# Dynamics for 2R manipulator

Potential Energy

$$\boxed{V = \frac{1}{2} m_1 l_1 g \sin q_1 + \frac{1}{2} m_2 l_2 g \sin q_2 + m_2 l_1 g \sin q_1}$$

Kinetic Energy. $\quad -(1)$

$$K = \frac{1}{2} \frac{m_1 l_1^2}{3} \dot{q}_1^2 + \frac{1}{2} m_2 v_{c_2}^2 + \frac{1}{2} \frac{m_2}{12} l_2^2 \dot{q}_2^2$$

$$v_{c_2} = \dot{x}_{c_2}^2 + \dot{y}_{c_2}^2$$

$$\dot{x}_{c_2} = -l_1 \sin q_1 \dot{q}_1 - \frac{l_2}{2} \sin q_2 \dot{q}_2$$

$$\dot{y}_{c_2} = l_1 \cos q_1 \dot{q}_1 + \frac{l_2}{2} \cos q_2 \dot{q}_2$$

substituting $\dot{x}_{c_2}$ & $\dot{y}_{c_2}$ in K

$$K = \frac{1}{2} m_1 \frac{l_1^2}{3} \dot{q}_1^2 + \frac{1}{2} m_2 \Big[ \left(-l_1 \sin q_1 \dot{q}_1 - \frac{l_2}{2} \sin q_2 \dot{q}_2\right)^2 + \left(l_1 \cos q_1 \dot{q}_1 + \frac{l_2}{2} \cos q_2 \dot{q}_2\right)^2 \Big] + \frac{1}{2} \frac{m_2 l_2^2}{12} \dot{q}_2^2$$

$$K = \frac{1}{2} \frac{m_1 l_1^2}{3} \dot{q}_1^2 + \frac{1}{2} m_2 \left( l_1^2 \sin^2 q_1 \dot{q}_1^2 + \frac{l_2^2}{4} \sin^2 q_2 \dot{q}_2^2 + 2 l_1 \frac{l_2}{2} \sin q_1 \sin q_2 \dot{q}_1 \dot{q}_2 \right) + \frac{1}{2} m_2 \Big( l_1^2 \cos^2 q_1 \dot{q}_1^2 + \frac{l_2^2}{4} \cos^2 q_2 \dot{q}_2^2 + 2 l_1 \frac{l_2}{2} \cos q_1 \cos q_2 \dot{q}_1 \dot{q}_2 \Big) + \frac{1}{2} m_2 \frac{l_2^2}{12} \dot{q}_2^2$$

using the property $\cos^2\theta + \sin^2\theta = 1$

$$\boxed{\begin{aligned} K = {} & \frac{1}{2} m_1 \frac{l_1^2}{3} \dot q_1^2 + \frac{1}{2} m_2 l_1^2 \dot q_1^2 + \frac{1}{2} m_2 \frac{l_2^2}{4} \dot q_2^2 \\ & + \frac{1}{2} m_2\, 2\left(l_1 \frac{l_2}{2}\right) \cos(q_1 - q_2)\, \dot q_1 \dot q_2 + \frac{1}{2} m_2 \frac{l_2^2}{12} \dot q_2^2 \end{aligned}}$$

— (ii)

## Equation of Motion using Lagragian Approach.

Lagragian $L = K - V$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = Q} \qquad Q = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot q}(K - V) - \frac{\partial}{\partial q}(K - V) = Q$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q} + \cancelto{0}{\frac{\partial V}{\partial \dot q}}\right) - \frac{\partial}{\partial q} K + \frac{\partial V}{\partial q} = Q$$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial K}{\partial \dot q}\right) - \frac{\partial K}{\partial q} + \frac{\partial V}{\partial q} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}} \quad \text{— (iii)}$$

$$\frac{\partial K}{\partial \dot q} = \left[ \begin{aligned} & \frac{1}{2} m_1 \frac{l_1^2}{3} 2\dot q_1 + \frac{1}{2} m_2 l_1^2 2\dot q_1 + \\ & \quad \frac{1}{2} m_2 (l_1 l_2) \cos(q_1 - q_2)\, \dot q_2 \end{aligned} \right.$$

$$
\begin{bmatrix}
\frac{1}{2} m_2 \frac{l_2^2}{4} 2\dot{q}_2 + \frac{1}{2} m_2 l_1 l_2 \cos(q_1 - q_2)\dot{q}_1 \\
\frac{1}{2} m_2 \frac{l_2^2}{12} 2\dot{q}_2
\end{bmatrix}
$$

$$
\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} =
\begin{bmatrix}
\frac{1}{2} m_1 \frac{l_1^2}{3} 2\ddot{q}_1 + \frac{1}{2} m_2 l_1^2 2\ddot{q}_1 + \\
+ \frac{1}{2} m_2 l_1 l_2 \cos(q_1 - q_2)\ddot{q}_2 \\
- \frac{1}{2} m_2 l_1 l_2 \sin(q_1 - q_2)\dot{q}_2 \dot{q}_1 \\
+ \frac{1}{2} m_2 l_1 l_2 \sin(q_1 - q_2)\dot{q}_2^2 \\
\\
\frac{1}{2} m_2 \frac{l_2^2}{4} 2\ddot{q}_2 + \frac{1}{2} m_2 l_1 l_2 (\cos q_1 - q_2)\ddot{q}_1 \\
+ \frac{1}{2} m_2 \frac{l_2^2}{12} 2\ddot{q}_2 - \frac{1}{2} m_2 l_1 l_2 \sin(q_1 - q_2)\dot{q}_1^2 \\
+ \frac{1}{2} m_2 l_1 l_2 \sin(q_1 - q_2)\dot{q}_1 \dot{q}_2
\end{bmatrix}
\quad \text{—(iv)}
$$

$$
\frac{\partial V}{\partial q} =
\begin{bmatrix}
m_1 \frac{l_1}{2} g \cos q_1 + m_2 l_1 g \cos q_1 \\
m_2 \frac{l_2}{2} g \cos q_2
\end{bmatrix}
\quad \text{—(V)}
$$

$$
-\frac{\partial K}{\partial q} =
\begin{bmatrix}
+\frac{1}{2} m_2 l_1 l_2 \sin(q_1 - q_2)\dot{q}_1 \dot{q}_2 \\
-\frac{1}{2} m_2 l_1 l_2 \sin(q_1 - q_2)\dot{q}_1 \dot{q}_2
\end{bmatrix}
\quad \text{—(vi)}
$$

$\sim\sim$ and $----$ cancel out

substituting (iv), (v) and (vi) in (iii),

we get,

$$\frac{1}{3}m_1 l_1^2\ddot{q}_1 + m_2 l_1^2\ddot{q}_1 + \frac{1}{2}m_2 l_1 l_2\cos(q_1-q_2)\ddot{q}_2$$
$$+\frac{1}{2}m_2 l_1 l_2\sin(q_1-q_2)\dot{q}_2^2 + \frac{1}{2}m_1 l_1 g\cos q_1$$
$$+ m_2 l_1 g\cos q_1 = \tau_1$$

$$\frac{1}{3}m_2 l_2^2\ddot{q}_2 + \frac{1}{2}m_2 l_1 l_2\cos(q_1-q_2)\ddot{q}_1 -$$
$$\frac{1}{2}m_2 l_1 l_2\sin(q_1-q_2)\dot{q}_1^2 + m_2\frac{l_2}{2}g\cos q_2 = \tau_2$$