1. Given X be a discrete random variable with the following PMF

- Find the range RX of the random variable X.

The range \( R\_X \) of \( X \) is the set of all possible values \( X \) can take.

- Find \( P(X \leq 0.5) \)

For a discrete random variable, \( P(X \leq 0.5) \) is the sum of probabilities for all values \( X \) that are less than or equal to 0.5.

- Find \( P(0.25 < X < 0.75) \)

For discrete \( X \), \( P(0.25 < X < 0.75) \) is the sum of probabilities for \( X \) values between 0.25 and 0.75.

- \( P(X = 0.2 | X < 0.6) \)

Conditional probability is \( P(X = 0.2 | X < 0.6) = \frac{P(X = 0.2)}{P(X < 0.6)} \).

2. Two equal and fair dice are rolled, and we observed two numbers X and Y.

- Find \( R\_X \), \( R\_Y \), and the PMFs of X and Y.

\( R\_X \) and \( R\_Y \) are both \( \{1, 2, 3, 4, 5, 6\} \). PMFs for each \( X \) and \( Y \) are uniform: \( P(X = k) = \frac{1}{6} \) for \( k = 1,2,3,4,5,6 \).

- Find \( P(X = 2, Y = 6) \)

Since dice are fair, \( P(X = 2, Y = 6) = P(X = 2) \cdot P(Y = 6) = \frac{1}{36} \).

- Find \( P(X > 3 | Y = 2) \)

\( P(X > 3 | Y = 2) = \frac{P(X > 3, Y = 2)}{P(Y = 2)} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2} \).

- If \( Z = X + Y \). Find the range and PMF of Z.

Range \( R\_Z \) is \( \{2, 3, ..., 12\} \). PMF of \( Z \) is computed by summing probabilities of all pairs \( (X, Y) \) that give the same sum.

- Find \( P(X = 4 | Z = 8) \)

Given \( Z = 8 \), find pairs \((X, Y)\) such that \( X = 4 \) and \( Y = 4 \). Probability is \( \frac{1}{5} \).

3. In an exam, there were 20 multiple-choice questions. Each question had 4 possible options.

- Find the PMF of \( X \). What is \( P(X > 15) \)?

\( X \) follows a binomial distribution \( X \sim \text{Binomial}(10, \frac{1}{4}) \). PMF is \( P(X = k) = \binom{10}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{10-k} \). \( P(X > 15) \) is 0 (since the max score is 10).

4. The number of students arriving at a college is a Poisson random variable.

- What is \( P(10 < Y \leq 15) \)?

\( Y \) follows \( \text{Poisson}(15) \). Calculate \( P(10 < Y \leq 15) = \sum\_{k=11}^{15} \frac{e^{-15} \cdot 15^k}{k!} \).

5. Two independent random variables, \( X \) and \( Y \), are given such that \( X \sim \text{Poisson}(\alpha) \) and \( Y \sim \text{Poisson}(\beta) \).

- State a new random variable \( Z = X + Y \). Find out the PMF of Z.

\( Z \sim \text{Poisson}(\alpha + \beta) \). PMF is \( P(Z = k) = \frac{(\alpha + \beta)^k e^{-(\alpha + \beta)}}{k!} \).

6. There is a discrete random variable X with the pmf.

- If we define a new random variable \( Y = (X + 1)^2 \) then

- Find the range of \( Y \).

The range of \( Y \) is derived from squaring \( (X + 1) \).

- Find the pmf of \( Y \).

Convert the pmf of \( X \) to find \( P(Y = y) \) based on \( Y = (X + 1)^2 \).

7. Assuming X is a continuous random variable with PDF

- Find \( E[X] \) and \( \text{Var}(X) \).

Compute expectation and variance using:

\[ E[X] = \int x f\_X(x) \, dx \]

\[ \text{Var}(X) = \int (x - E[X])^2 f\_X(x) \, dx \]

- Find \( P(X \geq a) \).

Compute using the cumulative distribution function \( F\_X(x) \):

\[ P(X \geq a) = 1 - F\_X(a) \]

- If X is a continuous random variable with pdf

- Find \( f\_Y(y) \) if \( Y = \sin(X) \).

Transform the PDF using the Jacobian method.

8. If X is a random variable with CDF

- What kind of random variable is X: discrete, continuous, or mixed?

The type depends on the form of the CDF \( F\_X(x) \).

- Find the PDF of X, \( f\_X(x) \).

Derive from the CDF:

\[ f\_X(x) = \frac{d}{dx} F\_X(x) \]

- Find \( E[e^X] \).

Compute using the moment generating function:

\[ E[e^X] = M\_X(1) \]

- Find \( P(X = 0 | X \leq 0.5) \).

Conditional probability is:

\[ P(X = 0 | X \leq 0.5) = \frac{P(X = 0)}{P(X \leq 0.5)} \]

9. There are two random variables X and Y with joint PMF given in Table below

- Find \( P(X \leq 2, Y \leq 4) \).

Sum probabilities for all \( (X, Y) \) pairs satisfying \( X \leq 2 \) and \( Y \leq 4 \).

- Find the marginal PMFs of X and Y.

Marginal PMFs are obtained by summing joint PMF over the other variable:

\[ P\_X(x) = \sum\_{y} P(X = x, Y = y) \]

\[ P\_Y(y) = \sum\_{x} P(X = x, Y = y) \]

- Find \( P(Y = 2 | X = 1) \).

Conditional probability is:

\[ P(Y = 2 | X = 1) = \frac{P(X = 1, Y = 2)}{P(X = 1)} \]

- Are X and Y independent?

X and Y are independent if \( P(X = x, Y = y) = P(X = x) P(Y = y) \) for all \( x \) and \( y \).

10. A box containing 40 white shirts and 60 black shirts.

- If we choose 10 shirts (without replacement) at random, find the joint PMF of X and Y, where X is the number of white shirts and Y is the number of black shirts.

Use hypergeometric distribution to find:

\[ P(X = x, Y = y) = \frac{\binom{40}{x} \binom{60}{y}}{\binom{100}{10}} \]

11. If A and B are two jointly continuous random variables with joint PDF

- Find \( f\_X(a) \) and \( f\_Y(b) \).

Marginal PDFs are:

\[ f\_X(a) = \int f\_{A,B}(a, b) \, db \]

\[ f\_Y(b) = \int f\_{A,B}(a, b) \, da \]

- Are A and B independent of each other?

A and B are independent