\tan^{-1}\left(\frac{2}{3}\right)^{r+1}$$

$$\text{then } S_k=\sum_{r=1}^{k}T_k=\tan^{-1}\left(\frac{2}{3}\right)-\tan^{-1}\left(\frac{2}{3}\right)^{k+1}$$

$$\lim_{k\to\infty}S_k=\tan^{-1}\left(\frac{2}{3}\right).$$

37. Answer (2)

$$\because \tan^{-1}\frac{1}{2r^2}=\tan^{-1}\left(\frac{2}{1+(4r^2-1)}\right)$$

$$=\tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right)$$

$$=\tan^{-1}(2r+1)-\tan^{-1}(2r-1)$$

$$\therefore \sum_{r=1}^{50}\tan^{-1}\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{50}\left(\tan^{-1}(2r+1)-\tan^{-1}(2r-1)\right)$$

$$\therefore p=\tan^{-1}(101)-\tan^{-1}1$$

$$=\tan^{-1}\left(\frac{101-1}{1+101\cdot 1}\right)$$

$$\therefore \tan p=\frac{50}{51}$$

38. Answer (1)

$$\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right)=\frac{\pi}{4} \quad \Rightarrow \quad \frac{5x}{1-6x^2}=1$$

i.e. $6x^2+5x-1=0$

$(6x-1)(x+1)=0$

$\Rightarrow x=\frac{1}{6}$ (as $x\geq 0$)

Hence A is a singleton set

39. Answer (1)

Let $\tan^{-1}x=t\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

$$\cot^{-1}x=\frac{\pi}{2}-t$$

$$f(t)=t^3+\left(\frac{\pi}{2}-t\right)^3 \Rightarrow f'(t)=3t^2-3\left(\frac{\pi}{2}-t\right)^2$$

$f'(t)=0$ at $t=\frac{\pi}{4}$

$$f(t)|_{\min}=\frac{\pi^3}{64}+\frac{\pi^3}{64}=\frac{\pi^3}{32}$$

Max will occur around $t=-\frac{\pi}{2}$

Range of $f(t)=\left[\frac{\pi^3}{32},\frac{7\pi^3}{8}\right)$

$$k\in\left[\frac{1}{32},\frac{7}{8}\right)$$

40. Answer (4)

$$-1\leq\frac{x^2-5x+6}{x^2-9}\leq 1 \text{ and } x^2-3x+2>0, \neq 1$$

$$\frac{(x-3)(2x+1)}{x^2-9}\geq 0 \;\Bigg|\; \frac{5(x-3)}{x^2-9}\geq 0$$

Solution to this inequality is

$$x\in\left[\frac{-1}{2},\infty\right)-\{3\}$$

For $x^2-3x+2>0$ and $\neq 1$

$$x\in(-\infty,1)\cup(2,\infty)-\left\{\frac{3-\sqrt{5}}{2},\frac{3+\sqrt{5}}{2}\right\}$$

Combining the two solution sets (taking intersection)

$$x\in\left[-\frac{1}{2},1\right)\cup(2,\infty)-\left\{\frac{3-\sqrt{5}}{2},\frac{3+\sqrt{5}}{2}\right\}$$

41. Answer (2)

Given $x * y=x^2+y^3$ and $(x*1)*1=x*(1*1)$

So, $(x^2+1)*1=x*2$

$\Rightarrow (x^2+1)^2+1=x^2+8$

$\Rightarrow x^4+2x^2+2=x^2+8$

$\Rightarrow (x^2)^2+x^2-6=0$

$\therefore (x^2+3)(x^2-2)=0$

$\therefore \boxed{x^2=2}$

$$\text{Now, } 2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)=2\sin^{-1}\left(\frac{4}{8}\right)$$

$$=2\cdot\frac{\pi}{6}=\frac{\pi}{3}$$

42. Answer (2)

$$\tan^{-1}\left(\frac{\cos\left(\frac{15\pi}{4}\right)-1}{\sin\frac{\pi}{4}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}-1}{\frac{1}{\sqrt{2}}}\right)$$

$$=\tan^{-1}\left(1-\sqrt{2}\right)=-\tan^{-1}\left(\sqrt{2}-1\right)$$

$$=-\frac{\pi}{8}$$

43. Answer (2)

$$f(x)=2\cos^{-1}x+4\cot^{-1}x-3x^2-2x+10\ \forall x\in[-1,1]$$

$$\Rightarrow\quad f'(x)=-\frac{2}{\sqrt{1-x^2}}-\frac{4}{1+x^2}-6x-2<0\ \forall x\in[-1,1]$$

So, $f(x)$ is decreasing function and range of $f(x)$ is $\left[f(1), f(-1)\right]$, which is $\left[\pi+5, 5\pi+9\right]$

Now $4a - b = 4(\pi + 5) - (5\pi + 9)$

$= 11 - \pi$

44. Answer (3)

$$\cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)+\frac{2}{5}\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right)$$

$$=\cos^{-1}\left(\frac{3}{10}\cdot\frac{3}{5}+\frac{2}{5}\cdot\frac{4}{5}\right)$$

$$=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

45. Answer (1)

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\tan^{-1}(-1)$$

$$=\frac{\pi}{3}+\frac{5\pi}{6}-\frac{\pi}{4}$$

$$=\frac{4\pi+10\pi-3\pi}{12}=\frac{11\pi}{12}$$

46. Answer (1)

$$\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right)$$

$$=\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)\right)$$

$$=\cot\left(\sum_{n=1}^{50}\left(\tan^{-1}(n+1)-\tan^{-1}n\right)\right)$$

$$=\cot\left(\tan^{-1}51-\tan^{-1}1\right)$$

$$=\cot\left(\tan^{-1}\left(\frac{51-1}{1+51}\right)\right)$$

$$=\cot\left(\cot^{-1}\left(\frac{52}{50}\right)\right)$$

$$=\frac{26}{25}$$

47. Answer (3)

$$\lim_{n\to\infty}6\tan\left\{\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{r^2+3r+3}\right)\right\}$$

$$=\lim_{n\to\infty}6\tan\left\{\sum_{r=1}^{n}\tan^{-1}\left(\frac{(r+2)-(r+1)}{1+(r+2)(r+1)}\right)\right\}$$

$$=\lim_{n\to\infty}6\tan\left\{\sum_{r=1}^{n}(\tan^{-1}(r+2)-\tan^{-1}(r+1))\right\}$$

$$=\lim_{n\to\infty}6\tan\left\{\tan^{-1}(n+2)-\tan^{-1}2\right\}$$

$$=6\tan\left\{\frac{\pi}{2}-\cot^{-1}\left(\frac{1}{2}\right)\right\}$$

$$=6\tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$=3$$

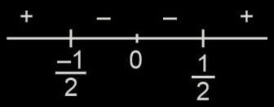
48. Answer (4)

$$-1\le \frac{2}{\pi}\sin^{-1}\left(\frac{1}{4x^2-1}\right)\le 1$$

$$-\frac{\pi}{2}\le \sin^{-1}\frac{1}{4x^2-1}\le\frac{\pi}{2}$$

$$-1\le\frac{1}{4x^2-1}\le 1$$

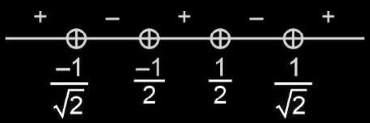
$$\frac{1}{4x^2-1}\ge -1 \Rightarrow \frac{4x^2}{(2n+1)(2x-1)}\ge 0$$



$$x\in\left(-\infty,\frac{-1}{2}\right)\cup(0)\cup\left(\frac{1}{2},\infty\right) \qquad \ldots(1)$$

$$\frac{1}{4x^2-1}\le 1 \Rightarrow 1-\frac{1}{4x^2-1}\ge 0\Rightarrow\frac{4x^2-2}{4x^2-1}\ge 0$$

$$\Rightarrow \frac{\left(x+\frac{1}{\sqrt2}\right)\left(x-\frac{1}{\sqrt2}\right)}{\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}\ge 0$$



$$x\in\left(-\infty,\frac{-1}{\sqrt2}\right]\cup\left(\frac{-1}{2},\frac{1}{2}\right)\cup\left(\frac{1}{\sqrt2},\infty\right) \qquad \ldots(2)$$

From (1) & (2),

$$x\in\left(-\infty,\frac{-1}{\sqrt2}\right]\cup\left[\frac{1}{\sqrt2},\infty\right)\cup\{0\}$$

49. Answer (29)

$$50\tan\left(\tan^{-1}\frac{1}{2}+2\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}(2)\right)$$
$$+4\sqrt2\tan\left(\frac{\tan^{-1}}{2}(2\sqrt2)\right)$$

$$\Rightarrow 50\tan\left(\pi+\tan^{-1}\left(\frac{1}{2}\right)\right)+4\sqrt2\tan\left(\frac{1}{2}\tan^{-1}2\sqrt2\right)$$

$$\Rightarrow 50\left(\frac{1}{2}\right)+4\sqrt2\tan\alpha\text{, where } 2\alpha=\tan^{-1}2\sqrt2$$

$$\Rightarrow \frac{2\tan\alpha}{1-\tan^2\alpha}=2\sqrt2 \qquad \ldots(i)$$

$$\Rightarrow 2\sqrt2\tan^2\alpha+2\tan\alpha-2\sqrt2=0$$

$$\Rightarrow 2\sqrt2\tan^2\alpha+4\tan\alpha-2\tan\alpha-2\sqrt2=0$$

$$\Rightarrow \left(2\sqrt2\tan\alpha-2\right)\left(\tan\alpha-\sqrt2\right)=0$$

$$\Rightarrow \tan\alpha=\sqrt2 \text{ or } \frac{1}{\sqrt2}$$

$$\Rightarrow \tan\alpha=\frac{1}{\sqrt2}$$

$\left(\tan\alpha=\sqrt2 \text{ doesn't satisfy (i)}\right)$

$$\Rightarrow 25+4\sqrt2\frac{1}{\sqrt2}=29$$

50. Answer (2)

$$\tan\left(2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{\sqrt5}{2}+2\tan^{-1}\frac{1}{8}\right)$$

$$=\tan\left(2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\cdot\frac{1}{8}}\right)+\sec^{-1}\frac{\sqrt5}{2}\right)$$

$$=\tan\left[2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}\right]$$

$$=\tan\left[\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{2}\right]$$

$$=\tan\left[\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{2}\right]$$

$$=\tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{8}}\right]=\tan\left[\tan^{-1}\frac{\frac{5}{4}}{\frac{5}{8}}\right]$$

$$=\tan\left[\tan^{-1}2\right]=2$$

51. Answer (12)

$$\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}\right)\right)\right)\right)\right)=k$$

$$\Rightarrow \cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\sqrt{1-x^2}\right)\right)\right)=k$$

$$\Rightarrow \cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)=k$$

$$\Rightarrow \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}}=k$$

$$\Rightarrow \frac{1-2x^2}{1-x^2}=k^2$$

$\Rightarrow$ $1-2x^2=k^2-k^2x^2$

$$\therefore\quad x^2-\left(\frac{k^2-1}{k^2-2}\right)=0 \begin{matrix}\nearrow\alpha\\ \searrow\beta\end{matrix}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=2\left(\frac{k^2-2}{k^2-1}\right)\qquad ...(1)$$

and $\frac{\alpha}{\beta}=-1$ ...(2)

$$\therefore\quad 2\left(\frac{k^2-2}{k^2-1}\right)(-1)=-5$$

$$\Rightarrow k^2=\frac{1}{3}$$

and b = S.R = $2\left(\frac{k^2-2}{k^2-1}\right)-1=4$

$$\therefore\quad \frac{b}{k^2}=\frac{4}{\frac{1}{3}}=12$$

52. Answer (2)

$$-1\le\frac{x^2-4x+2}{x^2+3}\le 1$$

$$\Rightarrow -x^2-3\le x^2-4x+2\le x^2+3$$

$$\Rightarrow 2x^2-4x+5\ge 0 \quad \& \quad -4x\le 1$$

$$x\in R \quad \& \quad x\ge-\frac{1}{4}$$

So domain is $\left[-\frac{1}{4},\ \infty\right)$.

53. Answer (1)

$\cos^{-1}x-2\sin^{-1}x=\cos^{-1}2x$

For Domain : $x\in\left[\frac{-1}{2},\ \frac{1}{2}\right]$

$$\cos^{-1}x-2\left(\frac{\pi}{2}-\cos^{-1}x\right)=\cos^{-1}(2x)$$

$\Rightarrow$ $\cos^{-1}x+2\cos^{-1}x=\pi+\cos^{-1}2x$

$\Rightarrow$ $\cos(3\cos^{-1}x)=-\cos(\cos^{-1}2x)$

$\Rightarrow$ $4x^3=x$

$$\Rightarrow x=0,\pm\frac{1}{2}$$

54. Answer (3)

$f(x)=\tan^{-1}(\sin x-\cos x),\quad [0,\ \pi]$

Let $g(x)=\sin x-\cos x$

$$=\sqrt{2}\sin\left(x-\frac{\pi}{4}\right) \text{ and } x-\frac{\pi}{4}\in\left[\frac{-\pi}{4},\ \frac{3\pi}{4}\right]$$

$$\therefore\quad g(x)\in\left[-1,\ \sqrt{2}\right]$$

and $\tan^{-1}x$ is an increasing function

$$\therefore\quad f(x)\in\left[\tan^{-1}(-1),\tan^{-1}\sqrt{2}\right]$$

$$\in\left[-\frac{\pi}{4},\tan^{-1}\sqrt{2}\right]$$

$$\therefore\quad \text{Sum of } f_{\max} \text{ and } f_{\min}=\tan^{-1}\sqrt{2}-\frac{\pi}{4}$$

$$=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)-\frac{\pi}{4}$$

55. Answer (3)

$$f(x)=\sin^{-1}\left(\frac{x^2-3x+2}{x^2+2x+7}\right)$$

$$-1\le\frac{x^2-3x+2}{x^2+2x+7}\le 1$$

$$\frac{x^2-3x+2}{x^2+2x+7}\le 1$$

$$x^2-3x+2\le x^2+2x+7$$

$$5x\ge -5$$

$$x\ge -1 \qquad \ldots(\text{i})$$

$$\frac{x^2-3x+2}{x^2+2x+7}\ge -1$$

$$x^2-3x+2\ge -x^2-2x-7$$

$$2x^2-x+9\ge 0$$

$$x\in R \qquad \ldots(\text{ii})$$

(i) $\cap$ (ii),

Domain $\in[-1,\infty)$

56. Answer (2)

Let $\dfrac{\sin^{-1}x}{\alpha}=\dfrac{\cos^{-1}x}{\beta}=k\Rightarrow \sin^{-1}x+\cos^{-1}x=k(\alpha+\beta)$

$$\Rightarrow\quad \alpha+\beta=\frac{\pi}{2k}.$$

Now, $\dfrac{2\pi\,\alpha}{\alpha+\beta}=\dfrac{2\pi\,\alpha}{\frac{\pi}{2k}}=4k\alpha=4\sin^{-1}x$

Here $\sin\left(\dfrac{2\pi\,\alpha}{\alpha+\beta}\right)=\sin\,(4\sin^{-1}x)$

Let $\sin^{-1}x=\theta\quad \because\ x\in\left(0,\dfrac{1}{\sqrt{2}}\right)\Rightarrow\theta\in\left(0,\dfrac{\pi}{4}\right)$

$\Rightarrow\quad x=\sin\theta$

$\Rightarrow\quad \cos\theta=\sqrt{1-x^2}$

$\Rightarrow\quad \sin 2\theta=2x\cdot\sqrt{1-x^2}$

$\Rightarrow\quad \cos 2\theta=\sqrt{1-4x^2(1-x^2)}=\sqrt{(2x^2-1)^2}=1-2x^2$

$$\left(\because\ \cos 2\theta>0 \text{ as } 2\theta\in\left(0,\frac{\pi}{2}\right)\right)$$

$\Rightarrow\quad \sin 4\theta = 2\cdot 2x\sqrt{1-x^2}\,(1-2x^2)$

$$=4x\sqrt{1-x^2}\left(1-2x^2\right)$$

57. Answer (3)

$-1\le 2x^2-3<2 \qquad \log_{\frac{1}{2}}(x^2-5x+5)>0$

or $2\le 2x^2<5 \qquad 0<x^2-5x+5<1$

or $1\le x^2<\dfrac{5}{2} \qquad x^2-5x+5>0\ \&\ x^2-5x+4<0$

$x\in\left(-\sqrt{\dfrac{5}{2}},-1\right] \qquad x\in\left(-\infty,\dfrac{5-\sqrt{5}}{2}\right)\cup\left(\dfrac{5+\sqrt{5}}{2},\infty\right)$

$\cup\left[1,\sqrt{\dfrac{5}{2}}\right) \qquad \&\ x\in(-\infty,1)\cup(4,\infty)$

Taking intersection

$$x\in\left(1,\ \frac{5-\sqrt{5}}{2}\right)$$

58. Answer (130)

$x=\sin\left(2\tan^{-1}\alpha\right)=\dfrac{2\alpha}{1+\alpha^2}\ \ldots(\text{i})$

and $y=\sin\left(\dfrac{1}{2}\tan^{-1}\dfrac{4}{3}\right)=\sin\left(\sin^{-1}\dfrac{1}{\sqrt{5}}\right)=\dfrac{1}{\sqrt{5}}$

Now, $y^2=1-x$

$$\frac{1}{5}=1-\frac{2\alpha}{1+\alpha^2}$$

$\Rightarrow\ 1+\alpha^2=5+5\alpha^2-10\alpha$

$\Rightarrow\ 2\alpha^2-5\alpha+2=0$

$\therefore\ \alpha=2,\dfrac{1}{2}$

$\therefore\ \displaystyle\sum_{\alpha\in S}16\alpha^3=16\times 2^3+16\times\frac{1}{2^3}$

$$=130$$

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